THE AMERICAN MATHEMATICAL MONTHLY

THE OFFICIAL JOURNAL OF
THE MATHEMATICAL ASSOCIATION OF AMERICA
(INCORPORATED)

DEVOTED TO THE INTERESTS OF COLLEGIATE MATHEMATICS

THE AMERICAN MATHEMATICAL MONTHLY, FOUNDED IN 1894 BY BENJAMIN F. FINKEL, WAS PUBLISHED BY HIM UNTIL 1913. FROM 1913 TO 1916 IT WAS OWNED AND PUBLISHED BY REPRESENTATIVES OF FOURTEEN UNIVERSITIES AND COLLEGES IN THE MIDDLE WEST

VOLUME 58
1951

PUBLISHED BY THE ASSOCIATION
MENASHA, WIS., AND ALBANY, N.Y.

THE AMERICAN

MATHEMATICAL MONTHLY

THE OFFICIAL JOURNAL OF

THE MATHEMATICAL ASSOCIATION OF AMERICA, INC.

VOLUME 58 NUMBER 1

CONTENTS

JANUARY 1951

The AMERICAN MATHEMATICAL MONTHLY

(Founded in 1894 by Benjamin F. Finkel)

EDITORIAL CORRESPONDENCE should be addressed to the Editor, C. V. Newsom, State Education Building, Albany 1, N. Y.

ADVERTISING CORRESPONDENCE should be addressed to J. F. Randolph, Morey Hall, University of Rochester, Rochester 3, N. Y.

NOTICE OF CHANGE OF ADDRESS by members of the Association as well as correspondence regarding subscriptions to the Monthly should be sent to the Secretary-Treasurer, H. M. Gehman, University of Buffalo, Buffalo 14, N. Y. Change of address must reach the Secretary-Treasurer about six weeks before the change can become effective.

THIS IS THE OFFICIAL JOURNAL OF THE

MATHEMATICAL ASSOCIATION OF AMERICA, INC.

(Devoted to the Interests of Collegiate Mathematics)

Entered as second class matter at the post office at Menasha, Wis. Acceptance for mailing at special rate of postage provided for in the Act of February 28, 1925, embodied in Paragraph 4, Section 538, P. L. and R., authorized April 1, 1926.

Subscription Price: To Members, $4 a Year, To Others, $5 a Year.

Published by the Association at Menasha, Wisconsin, and Albany, N. Y. during the months of January, February, March, April, May, June-July, August-September, October, November, December.

PROFESSIONAL OPPORTUNITIES IN MATHEMATICS

*A Report for Undergraduate Students of Mathematics**

CONTENTS

---

* Prepared by a committee of the Association, composed of H. W. Brinkmann, Z. I. Mosesson, S. A. Schelkunoff, S. S. Wilks, and Mina Rees, Chairman. Reprints of this article may be obtained from Professor H. M. Gehman, Mathematical Association of America, University of Buffalo, Buffalo 14, N. Y.; the cost is 25¢ for single copies and 10¢ each for orders of ten or more.

INTRODUCTION

The young student approaching the end of his preparatory or high school work, as well as the student beginning his work at college, is confronted with the difficult task of choosing a college course which will give him the best chance of a happy college experience combined with adequate preparation for a career which is at once satisfying and rewarding. Young men and women who have been particularly successful in high school and early college mathematics, who have found problem-solving fun, will often ask what kinds of jobs there are for college graduates with mathematical training, and what kinds of courses they should take to qualify for these jobs.

This report is designed to aid the young student who is considering mathematics as a career; to give him some idea of the types of work mathematicians or persons with considerable mathematical training do in universities, in business, and in government; and to suggest to him the types of college preparation he will need for these various positions.

For a student thinking of choosing mathematics as a career it is important to point out that early undergraduate mathematics with its emphasis on solving problems is quite different from the pure mathematics studied in a university and pursued by research workers who are professors at universities. This latter mathematics is largely concerned with logic and abstract ideas; and the solution of problems plays very little part. People of real stature in this kind of mathematical research find their place as professors at our great universities; for it is in teaching particularly that advancement to the highest levels requires distinguished performance in research. For people of less creative talent there are, however, many opportunities for interesting and rewarding work as secondary school and college teachers. In fact the greatest number of openings for college graduates will be found here. But the professional opportunities for students with training in mathematics are very broad, and an attempt will be made in this report to discuss the major areas in which young mathematicians may find employment. The space which is given to the various types of work is in no sense an indication of their importance. It does reflect the amount of information which may be useful and which is not usually available to college undergraduates and their advisers.

For a college graduate with only a bachelor's degree in mathematics it will be seen that there are substantial professional opportunities in industrial and government positions, as well as in secondary school teaching. Experience on the job, accompanied by some formal training, will frequently equip such a college graduate to qualify as an assistant to a senior worker in industrial statistics or in government mathematical research, while the actuarial profession provides a regular series of steps, described in Part V of this report, by which one can qualify as a full-fledged actuary. In industrial laboratories, however, only the relatively uninteresting computing jobs are usually open to a person

without the Ph.D.; and university professorships, as well as research jobs in industry and government, virtually always require advanced graduate training and the equivalent of post Ph.D. research experience.

In the separate parts of this report, stress will be laid on the importance of college training in fields other than mathematics, related to the particular work to be undertaken. In this general introduction it will be remarked only that the mathematician, to be effective in industry, government, or university, must be a well-rounded person; that while courses in analytic geometry, calculus, and differential equations constitute a common mathematical background of all scientists and engineers, a selection of courses from physics, chemistry, engineering, biology, psychology, statistics, and economics will usually be necessary to make the mathematician effective in applied work. In industry and in government the scientist and the engineer work in close cooperation; in actuarial work, and in many fields which offer jobs to young people with statistical training, a sound background in economics is of great value.

Thus the purpose of this report is two-fold: to suggest the types of positions available to young people with mathematical aptitude and training; and to urge such young people to broaden their horizons, and secure as sound and as deep a general education as the opportunities available to them will permit.

The report is divided into five parts. Part I is devoted to teaching in high school, college, or university. Part II explains the character of mathematical and applied statistics, and tells about career opportunities for young people who have suitable training at college to work in fields which rely heavily on statistical procedures. Part III discusses the role of the mathematician in an industrial laboratory; and Part IV tells about the work of mathematicians in government. In Part V a description is given of the work which is done by actuaries, and of the special preparation which many insurance companies make available to assist college graduates to qualify for the profession.

In each part, the character of the work is described, suggestions are made concerning the college courses which will assist the student to prepare for professional work in the field under discussion, and a rough estimate is given of the ranges of salaries which are now paid to those who are successfully employed in that kind of work.

The separate parts of this report were written by experts in the activities covered. No attempt has been made to secure uniformity at the expense of suppressing the valuable experience or judgment of the writer. The bibliography at the end of the report provides material which may assist the reader who wishes to secure further information; the number of items in the bibliography on a given subject reflects only the available additional reading which was known to the committee. Numbers in parentheses in the text refer to items in the bibliography.

PART I: THE TEACHER OF MATHEMATICS

Not so long ago it could be said truthfully that the main opportunity for a life work as a mathematician was as a teacher of the subject. Although this is no longer true to the degree it once was, it is still true that most students in college who are training themselves for a career in mathematics are planning to teach it, either in a secondary school or in a college or in a university. It will be convenient to consider the opportunities for such a career under the two corresponding headings.

1. Teaching mathematics in school. There is a great demand for properly qualified teachers of mathematics in our schools. The obvious need for the subject in the natural sciences, the increasing demand for it in the social sciences, as well as the realization that it forms an integral part of our cultural heritage make it evident that many properly trained teachers will be needed. It is clear that such a teacher will have to know much more than merely the subjects that he expects to teach in school. To be properly prepared the prospective teacher must therefore take such college work in mathematics as will enable him to understand the fundamental principles of mathematics; he should also take courses that will broaden his mathematical outlook as much as possible. Theory of equations should be included; and courses in projective geometry and non-euclidean geometry will enable the teacher to give the student a true appreciation of geometry (and indeed of mathematics in general); perhaps experience with this broader viewpoint toward geometry will even encourage the teacher to improve the traditional courses in geometry. A good course in the history of mathematics might also be desirable. The standard courses in calculus and differential equations are naturally essential, not only because they introduce the student to the fundamentals of analysis but also because they enable him to study and appreciate the application of mathematics to the natural and other sciences.

In this connection it is necessary to urge the prospective teacher to take as much work as he can in other sciences, as well as in engineering, economics and the social sciences. For it is essential that a teacher have an appreciation of the relation of mathematics to other fields and that he be a well-rounded individual who can take his place in the community as an educated person. It may also be of immediate practical value, for the teacher of mathematics may be asked to teach a subject other than mathematics itself—indeed, he may find that one of his duties is coaching, say, the football team. More frequently, perhaps, a secondary school teacher of mathematics is expected to teach physics or chemistry. In addition to the work that has just been described the student will have to take certain required technical courses in educational methods; what these are depends on the laws of the state in which he wishes to teach. In certain private schools such courses are not required but the beginning teacher may be given the status of an apprentice teacher for a year or so.

It should be pointed out that a successful teacher in any field must have a

real liking for his job, he must be genuinely fond of young people and he must be willing to teach his students as well as his subject. The teaching of mathematics is no exception to this general statement. Indeed it is probably true that a teacher of mathematics must be especially on the lookout to overcome the well-established misconceptions about his subject and its teachers. But the teacher who is well-equipped with an understanding of the subject will have the excitement of bringing to young students new adventures in mathematics that will kindle in some, at least, a real devotion to the subject.

Although teaching has, in general, been poorly paid, the situation has been gradually improving; and some states now set a minimum salary of $2400 (or even higher) for the beginning high school teacher who has an A.B. degree. Young graduates now entering teaching often find that salaries there are better than any others available. Moreover the teaching profession, through its rules of tenure and possible pension systems, offers a certain measure of security that is lacking in some other professions.

2. Teaching mathematics in a college or university. In order to become a teacher in a college or university it will be necessary for the student to do work in a graduate school and, as a general rule, he will be required to obtain the degree of Ph.D. Such a person should be well grounded in the various branches of mathematics and he should have a thorough understanding of its relations to other subjects as well. His mathematical work in college should be sufficiently diversified so that he will have at least a background in the disciplines of analysis, geometry and algebra. He will then be prepared to continue this work in graduate school and select some special field in which to do research for his doctorate.

The college teacher of mathematics has the responsibility of training the mathematician of the future, namely, the very person who will be interested in the opportunities that are described in this report. He will also teach students of the natural sciences and engineering who look upon mathematics as a tool subject, and he will teach students who study mathematics for its cultural value or perhaps just for the fun of it. Although a college teacher of mathematics is not likely to be called upon to teach subjects other than his own, he will be a better teacher of his subject if he has benefited from as broad and liberal an education as possible. In this connection prospective teachers of mathematics should keep in mind the present trend toward "general education" in the liberal arts colleges. It is clear that mathematics should play a vital role in this important development, and especially well prepared teachers will be needed to handle the courses involved in such a program.

In addition to doing his teaching, the college teacher of mathematics is generally expected to be a productive mathematician, that is, he is expected to combine research work in mathematics with his teaching. The relative importance that will be attached to these two aspects of his work will be different for different institutions and will vary from one individual to another. Thus a pro-

fessor of mathematics in a university will probably consider his research to be the most important part of his job; he will perhaps do very little teaching at all and will do most of that in graduate courses on his own specialty and in guiding graduate students in their research work. For a university has a two-fold obligation to the world. In addition to the important duty of disseminating accumulated knowledge, it has a still more important function, namely, the extension of knowledge beyond its present boundaries by means of the ever continuing search into the unknown. A field of knowledge in which there is not a continuous growth and expansion soon becomes sterile and dies. Every time a problem is answered, a host of new questions spring up to trouble the inquiring mind of the scientist. The university must offer opportunities for the continual search for the answers to these questions.

In a college, on the other hand, a professor of mathematics will often think of himself primarily as a teacher who will, to be sure, try to keep abreast of new mathematical developments and who may possibly also do some research of his own. It is sometimes hard for the beginning student of mathematics to realize that the subject is by no means a closed one and that a great many questions remain to be investigated. Many of these problems are of great difficulty, while others are of more routine character. In any case large quantities of new mathematical results are being published in the United States and abroad as the student can verify by taking a casual look at any one of the standard mathematical journals or at Mathematical Reviews. The attainment of the degree of Ph.D. involves the completion of a piece of original research and the prospective college teacher should consider carefully whether he is fitted for a career in which such research plays a more or less important part.

The financial remuneration of a college or university professor varies greatly with the institution. The beginning salary of an instructor with a doctor's degree might be about \$3000 while the maximum salary of a full professor may range from \$5000 to about \$15,000, although only very few will be found in the higher brackets. Although a college professor is not likely to become excessively rich, he will enjoy the pleasures and advantages of an academic life and he will have the security that is afforded by tenure rules and pension plans.

PART II: OPPORTUNITIES IN MATHEMATICAL AND APPLIED STATISTICS

1. Introductory comments. Mathematical statistics is a branch of mathematics which has received a great deal of attention in this country during the last 25 years. It has been developing in Europe for more than a century. It deals with the mathematical examination and study of various kinds of statistical problems which arise in scientific research, in social and economic investigations, in business and industry, and in government work. Statisticians who deal with those problems are of two main classes. One is the class of mathematical statis-

ticians who deal with the general mathematical theory of the combination of observations, testing hypotheses and estimating unknown quantities with accuracy specified in terms of probability, and the design of efficient experiments for obtaining such tests and estimates. Mathematical statisticians also work out methodological procedures for applying the theory. The other class is represented by the applied statistician who concerns himself with the application of statistical methods already known in one or more fields. He is often an expert in some field of application (a biologist, an economist, a sociologist, an industrial quality control engineer, etc.) who has enough statistical training to apply or adapt statistical methodology to his own field. Often the same individual will function both as a mathematical and an applied statistician; this is especially the case on the higher levels.

The best index of the recent growth of interest in mathematical and applied statistics is provided by the growth of membership in national societies concerned with mathematical statistics and its applications. The American Statistical Association, founded over 100 years ago, had 1700 members in 1935; now it has nearly 5000. The Institute of Mathematical Statistics, founded in 1935, now has over 1100 members. The American Society for Quality Control, an organization founded in 1946 for fostering the application of statistical methods in industry, now has nearly 3000 members. The Biometric Society, formed in 1947 for promoting the use of mathematical and statistical methods in the biological sciences, has about 900 members. Similar organizations for economics and psychology, namely, the Econometric Society and the Psychometric Society, were formed in the early 1930's and now have approximately 900 and 300 members, respectively. Other societies have varying degrees of interest in the application of statistical methods to their problems, such as the American Public Health Association, the American Marketing Association, the American Sociological Society, and the American Association for Public Opinion Research.

2. Opportunities for personnel trained in mathematical and applied statistics. Since the war there has been a considerable number of opportunities for the teaching of mathematical and applied statistics in the colleges and universities of the country. In an inquiry made in 1946 by the National Research Council Committee on Applied Mathematical Statistics among academic statisticians at 27 leading universities, 135 requests had been received since the end of the war for persons to fill academic positions in statistics. These requests were for men at the Ph.D. level with a major in mathematical statistics or with a major in such fields as agronomy, biology, economics, or psychology, and a minor in mathematical statistics. The positions to be filled ranged from instructorships to full professorships. Although no systematic inquiry has been made since 1946, the scattered evidence indicates that the demand for such persons has hardly diminished.

The training of mathematical statisticians involves a considerable amount of advanced mathematics. The requirements in mathematics include real and complex variables, linear and quadratic forms and matrix algebra, n-dimensional Euclidean geometry, measure and integration theory. These mathematical subjects are essential for a full understanding of probability theory, which is the foundation for advanced mathematical statistics. There are very few universities which give adequate graduate training in mathematical statistics at the present time. To prepare for these advanced courses, the student who hopes to become a mathematical statistician should be sure to include in his early undergraduate courses a strong group of mathematics courses. Moreover the student who is interested in applied statistics as a career would be well advised to include at least mathematics through the calculus.

The training of the applied statistician is based on various courses in applied statistics together with substantial training in the fields to which he expects to apply his statistics. Courses in applied statistics beyond the usual introductory courses in statistics include analysis of variance, design of experiments, quality control and engineering statistics, biometry, survey sampling and its applications, economic statistics, and psychometrics. Various combinations of these applied statistics courses are being given in a considerable number of colleges and universities at present. In a few cases they are given in statistics departments, but in most cases they are given in the departments whose students have the greatest interest in them.

The demand for statistically trained persons in business, industry, government and other non-academic fields has been no less than in the academic field. The non-academic fields which account for most of the recent growth of interest in statistical methods are (a) industrial statistical quality control, (b) research in the biological sciences, (c) collection and analyses of government statistics, (d) market research and commercial sample surveys, and (e) psychological testing. There are also other fields in which there is an increased need for statisticians, such as finance and taxes, labor and employment, prices, production, and national income analysis.

Of all these fields, industrial quality control has grown fastest. Statistical quality control methods were initiated about 25 years ago by Bell Telephone engineers for the purpose of maintaining uniform quality of the thousands of kinds of pieces of telephone equipment required by the telephone system of the country. These statistical methods were introduced widely and rapidly into many mass production industries during the war. At the present time they are being introduced into chemical and other industries. The training requirements for personnel to meet the needs of statistical quality control work consist of one or two courses in engineering statistics in addition to the usual chemical, electrical, and mechanical engineering curricula.

Interest in statistical methods in the biological sciences has also grown rapidly in recent years. Statistical analysis is being more and more widely applied in agricultural experimentation, biological assay, public health studies,

and medical research. The training required for this work is at the graduate level in one of the biological sciences, with a minor in statistical methods.

In the field of government there has been a considerable increase of statistical activity in such fields as sampling surveys in census work, economics, social security, and labor statistics. Sampling methods are becoming more and more widely used as an effective and economical way for obtaining information needed in government work. Experts in sampling methods require graduate training in mathematical statistics.

Statistical methods have long been used in business operations. One of the most important recent developments has been the application of sampling survey methods to business problems as a means of gathering many kinds of social and economic information for commercial purposes. Manufacturers and market research firms spend large sums of money every year to determine consumers' buying habits, brand preferences, opinions, etc., for use in making business decisions. Magazine and newspaper research organizations, radio and television broadcasters and advertising agencies make studies of reading and listening habits for use in guiding editorial and advertising policy. Sampling methods are playing a fundamental role in obtaining this kind of information. Statisticians with training in economics and the social sciences and in the applications of sampling theory are being sought for this kind of work.

Psychological testing is another rapidly growing field strongly dependent on statistical methods. This field is concerned with the development of tests for the selection of personnel for various purposes. Tests are becoming widely used not only in the selection of candidates for schools of all kinds, but also for the selection of personnel for many kinds of positions in the trades and professions. The modern theory of test construction is broadly based on statistical methodology. The training required is graduate-level training in psychology and statistics.

There are probably more opportunities in business and government for women with statistical training than there are in college and university teaching. For example, about 10 per cent of the members of the American Statistical Association are women. There are some opportunities for women with such training in industry. For example, approximately 4 per cent of the members of the American Quality Control Society are women.

3. Financial remuneration in mathematical and applied statistics. Salaries for personnel in mathematical and applied statistics at the B.A., M.A. and Ph.D. levels of training are very similar to those in other branches of mathematics and the physical sciences. Academic salaries for beginning instructors in mathematical statistics with Ph.D. degrees range between \$3000 and \$4500 per year. Salaries for these same persons in government and industry are somewhat higher. Maximum salaries for professorships in mathematical statistics in various universities range up to \$18,000 or \$20,000 per year depending on the institution, scientific achievement, etc. Maximum salaries in the government

are approximately $11,000 per year while the maximum for positions in business and industry may be somewhat higher. Further information about salaries in government will be found in Part IV.

Beginning salaries for students trained to the B.A. or M.A. level in mathematical or applied statistics range from $2400 to $3600 per year, and the positions are usually in government, business or industry. These persons, especially those trained to the B.A. level, often play the role of assistants to more highly trained persons.

PART III: THE MATHEMATICIAN IN AN INDUSTRIAL LABORATORY

1. Introductory comments. In industry the number of mathematicians is relatively small even though the use of mathematics is extensive. Engineers and physicists often use mathematics in the course of their experimental work. Most of this mathematics is elementary, including calculus and perhaps differential equations; some is decidedly advanced. Nevertheless these men are primarily engineers and physicists and not mathematicians. Those who major in mathematics are unlikely to have sufficient training in physics or engineering to qualify them for jobs in this category. There are also engineers and physicists who are doing little experimental work or none at all; and some of these are thought of by their experimental colleagues as "mathematicians." In fact, the dividing line between theoretical physicists and engineers on the one hand and industrial mathematicians on the other is not sharp. The former may be described as physicists or engineers strong in mathematics, and the latter as mathematicians strong either in physics or engineering or both.

2. The activities of an applied mathematician in industry. Industrial mathematicians act primarily as consultants. They are seldom responsible for engineering projects, for such a responsibility does not require the special talents and knowledge they possess and does require the varied experience which they usually do not possess. Whenever a mathematician becomes responsible for an engineering project, he ceases to function as a mathematician and becomes an engineer. As a consultant he works with engineers and must be able to understand their language since he is a specialist whose language is difficult in the opinion of his engineering colleagues. He can learn their language and point of view simply by reading elementary engineering textbooks which they study in schools. As a consultant he has to work on the more difficult problems. He should be able to formulate them, and he should be able to suggest experiments when necessary. In fact, his help in design of experiments is often sought. Thus in addition to being strong in analysis he should acquire a supporting background in mechanics, electrodynamics, aerodynamics, etc., depending on the particular field of his activities. Since he is most likely to work in a research laboratory, imagination and originality are even more important than specialized training. Most industrial mathematicians are Ph.D.'s or possess an equivalent training.

Opportunities for B.A.'s and M.A.'s in mathematics alone are limited

largely to the field of computing, since such persons do not normally possess the qualifications for employment either as engineers or as physicists or as industrial mathematicians. Frequently the work consists solely of routine computing. In some computing groups, however, the work is more varied and interesting, perhaps involving the evaluation of integrals and solution of differential equations by numerical methods, the analytical solution of simple differential equations and systems of equations, or the solution of more difficult mathematical problems under guidance of a senior member of the staff. Opportunities for advancement (in this type of work) are restricted, and anyone who likes collegiate mathematics but does not wish to pursue a complete graduate course and who wants to work in industry rather than be a teacher would be well advised to major in physics or engineering and take as much mathematics as the curriculum permits. In this way he is more likely to find a job to his liking. Alternatively a B.A. or M.A. in mathematics is advised to acquire a B.S. in engineering or physics. This is particularly true of men since many industrial laboratories employ only women to fill positions in their computing groups.

One who is thinking of choosing mathematics as a career should be warned that there is a vast difference between the points of view in early collegiate mathematics and in postgraduate pure mathematics. The first is largely "solving problems," while the second is concerned primarily with logic and philosophy. It is quite possible that he who likes one type of mathematics may dislike the other. Advanced applied mathematics is more like the early collegiate mathematics in that it concerns itself with the solution of problems. This difference in the points of view may help the mathematics student to decide in his senior or first postgraduate year whether he would like to continue studying pure mathematics and be a teacher, or applied mathematics and be either a teacher or an industrial mathematician. If he wants to be the latter he should also concentrate on further studies in physics.

A prospective mathematician considering an industrial career will wish to know something more specific about the kind of activities he is likely to engage in. These vary from place to place and from individual to individual, but solving problems arising in the course of his consultations with engineers and physicists is a typical daily activity, just as teaching is a routine activity of a teacher. Some teachers do little teaching and are engaged in extending mathematical knowledge. Likewise some industrial mathematicians do not merely participate in current engineering research but engage in evolving new mathematical methods for solving problems and in other creative research on a level equal to and in some fields surpassing the best academic research. In many instances the mathematician has considerable latitude and freedom in his work. The industrial mathematician writes reports for internal consumption which contain solutions of his problems and results of his other work. The most important of these he publishes in engineering or scientific journals. In larger laboratories he may occasionally give courses of lectures to his engineering colleagues.

3. The kinds of ability and preparation needed for industrial mathematics. What kind of mathematics do industrial mathematicians use? It would be a mistake to say that they use only advanced mathematics. It is true that, in assisting engineers in solving their problems, mathematicians are most likely to use advanced mathematics since the more elementary problems would be solved without their assistance. But in research "such simple processes as algebra, trigonometry and the elements of calculus are the most common and the most productive. They frequently lead to results of the greatest practical importance. The single sideband system of carrier transmission, for example, was a mathematical invention. It virtually doubled the number of long distance calls that could be handled simultaneously over a given line. Yet the only mathematics involved in its development was a single trigonometric equation, the formula for the sine of the sum of two angles." [17] Next in the order of importance are linear differential equations with constant coefficients which are used extensively in studying vibrations of linear mechanical, acoustical, and electrical systems consisting of "lumped" elements. The theory of functions of a complex variable is employed in electric circuit theory, propagation of waves on wires, gravitational and electric fields as used in prospecting for oil, aerodynamics, etc. Partial differential equations and their solutions in terms of orthogonal functions such as Bessel functions and Legendre functions are also widely used. To these may be added Fourier and Laplace transforms. Among the less frequently used subjects are: calculus of variations, integral equations, matrix algebra, Boolean algebra, etc. These are the basic types of applied mathematics common to all fields of application. In addition there are types with a special physical flavor: mechanics, dynamics, elasticity, potential theory, fluid flow, electrodynamics, etc.

An applicant for a job is not expected to be conversant with specific fields of activity in industrial laboratories, but he is expected to be equipped for attacking these problems effectively. Here, imagination, originality, and ability to make simplifying assumptions without sacrificing the practical value of the results count more than specialized training. To indicate the kind of problems industrial mathematicians are working on we shall mention a few. In the aeronautical industry, for example, the problems are in aero-elasticity, turbulence, viscous flow and boundary layer problems, vibrations and stability, supersonic aerodynamics, jet propulsion and rocket propulsion. In the oil industry a mathematician may work flow problems in reservoir studies, circuit problems in seismology, interpretation problems in seismology, potential problems in gravimetry and magnetics. In electrical manufacturing there are structural and dynamic problems, such as strain, creep and fatigue in machine parts, vibration and instability in turbines and other rotating machinery. There are also electrical problems in refining the design of generators, motors, transformers, and miscellaneous instruments. The communication field is the one in which mathematical methods of research have been most widely used. The problems are

varied in character: electrical, acoustical and mechanical. Mathematical activity is most intense in designing wave filters and equalizers, in studying transmission by wire and radio, in providing a rational basis for the design of instruments such as transmitters and receivers, vacuum tubes, television scanning devices, etc. Most activities consist in solving specific problems. In addition there are long range activities in developing more effective methods of handling various classes of problems. Mathematics is invading general thinking about communication; a basic theory of information and its transmission is being developed.

4. Salaries of mathematicians in industry. Salaries for computers range from $200 to $350 per month depending on experience. Starting salaries for industrial mathematicians (with a Ph.D. degree) are between $4500 and $6000 per year. Mean salaries in industry are higher than in teaching and in government laboratories. Maximum salaries are also higher than in government laboratories and in most universities. Some universities, however, compare vary favorably.

PART IV: MATHEMATICIANS IN GOVERNMENT

1. Similarities and differences between government and industrial laboratories. The government, like industry, has its laboratories where it carries on research programs related to its special needs. Thus the Mint must concern itself, in emergencies like a war, with the development of alloys which will preserve the important properties-in-use of coins (*e.g.*, usability in turnstiles and other coin boxes), while using little or no scarce metals. Many agencies of the government have some research activities, and in great numbers of these mathematicians play a part. As in industrial laboratories, the trend is toward an increasing use of mathematicians and mathematical statisticians.

Much that has been said about the role of the mathematician in industrial laboratories applies here. The essential differences are those which arise primarily from the fact that the mathematician in government has a civil service position, with all its strengths and weaknesses; and from the fact that the total spread of government activity covers a very wide range, so that a greater variety of opportunity exists. One additional point should be mentioned. Many mathematician posts in the government are not in laboratories where a major research activity is in progress, but are in an executive branch of the government like the Census Bureau. Such posts may be largely scientific. They may, however, be devoted primarily to what has come to be known as "scientific administration," the administration of contracts for research which is actually carried out by industry and by the universities. Of the $625 million spent for research by the Federal government in fiscal 1947, less than a third was spent for investigations within Federal laboratories.

This investment by the Federal government in research which is actually carried out sometimes by industry and frequently by the universities, has created a kind of university employment for scientists (including mathemati-

cians) which cannot be covered adequately by the discussions of the activities of teachers given in Part I of this report. Much, but not all, of this type of work is under contract to the Department of Defense. It is frequently carried out in a specially constituted part of the university (like a "Research Institute") by persons having no teaching assignments; and there are often opportunities in such activities for junior persons who are interested in applying their mathematics to military and other problems of interest to the government and who may spend part of their time in graduate study. A few industrial laboratories have been set up specially to handle this type of work. In certain of these there is somewhat more emphasis on mathematics apart from its engineering applications than in most industrial laboratories, but there are relatively few opportunities here for junior personnel. When special research problems are undertaken by well established industrial laboratories for the government, the opportunities for mathematicians are in no essential way different from those described in Part III.

2. Levels of work carried on by mathematicians in the government. If a young man or woman enters government service (or the service of a university working on a government contract) as a mathematician or statistician immediately after graduation from college, his job will almost surely have some computational aspect. Finding numerical solutions of differential equations, or some very similar problem, is likely to be assigned to him. In general it is expected that the beginning mathematician will perform miscellaneous duties which are specifically assigned and which involve a variety of standard mathematical techniques. Assignments typically are confined to a few related processes and require a minimum of mathematical judgment, although they require sound collegiate training in mathematics. The usual courses through differential equations and some work in mathematical statistics are desirable.

As one proceeds to positions in higher grades—and one can qualify usually by a combination of experience and study—the level of work becomes more difficult, and the responsibility greater. Most frequently, as in industrial laboratories, a combination of mathematical competence with sound training in some field of physical science or engineering is needed for promotion beyond the GS-9 level (see p. 17). A student of mathematics would be well advised to take account of this fact in choosing his courses at college.

It should also be pointed out that many Federal agencies have, as is true of many effective industrial research organizations, recognized the economic value of providing educational opportunities for scientists, particularly for the younger workers. There is an increasing tendency toward in-service training directed to improving the technical competence of scientists in fields related to their jobs. Extensive cooperative arrangements with educational institutions for evening and other part-time technical courses for Federal scientific employees are being made [18], and some agencies have statutory authority to detail their employees to colleges and universities for training.

3. Types of assignment, and mathematical background required. The types of mathematical work carried on in government posts are extremely varied, and call on the widest background of training in mathematics. For an assignment on one of the senior levels, a Ph.D. in mathematics or mathematical statistics and a good deal of maturity in mathematics and in applying mathematics to physical or engineering problems are practically essential. Some indication of the types of problems which are dealt with will be attempted.

Mention was made earlier of computing as an activity of a junior mathematician. Actually, there are phases of computing appropriate to all the levels of professional activity. The need to get numerical answers to real problems which can be solved only approximately (not exactly, like many of the problems in school where the exact answer is in the back of the book) makes extensive computing necessary. Many problems have in the past been solved by very rough approximations because the amount of computing was prohibitive. Now, a new development in large high-speed computing machines (the "mechanical brain" sometimes mentioned in the newspapers) promises to make it possible to improve greatly the accuracy of our approximations to some of the most difficult problems. But new mathematical results are needed in order that these machines may be used effectively; and the government has a research program involving professional mathematicians at the highest levels who are trying to secure some results in this direction. Thus computing involves mathematical work at all professional levels.

Similarly there are applications of mathematics in preliminary studies of designs of all types of engineering equipment. These studies extend from relatively simple equipment to complicated structures like airplanes and ships. And there is a corresponding variety in the professional levels of the mathematicians working on such studies. Activity of this sort is carried on in many military development programs in an attempt to secure the best operating characteristics with the least experimental cost.

Mathematical statistics is playing an increasingly important role in government establishments in the design and analysis of experiments, and in the acceptance of materials purchased for the government. Thus there are positions at all levels for well-trained persons in this field.

A new activity in government which has a considerable appeal to well-trained mathematicians is referred to as "operations research." This attempts to provide a scientific basis for operations and management decisions. Although many problems in operations and management are not subject to mathematical solution, mathematical and statistical methods can frequently give a decisive insight into the essential issues which are involved. The primary government activities in this connection are concerned with problems of military operations, logistics, and strategy, and in the study of optimum methods for employing the complex equipment of modern warfare. The possibilities of this technique extend to program planning and management also, and adaptation of mathematical methods to a wide variety of such problems is under way.

What of the position of women? As in industry, there are many women in the lowest levels, where the emphasis is on computing under relatively close supervision. In the higher levels, because it is relatively hard to secure a well-qualified mathematician for many government positions, the heads of agencies usually welcome women if their qualifications are much better than those of the available men. There are heads of units in government, as well as in business, who consider a woman a bad risk at any except the lowest levels. But once a woman is established in a higher level job, she is usually fully accepted.

Clearly there are counterparts in government of many of the activities carried on by private industry, and there is a legally protected security in the job. Working conditions are in most cases comparable with those in industry. For the supervisor the largeness of the government's operations sometimes is responsible for formalism and red-tape, but for most persons in non-supervisory positions this is not troublesome.

4. How the Civil Service operates [19] [21]. Before closing this part an attempt will be made to give some idea of the number of positions available to mathematicians in the Federal government, the range of salaries covered, and the operation of the Civil Service with respect to these positions.

On July 1, 1947 there was a total of 518 professional mathematicians and 1699 statisticians employed by the Federal government, the largest number of these being in the Army, the Navy, and the Commerce Department. (The Air Force was not at the time a separate department.) Since that time the number has increased considerably. In the Departmental (Washington) Service alone there are now about 200 mathematician jobs.* The large majority of these are concerned with applications of mathematics, for example, in physics, engineering statistics, and operations research, while the remainder are chiefly in scientific administration. There are also a few concerned with pure research in mathematics, for example, at the National Bureau of Standards.

Except in certain beginning positions, examinations for scientific positions in the Federal government are "unassembled," that is, they are based on a review of the candidate's experience, education, and training. The scientific level of activity expected in a position is indicated by a series of ratings, which, for professional personnel, range from GS-5 to GS-18. The GS-5 grade is the position for which a young college graduate with a Bachelor's degree may hope to qualify. GS-7 is the normal beginning grade for a person with a Master's degree; and a Ph.D. without further experience is considered qualification for a GS-9. (Grades GS-16, 17, and 18 were created by Act of Congress in 1949. The total number of persons in these grades is limited by Congress, and the number of mathematicians appointed to these grades will presumably be very small.)

---

* Less than one-tenth of all the government's employees are in Washington, including about one-fourth of the mathematicians. There are 14 Regional Civil Service Offices in the country through which Civil Service information can be obtained [19].

The percentage of mathematicians in the grades which were in existence in 1947 and which corresponded to the present GS (General Schedule) grades is given below, together with the salaries established in 1949:

Grade	*Percentage in Grades in 1947*	*Salaries established in 1949*	
GS-5	27	GS-5	$ 3,100–$ 3,850
GS-7	31	GS-6	3,450– 4,200
GS-9	15	GS-7	3,825– 4,575
GS-11	13	GS-8	4,200– 4,950
GS-12	7	GS-9	4,600– 5,350
GS-13	5	GS-10	5,000– 5,750
GS-14	1	GS-11	5,400– 6,400
GS-15	1	GS-12	6,400– 7,400
		GS-13	7,600– 8,600
		GS-14	8,800– 9,800
		GS-15	10,000– 11,000
		GS-16	11,200– 12,000
		GS-17	12,200– 13,000
		GS-18	14,000

This part has been devoted to opportunities for mathematicians in the Federal government. In a letter dated February 7, 1950, Kenneth O. Warner, Director of the Civil Service Assembly of the United States and Canada, states:

> Most straight mathematics positions in the government service are found at the Federal level. All states and most cities of any size have statistical positions which require varying degrees of mathematical understanding. Other applications for mathematical training may be found in many positions in state and local governments, but there are very few, if any, positions of the mathematical research type. We do not have precise information on this point, but I am sure that research programs, outside the universities, are rare.

PART V: OPPORTUNITIES IN THE ACTUARIAL PROFESSION

The following description of the actuarial profession will deal mostly with the work of the life insurance company actuary, since the majority of the members of the profession in the United States and Canada are employed in that capacity. The work of actuaries otherwise employed will be treated more briefly.

1. The work of the actuary. The actuary of a life insurance company is responsible for calculating the premium rates his company charges and for preparing the tables of death rates upon which such calculations are based. He is concerned with the determination of what benefits shall be included in the

company's policies and how much money must be set aside from year to year to guarantee the payment of such benefits many years in the future. He must analyze the sources of earnings under policy contracts so that he may recommend proper rates of dividends. He investigates the effect on mortality of various physical impairments, hazardous occupations, and other unusual risks, and in collaboration with the medical officer of his company determines the basis for accepting or rejecting applicants for insurance. He is concerned with many theoretical and practical problems which have arisen as a result of the rapid development of the group insurance and pension fields and social insurance of various types.

Although the actuary cannot operate without a thorough knowledge of the mathematical basis of life insurance, he is essentially a technically trained businessman rather than a mathematician. Because of his broad fundamental training, he is particularly well fitted to secure a proper perspective of the problems of a life insurance company, and he usually has an important part in the development of general company policy.

The casualty actuary performs similar functions for his company. Since rates for many coverages, such as automobile insurance and workmen's compensation, are revised annually, a considerable part of the casualty actuary's time is spent on the investigation of current experience and the calculation of rates.

The consulting actuary is often asked to prepare cost estimates for proposed pension plans for industry. Many industrial, municipal, and state retirement and welfare plans are established and supervised by independent consultants. Consulting actuaries serve the needs of small insurance companies which are not large enough to employ their own actuaries. The consulting actuary who is associated with a brokerage firm has extensive dealings with the sales and actuarial departments of the various insurance companies with which he places his clients' business.

The student can obtain some idea of the scope of the actuary's work by consulting recent issues of the Transactions of the Actuarial Society of America (discontinued in 1949), the Record of the American Institute of Actuaries (discontinued in 1949), the Transactions of the Society of Actuaries (first number in 1949) and the Proceedings of the Casualty Actuarial Society. If these are not available in his college library, he can probably see them in the office of any insurance company.

2. Employment of actuaries. All life insurance companies need the services of actuaries. In fact, the large companies need large staffs of actuaries. Moreover, there is a growing tendency for men initially trained as actuaries to move on to other spheres of activity in insurance offices, particularly in the investment, administrative, and accounting fields. A considerable number of actuaries fill high executive positions.

There are many needs for actuaries outside the life insurance companies. The casualty and fire insurance companies require actuaries trained specifically

for those positions. The rapidly increasing demand for pension and retirement plans in business and industry creates many opportunities in the various firms of consulting actuaries. Some of the larger industrial corporations employ actuaries to advise management in connection with their insurance and pension needs. There are actuarial positions in many of the state insurance departments, which supervise and regulate the insurance business. The Federal governments of both the United States and Canada use actuaries in several departments, such as the Social Security Board, the Veterans Administration, the Railroad Retirement Board, and the Bureau of the Census.

The actuarial profession is not crowded and there appears to be no prospect of its becoming so for many years. There are more than 600 life insurance companies in the United States and Canada. Many of these companies require more than one actuary. Most of them are growing in size, as are the casualty and fire companies, and their business is continually becoming more complex. Thus their need for actuaries is constantly growing. Moreover, the need for actuaries in the consulting field is increasing rapidly.

In spite of these needs the number of professionally qualified actuaries is not large. In March of 1950 the Society of Actuaries (see section 4, below) had only 684 Fellows and 430 Associates. The Casualty Actuarial Society had, in November 1949, 151 Fellows and 119 Associates, many of whom were also members of the Society of Actuaries.

In recent years the number of women in the actuarial profession has increased and a number of women have attained prominence in the field. The limitations in opportunities that exist for women arise mostly from the present reluctance of most life insurance companies to assign women to executive duties and to positions requiring frequent contacts outside their own department or company.

3. **Qualifications needed for success as an actuary.** A prominent contemporary actuary has described the qualities required for success in the profession as "competent statistical and mathematical capacity, adequate economic and financial knowledge, and wide social information." The successful actuary has been further characterized as a person "with a determined, lively and ingenious mind and a broad outlook."

There is currently an increasing demand for the actuary to assume many executive responsibilities. Thus, in addition to being consulted on technical matters, he is often charged with the administration of fairly large departments, and shares in the responsibility for the relations of his organization with policyholders and the general public. Therefore, his education and vision should be broad and well rounded.

College students majoring in mathematics and considering the actuarial profession as a possible career should bear in mind that it is not primarily a mathematical profession. If the student wants a profession which calls for the application of advanced mathematics, he should not become an actuary. If, on

the other hand, he desires a profession in which his early training will be based on mathematics and in which all of his work will require the exercise of reasoning and judgment similar to that used in his study of mathematics, then he should consider the actuarial profession.

4. The training of an actuary. Professional status as a life insurance actuary is obtained by becoming a Fellow of the Society of Actuaries. The Society, a private organization, was formed for the purpose of promoting the theory and practice of actuarial science. Its publications and the regular meetings of its members give opportunity for the interchange and development of professional knowledge.

The main requirement for the Fellowship is the passing of eight examinations set by the Society and given annually. The first three of these, known as preliminary examinations, consist of

(1) a language aptitude examination
(2) a general mathematics examination covering algebra, trigonometry, analytical geometry and calculus, and
(3) a special mathematics examination covering finite differences, probability and statistics.

The remaining five examinations, which are more advanced in character, cover such topics as construction of mortality tables and monetary tables, selection of risks, calculation of insurance premiums, investments, the equitable allocation of dividends, life insurance accounting, life insurance law, group insurance, social insurance, and pension plans. Upon completion of the first five examinations, the student becomes an Associate of the Society of Actuaries, and upon completion of all eight of the examinations a Fellow.

The number of years required to complete the examinations normally ranges from four to ten, depending upon the student and the amount of time he devotes to study. The examinations are difficult and exacting, and the attainment of the fellowship is in many respects as difficult a task as obtaining the degree of Ph.D.

It is a great advantage to the student to be able to take the preliminary actuarial examinations while he is still in college, as these examinations cover subjects which are ordinarily taught there. Moreover, if the student starts his examinations while still in college, he has a chance to estimate his aptitudes and probability of passing, and thus determine before he graduates whether it is wise to pursue an actuarial career or to enter some other field. The later examinations are usually taken by the candidate while he is employed in some actuarial capacity, since practical experience is very valuable, if not essential, in preparing for such examinations.

Many of the larger insurance companies have definite training programs for actuarial students. They select college graduates on the basis of their college work and other qualifications for the profession. They arrange to give their

students training in many sections of the actuarial department, as well as some experience in other departments. This enables the student to get information regarding the practices and procedures of his own company while he is studying for the examinations. The student is encouraged to pass the actuarial examinations. In many cases automatic salary raises are given for examinations passed. As the student progresses in his office work, from routine clerical jobs to positions of gradually increasing responsibility, he obtains varied experience in many of of the problems with which his company and the industry are concerned. In many of the smaller companies similar experience is automatically available without any formalized training program.

The importance of individual study in preparing for the actuarial examinations can hardly be over-estimated. Many hours of home study are required for each examination, and no training program can eliminate the necessity for such study.

Professional status as a casualty actuary is obtained by becoming a Fellow of the Casualty Actuarial Society. The usual method of becoming a Fellow is to pass the eight examinations set by the Society. Although the casualty companies have no formal training programs similar to those of some of the life insurance companies, many of them hire college graduates as actuarial students.

5. The salary of an actuary. The salary of an actuary varies with the organization which employs him and the position which he holds in that organization. Salaries depend on general economic conditions and on living costs in the surrounding area, so that the following figures should be interpreted merely as examples, not as the complete picture. All salaries quoted are as of May 1, 1950.

In one of the larger life insurance companies in a large Eastern city, a company which has high qualification standards, an actuarial student earns about \$3,000 during his first year of employment. The starting salary is larger if at least two of the actuarial examinations have already been passed. With satisfactory progress in his office work and his examinations, the student in the company described should be earning about \$5,000 in his fourth or fifth year of service. Within a few more years, if by then he has become a Fellow of the Society, he might reasonably expect to be earning about \$7,000 a year. Beyond this point, there is a wide range of possible salaries, which depend on the ability of the actuary and his value to his company. A number of actuaries, who have reached the top of their profession and have become executives in their companies, earn \$20,000 or more yearly.

Starting salaries for actuarial students vary with the steepness of the salary scale and with the level of qualification standards. For this reason, as well as because of the dependence on living costs in the surrounding area, the starting salaries of different companies often show substantial differences.

In the casualty companies actuarial students begin at salaries ranging from \$2,400 to \$3,000. As Associates they earn between \$4,000 and \$5,000. Fellows

with primarily actuarial duties earn $6,000 or more; those with executive duties generally earn $9,000 or more.

Consulting actuarial organizations and brokerage firms usually start college graduates with an interest in actuarial work at salaries comparable to those of insurance companies. Substantial raises are given, as in the case of insurance companies, as soon as the student demonstrates that he can pass the actuarial examinations.

6. Preliminary preparation for the actuarial profession in college. During his college career, the prospective actuary should acquire a knowledge of mathematics through the calculus. A course in college algebra will be valuable. Courses in mathematical statistics should be taken if offered. Courses in such subjects as differential equations and the theory of functions do not contribute directly to actuarial training but add to the student's general background. Some universities offer undergraduate and graduate work in actuarial science. The courses so offered cover the material of the earlier actuarial examinations and related fields.

A thorough grounding in English composition is essential. If the student has the opportunity, he should take at least one full year's course in economics. Courses in accounting and in banking and finance are useful but not essential.

Aside from these subjects, the prospective actuary should pursue studies which will give him a broad cultural foundation. It is not necessary for him to take graduate work in pure mathematics. In fact, it is not desirable in general, because the time so spent would be better spent in getting practical insurance experience and in passing the actuarial examinations.

As explained above, the student should attempt to take some or all of the preliminary actuarial examinations while he is still in college. In addition, he would be well advised to seek employment in an insurance company during the summer months (openings are often available to promising students) in order to acquaint himself with the practical side of an insurance company's operations and to ascertain if he likes office work.

For information about the preliminary actuarial examinations the student should send for the booklet, *Preliminary Actuarial Examinations*, published by the Society of Actuaries, 208 South LaSalle Street, Chicago 4, Illinois. Information about the examinations of the Casualty Actuarial Society can be obtained by addressing the Secretary-Treasurer of the Society at 60 John Street, New York 7, New York.

The student can best obtain information about the training programs and actuarial student opportunities offered by the insurance companies by writing to the companies in which he is interested. The Secretary of the Society of Actuaries and the Secretary of the Casualty Actuarial Society are able to furnish information about the location of insurance companies and other organizations which may be seeking actuarial students.

SELECTED REFERENCES FOR FURTHER READING

1. *Why Study Mathematics?* Prepared for the use of teachers and students by the Canadian Mathematical Congress, 33 pp.

2. *Guidance Pamphlet in Mathematics for High School Students.* The Mathematics Teacher, 1948, 25 pp.

3. *Do You Want to Major in Mathematics?* Mathematics Department, University of Georgia, 3 pp.

4. *Do You Want to be a Mathematician?* Mathematics Department, University of Georgia, 4 pp.

5. *A Mathematics Student—To Be or Not To Be?* A note to Michigan High School students from a committee of the Michigan Section of the Mathematical Association of America. June, 1948, 9 pp.

6. *The Guidance Report of the Commission on Postwar Plans.* National Council of Teachers of Mathematics, *The Mathematics Teacher*, November, 1947.

7. Catlin, John B. "Opportunities for Statisticians in Industry," *Journal of the American Statistical Association*, Vol. 40 (1945) pp. 75–79.

8. Cochran, W. G. "Training at the professional level for statistical work in Agriculture and Biology," *Journal of the American Statistical Association*, Vol. 40 (1945) pp. 160–166.

9. Cochran, W. G. "Graduate Training in Statistics," *The American Mathematical Monthly*, Vol. 53 (1946) pp. 193–199.

10. Cox, Gertrude M. "Opportunities for (Statisticians in) Teaching and Research," *Journal of the American Statistical Association*, Vol. 40 (1945) pp. 71–74.

11. Leonard, William R. "Opportunities for Statistical Work in the Federal Government," *Journal of the American Statistical Association*, Vol. 40 (1945) pp. 57–61.

12. *Personnel and Training Problems Created by the Recent Growth of Applied Statistics in the United States.* A report by the Committee on Applied Mathematical Statistics, National Research Council, Washington, D. C. Reprint and Circular Series No. 128, 1947, 17 pp.

13. Schneider, David M. "Opportunities for Statistical Work in State and Local Governments," *Journal of the American Statistical Association*, Vol. 40 (1945) pp. 62–70.

14. *The Outlook for Women in Mathematics and Statistics*, Bulletin No. 223–4, Women's Bureau, U. S. Department of Labor, 1947, 21 pp.

15. The Teaching of Statistics, Report of the Institute of Mathematical Statistics Committee on the Teaching of Statistics, *Annals of Mathematical Statistics*, Vol. 19 (1948) pp. 95–115.

16. Wilks, S. S. "Statistical Training for Industry," *Analytical Chemistry*, Vol. 19 (1947) pp. 953–955.

17. Fry, Thornton C. "Industrial Mathematics," *Research—A National Resource*, II, Section VI, Part 4, pp. 268–288. A House Document, 77th Congress. Reprinted as a supplement to *American Mathematical Monthly*, June–July, 1941.

18. *Scientific Personnel Bulletin.* Graduate and Undergraduate Courses in Engineering, Science and Mathematics Available in the Washington Area, Fall 1949. Personnel Division, Office of Naval Research, Department of the Navy, Washington, D. C., September, 1949, 17 pp.

19. *Facts about Jobs in Government for Veterans.* Prepared by *Civil Service Assembly* of the United States and Canada and *Public Administration Clearing House*, 1946, 16 pp. Contains list of all Regional Federal Civil Service Offices and large City and County Civil Service Systems.

20. *Physical Sciences.* Description of Professions Series, Pamphlet No. 6, National Roster of Scientific and Specialized Personnel, U. S. Employment Service, Department of Labor, 1947, 20 pp.

21. *Civil Service Agencies in the United States, A 1940 Census.* Pamphlet No. 16, Civil Service Assembly of the United States and Canada, July 1940, 59 pp.

22. Examination Announcement for *Junior Professional Assistant* and *Junior Management Assistant*, Announcement No. 192, Issued October 11, 1949; Closing Date, November 8, 1949. United States Civil Service Commission, Washington, D. C., 30 pp.

23. Examination Announcement for *Junior Scientist*, Announcement No. 128, Issued October 26, 1948; Closing Date, June 30, 1949. U. S. Civil Service Commission, Washington, D. C., 13 pp.

24. *Research Opportunities for Student Aids* (Chemistry, Engineering, Mathematics, Metallurgy, Physics), Announcement No. 4-34-1 (1948), The Board of U. S. Civil Service Examiners for Scientific and Technical Personnel of the Potomac River Naval Command and The Committee of Expert Examiners of the National Bureau of Standards, 11 pp.

25. *Preliminary Actuarial Examinations*. Society of Actuaries, Chicago, Illinois.

26. Gerhard, F. B. "The Actuarial Profession," *American Mathematical Monthly*, Volume 55, Number 8, October 1948.

27. Rotter, Paul T. "What Is An Actuary?" *School and College Placement*, Vol. 9, No. 2, December, 1948.

28. Vocational Guidance Center Monograph, *Actuary*, Ontario College of Education, University of Toronto, Toronto 5, Ont., Canada.

29. *Year Book of the Society of Actuaries*, Society of Actuaries, Chicago, Illinois.

30. *1949 National Survey of Professional Scientific Salaries*, Los Alamos Scientific Laboratory of the University of California, Los Alamos, New Mexico. Conducted and compiled by the Personnel Department of the Laboratory, 25 pp.

MATHEMATICAL NOTES

EDITED BY E. F. BECKENBACH, *University of California, Los Angeles*

Material for this department should be sent to F. A. Ficken, Institute of Mathematics and Mechanics, New York University, 45 Fourth Ave., New York 3, New York.

ON A THEOREM OF STEINER*

VICTOR THÉBAULT, Tennie, Sarthe, France

1. Introduction. In his excellent book *Modern Geometry*, R. A. Johnson has recalled some circles which are associated with a triangle and which arise from the following theorem, announced without demonstration by Steiner [**1**]: *If a circle with center at the orthocenter H of a triangle ABC cuts the lines joining the midpoints of the sides AB and AC, BC and BA, CA and CB in the points A_1 and A_2, B_1 and B_2, C_1 and C_2, then*

$$AA_1 = BB_1 = CC_1 = AA_2 = BB_2 = CC_2.$$

Johnson mentions the circles of Droz-Farny [**2**], but does not give references for the circles considered in §§ 426 and 427. Some time ago the present writer developed all of the circles of these sections [**3**].

The object of this note is to extend Steiner's theorem, as well as our aforementioned results, to the orthocentric tetrahedron, and to recall other earlier results.

2. Theorem. We shall establish the following result.

THEOREM 1. *If a sphere with center at the orthocenter H of an orthocentric tetrahedron $T \equiv ABCD$ cuts the lines joining the midpoints of the edges AB and AC, BC and BD, CD and CA, DA and DB, $\cdots$, in the points A_1 and A_2, B_1 and B_2, C_1 and C_2, D_1 and D_2, $\cdots$, then*

$$AA_1 = BB_1 = CC_1 = DD_1 = \cdots = AA_2 = BB_2 = CC_2 = DD_2 = \cdots .$$

Proof: If one designates by M, N, A'', A_1 the midpoints of the edges AB, AC, of the altitude AA', and an arbitrary point on the line MN, then, by considering the triangle AA_1H, we have

$$\begin{aligned}(AA_1)^2 &= (AH)^2 + (HA_1)^2 - 2(AH)(A''H) = (HA_1)^2 + AH(AH - 2A''H)\\ &= (HA_1)^2 + (AH)(HA') = (HB_1)^2 + (BH)(HB')\\ &= (HC_1)^2 + (CH)(HC') = (HD_1)^2 + (DH)(HD') = \cdots .\end{aligned}$$

Thus the points A_1, B_1, C_1, D_1, $\cdots$ and, by analogy, the points A_2, B_2, C_2, D_2, $\cdots$, equidistant from H, are also equidistant from the vertices A, B, C, D of the tetrahedron. The sphere (H, HA_1) then cuts the twelve lines joining the midpoints of the edges in twenty-four points whose distances from the corresponding vertices of T are equal.

---

* Translated from the French by Howard Eves.

Conversely, *four equal spheres described with the vertices of an orthocentric tetrahedron* T (or, what amounts to the same thing, with the feet of the altitudes) *as centers cut the lines joining the midpoints of the edges in twenty-four points situated on a common sphere of center* H[4].

For, since

$$(AH)(HA') = -(HA)(HA') = -\rho^2,$$

one obtains

$$(HA_1)^2 = R_0^2 + \rho^2,$$

R_0 being the radius of the equal spheres, and ρ that of the conjugate sphere of the tetrahedron.

3. Another result. We shall establish also the following result.

THEOREM 2. *In an orthocentric tetrahedron, of orthocenter* H, *the circles intercepted by the planes of the faces on the corresponding spheres having for centers the feet of the altitudes and passing through the orthocenter* H' *of the medial tetrahedron are situated on a common sphere* (H, σ), *of center* H.

Proof: If A_3 is any point of the circle common to the plane BCD and the sphere $(A', A'H')$, ω the midpoint of HH', then

$$\begin{aligned}(HA_3)^2 &= (HA')^2 + (A'A_3)^2 = (HA')^2 + (A'H')^2 = 2(A'\omega)^2 + (HH')^2/2 \\ &= 2[5R^2 - (3/4)(a^2 + a'^2)]/9 = \sigma^2,\end{aligned}$$

where $a = BC$, $a' = DA$, and R is the circumradius; for, the orthocenter H' of the tetrahedron having for vertices the centroids of the faces of T (the medial tetrahedron) coincides with the isogonal conjugate, in the tetrahedron T, of the orthocenter H [5], and consequently the point ω coincides with the center of the sphere $A'B'C'D'$ through the feet of the altitudes of T (the *second* twelve point sphere). This gives

$$A'\omega = R/3, \qquad (HH')^2 = 4(HO)^2/9 = 4[4R^2 - (3/4)(a^2 + a'^2)]/9.$$

COROLLARY 2.1. *The circles that the planes of the faces intersect on the spheres having for centers the corresponding vertices of the medial tetrahedron and passing through the orthocenter* H *of* T *are on a sphere of center* H' *equal to the sphere* (H, σ).

COROLLARY 2.2. *The spheres orthogonal to the sphere* (H, σ) *and with centers at the centroids of the faces of* T *cut the planes of these faces in circles situated on a sphere, of center* H', *which is coaxial with the sphere* (H, σ) *and the second twelve point sphere of* T [4].

4. Note. Theorem 2 and corollary 2.1 are particular cases of the following [6]:

THEOREM 3. *In any tetrahedron, the circles that the planes of the faces cut on the spheres having for centers the corresponding vertices of the pedal tetrahedron of a point P, and which cut orthogonally a sphere with center at the isogonal conjugate P′ of P, are on the same sphere of center P.*

References

1. Journal de Crelle, vol. 30, p. 273; Oeuvres, vol. II, p. 345.
2. Mathesis, 1901, p. 22; J. Neuberg, Mathesis, 1910, p. 152.
3. Journal de Vuibert, 1910, p. 65; Nouvelles Annales de Mathématiques, 1910, p. 276.
4. V. Thebault, Mathesis, 1922, p. 334, question 2074.
5. V. Thebault, Mathesis, 1932, p. 152.
6. V. Thebault, Annales de la Société Scientifiques de Bruxelles, 1920, p. 60.

A CLASS OF ALGEBRAIC INTEGERS

IVAN NIVEN, University of Oregon

1. Introduction. In a recent paper A. A. Trypanis[1] proved that

$$\left\{a^{(p-1)/p^n} - 1\right\}/p^{1/p^n} \tag{1}$$

is an algebraic integer if a is any integer not divisible by the positive odd prime p. This result is true also in case $p=2$, and can be established by observing that (1) is a root of

$$\left\{x + \frac{1}{p^{1/p^n}}\right\}^{p^n} - \frac{a^{p-1}}{p} = 0, \tag{2}$$

whose coefficients, and consequently also whose roots, are algebraic integers.[2] That the coefficients of (2) are algebraic integers can be concluded from the fact that p divides the binomial coefficients

$$\binom{p^n}{j} \qquad \text{for } j = 1, 2, \cdots, p^n - 1,$$

and, in the case of the constant term of (2), from Fermat's theorem.

It is clear from this proof that, although (1) can be regarded as a multiple-valued expression because of the operation of taking roots, nevertheless all values of (1) are algebraic integers. It is also clear that

$$\left\{b^{1/p^n} - 1\right\}/p^{1/p^n} \tag{3}$$

is an algebraic integer if $b \equiv 1 \pmod{p}$, and that this constitutes a generalization of (1).

[1] An extension of Fermat's theorem, this MONTHLY, vol. 57 (1950), pp. 87–89.

[2] Cf. E. Hecke, Theorie der algebraischen Zahlen, Leipzig (1923), p. 79, Theorem 62.

2. The problem. This suggests the following question. For what integral values of a, b, c and $m>1$ is

$$(4) \qquad (b^{1/m} - a^{1/m})/c^{1/m}$$

an algebraic integer for all interpretations of the m-th roots involved? If $c=\pm 1$ it is clear that (4) is an algebraic integer for all values of b and $m>0$. Hence we consider $|c|>1$, and there is no loss of generality in assuming $(a, b, c)=1$.

THEOREM. *Let a, b, c, m be integers with $m>1$, $|c|>1$, and $(a, b, c)=1$. The expression(4) is an algebraic integer for all cases of the multiple values of the mth roots if and only if $c\,|\,(b-a)$, m is a power of a prime, say $m=p^n$, and $|c|=p$ if p is odd, $|c|=2$ or 4 if $p=2$.*

We begin the proof by observing that there is no loss of generality in presuming that $c^{1/m}$ is single-valued, since the quotient of any two of the values of $c^{1/m}$ is a root of unity, and hence an algebraic integer. For the same reason we shall regard $a^{1/m}$ as single-valued, and so we consider (4) to have m values corresponding to the values of $b^{1/m}$.

These m values of (4) are the zeros of the polynomial

$$(5) \qquad \left\{x + \left(\frac{a}{c}\right)^{1/m}\right\}^m - \frac{b}{c}\cdot$$

As noted previously about (1) and (2), we observe that if the coefficients of (5) are algebraic integers, then so are the values (4). The converse is also true because of the relation of the coefficients of (5) to the values (4), and the fact that the set of algebraic integers is closed under addition and multiplication. Hence we can confine our discussion to the coefficients of (5).

3. The sufficiency of the conditions. Because of the remark in §1 about the divisibility of the binomial coefficients, we see that the coefficients of (5) are algebraic integers in case $c\,|\,(b-a)$, $c=p$, $m=p^n$. Thus we need discuss only the cases $c=4$, $m=2^n$, $b\equiv a \pmod 4$. For $n=1$ it can readily be verified that (5) has coefficients which are algebraic integers. For $n\geqq 2$ the binomial coefficients

$$\binom{2^n}{j}$$

are divisible by 4 if $j=1, 2, \cdots, 2^{n-1}-1$, but divisible only by 2 if $j=2^{n-1}$. Then it is clear that the coefficients of (5) are algebraic integers, including the coefficient of the term in $x^{2^{n-1}}$, namely

$$\binom{2^n}{2^{n-1}} a^{1/2} 4^{-1/2}.$$

4. The necessity of the conditions. The constant term in (5) is $(a-b)/c$, hence the condition $c\,|\,(a-b)$. It follows that

$$(a, c) = 1, \tag{6}$$

for otherwise we would have $(a, b, c) > 1$, contrary to hypothesis. The coefficient of the linear term in (5) is

$$m\left(\frac{a}{c}\right)^{1-1/m}$$

Because of (6), this coefficient being an algebraic integer implies

$$c \mid m \quad \text{if} \quad m \geqq 3; \qquad c \mid 4 \quad \text{if} \quad m = 2. \tag{7}$$

Now let p be any prime divisor of c, so that $p \mid m$, say $m = p^n M$ where $(M, p) = 1$. We prove that $M = 1$. For if $M > 1$, the coefficient of x^{m-p^n} in (5) is

$$\binom{m}{p^n}(a/c)^{p^n/m}. \tag{8}$$

But

$$\binom{m}{p^n}$$

is prime to p, and $p \nmid a$, but $p \mid c$, so that (8) is not an algebraic integer. Hence $M = 1$, $m = p^n$, and because of (7), $c = p^k$, say. (There is no loss of generality in presuming c to be positive.)

Finally we must prove that $k = 1$ when p is odd, and that $k = 1$ or 2 when $p = 2$. The coefficient of $x^{p^{n-1}}$ in (5) is

$$\binom{p^n}{p^{n-1}} p^{-k(p^n - p^{n-1})/p^n} a^{(p^n - p^{n-1})/p^n}. \tag{9}$$

The binomial coefficient

$$\binom{p^n}{p^{n-1}}$$

is divisible by p but not by p^2. Thus if (9) is to be an algebraic integer, we must have, since $p \nmid a$,

$$1 - k(p^n - p^{n-1})/p^n \geqq 0 \quad \text{or} \quad p \geqq k(p - 1).$$

If p is odd, this implies that $k = 1$, and if $p = 2$, that $k \leqq 2$. This completes the proof of the theorem.

In cases where the conditions of the theorem are not satisfied, it is possible that some of the values of (4) are algebraic integers; for example, this is the case if $m = 4$, $b = 81$, $c = 16$, $a = 1$.

CLASSROOM NOTES

EDITED BY C. B. ALLENDOERFER, Haverford College,

All material for this department should be sent to C. B. Allendoerfer, Haverford College, Haverford, Pa.

ON PRIMITIVE PYTHAGOREAN TRIANGLES

P. J. SCHILLO, University of Buffalo

1. Introduction. A primitive Pythagorean triangle (*ppt*) is a right triangle whose sides are integers which are relatively prime in pairs. It can be shown* that, if u and v are any two relatively prime positive integers such that $u<v$, there is either a *ppt* with sides: $x=2uv$, $y=v^2-u^2$, $z=v^2+u^2$, or one with sides: $x=uv$, $y=(v^2-u^2)/2$, $z=(v^2+u^2)/2$, according as $u+v$ is odd or even. Therefore, every reduced proper fraction, u/v, characterizes a *ppt*. Furthermore it can be shown† that every *ppt* is characterized by two such fractions, u/v and u'/v', where $u+v$ is odd and $u'+v'$ is even.

It is always possible to find a *ppt* which is as nearly similar as desired to a given right triangle. Let θ be one of the acute angles of a given right triangle and let u/v be the ratio of two relatively prime integers which approximates $\tan(\theta/2)$ as closely as desired; then this ratio characterizes some *ppt*. Since $2uv/(v^2-u^2)$ is approximately equal to $\tan\theta$, this *ppt* is nearly similar to the given right triangle.

2. Approximating an Isosceles Right Triangle. In illustrating the use of these facts, *ppt's* which approximate isosceles right triangles are sought. The infinite, repeating, simple continued fraction (s.c.f.),‡ $(0, \overline{2})=\sqrt{2}-1=\tan 22\frac{1}{2}°$, has nth convergent, u_n/u_{n+1}, where $u_n=2u_{n-1}+u_{n-2}$ for $n>2$, is the nth integer in the sequence: $0, 1, 2, 5, \cdots$. Because of well-known properties of the convergents of a s.c.f., u_n and u_{n+1} are relatively prime and u_n/u_{n+1} is the best approximation of $\tan 22\frac{1}{2}°$ with respect to the magnitudes of u_n and u_{n+1} for all values of $n>1$. Also it can be seen that u_n+u_{n+1} is odd, so there is a *ppt* with sides: $x_n=2u_nu_{n+1}$, $y_n=u_{n+1}^2-u_n^2$, $z_n=u_{n+1}^2+u_n^2$. This *ppt* approximates more and more nearly an isosceles right triangle as n increases.

Now, by direct formula, $u_n=(s^{n-1}-t^{n-1})/2\sqrt{2}$, where $s=1+\sqrt{2}$ and $t=1-\sqrt{2}$, so $x_n=(u_{2n+2}-u_{2n-2})/8+(-1)^n/2$, $y_n=x_n-(-1)^n$ and $z_n=u_{2n}$. Hence, in order to compute the sides for any $n>1$, it is sufficient to obtain z_{n-1}, z_n and z_{n+1} from the sequence: $1, 5, 29, 169, \cdots$, where $z_n=6z_{n-1}-z_{n-2}$ for $n>2$, is the nth integer of this sequence, and then to determine $x_n=(z_{n+1}-z_{n-1})/8+(-1)^n/2$ and $y_n=x_n-(-1)^n$. For example, since z_8, z_9 and z_{10} are 195025, 1136689 and 6625109 respectively, $x_9=803760$ and $y_9=803761$.

It is not surprising that these *ppt*'s are the only solutions of the classical

* H. N. Wright: First Course in Theory of Numbers, pp. 92–95.

† Wright: *loc. cit.*

‡ The symbol, $(a_1, a_2, a_3, \cdots)$, represents $a_1+1/a_2+1/a_3+\cdots$. In $(a_1, a_2, \cdots, a_n, \overline{\alpha_1, \alpha_2, \cdots, \alpha_\nu})$, the sequence under the bar is infinitely repeated.

problem of finding *ppt*'s whose legs are consecutive integers. These *ppt*'s alone meet the criterion, $x_n+y_n+z_n\sqrt{2}=s^{2n-1}$, where $s=1+\sqrt{2}$, which P. Bachmann** showed such triangles must meet.

3. Approximating a Right Triangle with a 30° Angle. In a manner somewhat analogous to that above, *ppt*'s are found, which are as nearly similar as desired to a right triangle whose least acute angle is 30°. The infinite, repeating, s.c.f., $(0,\ 1,\ \overline{1,\ 2})=\sqrt{3}/3=\tan 30°$ has nth convergent, u_n/v_n, where, letting $p_n=(3+(-1)^n)/2$, $u_n=p_nu_{n-1}+u_{n-2}$ for $n>2$, is the nth integer in the sequence: 0, 1, 1, 3, 4, $\cdots$, and $v_n=p_nv_{n-1}+v_{n-2}$ for $n>2$, is the nth integer in the sequence: 1, 1, 2, 5, 7, $\cdots$. Since for all values of $n>1$, u_n and v_n are relatively prime and u_n+v_n is even when, and only when, n is even, there is a *ppt* with sides: $x_n=2u_nv_n/p_n$, $y_n=(v_n^2-u_n^2)/p_n$, $z_n=(v_n^2+u_n^2)/p_n$. This *ppt* approximates more and more nearly a right triangle whose least acute angle is 30° as n increases.

Now, by direct formulas, $u_n=(\sigma^{n-1}-\tau^{n-1})/2^k\sqrt{3}$ and $v_n=(\sigma^{n-1}+\tau^{n-1})/2^k$, where $\sigma=1+\sqrt{3}$, $\tau=1-\sqrt{3}$ and k is the greatest integer contained in $(n+1)/2$, so $x_n=u_{2n-1}$, $y_n=(u_{2n+1}-2u_{2n-1})/3-2(-1)^n/3$ and $z_n=2y_n+(-1)^n$. In order to compute these sides for any $n>1$, we may take x_n and x_{n+1} from the sequence: 0, 1, 4, 15, 56, $\cdots$, where $x_n=4x_{n-1}-x_{n-2}$ for $n>2$ is the nth integer of the sequence, and then determine $y_n=(x_{n+1}-2x_n)/3-2(-1)^n/3$ and $z_n=2y_n+(-1)^n$. For example, since x_{12} and x_{13} are 564719 and 2107560 respectively, $y_{12}=326040$ and $z_{12}=652081$.

4. Approximating a Given *ppt*: First Method. A slightly different problem is that of finding *ppt*'s which are as nearly similar as desired to another *ppt*. One method of approximating a given *ppt* depends upon the following fact: if f, an even number, and g are relatively prime integers, and if w is the greatest odd divisor of f such that w is prime to some integer, $r>1$, then, for any positive integer, n, each of the odd prime factors of fr^n divides either gr^n or w but not both. Consequently either fr^n and $gr^n\pm 2w$, or fr^n and $gr^n\pm w$, according as r is odd or even, is a pair of relatively prime integers whose sum is odd. The ratio of integers in such a pair approximates the ratio of f and g as n increases. Hence, if f/g or g/f characterizes a given *ppt*, the appropriate ratio of fr^n and either $gr^n\pm 2w$ or $gr^n\pm w$, characterizes an approximation to the given *ppt*.

As an example, let the given *ppt* have sides: 4, 3, 5. It is characterized by 1/2, so $f=2$, $g=1$, $w=1$, and r is any positive integer. If r is even, the nth approximate *ppt* is characterized by $(r^n\pm 1)/2r^n$ and has sides: $x_n=4r^{2n}\pm 4r^n$, $y_n=3r^{2n}\mp 2r^n-1$, $z_n=5r^{2n}\pm 2r^n+1$. By rejecting the lower signs of those terms where an alternative exists and by letting $r=2$, the radix of the binary system, the resulting expressions are in effect those used by Uhler* in developing a colossal *ppt*. If $r=10$, the radix of the decimal system, $x_n=4\cdot 10^{2n}+4\cdot 10^n$, $y_n=3\cdot 10^{2n}$

** L. E. Dickson: History of the Theory of Numbers, Vol. 2, Chap. 4, p. 182.

* H. S. Uhler: A Colossal Primitive Pythagorean Triangle, this MONTHLY, Vol. 57, No. 5, p. 331.

$-2\cdot 10^n-1$ and $z_n=5\cdot 10^{2n}+2\cdot 10^n+1$, which values are readily expressed in decimal notation for any n. For example, $x_4=400040000$, $y_4=299979999$ and $z_4=500020001$.

5. Approximating a Given *ppt*: Second Method. Another method of making such an approximation depends upon this fact: if one *ppt* has sides: a, b, c, and a second *ppt* has sides: A, B, C, where the legs are so designated that $Bb>Aa$, and the hypotenuses, C and c, are relatively prime, then there is a third *ppt* with sides, $x=Ab+Ba$, $y=Bb-Aa$ and $z=Cc$. It can be shown that $x^2+y^2=z^2$ by substitution. In order to show that x, y and z are relatively prime in pairs it is sufficient to show that x and y are relatively prime. Since $Ax+By=(A^2+B^2)b=C^2b$ and $Bx-Ay=(A^2+B^2)a=C^2a$, any common divisor of x and y must divide C^2 for, if it did not, it would have a factor which divides both a and b, which is impossible because a and b are relatively prime. Similarly, since $ax+by=(a^2+b^2)B=c^2B$ and $bx-ay=(a^2+b^2)A=c^2A$, any common divisor of x and y must divide c^2. But c is prime to C and hence x must be prime to y.

If a given *ppt* has sides: a, b, c, let $A_n=2hr^n$, $B_n=r^{2n}-h^2$ and $C_n=r^{2n}+h^2$, where r and h are relatively prime, $r>1$ and $r+h$ is odd, and let h be such that each prime factor of c divides either r or h but not both so that C_n and c are relatively prime. Then the nth approximate *ppt* has sides:

$$x_n = ar^{2n} + 2bhr^n - ah^2, \qquad y_n = br^{2n} - 2ahr^n - bh^2, \qquad z_n = cr^{2n} + ch^2.$$

As an example, let the given *ppt* have sides: $a=4$, $b=3$, $c=5$. If $r=10$, let $h=1$; then $x_n=4\cdot 10^{2n}+6\cdot 10^n-4$, $y_n=3\cdot 10^{2n}-8\cdot 10^n-3$ and $z_n=5\cdot 10^{2n}+5$. For example, $x_4=400059996$, $y_4=299919997$ and $z_4=500000005$.

THE MAXIMUM VALUE OF A CONTINUOUS FUNCTION

M. K. Fort, Jr., University of Mississippi and University of Illinois

1. Introduction. In this note we present a proof (believed to be new) of the following well known theorem:

Theorem A. *If a function f is continuous on a closed interval I, then there exists a point of I at which f has a maximum for I.*

A rather large part of calculus is based upon this theorem (e.g., Rolle's Theorem, the Mean Value Theorem, l'Hospital's Rule, Taylor's Theorem, etc.). Consequently it is unfortunate that the usual proofs of Theorem A are of such a nature that they are unsuitable for presentation in an elementary course.

All of the proofs of Theorem A which the author has seen follow the same general pattern. One first proves that the set of functional values $f(x)$, for x in I, is bounded; next M is defined to be the least upper bound of this set of functional values; finally one proves that there is a point p of the interval for which $f(p)=M$. Thus, the usual proofs of Theorem A require an acquaintance with the

concept of least upper bound. This is undesirable, since the concept of least upper bound is difficult for the average calculus student to grasp and hence is not ordinarily discussed in a first course. Our proof of Theorem A differs fundamentally from the usual proofs in that we proceed directly to find a point p at which f has a maximum and we do not need to first prove that f is bounded on I.

The proof given in this note was designed for presentation to an advanced calculus class whose members were unacquainted with the concept of least upper bound, the Borel Theorem, the Weierstrass Theorem, etc. This proof could be presented equally as well to a beginning calculus class. Our proof is based upon the following proposition:

PROPOSITION A. *Every non-decreasing sequence of numbers which is bounded above converges to a limit.*

This proposition is stated in most calculus books in the chapter on infinite series. In a beginning calculus course it seems reasonable to assume this proposition as an axiom.

2. Proof of Theorem A. If J_1 and J_2 are sub-intervals of I and J_1 contains J_2, then we define $J_1 \gg J_2$ to mean that corresponding to each x in J_1 there exists y in J_2 for which $f(y) \geqq f(x)$. The student's intuitive ideas about the relation $\gg$ will be clarified if it is pointed out to him that $J_1 \gg J_2$ implies "if f has a maximum for I at some point of J_1 then f has a maximum for I at some point of J_2." The following two properties of the relation $\gg$ are trivial to verify:

(i) If $J_1 \gg J_2$ and $J_2 \gg J_3$, then $J_1 \gg J_3$.

(ii) If K is a closed sub-interval of I and K is bisected to form closed sub-intervals K_1 and K_2, then $K \gg K_1$ or $K \gg K_2$.

We now define a sequence $I_1, I_2, I_3, \cdots$ of closed sub-intervals of I. We first define $I_1 = I$. If I_n has been defined, we bisect I_n to form two closed sub-intervals $I_{n,1}$ and $I_{n,2}$. We see from (ii) that $I_n \gg I_{n,j}$ for at least one value of j. I_{n+1} is defined to be one of the intervals $I_{n,j}$ for which this is true. It follows from (i) that $I \gg I_n$ for each n. Since each interval of our sequence contains the next one, the left endpoints of the intervals form a non-decreasing sequence of numbers. This sequence of numbers is bounded above by the right endpoint of I, and hence by Proposition A this sequence converges to a number p which is seen to be in I.

Suppose that there exists a number q in I such that $f(q) > f(p)$. Then, by the continuity of f there is a neighborhood N of p such that $f(q) > f(x)$ for all x in N. However, since the length of I_n tends to 0 as n tends to infinity, there is an integer m such that N contains I_m. This implies that $f(q) > f(x)$ for all x in I_m, and this is impossible since $I \gg I_m$. Therefore, $f(p) \geqq f(q)$ for all q in I and we see that f has a maximum at p.

We obtain as an immediate corollary to Theorem A the fact that the function f is bounded on I.

TRIGONOMETRY OF RIGHT SPHERICAL TRIANGLES AND THE GNOMONIC PROJECTION

A. D. BRADLEY, Hunter College

A method for deriving the formulas for right spherical triangles from a single plane triangle is suggested by Nunn's proof of $\cos A = \tan b \cot c$.*

Consider the right spherical triangle ABC and the plane triangle AB_1C_1, obtained by gnomonic projection of ABC on a plane tangent to the sphere at A (Figure 1). We assume that $a < 90°$ and $b < 90°$, and so the formulas obtained by

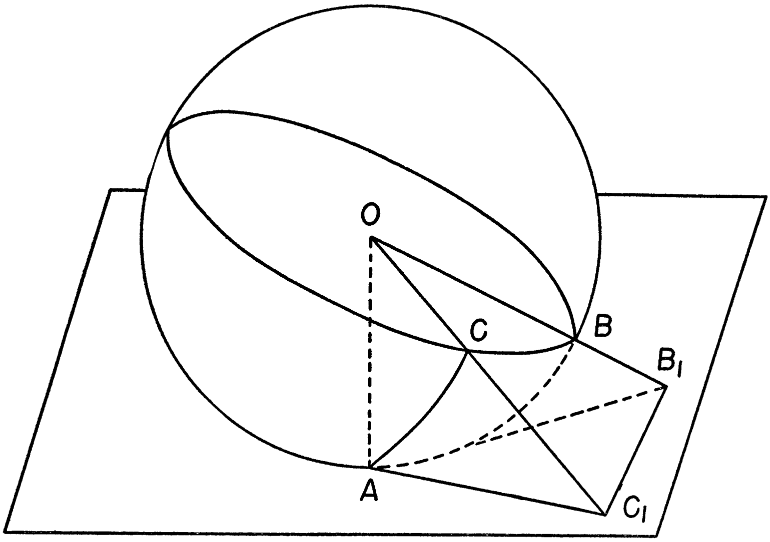


FIG. 1

this method require the usual extension to be valid for all right spherical triangles. Since the planes OB_1C_1 and AC_1B_1 are both perpendicular to OAC_1, B_1C_1 is perpendicular to OAC_1 and triangles OC_1B_1 and AC_1B_1 are right-angled at C_1.

The sides of triangle AB_1C_1 are easily determined:

$$OC_1 = r \sec b,$$

$$C_1B_1 = OC_1 \tan a = r \tan a \sec b,$$

$$AC_1 = r \tan b,$$

$$AB_1 = r \tan c.$$

Angle A of triangle AC_1B_1 equals angle A of the spherical triangle. Projection of ABC on a plane tangent at B would of course produce the analogous triangle

* T. P. Nunn, The Teaching of Algebra including Trigonometry, London, 1927, p. 456.

A_2BC_2, whose sides are $BA_2=r$ tan c, $BC_2=r$ tan a, and $C_2A_2=r$ tan b sec a (Figure 2).

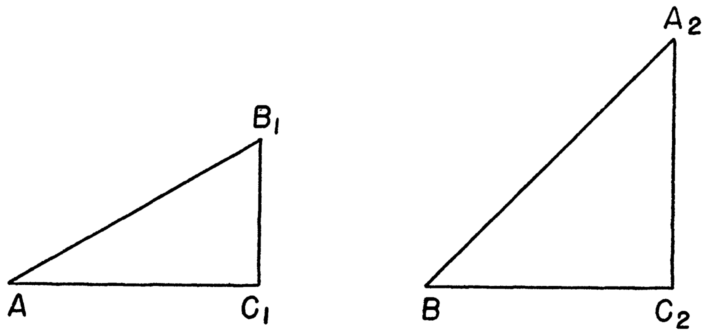


FIG. 2

The formula cos $c=$cos a cos b results from applying the Pythagorean theorem to AB_1C_1:

$$\tan^2 c = \tan^2 a \sec^2 b + \tan^2 b$$
$$\sec^2 c - 1 = \sec^2 a \sec^2 b - \sec^2 b + \sec^2 b - 1$$
$$\sec^2 c = \sec^2 a \sec^2 b.$$

Since a, b, and c are each less than 90°,

$$\cos c = \cos a \cos b \tag{1}$$

$$\sin A = \frac{\tan a \sec b}{\tan c} = \frac{\sin a \cos c}{\sin c \cos a \cos b}$$

$$\sin A = \frac{\sin a}{\sin c} \tag{2}$$

$$\sin B = \frac{\sin b}{\sin c} \tag{3}$$

$$\cos A = \frac{\tan b}{\tan c} \tag{4}$$

$$\cos B = \frac{\tan a}{\tan c} \tag{5}$$

$$\tan A = \frac{\tan a \sec b}{\tan b}$$

$$\tan A = \frac{\tan a}{\sin b} \tag{6}$$

$$\tan B = \frac{\tan b}{\sin a} \tag{7}$$

$$\frac{\sin A}{\cos B} = \frac{\sin a \tan c}{\sin c \tan a} = \frac{\cos a}{\cos c} = \frac{1}{\cos b}$$

$$\sin A = \frac{\cos B}{\cos b} \tag{8}$$

$$\sin B = \frac{\cos A}{\cos a} \tag{9}$$

$$\frac{\tan a \tan b}{\sin b \sin a} = \tan A \tan B$$

$$\cos a \cos b = \cot A \cot B$$

$$\cos c = \cot A \cot B. \tag{10}$$

ELEMENTARY PROBLEMS AND SOLUTIONS

EDITED BY HOWARD EVES, Oregon State College

Send all communications concerning Elementary Problems and Solutions to Howard Eves, Mathematics Department, Oregon State College, Corvallis, Oregon. This department welcomes problems believed to be new, and demanding no tools beyond those ordinarily furnished in the first two years of college mathematics. To facilitate their consideration, solutions should be submitted on separate, signed sheets, within three months after publication of problems.

PROBLEMS FOR SOLUTION

E 946. *Proposed by C. S. Ogilvy, Columbia University*

It is well known that the distribution of six points on the surface of a given sphere which makes the least distance between any pair a maximum is that of the vertices of the regular inscribed octahedron. What is the corresponding distribution for five points?

E 947. *Proposed by Victor Thébault, Tennie, Sarthe, France*

Find the locus of a point in the plane of a given triangle such that the pedal triangle of its isogonal conjugate is right angled.

E 948. *Proposed by Roy Dubisch, Fresno State College*

Find the inverse of the general even ordered skew-symmetric matrix all of whose elements above the principal diagonal are equal to 1.

E 949. *Proposed by Robert Oeder, Oregon State College*

Show that if D_n is a determinant whose elements are a_{ij}, $i, j=1, 2, \cdots, n$, and, for all i,

$$a_{ii} \geqq (1/2)\sum_{k=1}^{n} | a_{ik} |,$$

then

$$D_n \geqq a_{11}\prod_{i=2}^{n}\left(a_{ii} - \sum_{k=1}^{i-1} | a_{ik} |\right).$$

SOLUTIONS

An Infinite Product

E 912 [1950, 259]. *Proposed by H. E. G. P.*

Show that the infinite product

$$\prod_{n=1}^{\infty} (n+1)^3/n(n+x)(n+2x)$$

represents, in all points of its set of convergence, the rational function $2x$. (By permission of Prof. E. P. B. Umbugio, April 1, 1950.)

I. *Solution by J. D. E. Konhauser, Pennsylvania State College.* The result is an immediate consequence of the following well known theorem on Gamma functions (cf., Whittaker and Watson, *Modern Analysis*, pp. 238–9):

If the a's and b's are non-negative integers, then

$$\prod_{n=1}^{\infty} \frac{(n+a_1)(n+a_2)\cdots(n+a_r)}{(n+b_1)(n+b_2)\cdots(n+b_s)}$$

converges if and only if $r=s$ and $\sum a_i = \sum b_j$, and the limit of convergence is then

$$\frac{\Gamma(1+b_1)\Gamma(1+b_2)\cdots\Gamma(1+b_s)}{\Gamma(1+a_1)\Gamma(1+a_2)\cdots\Gamma(1+a_r)}.$$

In the present problem the first condition ($r=s$) is satisfied, and in order that the second condition ($\sum a_i = \sum b_j$) be satisfied, it is necessary that $x=1$. Thus the infinite product converges only for $x=1$, and in this case the value of the product is

$$\Gamma(1)\Gamma(2)\Gamma(3)/\Gamma(2)\Gamma(2)\Gamma(2) = 2 = 2x.$$

II. *Solution by O. E. Stanaitis, St. Olaf College, Minnesota.* If we write the product in the form

$$\prod_{n=1}^{\infty} (1+a_n), \qquad a_n = \frac{3n^2(1-x) + n(3-2x^2) + 1}{n(n+x)(n+2x)},$$

a necessary and sufficient condition for the convergence of the product is the convergence of the series $\sum_{n=1}^{\infty} a_n$. Since $a_n = O(1/n)$ for $x \neq 1$ and $a_n = O(1/n^2)$ for $x=1$, the series converges only for $x=1$. It is easily seen that for $x=1$

$$\prod_{n=1}^{m} (n+1)^3/n(n+1)(n+2) = 2(m+1)/(m+2),$$

which approaches 2 when $m \to \infty$.

Also solved by N. J. Fine, Emil Grosswald, M. S. Klamkin, Roger Lessard, and F. Underwood.

The Isosceles Triangle Teaser

E 913 [1950, 260], *Proposed by C. S. Ogilvy, Columbia University*

An isosceles triangle ABC has vertex angle $C=20°$. Points M and N are taken on AC and BC such that angle $ABM=60°$ and angle $BAN=50°$. Prove that angle $BMN=30°$. (Attention Prof. E. P. B. Umbugio.)

I. *Solution by S. T. Thompson, Tacoma, Washington.* Consider a regular 18-gon $ABA_3A_4 \cdots A_{18}$ with center C. Draw A_3A_{15}. By symmetry A_3A_{15} and AA_7 intersect on CB at N. Also, A_3A_{15} perpendicularly bisects CA_{18} (since angle $A_{15}CA_{18}=60°$), and therefore passes through M (since $MC=MB=MA_{18}$). It now follows that angle $BMN=30°$.

II. *Solution by A. Sisk, Maryville, Tennessee.* We see that $BM=MC$, $AB=BN$, and, applying the law of sines to triangle MAB,

$$AB/\sin 40° = BM/\sin 80° = BM/2 \sin 40° \cos 40°,$$

or

$$BN/BM = 1/2 \sin 50° = \sin 30°/\sin 130°.$$

It now follows that angle $BMN=30°$.

III. *Solution by J. H. Braun, Illinois Institute of Technology.* Draw AD parallel to BC and mark off on AC, $AQ=BN$. Let BQ cut AD in P. Then angle $ABQ=50°$, angle $APB=30°$, angle $AMB=40°$. By the law of sines

$$BM = AB \sin 80°/\sin 40° = 2AB \cos 40°,$$

$$AP = AB \sin 50°/\sin 30° = 2AB \cos 40°.$$

Therefore $BM=AP$ and triangles BMN, APQ are congruent. It follows that angle $BMN=$ angle $APQ=30°$.

Also solved by Alan Berndt, H. W. Carter, W. J. Cherry, Rowland Cross, Ragnar Dybvik, D. E. Freeland, Vern Hoggatt, B. C. Keeler, M. S. Klamkin, Sam Kravitz, Roger Lessard, Octave Levenspiel, W. O. Pennell, J. W. Ross, N. T. Seely, Jr., W. D. Serbyn, O. E. Stanaitis, E. B. Staub, C. W. Trigg, F. Underwood, G. W. Walker, and the proposer.

Most of the solutions were trigonometrical in nature. Pennell established the following generalization: *A triangle ABC has a vertex angle $C<60°$. The base angle $A=C+60°$. Points M and N are taken on AC and BC such that angle $ABM=90°-3C/2$ and angle $BAN=30°+C$. Then angle $BMN=30°$. When* $C=20°$ this reduces to the given problem.

Trigg pointed out that the problem has appeared in *School Science and Mathematics*, April 1939, p. 379. It was earlier proposed by E. M. Langley in *Mathematical Gazette*, XI (1922–3), p. 173. In the same volume, p. 321, a number of solutions can be found.

Professor Umbugio wrote that he was afraid to tackle the problem, because 913 has 13 for the last two digits. Also, the sum of the digits is equal to 13, and the sum of the squares of the digits is 91, which is divisible by 13.

Folding Tetrahedra

E 914 [1950, 260]. *Proposed by C. W. Trigg, Los Angeles City College*

It is desired to arrange congruent equilateral triangles, with at least one side of each triangle coinciding with a side of another triangle, in a plane configuration which can be folded along the common sides into a tetrahedron with no open edges. (An open edge is one through which there is straight line access to the centroid of the solid figure.)

(1) What is the smallest number of triangles necessary?

(2) How many distinct configurations containing this smallest number may be folded into a closed tetrahedron?

Solution by the Proposer. (1) Let the edges issuing from one vertex of a regular tetrahedron be a, b, c and let d be the other edge of the face common to b and c. In order to free one face and unfold it into the plane of the base, two edges, say a and b, must be cut. The one additional cut necessary to destroy the fourth trihedral angle and permit all faces to be folded into the plane of the base may be made either (a) along c, so that the faces unfold into an equilateral triangle with sides $2a$, $2b$, $2c$; or (b) along d, so that the faces unfold into a parallelogram with sides $2a$, b, $2d$, b. When either of these figures is folded back into a tetrahedron, the three cut edges must be covered. This will require three triangles properly arranged in addition to the four face triangles. Thus the smallest number of triangles necessary in the configuration is seven.

(2) We indicate the two sides of the triangles along a cut edge by the same letter as the edge. The three additional triangles may be added to the unfolded equilateral triangle in the following ways:

(a) A properly oriented group of three attached to a to form a five-triangle trapezoid from which a triangle extends from the shorter base and another extends from the opposite end of the longer base. (Configuration A)

(b) A group of two attached to a and one attached to the adjacent b to form a five-triangle trapezoid from which a triangle extends from the shorter base and another from the middle of the longer base. (Configuration B)

(c) A group of two attached to a and one attached to the non-adjacent b, to form a five-triangle trapezoid from which a triangle extends from the shorter base and another from the same end of the longer base. (Configuration C)

(d) One each attached to non-adjacent a, b, and c. (Configuration D)

(e) One each attached to adjacent a and b and one attached to c to give Configuration B. (Rotations and reflections of a configuration are not considered to be distinct configurations.)

The three additional triangles may be added to the unfolded parallelogram in the following ways:

(a) A group of two attached to b and one to the adjacent a. (Configuration C)

(b) A group of two attached to b and one to the non-adjacent a. (Configuration A)

(c) One each attached to adjacent a and b (or b and d) and one to the adjacent d (or a). (Configuration B)

(d) One each attached to adjacent a and b (or b and d) and one to the non-adjacent d (or a). (Configuration A or B)

(e) One each attached to non-adjacent a, b, and d. (Configuration C)

Thus there are four distinct plane configurations of seven joined equilateral triangles which may be folded into a tetrahedron with no open edges. With configurations A, C, and D, the edge-closing triangular flaps may be tucked in to give stability to the tetrahedron without using adhesive. Other methods than those given above for attaching the three additional triangles yield some of the other twenty seven-triangle configurations. One of these, that consisting of a regular hexagon with one triangle attached, cannot be folded into a closed tetrahedron. The other nineteen fold into a tetrahedron with one open edge.

Also solved by Roger Lessard and G. W. Walker.

Twenty Questions

E 916 [1950, 334]. *Proposed by H. D. Larsen, Albion College*

As a variation of the popular game of *Twenty Questions*, suppose I think of a number which you are to determine by asking me not more than twenty questions, each of which can be answered by only "yes" or "no." What is the largest number that I should be permitted to choose that you may determine it in twenty questions?

Solution by H. M. Gehman, University of Buffalo. If a questioner wishes to determine an object previously chosen from a finite set of objects, his most efficient procedure is to ask at each stage whether the chosen object has a property which is possessed by exactly half of the members of the set. Regardless of which answer he receives, the set containing the chosen object is thereby cut in half. By this procedure the questioner can determine a chosen number in twenty questions, provided that the number chosen is a positive integer not greater than 2^{20}.

The ith question for $i=1, \cdots, 20$, could be: "If the number is written to

the base 2, is the ith digit the number 1?" Notice that if the answer is always "no" then the number (base 2) is the 21 digit number consisting of 1 followed by twenty zeros. If at least one answer is "yes" then the number contains at most twenty digits, each of which has been determined.

Also solved by N. Balasulxamanian, D. H. Browne, D. G. O. Connor, R. E. Ekstrom, N. J. Fine, Daniel Finkel, L. O. Heflinger, Ray Jurgensen, H. C. Kranzer, Sidney Kravitz, Roger Lessard, C. S. Ogilvy, M. W. Oliphant, L. A. Ringenberg, Azriel Rosenfeld, C. M. Sandwick, Sr., P. J. Schillo, R. E. Shafer, E. H. Vance, G. W. Walker, and the proposer.

Connor called attention to *Information Theory*, by C. E. Shannon, Bell System Technical Journal, July and October, 1948.

The Diophantine Equation $1/x^2+1/y^2=1/z^2$

E 917 [1950, 334]. *Proposed by Lawrence Ringenberg, Eastern Illinois State College*

Find all solutions of the equation $1/x^2+1/y^2=1/z^2$, where x, y, z are relatively prime positive integers.

Solution by H. Ohbayashi, Nagoya University, Japan. Let m be the positive integer such that $x^2=z^2+m$. Then

$$1/y^2 = 1/z^2 - 1/x^2 = m/z^2x^2,$$

and m is a square, say s^2. Thus x, z, s form a set of Pythagorean numbers. Applying the general solution for Pythagorean numbers we have

$$x = (p^2 + q^2)A, \qquad z = (p^2 - q^2)A, \qquad s = 2pqA,$$

where p and q are positive integers with $p>q$, $(p, q)=1$, $pq\equiv 0 \pmod 2$, and A is any positive integer. Then

$$y = zx/s = (p^2 - q^2)(p^2 + q^2)A/2pq.$$

Since y is integral it follows that A must be a multiple of $2pq$, and since x, y, z are relatively prime, we actually have $A=2pq$. Thus the required solution is

$$x = 2pq(p^2 + q^2), \qquad y = (p^2 - q^2)(p^2 + q^2), \qquad z = 2pq(p^2 - q^2),$$

with $p>q$, $(p, q)=1$, $pq\equiv 0 \pmod 2$.

Also solved by H. W. Becker, David Berkowitz, Alan Berndt, D. H. Browne, Ray Jurgensen, M. S. Klamkin, H. C. Kranzer, Sam Kravitz, Sidney Kravitz, Roger Lessard, C. M. Sandwick, Sr., P. J. Schillo, Elijah Swift, G. W. Walker, and the proposer.

Editorial Note. If x and y are the legs of a right triangle, w the hypotenuse, and t the projection of y on w, then z is the altitude on the hypotenuse and s is the projection of x on w. All six of x, y, z, w, s, t are integers.

ADVANCED PROBLEMS AND SOLUTIONS

EDITED BY E. P. STARKE, Rutgers University

Send all communications concerning Advanced Problems and Solutions to E. P. Starke, Rutgers University, New Brunswick, New Jersey. All manuscripts should be typewritten, with double spacing and margins at least one inch wide. Problems containing results believed to be new or extensions of old results are especially sought. Proposers of problems should also enclose any solutions or information that will assist the editor. In general, problems in well known textbooks or results in readily accessible sources should not be proposed for this department.

PROBLEMS FOR SOLUTION

4420. *Proposed by F. S. Acton, National Bureau of Standards, Institute for Numerical Analysis*

Evaluate

$$g(m) = \sum_{j=0}^{m-1} (-1)^j \tan\left[\frac{\pi}{2}\left(\frac{2j+1}{2m}\right)\right].$$

4421. *Proposed by N. C. Ankeny, Princeton University*

Prove that there are infinitely many polynomials of degree n with integer coefficients, the first being unity, which have n real roots and are irreducible in the field of rational numbers.

4422. *Proposed by Jacques Dutka, Rutgers University*

n elements numbered $1, 2, \cdots, n$, stand in order. In how many ways can they be rearranged in a line so that no two are together which originally stood together?

4423. *Proposed by Victor Thébault, Tennie, Sarthe, France*

Determine systems of numeration such that there exist pairs of consecutive three-digit integers each of which equals the sum of the cubes of its digits. (For example, 370 and 371 in the decimal scale.)

4424. *Proposed by Rufus Isaacs, the Rand Corporation, Santa Monica, California*

Describe the most general continuous real-valued function $f(x)(-\infty \leqq x \leqq +\infty)$ which satisfies

$$f[1-f(x)] = 1 - f(x).$$

SOLUTIONS

Spheres Tangent to a Quadrilateral

4316 [1946, 586]. *Proposed by Victor Thébault, Tennie, Sarthe, France*

Let there be a skew quadrilateral having the sum of one pair of opposite

sides equal to the sum of the other pair. Then, (1) there are infinitely many spheres tangent to all four sides of the quadrilateral, the locus of the centers being a straight line Δ, (2) the points of contact of any one of the spheres lie on a plane perpendicular to Δ, (3) the sides of the quadrilateral belong to a hyperboloid of revolution which envelops all the spheres and has Δ for its axis.

Solution by L. M. Kelly, Michigan State College. We prove first the

LEMMA. *If in any quadrilateral $ABCD$, P, Q, R, S are four points on the segments AB, BC, CD, DA, respectively, such that $AP=AS$, $BP=BQ$, $CQ=CR$, $DR=DS$, then the points P, Q, R, S lie on a circle.*

Proof. Suppose $ABCD$ is a plane, convex quadrilateral. It is evident from a diagram that the sum of the angles at P and R equals the sum of the angles at Q and S in the quadrilateral $PQRS$, and hence the quadrilateral is cyclic. A similar proof holds if the quadrilateral is in a plane but not convex.

To extend the proof to a skew quadrilateral, we first observe that by Carnot's theorem (Court, *Modern Pure Solid Geometry*, p. 111) the four points P, Q, R, S are coplanar. Now project A, B, C, D onto the plane of $PQRS$ obtaining the plane quadrilateral $A'B'C'D'$. From the congruent triangles APA' and ASA' it follows that $A'P=A'S$, etc. It is thus clear that the plane quadrilateral and the points P, Q, R, S satisfy the conditions of the preceding paragraph, and that the points P, Q, R, S are thus concyclic.

It should be noted from the congruent triangles APA' and ASA' that angles ABA' and ASA' are equal. Thus the four sides of the skew quadrilateral make equal angles with the plane $PQRS$ or with its normal.

Proof of Main Theorem. Let AB be the shortest side of the quadrilateral. We will concern ourselves with internal division of segments although the results can be extended to include both internal and external division. Corresponding to any point P on the segment AB there will be a quadruple of points P, Q, R, S which satisfy the conditions of the above lemma. Thus the points P, Q, R, S lie on a circle. If P', Q', R', S' are a second quadruple, their plane is parallel to that of $PQRS$ since PS is parallel to $P'S'$, etc. Furthermore the center of circle $PQRS$ is on the perpendicular bisecting planes of the segments PQ, QR, RS, PS. But these planes are also the perpendicular bisectors of the segments $P'Q'$, $Q'R'$, $R'S'$, $P'S'$. Thus the centers of all such circles lie on a line. This is the line Δ. It is now clear that if the line AB be revolved around Δ generating a hyperboloid of revolution, the surface will contain the other three lines.

Now if $PQRS$ is one such quadruple of points, select a point O on Δ such that angle OPA is a right angle. From the congruent triangles OPA and OSA it follows that ASO is also a right angle, etc. Thus O is the center of a sphere tangent to the four sides of the skew quadrilateral.

Finally it is certainly true that the normals to the surface at P, Q, R, S (these points being on a "latitude circle") meet on the axis Δ and that they are perpendicular to the sides of the quadrilateral. Thus OP is both a normal to the

sphere and the surface, and the hyperboloid envelopes the spheres.

Also solved by Robert Bouvaist, Joseph Langr, and the Proposer.

Circle and Circumscribed Quadrangle

4323 [1948, 643]. *Proposed by Victor Thébault, Tennie, Sarthe, France*

Let $ABCD$ be a convex quadrangle circumscribed about a circle with center O, and let A', B', C', D' be the points of tangency of the sides BC, CD, DA, AB. Consider the circles (OA, B), (OA, D) tangent to OA at O and passing, respectively, through the vertices B, D neighboring A. Consider also the analogous circles tangent to OB, OC, OD at O. (1) The pairs of circles (OA, B) and (OD, C), (OB, C) and (OA, D), (OC, D) and (OB, A), (OD, A) and (OC, B) respectively intersect at N on BC, P on CD, Q on DA, M on AB. (2) The quadrangle $MNPQ$ is a parallelogram with center O having its sides parallel to the diagonals of $ABCD$ and its diagonals parallel to those of $A'B'C'D'$. (3) The centers of the circle and of the equilateral hyperbola circumscribing $A'B'C'D'$, and the Miquel point of the complete quadrilateral formed by the sides of the quadrangle $ABCD$, are collinear.

Solution by J. W. Clawson, Ursinus College, Collegeville, Pennsylvania. Let AB and DC intersect at R and AD and BC at S. The given circle will be used as the basic circle for an inversion. The inverses of A, B, C, D, R, S are the middle points of $C'D'$, $D'A'$, $A'B'$, $B'C'$, $B'D'$, $A'C'$. We shall call these A_1, B_1, C_1, D_1, R_1, S_1, respectively.

The circles (OA, B), (OD, C) invert into straight lines through B_1 parallel to OA_1 and through C_1 parallel to OD_1. Let these intersect at N_1. Now angle $B_1N_1C_1$ = angle A_1OD_1 = supplement of angle $B'C'D'$. Again angle B_1OC_1 = supplement of angle $B'A'D'$. Hence, N_1 lies on the circle B_1C_1O, and so its inverse N lies on BC.

Further $A_1B_1C_1D_1$ is a parallelogram. Hence sides of triangles OA_1B_1 and $P_1D_1C_1$ are respectively parallel and the corresponding sides are equal. Thus D_1P_1 (and also B_1N_1) is equal and parallel to OA_1, with similar results for analogous lines. Again, since $OA_1B_1N_1$ is a parallelogram, ON_1 is equal and parallel to A_1B_1. Similarly OQ_1 is equal and parallel to A_1B_1. Hence O bisects N_1Q_1. In the same way O bisects M_1P_1. Now A_1B_1 is parallel to $A'C'$. Hence the diagonals of $MNPQ$ are parallel to those of $A'B'C'D'$. Again, since triangles $M_1B_1N_1$ and C_1OA_1 have two pairs of sides equal and parallel, the third pair M_1N_1 and A_1C_1 are also equal and parallel. Thus the sides of $MNPQ$ are parallel to the diagonals of $A_1B_1C_1D_1$ and hence of $ABCD$.

Now the inverses of the circles ABS, $\cdots$ which meet at F, the Miquel (focal, Wallace) point of the circumscribed quadrilateral, are the circles $A_1B_1S_1$, $\cdots$. But these are the nine-point circles of $A'C'B'$, $\cdots$, since they pass through the midpoints of the sides of these triangles. It is well known that these four nine-point circles are concurrent at H, the orthic center of the quadrangle $A'B'C'D'$, at which point perpendiculars from the midpoints of the sides to the opposite

sides also meet. It is also well known that the center of a rectangular hyperbola circumscribed about a triangle lies on its nine-point circle. It follows immediately that the inverse point of F is the center of the rectangular hyperbola which circumscribes $A'B'C'D'$. It is thus established that O, F, H are collinear.

Also solved by Robert Bouvaist, and the Proposer.

Euler Lines

4328 [1949, 39]. *Proposed by Victor Thébault, Tennie, Sarthe, France.*

Given a triangle ABC whose altitudes are AA', BB', CC'. Prove that the Euler lines of the triangles $AB'C'$, $A'BC'$, $A'B'C$ are concurrent on the nine-point circle at a point P which is such that one of the distances PA', PB', PC' equals the sum of the other two.

I. *Solution by O. J. Ramler, Catholic University, Washington, D. C.* Triangles $AB'C'$ and $A'B'C$ are directly similar with B' as center of similitude. The triangles $BC'A'$, $B'C'A$ and $CA'B'$ are similar in pairs with C' and A' as centers of similitude respectively. Hence $A'B'C'$ is the triangle of similitude and the nine-point circle is the circle of similitude. The perpendicular bisectors of $B'C'$, $C'A'$, $A'B'$ are concurrent and are corresponding lines. Hence they cut the circle of similitude in the three invariable points, A'', B'', C''. Moreover the orthocenters H_a, H_b, H_c, of $AB'C'$, $A'BC'$, $A'B'C$, respectively, are corresponding points. Hence the triangles $A''B''C''$ and $H_aH_bH_c$ are perspective from a point on the circle of similitude, i.e., the Euler lines $AB'C'$, $A'BC'$, $A'B'C$ are concurrent on the nine-point circle at a point P.

The property $PA'+PB'+PC'=0$ was proved by the Proposer in an article in this MONTHLY [1947, 448].

II. *Note by R. Goormaghtigh, Bruges, Belgium.* The theorem may be generalized as follows:

The triangles $AB'C'$, $BC'A'$, $CA'B'$, formed by the vertices of a triangle ABC and the feet A', B', C' of the altitudes AA', BB', CC' are similar to ABC; the circumdiameters of these triangles homologous to a given circumdiameter Δ of ABC are concurrent at a point P on the nine-point circle, and PA', PB', PC' equal the distances from A, B, C to Δ.

The point P is the orthopole of Δ as to the triangle ABC and both properties contained in this generalization appear in my article on *The Orthopole.* Tôhoku Mathematical Journal, Sendai, 1926, p. 93. The first part of the problem, about Euler lines meeting in a point, was mentioned by Visschers in *Mathesis*, 1922, p. 187, and I gave the above generalization of that part in *Mathesis*, 1922, p. 403.

When Δ is the Euler line of ABC, as considered in the present problem, Δ passes through the centroid of ABC and one of the distances from A, B, C, to Δ equals the sum of the two others.

Also solved by J. W. Clawson, Joseph Langr, and the Proposer.

Two Triangles Inscribed in a Circle

4338 [1949, 186]. *Proposed by R. Bouvaist, Vincelles, Saône-et-Loire, France*

For any given triangle ABC inscribed in a circle (O), there are three points α, β, γ on (O) such that the segments determined by the sides of angle BAC on the tangents to (O) at α, β, γ have α, β, γ, respectively, for their midpoints. Show that the orthocenter of triangle $\alpha\beta\gamma$ is the midpoint of BC.

Solution by J. W. Clawson, Ursinus College, Collegeville, Pa. We shall represent A, B, C as turns t_1, t_2, t_3 on a unit circle (O) in the complex plane. If T, any point on (O), is given by turn t, the equations of AB and of the tangent at T are

$$z + t_1t_2\bar{z} = t_1 + t_2, \qquad z + t^2\bar{z} = 2t.$$

The point C' in which the tangent at T intersects AB is given by

$$t' = t(2t_1t_2 - tt_1 - tt_2)/(t_1t_2 - t^2),$$

while the analogous point B' has the representation t'' which is the same except that t_2 is replaced by t_3. The midpoint of $B'C'$ is given by $(t'+t'')/2$. Equating this to t and removing the factor $(t-t_1)$, we obtain the cubic

$$2t^3 - (t_2 + t_3)t^2 - t_1(t_2 + t_3)t + 2t_1t_2t_3 = 0.$$

Now for any three points on (O) the turn representing the orthocenter is the sum of the turns corresponding to the three points. Hence, if these points are α, β, γ, the roots of the above cubic, then their orthocenter is $(t_2+t_3)/2$, which is the midpoint of BC.

Also solved by R. Goormaghtigh, Roger Lessard and by the Proposer.

Editorial Note. Goormaghtigh remarks that, if we further introduce the notion of mean triangle connected with a given triangle in the sense defined by Delens, *Mathesis*, vol. 51, 1937, p. 264 (see also this MONTHLY, vol. 47, 1940, p. 140; vol. 53, 1946, p. 200), we find the following theorem:

The triangles $\alpha\beta\gamma$ and ABC have opposite means.

Number of Equivalence Relations for n Elements

4340 [1949, 187]. *Proposed by N. S. Mendelsohn, University of Manitoba, Canada*

Let $f(n)$ be the number of distinct equivalence relations connecting n elements. Show that

$$f(n) = \sum_{k=1}^{n}\sum_{j=0}^{k}(-1)^n\frac{(k-j)^n}{j!(k-j)!} = \frac{n^n}{n!}\left(\frac{1}{0!}\right) + \frac{(n-1)^n}{(n-1)!}\left(\frac{1}{0!} - \frac{1}{1!}\right) + \cdots$$
$$+ \frac{(n-r)^n}{(n-r)!}\left(\frac{1}{0!} - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \cdots + (-1)^r\frac{1}{r!}\right) + \cdots,$$

and find an asymptotic formula for $f(n)$ as $n\to\infty$.

Note. For equivalence relations and their connection with partitions see Birkoff and MacLane, *Survey of Modern Algebra*, pp. 159 ff.

Solution by John Riordan, Bell Telephone Laboratories, New York City. The number of equivalence relations for n elements is, by the noted connection with partitions, the number of ways of distributing n things into $1, 2, \cdots, n$ like boxes. By proposition XXIV of Whitworth, *Choice and Chance*, London, 1901, this latter has the exponential generating function $\exp(e^t-1)$; that is

$$\sum_{n=0}^{\infty} f(n) \frac{t^n}{n!} = \exp(e^t - 1).$$

Hence $f(n)$ is an exponential number or Bell number of order 1, and is calculable by the relation

$$f(n+1) = (f+1)^n = \sum_{k=0}^{n} \binom{n}{k} f(n-k).$$

Also, by Herschel's theorem applied to $\exp(e^t-1)$,

$$f(n) = \sum_{x=0}^{n} \Delta^x 0^n / x!, \tag{1}$$

where

$$\Delta^x 0^n = (E-1)^x 0^n = \sum_{k=0}^{x} (-1)^k \binom{x}{k} (x-k)^n.$$

Rearrangement of this series leads to the result stated, namely

$$f(n) = \sum_{m=0}^{n} \frac{(n-m)^n}{(n-m)!} \frac{d_m}{m!},$$

where $d_m = \Delta^m 0!$ is a rencontres or displacement number:

$$d_m = m d_{m-1} + (-1)^m.$$

The asymptotic relation for the exponential numbers given by Knopp, *Theory and Application of Infinite Series*, London, 1928, p. 563, is

$$\frac{f(n)}{n!} = \left(\frac{1+\eta_n}{\log n}\right)^n, \qquad \eta_n \to 0.$$

Also solved by Max LeLeiko and by the Proposer.

Editorial Note. In a paper by G. T. Williams, Numbers Generated by the Function $\exp(e^x-1)$, this MONTHLY, 1945, pp. 323–327, the present $f(n)$ is called G_n and the "number of distinct equivalence relations connecting n elements" is referred to as the "number of ways in which a product of n distinct

primes can be factored"; the identity of these statements can be seen when the convention is made, in the latter assertion, that any two primes which occur in the same divisor in the factorization in question shall be considered equivalent. The statement that G_n is the sum of all the Stirling numbers of the second kind is equivalent to (1) in the above solution.

Williams refers further to the extensive critical bibliography given in L. F. Epstein, A Function Related to the Series for exp (e^x), *Journal of Mathematics and Physics*, July 1939, and to his proofs of a number of asymptotic expressions for G_n (which he calls K_n), the simplest of which would seem to be

$$K_n \sim \left(\frac{ne^{1/\log n}}{\log n}\right)^n.$$

Division Ring

4345 [1949, 269]. *Proposed by Irving Kaplansky, University of Chicago.*

An element x in a ring is said to be right quasi-regular if there exists an element y with $x+y+xy=0$. It is evident that in a division ring, every element except -1 is right quasi-regular. Prove the converse: if every element in a ring A is right quasi-regular, with exactly one exception, then A is a division ring.

Solution by Robert Steinberg, University of California, Los Angeles. Let e be the one exception. Evidently $e\neq 0$.

(1) Since $e+(-e+y-ey)+e(-e+y-ey)=-e^2+y-e^2y\neq 0$ for all y, it follows that $-e^2=e$ or $e^2+e=0$.

(2) $x+ex=0$ for all x. For, $e+x+ex\neq e$ implies the existence of a y such that $e+x+ex+y+(e+x+ex)y=0$; and left multiplication by e gives, by (1), $e^2=0$ or $e=0$ which is impossible. Similarly, and using the result just obtained, $x+xe=0$ for all x. Thus $-e$ is the unit element of A.

(3) The equation $x+y+xy=0$ can now be written

$$(x-e)(y-e) = -e \qquad \text{for } x\neq e$$

or $XY=-e$ for $X\neq 0$, where $X=x-e$, $Y=y-e$. Thus every X of A different from 0 has an inverse, so that A is a division ring.

Also solved by R. C. Buck, J. W. Gaddum, I. N. Herstein, N. D. Lane, J. E. McLaughlin, Alex Rosenberg, R. D. Schafer, Daniel Sokolowsky, W. L. Stamey, Olga Taussky, and the Proposer.

Editorial Note. The Proposer suggests as generalization the consideration of rings in which all elements are right quasi-regular with a finite number n of exceptions. If n is prime the analysis is relatively easy in terms of the Jacobson radical; for n composite he gives no results. See also N. H. McCoy, *Rings and Ideals*, 1948, pp. 132, ff.

Greatest Integer Function

4346 [1949, 343]. *Proposed by N. S. Mendelsohn, University of Manitoba.*

Prove that

$$n - 1 = \sum_{r=1}^{\infty} \left[\frac{n + 2^{r-1} - 1}{2^r} \right],$$

for any positive integer n. The brackets denote, as usual, the greatest integer function.

I. *Solution by R. C. Buck, University of Wisconsin.* This is a special case of a previous problem, no. 4199 [1947, 487], with $r=2$, $\alpha=1$, $n-1$ in place of n, and $\nu=\sigma$.

A direct proof can be given quite briefly. Let

$$n - 1 = \sum_{k=0}^{\infty} 2^k a_k, \qquad a_k = 0, 1.$$

Then

$$\left[\frac{n-1}{2^r} + \frac{1}{2} \right] = a_{r-1} + \sum_{m=r}^{\infty} 2^{m-r} a_m,$$

so that by summing we have

$$\sum_{r=1}^{\infty} \left[\frac{n - 1 + 2^{r-1}}{2^r} \right] = \sum_{m=0}^{\infty} a_m + \sum_{m=1}^{\infty} \sum_{r=1}^{m} 2^{m-r} a_m$$

$$= \sum_{m=0}^{\infty} 2^m a_m = n - 1.$$

II. *Solution by H. S. Zuckerman, University of Washington,* Seattle. The integers $1, 2, 3, \cdots, m$ are given, each just once, by

$$2^{r-1}(2s - 1), \qquad s = 1, 2, \cdots, \left[\frac{m + 2^{r-1}}{2^r} \right], \quad r = 1, 2, 3, \cdots.$$

Counting these integers we find

$$m = \sum_{r=1}^{\infty} \left[\frac{m + 2^{r-1}}{2^r} \right]$$

which is the desired result with $m = n - 1$.

Since $[x + \frac{1}{2}]$ is the nearest integer to x, taking the larger one if there are two, the result can be expressed in the following interesting form: The true equation

$$m = \frac{m}{2} + \frac{m}{2^2} + \frac{m}{2^3} + \cdots$$

remains correct if we replace each fraction by its nearest integral approximation, again using the larger one if two are equally close.

Also solved by Michael Aissen, Murray Barbour, W. D. Berg, D. H. Browne, W. M. Frank, L. S. Kennison, M. S. Klamkin, M. S. Knebelman, N. D. Lane, D. H. Lehmer, Roger Lessard, Leo Moser, Ivan Niven, S. T. Parker, Alex Rosenberg, and the Proposer.

Editorial Note. Lehmer gave the present result as an application of the main theorem in *An Inversive Algorithm*, Bulletin of the American Mathematical Society, 1932, p. 693. He also notes the following generalization

$$m = \sum_{r=1}^{\infty} \sum_{k=1}^{a-1} \left[\frac{m + ka^{r-1}}{a^r} \right].$$

Knebelman considers the generalization where n is not necessarily an integer: If $m > 0$, then

$$\sum_{r=1}^{\infty} \left[\frac{m}{2^r} + \frac{1}{2} \right] = [m], \qquad \sum_{r=1}^{\infty} \left[\frac{-m}{2^r} + \frac{1}{2} \right] = [-m] + 1.$$

RECENT PUBLICATIONS

EDITED BY E. P. VANCE, Oberlin College

All books for review should be sent directly to the editor of this department, American Mathematical Monthly, 531 West 116th Street, New York 27, N. Y. and not to any of the other editors or officers of the Association.

Introduction to Algebraic Geometry. By J. G. Semple and L. Roth. Oxford University Press, 1949. 16+446 pages. $7.50.

The number of introductory books in English on Algebraic Geometry is unfortunately quite small. Such introductory material as does exist treats, for the most part, with the theory of curves: there is hardly a single introductory work on surfaces. The present textbook is therefore particularly welcome to a scanty literature.

Semple and Roth have attempted to give a survey of the development of Algebraic Geometry over the past 100 years; some of the topics covered are: 1) the projective theory of curves and surfaces in r-space, including a discussion of these as partial intersections of $r-1$ and $r-2$ hypersurfaces; plenty of illustrations are given; 2) the birational theory of curves, including linear and algebraic series; and the theory of correspondences between points on two curves; 3) the representation of rational 3-folds and rational surfaces respectively by linear systems of surfaces and curves, illustrated, in the case of surfaces, by a large number of special cases; for example, it is in this way that the 27 lines upon the general cubic surface are studied; 4) line geometry; 5) an exposition of the incidence

calculus of Schubert, that is, of that method invented by Schubert which is intended to answer the question of how many configurations of a specified type fulfill a specified condition; and 6) a final chapter on surfaces.

It is clear from this quite brief list that no complaint of paucity of topics can be lodged against the *Introduction*. If anything, the book covers too much material. This in itself would be no criticism, but the inclusion of so much material in so few pages presumably has led to a treatment "which may trouble the minds of some readers" (p. iii). Theorem 1, for example, asserts that in r-space there is a unique hyperplane through r general points. Actually there is no lack of precision in that statement, but the few words needed to make it clear are not given. (The authors use the term "general point" not in the usual sense of van der Waerden, but to refer to a point in non-special position.) Another theorem (p. 123) states that the "general" 3-space cubic surface can be written as a 3×3 determinant set equal to zero, where the nine entries are linear forms: here again "general" means "non-special," but the fact that a sketch of the proof refers to the principal of counting constants, so that tacitly the term "general" is used in two senses, might easily confuse a reader not familiar with either sense.

A more serious criticism is, however, the fact that in place of a definite algebraic technique there is substituted a tacit and vague assumption that the various elements entering into an algebro-geometric configuration must somehow be algebraically related, an assumption which no doubt has heuristic value. Consider, for example, the situation in the theorem on p. 61; there one has a series of sets G^n of n points on a line (that is, the n solutions x of a polynomial equation $H(t, x)=0$, t, a parameter) of which it is assumed that i) the points of a generic set G^n of the series are all distinct and all variable, and ii) an arbitrary point of the line belongs to precisely one set of the series. If one makes an arbitrary point x' on the line correspond to the other $n-1$ points x of the G^n containing it, then one obtains in this way an $(n-1, n-1)$ correspondence $x'\leftrightarrow x$, and it is very plausible to suppose that the correspondence is algebraic, that is, that there exists a polynomial relation $F(x, x')=0$. But what is the proof? The text does not even hint that there is a question here. Actually the amount of algebra required in this connection, and for the most part throughout, is quite limited, and it seems unfortunate that this minimal quantity of algebra was not exploited, or at any rate illustrated in a brief fashion at, say, two or three places in the book, so that the reader might know more or less what is involved and have a confident feeling that the gaps in the argument could be removed.

There is no question that the study of this book can be richly rewarding. In extenuation of any of its possible defects, it may be emphasized that no comparable exposition of the material covered is available in any other text in English.

ABRAHAM SEIDENBERG

Advanced Calculus For Engineers. By F. B. Hildebrand. New York, Prentice Hall, 1949. 14+594 pages. $6.00.

This book is not written from the standpoint of analysis, as the Advanced Calculus part of its title might lead one to suppose. Instead, its scope and manner of presentation are more nearly what one would expect to find in a book on applied or engineering mathematics. In fact, it could almost be considered as a book on differential equations from the applied viewpoint.

There are four chapters on ordinary differential equations, Chapters 1, 3, and 4 being on analytical, numerical, and series methods of solution, and Chapter 5 on boundary value problems. In addition, there are the following chapters: Chapter 2 on the Laplace transform, Chapter 6 on vector analysis, Chapter 7 on partial differential equations, Chapter 8 on solution of partial differential equations in mathematical physics, and Chapter 9 on complex variables. The chapters are largely independent of each other. For example, the Laplace transform, though introduced in Chapter 2, is used only in one isolated case in the rest of the book. The reviewer was most favorably impressed by Chapters 4, 5, 7, and 8. The book contains a wealth of material concisely and accurately presented, with examples from time to time to illustrate the theory. The many fine problems at the end of each chapter (an average of 32 per chapter) serve to illustrate portions of the text and to introduce additional material. Answers are given to all problems.

The reviewer feels that the author should have made more use of diagrams to aid and illustrate his explanations. Also, it would seem better to use less frequently such phrases as "Thus it might be expected that. . . . ", "The appearance of the equation suggests. . . . ", "It is clear that. . . . ", to imply that something follows obviously, when often it does not—especially not to the beginner. The reader will probably find a need for working out missing steps occasionally, as the author often leaves out details of manipulation. Then too, it is recommended that one read carefully and ponder as he reads, since otherwise the significance of many comments could be missed.

Since the book is more informative than instructive and is rather condensed, it does not seem desirable for use as a textbook except for very good students, or ones who have had a considerable amount of applied mathematics. Because the topics are somewhat independent of each other, however, and many of them very fine, portions of the book might be used to advantage in some courses. As a reference book it should be excellent. In fact, the reviewer recommends it very highly for the library of every person who uses or expects to use mathematics to tackle physical problems. When using it for reference, the reader should be sure to look at the examples at the end of each chapter, since much new material is given there in the form of problems.

W. P. Reid

First-Year Mathematics for Colleges. By P. R. Rider. New York, The Macmillan Company, 1949. 15+714 pages. $5.00.

This book contains those topics in algebra, plane trigonometry, and plane and solid analytic geometry usually given in a first year college course in mathematics. In general, it reprints within the compass of a single volume the contents of its author's *College Algebra*, *Plane Trigonometry*, and *Analytic Geometry* with occasional changes in wording and some rearrangement of material. The only topics from these books which are omitted from the present volume seem to be the methods of solution of cubic and quartic equations of Cardan and Ferrari and finite differences. The exercises in trigonometry and analytic geometry are essentially the same as in the parent volumes, while those from algebra are new and somewhat more numerous. The order of topics is as follows: elementary algebra through quadratics and progressions (158 pages), logarithms and the graphs of exponential, logarithmic, and power functions (40 pages), plane trigonometry (139 pages), polynomials and the theory of equations (29 pages), plane analytic geometry (159 pages), probability, determinants, partial fractions, and infinite series (58 pages), and solid analytic geometry (49 pages). There are 19 pages of tables, a complete index, and answers to odd-numbered problems.

Besides having the very real merits of clear exposition, well-chosen illustrative material, attractive diagrams, and numerous exercises possessed by the original texts of which it is composed, the book is attractively printed and highly readable. Practically no errors were found, except that Figure 38.17, noted by a previous reviewer, remains uncorrected. The book should be well suited for effective teaching of a course which combines materials from algebra, trigonometry, and analytic geometry, and should also serve as a useful reference book for elementary college mathematics.

M. P. FOBES

NEW BOOKS RECEIVED

Calcul Operationnel. By Edouard Labin. Paris, Masson and Cie, 1949. 150 pp. 780 fr.

Introduction to the Theory of Statistics. By A. M. Mood. New York, McGraw-Hill, 1950. 9+433 pp. $5.00.

A Course of Mathematical Analysis. Second Edition. By Shanti Narayan. Delhi, Chand, 1949. 5+304 pp. Rs. 15/.

Lecons sur Quelques Equations Fonctionnelles. (Number III.) By E. Picard. Paris, Gauthier-Villars, 1950. 3+187 pp. 400 fr.

Lecons sur quelques types simples d'equations aux derivees partielles. (Number I). By E. Picard. Paris, Gauthier-Villars, 1950. 4+214 pp.

Gesammelte mathematische Abhandlungen. (Vol. I.) By Schlafli. Basel, Birkhauser, 1950. 392 pp. 54 fr.

Die Entwicklung der Infinitesimalrechnung. (Vol. LVI.) By Otto Toeplitz. Berlin, Springer-Verlag, 1949. 10+180 pp. DM 22.60.

Arithmetic for Colleges. By H. D. Larsen. New York, McGraw-Hill, 1950. 11+275 pp. $3.75.

Classical Mechanics. By Herbert Goldstein. Cambridge, Mass., Addison Wesley Press, 1950. 399 pp. $6.50.

The Foundations of Arithmetic. Translated by J. L. Austin from *Die Grundlagen der Arithmetik* by Dr. G. Frege, Breslau Verlag von Wilhelm Koebner, 1884). Philosophical Library, New York, 1950. xi+119 pp.

Grundlagen und Analytischer Aufbau der Geometrie. By L. Heffter. Leipzig, B. G. Teubner, 1950. 190 pp. $3.00.

La Theorie de la Relativite Restreinte. By O. C. de Beauregard. Paris, Masson, 1949. 6+174 pp. 800 fr.

Enciclopedia delle matematiche elementari. Vol. 3, Part 2. By L. Berzolari. Milano, Ulrico Hoepli, 1950. 19+1037 pp. L. 3800.

La Geometrie Integrale du Contour Gauch. By A. Bloch and G. Guillaumin. Paris, Gauthier-Villars, 1949. 6+141 pp. 1500 fr.

Les Principes de l'analyse Geometrique. Vol. I: *Lecons de Geometrie Vectorielle.* Third Edition. By Georges Bouligand. Paris, Vuibert, 1949. 436 pp. 1.500 fr.

Theoretische Mechanik. Vol. LVII. By G. Hamel. Berlin, Springer-Verlag, 1949. 16+796 pp. DM 66.

Description of a Relay Calculator. (The Annals of the Computation Laboratory of Harvard University, vol. 24.) Cambridge, Harvard University Press, 1949. 18+366 pp. $8.00.

Cours de Mecanique Rationnelle. Vol. IV: Problemes et Exercices. By L. Roy. Paris, Gauthier-Villars, 1950. 12+276 pp.

Geometrical Constructions with a Ruler Given a Fixed Circle with Its Center. By J. Steiner. Translated by M. E. Stark; edited with an introduction and notes by R. C. Archibald. The Scripta Mathematica Studies, No. 4. New York, Scripta, 1950. 3+88 pp. $2.00.

Nonlinear Vibrations. Vol. 2: Pure and Applied Mathematics. By J. J. Stoker. New York, Interscience, 1950. 20+273 pp. $5.00.

The Mathematical Theory of Communication. By C. Shannon and W. Weaver. The University of Illinois Press, 1949. 117 pp. $2.50.

Primer of College Mathematics. By J. F. Randolph. New York, MacMillan, 1950. 14+545 pp. $4.75.

The Anatomy of Mathematics. By R. B. Kershner and L. R. Wilcox. New York, The Ronald Press, 1950. xi+416 pp. $6.00.

Theory of Equations. By R. Behari and H. Gupta. Delhi, India, Chand, 1947. 4+169 pp. Rs. 7/8.

Analytic Geometry and Calculus. By H. J. Gay and R. K. Morley. New York, McGraw-Hill, 1950. 8+524 pp. $5.00.

Plane Trigonometry. By J. Corliss and W. Berglund. New York, Houghton Mifflin, 1950. 12+388 pp. $3.00.

Probability and the Weighing of Evidence. By I. J. Good. New York, Hafner Publishing Company, 1949. 119 pp. $3.00.

Basic Mathematics for General Education. By Trimble, Bolser and Wade. New York, Prentice-Hall, 1950. 13+313 pp. $3.25.

Measure Theory. By P. Halmos. New York, Van Nostrand, 1950. 12+304 pp. $5.90.

Anwendung der elliptischen Funktionen in Physik und Technik. Die Grundlehren der Mathematischen Wissenschaften, vol. 55. By F. Oberhettinger and W. Magnus. Berlin, Springer, 1949. 7+126 pp. 15.60 DM.

The Meaning of Relativity. By A. Einstein. Princeton University Press, 1950 (Third Edition). 5+150 pp. $2.50.

Elements de Calcul Tensoriel. By A. Lichnerowicz. Paris, Colin, 1950. 216 pp. 180 fr.

Differential Algebra. By J. F. Ritt. American Mathematical Society Colloquium Series, Vol. 33. New York, 1950. 8+184 pp. $4.40.

An Introduction to Vector Analysis. By B. Hague. New York, John Wiley and Sons, 1950. viii+122 pp. $1.25.

Die Welt der Vektoren. By F. Ollendorff. Springer-Verlage in Wien 1, Mölkerbastei 5, 1950. Mit 68 Text-abb. viii, 470 S. 1950. $9.00, geb $9.50.

Atomic Energy. By Frank Gaynor. Pocket Encyclopedia of New York, Philosophical Library, 1950. 204 pp. $7.50.

An Introduction to Probability Theory and Its Applications. By William Feller. New York, John Wiley, 1950. xii+419 pp. $6.00.

Normal Elliptic Functions. By A. R. Low. Toronto, University of Toronto Press, 1950. 27 pp.+6 tables. $1.25.

Methods of Logic. By W. V. O. Quine. New York, Henry Holt, 1950. xx +264 pp. $2.90.

Acceptance Sampling, A Symposium. By the American Statistical Association. Washington, 1950. 4+155 pp. $1.50.

Electromagnetic Fields, Theory and Application, Mapping of Fields. By Ernst Weber. John Wiley and Sons, Inc., 1950. xiv+590 pp. $10.00.

Some Theory of Sampling. By W. E. Deming. John Wiley and Sons, 1950. xvii+602 pp. $9.00.

Commercial Algebra, 3rd ed. By Simpson, Pirenian and Crenshaw. Prentice-Hall, 1950. pp. 12+173+19 (tables)+9 (answers). $2.50.

Intermediate College Algebra. By E. M. J. Pease. Prentice-Hall, 1950. pp. 7+420+36 (answers). $2.85.

Carus Monograph No. 9. The Theory of Algebraic Numbers. By Harry Pollard. Mathematical Association of America and distributed by John Wiley and Sons, New York, 1950. xiii+143 pp. $3.00.

Carus Monograph No. 10. The Arithmetic Theory of Quadratic Forms. By B. W. Jones. Mathematical Association of America and distributed by John Wiley and Sons, New York, 1950. x+212 pp. $3.00.

Methods of Mathematical Physics. By H. Jeffreys and B. S. Jeffreys. New York, Cambridge University Press, 1950. 12+708 pp. $15.00.

CLUBS AND ALLIED ACTIVITIES

EDITED BY L. F. OLLMANN, Hofstra College

Send reports of all activities, such as club reports, special features, topics with references, student papers, and other material of interest to L. F. Ollmann, Hofstra College, Hempstead, New York.

CLUB REPORTS, 1949–50

Pi Mu Epsilon, University of Arkansas

The following papers were presented before the *Pi Mu Epsilon* fraternity during the year 1949–50:

Opportunities in Cuba for mathematicians, engineers, and scientists, by Prof. B. H. Gundlach

Report on the Christmas mathematics meetings, by Prof. D. P. Richardson

Nuclear research and technology, by Prof. H. W. Schwartz of the Physics department

Operation of the automatic Friden calculator, by Mr. R. H. McCarrol.

The chapter sponsored a very successful tutoring service to aid the slower students in mathematics. Members of the chapter volunteered their services, and a chairman was named.

The annual fall initiation banquet was held in December, when 23 students were initiated, and an interesting program arranged.

At each of the monthly meetings, the faculty advisers distributed interesting and unique problem sheets to members, to be discussed at the following meeting.

The officers to serve for the coming year are: President, Bill Spinelli; Vice-President, Eric Li; Secretary, Jess Olive; Treasurer, Bill Robinson; Publicity Director, Bob Doyle; Faculty Advisers, B. H. Gundlach and S. L. Hull.

Pi Mu Epsilon, New York University

The following topics were presented at meetings of the *Delta Chapter* of *Pi Mu Epsilon* at Washington Square College of New York University during 1949–50:

Convex sets, by Rudolph van Heijenoort of the Mathematics Department of Washington Square College

Theory of relativity, by Prof. B. Hoffman of Queens College

Computing machines, by Eleanor Krawitz of the Watson Scientific Laboratories of Columbia University

Mathematical applications to some sensory constructs, by Mr. Victor Twersky of the Institute for Mathematics and Mechanics.

A business meeting was held to discuss the revision of the constitution and election of new officers.

The annual *Pi Mu Epsilon* dinner took place at the Fifth Avenue Hotel. Prof. Albert Hofstadter of the Philosophy department spoke on *Mathematics and philosophy*.

At an April meeting Dr. B. MacMillan of the Bell Telephone Laboratories

spoke on *An introduction to information theory.*

The following officers and committees were elected for the school year 1950–1951: Chapter Director, Elaine Weiss; Vice-Director, Alvin Saperstein; Secretary-Treasurer, Mordecai Schwartz; Faculty Advisor and Permanent Secretary, Dr. Gottfried Noether.

Pi Mu Epsilon, University of Missouri

Forty-eight members were initiated in the *Missouri Alpha* Chapter of *Pi Mu Epsilon* during the year. Programs of interest were the following:

Mathematics among the Babylonians, by Prof. Herman Betz
Some Diophantine problems, by Prof. W. R. Utz
Coverage theorems, by Prof. L. M. Blumenthal
Examples of non-linear vibrations, by Prof. George Ewing
Mechanism of calculating machines, by Prof. Paul Burcham.

The "Problem of the Month" was posted at the beginning of each month, and a one-dollar prize was posted for the first person submitting a contest solution.

Winners in the annual calculus competition were: Delbert Calvert, first prize $15.00; Harley Newsom, second prize $10.00; and Charles Floyd, third prize, $5.00.

The annual banquet was the high light of the year. Ninety-two members and guests were present. Prof. Herman Betz served as toastmaster, calling on the following persons for short talks, Prof. G. Ewing, Miss Mary Cummings, Sidney Minnick, Bill Nichols and David Neebe.

Officers for 1950–51 are: President, David Neebe; Vice-President, Richard Wood; Secretary, Raymond Poyner; Treasurer, Aldo Linsenbardt; Faculty Adviser, Miss Mary Cummings.

Graduate Mathematics Club, Indiana University

The *Graduate Mathematics Club* instituted a policy of having short talks by graduate students in addition to the talks by members of the department of mathematics. The topics and speakers were:

Squaring the circle, by Dr. V. Hlavaty
On the arrangement of squares, by Dr. W. Gustin
The sandwich problem, by Dr. J. W. T. Youngs
Helly's theorem in the plane, by I. Reff
Every integer is the sum of four squares, Mrs. Thompson
e is transcendental, by J. A. Sullivan
The probability that two integers are relatively prime, by W. Perel
An isoperimetric problem, by J. B. Serrin
Weierstrass approximation theorem, by D. E. Van Tijn
A proof of Bertrand's postulate, by N. Shklov
The existance of a free group, by L. C. Graue
The derived congruences of Darboux, by L. K. Frazer

Poincare's recurrence theorem, by D. M. Nead
A paradox, a most ingenious, by W. F. Brown
The Banach-Tarski paradox, by B. H. McCandless.

The executive committee for this year consists of: David Van Tijn, Lowell Frazer, and Israel Reff.

Kappa Mu Epsilon, Upsala College

The *New Jersey Alpha* Chapter of *Kappa Mu Epsilon* had six regular meetings and three special meetings during the year 1949–50. The regular meetings were devoted to the background and foundation of elementary mathematics by means of student papers. These covered algebra, geometry, analytics, and calculus, and were given by Victor Valentino, James Christakos, William Stachel, Edward Alquist, Ethel Larson, and Robert Ruppert. In addition, Prof. M. A. Nordgaard discussed the contribution of the Arabs to our present type of algebra, and Prof. Robert Reed and Prof. Donald Lindtvedt gave two talks entitled *Previews of topics in graduate mathematics.*

The *New Jersey Beta* Chapter of *Kappa Mu Epsilon* at Montclair Teachers College, invited the fraternity to a social hour and a talk by Prof. Howard Fehr on *Appreciation of elementary mathematics.* Upsala's *Mathematics Club* and *Kappa Mu Epsilon* Chapter had a joint social meeting at which Prof. Virgil Mallory spoke on the *Foundation of our enumeration and notation.* The annual banquet and initiation of new members was held; Prof. Edward Molina was the honor guest and speaker.

The officers for 1950–51 are: President, William Stachel; Vice-President, Lloyd Johnson; Treasurer, Robert Ruppert; Secretary, Ethel Larson; Recording Secretary, Dr. M. A. Nordgaard; Historian, Prof. D. Lindtvedt.

Pi Mu Epsilon, University of Wisconsin

The chapter's program for 1949–50 included the following talks:
A uniqueness theorem, by Prof. O. G. Owens
Application of elliptic functions to geometry, by Joseph Zemmer
Applications of matrices to linear algebras, by Jacob Goldhaber
The endomorphisms of a finite Abelian group, by Donald Morrison
An extension of the Hermite canonical form for a matrix, by Leonard Fuller.

In the fall a get-acquainted open house was held for the mathematics faculty, graduate students and guests. A picnic and the initiation banquet were held in the spring.

Officers for 1949–1950 were: President, Gerald P. Dinneen; Vice-President, Leonard E. Fuller; Secretary-Treasurer, Doris Efram; Faculty Adviser, Prof. R. E. Fullerton.

Pi Mu Epsilon, University of California, Los Angeles

The *California Alpha* chapter of *Pi Mu Epsilon* has had eight meetings during 1949–50. Of these, two were purely business meetings, two were to initiate

new members, and four had speakers or other programs. The programs included:

Summability and Tauberian theorems, by Dr. G. Milton Wing

The theory of plasticity, by P. G. Hodge

Unified field theory, by Dr. E. G. Straus

The Putnam prize examination: Solutions were presented by the team composed of John Juhns, Genevevo Lopez, John Harrington, William Sibley and William Stalder.

The chapter sponsors an annual contest in calculus. First prize went to Leonard Ross, second prizes to Seymour Singer and to Jack Sandweiss.

Twenty-eight new members were initiated at the two initiation meetings. As is the chapter custom, the first was held in January at a local restaurant, and the May meeting was held at the home of one of the members of the faculty.

The officers for 1950–51 are: Director, Irving Glicksberg; President, Rhodo Cagan; Secretary, James Jackson; Treasurer, Dr. W. Puckett, Jr.; Faculty Adviser, Dr. G. Milton Wing; Membership Committee, Erick McAllister, David Pope, Richard Tibbitts; Library Committee, William Bade, Milton Galper, Edward Law.

Michigan Undergraduate Mathematics Conference

At the spring meeting of the Michigan Section of the Mathematical Association of America, a special undergraduate conference was held. The Wayne University chapter of *Kappa Mu Epsilon* and the Officers of the Michigan Section of the Association were joint sponsors of this conference. Seventy-nine students representing eight colleges in Michigan and Ohio attended.

The students heard papers presented by members of the Michigan Section during the morning session and, after a luncheon at the Michigan Union, visited the rare books collection and statistics laboratories on the University of Michigan campus. During the special afternoon session for undergraduates the following papers were presented:

Solutions of the quadratic equation, by Raymond Gillespie, Albion College.

Mechanical brains, by Richard Little, Central Michigan College of Education

The inverse integral of the LaPlace transformation, by Christoph Neugebauer, University of Dayton

Notes on the beginnings of calculus taken from original sources, by students of the University of Michigan:

(a) *Isaac Barrow's differential triangle*, by David Lincoln

(b) *Newton's approaches to the calculus*, by Cecelia Woodworth

(c) *Leibniz' first paper in Acta Eruditorum*, by Charles Federspiel

Gibb's phenomenon, by Marvin Snider, Wayne University

Complex Fourier series, by Richard Segers, University of Dayton.

The *Kappa Mu Epsilon* chapter at Central Michigan College of Education, Mt. Pleasant, is sponsoring another undergraduate conference which will be held in the spring of 1951.

NEWS AND NOTICES

EDITED BY EDITH R. SCHNECKENBURGER, University of Buffalo

Readers are invited to contribute to the general interest of this department by sending news items to Edith R. Schneckenburger, University of Buffalo, Buffalo 14, New York. Items must be submitted at least two months before publication can take place.

POSTDOCTORAL FELLOWSHIP OF SIGMA DELTA EPSILON

Sigma Delta Epsilon, graduate women's scientific fraternity, announces that it will award a postdoctoral fellowship for the year 1951–52. The stipend will be $1,600. Applications should be submitted before February 1, 1951, to the Fellowship Awards Board authorized to make the award.

Women with the equivalent of a Ph.D. degree, carrying on research in the mathematical, physical, or biological sciences, who need financial assistance and give evidence of high ability and promise are eligible. During the term of her appointment the appointee must devote the major part of her time to the approved research project and not engage in other work for remuneration (unless such work shall have received the written approval of the Awards Board before the award of the fellowship).

Application blanks may be secured from Dr. Mayme I. Logsdon, The University of Miami, Coral Gables 46, Florida. Announcement of the award will be made early in March.

PERSONAL ITEMS

Mrs. Elsie M. Buck of Boise Junior College was the representative of the Association at the inauguration of President P. M. Pitman of the College of Idaho on October 14, 1950.

Professor H. V. Craig of the University of Texas represented the Association at the inauguration of President M. T. Harrington of Texas Agricultural and Mechanical College on November 9, 1950.

Professor I. O. Horsfall, University of Utah, was appointed to represent the Association at the Diamond Jubilee in Commemoration of the Founding of Brigham Young University on October 16–17, 1950.

Mrs. L. S. Hunter of Virginia State College served as representative of the Association at the inauguration of President R. P. Daniel of Virginia State College on October 14, 1950.

Associate Professor J. H. Roberts, Duke University, was the representative of the Association at the inauguration of President Gordon Gray of the consolidated University of North Carolina on October 8–10, 1950.

Assistant Professor Edith R. Schneckenburger of the University of Buffalo represented the Association at the Convocation for the formal presentation of the University Charter to St. Bonaventure University on October 4, 1950.

Canisius College announces the promotions of Assistant Professor James McGowan to an associate professorship and Instructor Frank Davenport to an assistant professorship.

Cornell University announces the appointments of Dr. W. H. J. Fuchs, lecturer at the University of Liverpool, to an associate professorship and of Dr. A. S. Shapiro, formerly A. E. C. fellow at the University of Chicago, to an instructorship.

Duquesne University reports: Assistant Professor Ruth E. Goodman has been promoted to an associate professorship; Mr. B. L. Schwartz has been appointed to an instructorship.

Florida State University announces the following appointments: Professor H. H. Hyman of Henderson State Teachers College to an associate professorship; Lecturer S. L. Jamison, University of California at Berkeley, and Dr. E. R. Keown, Massachusetts Institute of Technology, to assistant professorships; Dr. H. E. Taylor, graduate fellow at Rice Institute, and Mr. W. W. Page, graduate student at North Carolina Institute of Statistics, to instructorships.

Georgia Institute of Technology announces: Professor D. M. Smith has retired as Chairman of the Department but continues as Professor of Mathematics; Professor H. K. Fulmer has been named Acting Head of the Department of Mathematics; Associate Professors A. H. Bailey and Walter Reynolds have been promoted to professorships.

Massachusetts Institute of Technology reports: Associate Professor G. B. Thomas, Jr. has been appointed Executive Officer of the Department; Associate Professor Raphael Salem has been promoted to a professorship; Assistant Professors Warren Ambrose, F. B. Hildebrand and G. B. Thomas, Jr. have been promoted to associate professorships; Dr. Walter Rudin and Dr. I. M. Singer have been appointed C. L. E. Moore Instructors.

At McMaster University: Dr. D. B. Sumner, formerly senior lecturer at the University of Witwatersrand, has been appointed to an assistant professorship; Mr. F. R. Britton has been promoted to the position of lecturer.

Newark College of Engineering announces the promotions of Assistant Professor Harold Wasson to an associate professorship and of Instructor Herbert Barkan to an assistant professorship.

Rensselaer Polytechnic Institute makes the following announcements: Instructors A. S. Hendler and M. R. Spiegel have been promoted to assistant professorships; Mr. W. R. Beck, instructor at Fort Wayne Center of Purdue University, has been appointed to an instructorship.

Texas Technological College announces the appointments of Professor Gordon Fuller of Alabama Polytechnic Institute to a professorship and Dr. Leo Moser of the University of North Carolina to an associate professorship.

At the University of Alaska: Assistant Professor W. R. Cashen has been promoted to the position of Associate Professor and Head of the Department of Mathematics; Professor W. E. Duckering, formerly head of the Department of Civil Engineering and Mathematics, has been promoted to the position of Dean of the University; Mr. E. L. Dolney, previously graduate student at Notre Dame, has been appointed to an instructorship.

The University of Florida reports: Dr. David Ellis of the University of

Missouri has been appointed to an assistant professorship; Dr. F. Virginia Rhode of the University of Kentucky has been appointed to an instructorship; Instructor E. J. Lytle, Jr. has been called to active military duty.

University of Maryland announces: Mr. W. T. Sharp, previously graduate student at Princeton University, has been appointed to an assistant professorship; Professor Alexander Weinstein has been transferred to the Institute for Fluid Dynamics and Applied Mathematics of the University.

At the University of Michigan: Assistant Professors R. C. F. Bartels and Wilfred Kaplan have been promoted to associate professorships; Instructors D. A. Darling and W. J. LeVeque have been promoted to assistant professorships; Dr. J. R. Büchi, Ripon College, Dr. H. C. Davis, Harvard University, Dr. J. R. Lee, Yale University, Mr. G. R. Livesay, University of Illinois, Dr. J. E. McLaughlin, California Institute of Technology, and Mr. Daniel Resch, Syracuse University, have been appointed to instructorships.

Professor Eugene Alliott of St. Edmund's Juniorate, Swanton, Vermont, is now at St. Michael's College, Winooski Park, Vermont.

Dr. D. F. Atkins of the University of Kentucky has been appointed to an assistant professorship at Bowling Green State University.

Dr. P. T. Bateman, previously at the Institute for Advanced Study, has been appointed to an assistant professorship at the University of Illinois.

Mr. E. H. Batho, formerly a student at Fordham University, is now a teaching assistant at the University of Wisconsin.

Mr. H. S. Berg, graduate assistant at the University of North Dakota, has been appointed to an instructorship at State Teachers College, St. Cloud, Minnesota.

Assistant Professor C. M. Bjork of Northern Michigan College of Education has been promoted to an associate professorship.

Dr. Archie Blake, who has been Senior Statistician in the Office of Army Surgeon General, Washington, D. C., has accepted a position as Treasurer and Mathematics Consultant for the Mechanical Research Corporation, Chicago, Illinois.

Professor C. W. Bruce of Wesleyan College has been appointed Head of the Physics Department at Tennessee Polytechnic Institute.

Associate Professor A. H. Clifford of Johns Hopkins University is now in the United States Naval Reserve.

Mr. Harold Glander, formerly a graduate student at the University of Chicago, has received an appointment as instructor at Superior State College, Superior, Wisconsin.

Mr. D. A. Gorsline, previously a graduate assistant at the University of Oklahoma, is teaching at Cambridge Central School, Cambridge, New York.

Mr. H. M. Hardy, who has been Head of the Mathematics Department of Hillsboro Junior College, has accepted a position as a teacher in Uvalde High School, Uvalde, Texas.

Mr. L. C. Hartsell has been appointed to an assistant professorship at State

Teachers College, Troy, Alabama.

Associate Professor L. D. Hemenway of Simmons College has been promoted to a professorship.

Assistant Professor Shizuo Kakutani of Yale University has been promoted to an associate professorship.

Mr. W. D. Krentel, formerly a graduate fellow at Oklahoma Agricultural and Mechanical College, is now in the United States Air Force, Brooks Air Force Base, San Antonio, Texas.

Associate Professor R. R. Kuebler, Jr., is on leave of absence from Dickinson College for the purpose of studying at Columbia University.

Dr. R. C. Lyndon, Princeton University, has been promoted to an assistant professorship.

Associate Professor D. D. Miller of the University of Tennessee has been recalled to active duty in the United States Navy.

Dr. Morris Morduchow of the Polytechnic Institute of Brooklyn has been promoted to an assistant professorship.

Dr. C. V. Newsom, who has been Assistant Commissioner for Higher Education in the State Education Department, New York, has been appointed Associate Commissioner for Higher and Professional Education.

Assistant Professor Abba V. Newton of Vassar College has been promoted to an associate professorship.

Mr. O. L. Phillips, formerly head of the Mathematics Department of Mississippi Southern College, is now Director of the Bureau of Field Service and Professor of Mathematics at East Carolina Teachers College, Greenville, North Carolina.

Assistant Professor W. P. Reid of the United States Air Force Institute of Technology, Wright Field, Dayton, Ohio, has been appointed a mathematician at the United States Naval Ordnance Test Station, China Lake, California.

Dr. Harold Shniad of the University of Southern California has been appointed to an assistant professorship at the University of Arkansas.

Sister M. Madeleine Rose has been appointed President of the College of the Holy Names, Oakland, California.

Associate Professor Henry Wallman of Massachusetts Institute of Technology has accepted a professorship in the Electrical Engineering Department of the Chalmers Institute of Technology, Gothenburg, Sweden.

Mr. R. W. Young of the University of Florida has been appointed Professor and Head of the Department of Mathematics of Henderson State Teachers College, Arkadelphia, Arkansas.

Professor C. K. Alexander of Occidental College died on October 4, 1950.

Associate Professor Joseph Clare of Knox College died on October 11, 1950.

Mrs. E. H. Moore, Chicago, Illinois, died on October 23, 1950.

Professor Emeritus A. J. Pyke of the University of Saskatchewan died on June 5, 1950.

Professor S. E. Rasor of Ohio State University died October 17, 1950. He was a charter member of the Association.

Professor Emeritus S. W. Reaves of the University of Oklahoma who was a charter member of the Association died on August 2, 1950.

Professor Emeritus F. H. Safford of the University of Pennsylvania died on October 29, 1950. He was a charter member of the Association.

Professor Emeritus C. C. Spooner, Northern Michigan College of Education, died October 7, 1950. He was a charter member of the Association.

Associate Professor Arthur Tilley, Washington Square College, New York University, died on October 4, 1950. He had been a member of the Association for twenty-eight years.

THE MATHEMATICAL ASSOCIATION OF AMERICA

NEW MEMBERS

Professor H. M. Gehman, Secretary-Treasurer, announces that the following one hundred and one persons have been elected to membership by the Board of Governors on applications duly certified:

T. M. Apostol, Ph.D. (California) Asst. Professor, California Institute of Technology, Pasadena, Calif.

P. H. Arnold, A.B. (Nebraska Wesleyan) Grad. Assistant, Kansas State College, Manhattan, Kans.

Miriam C. Ayer, Ph.D. (Ohio State) Asst. Professor, Wellesley College, Mass.

Mr. Souren Babikian, B.S. (London) Instructor, Aleppo College, Syria

W. R. Baum, Dr.Sc.Math. (Swiss Fed. Inst. of Tech.) Asst. Professor, University of Tennessee, Knoxville, Tenn.

Walter Beerman, B.S.M.E. (Michigan) Consulting Engineer, 332 West 46th Street, New York, N. Y.

Z. W. Birnbaum, Ph.D. (Lwow) Professor, University of Washington, Seattle, Wash.

J. A. Boose, B.S. (St. Ambrose) Instructor, Spalding Institute, Peoria, Ill.

J. H. Braun, Student, Illinois Institute of Technology, Chicago, Ill.

Albert Brown, Student, Wayne University, Detroit, Mich.

H. W. Burnette, M.S. (Georgia) Instructor, Alabama Polytechnic Institute, Auburn, Ala.

W. O. Buschman, M.Ed. (Oregon) Instructor, Vanport Extension Center, Portland, Ore.

J. G. Campbell, Student, University of Kentucky, Lexington, Ky.

H. P. Carter, M.A. (Vanderbilt) Instructor, David Lipscomb College, Nashville, Tenn.

E. W. Coffin, B.S. (Maryland) 705 18th Street N. W., Washington, D. C.

Carl Cohen, M.A. (Harvard) Instructor, Cambridge Junior College, Mass.

G. J. Corley, M.A. (Columbia) Asst. Professor, Northwestern State College, Natchitoches, La.

Pauline E. Crow, M.A. (Pittsburgh) Instructor, Carnegie Institute of Technology, Pittsburgh, Pa.

D. A. Darling, Ph.D. (C.I.T.) Asst. Professor, University of Michigan, Ann Arbor, Mich.

S. L. Day, M.E. (Pratt) Senior Draftsman, Century Projector Corporation, Long Island City, N. Y.

R. C. DiPrima, B.S. (Carnegie) Grad. Student and Part-time Instructor, Carnegie Institute of Technology, Pittsburgh, Pa.

Genevieve H. Dixon, M.A. (Buffalo) Computer, Bell Aircraft Corporation, Niagara Falls, N. Y.

M. P. Dossey, M.S. (Texas) Asso. Professor, Southwestern Louisiana Institute, Lafayette, La.

DAVID ELLIS, Ph.D. (Missouri) Asst. Professor, University of Florida, Gainesville, Fla.

FRANK EPSTEIN, M.A. (Columbia) Instructor, New York State Institute of Applied Arts and Sciences, Brooklyn, N. Y.

J. J. FISCHER, M.S. (Illinois Inst. of Tech.) Instructor, Morgan Park Junior College, Chicago, Ill.

JOYCE B. FRIEDMAN, B.A. (Wellesley) Mathematician, George Washington University, Washington, D. C.

W. H. FROM, B.A. (Virginia) Grad. Student, University of Virginia, Charlottesville, Va.

R. A. GAMBILL, Student, Butler University, Indianapolis, Ind.

C. F. GAYTON, Jr., B.S.E.E. (Nebraska) Instructor, Electronic Radio Television Institute, Omaha, Nebr.

MABEL E. GREGG, B.S. (East Tennessee S.C.) Teacher, Church Hill School, Tenn.

VIOLET G. HACHMEISTER, Ph.D. (Wisconsin) Asst. Professor, Kansas State College, Manhattan, Kans.

HELEN V. HAMMAR, B.S. (Illinois Inst. of Tech.) Mathematician, Navy Department, David Taylor Model Basin, Washington, D. C.

C. V. HANNAN III, Maintenance Technician, Civil Aeronautics Administration, Kansas City, Mo.

LENA L. HAYS (Mrs. A. Z.), M.E. (Texas Tech.) Instructor, Abilene Christian College, Tex.

P. W. HEALY, Ph.D. (Kentucky) Asso. Professor, Southwestern College, Winfield, Kans.

J. J. HENRICK, Student, Texas Christian University, Ft. Worth, Tex.

MELVIN HENRIKSEN, M.S. (Wisconsin) Grad. Assistant, University of Wisconsin, Madison, Wis.

H. A. HILL, B.S. (U. of Washington) R.F.D. 1, Box 41, Freewater, Ore.

H. H. HINMAN, A.M. (Columbia) Asst. Professor, City College of the City of New York, N. Y.

D. B. HOAGLAND, M.A. (Missouri) Instructor, Trenton Junior College, Mo.

DOROTHY S. HOOPER, (Mrs. W. J.), M.A. (California) Professor, The Principia College, Elsah, Ill.

J. D. HOUSTON, JR., B.S. (Clark C.) Teacher, Westside Senior High School, Talladega, Ala.

W. H. INGRAM, M.Sc. (Cantab) 150 Claremont Avenue, New York, N. Y.

S. L. JAMISON, Ph.D. (California) Asst. Professor, Florida State University, Tallahassee, Fla.

T. M. JASPER, M.A. (Texas Tech.) Asst. Professor, Arlington State College, Tex.

R. H. JOHNSTON, B.S.Ed. (Miami U.) Instructor, Miami University, Oxford, Ohio

R. M. JOHNSTON, M.A. (Tulsa) Teacher, Tulsa Public Schools, Okla.

L. H. KANTER, Ph.D. (Stanford) Asso. Professor, Mississippi State College, Miss.

J. S. KLEIN, M.S. (M.I.T.) Instructor, Williams College, Williamstown, Mass.

W. S. KNIGHT, Student, University of Georgia, Athens, Ga.

J. H. LAWRENCE, M.ED. (Oklahoma) Teacher, Okemah High School, Okla.

W. D. LINDSTROM, B.A. (Iowa) Instructor, University of Iowa, Iowa City, Iowa

ROY LONG, Student, University of Georgia, Athens, Ga.

R. C. LYNDON, Ph.D. (Harvard) Asst. Professor, Princeton University, N. J.

B. W. MARKS, B.S. (Texas Western C.) Instructor, Midland High School, Tex.

M. M. MCFARLAND, A.A. (Santa Monica City C.) Party Chief (Surveyor), Engineering Service Corporation, Los Angeles, Calif.

A. G. MCGLASSON, B.S. (Eastern Kentucky S.C.) Instructor, Eastern Kentucky State College, Richmond, Ky.

J. E. MEJIA, E. E. (Rensselaer) Chief, Bridge Division, Highway Department, San Salvador, El Salvador

A. V. MELGAR, Secretary, Banco Internacional del Peru, Arequipa Branch, Peru

D. S. MERRILEES, B.A. (Syracuse) Grad. Student, Indiana University, Bloomington, Ind.

N. H. MEWALDT, M.A. (Wisconsin) Asso. Professor, Northern State Teachers College, Aberdeen, S. D.

HERMAN MEYER, Ph.D. (Chicago) Chairman, Department of Mathematics, University of Miami, Coral Gables, Fla.

V. A. MICULKA, M.S. (Texas Tech.) Instructor, Frank Phillips College, Borger, Tex.

B. E. MITCHELL, M.A. (Louisiana) Grad. Assistant, University of Wisconsin, Madison, Wis.

L. J. MORDELL, M.A. (Cambridge, England) Visiting Professor, University of Pennsylvania; Sadleirian Professor of Pure Mathematics, Cambridge University, England.

MORRIS MORDUCHOW, D.Ae.E. (Poly. Inst. of Brooklyn) Asst. Professor, Polytechnic Institute of Brooklyn, N. Y.

J. T. MORSE, Student, Case Institute of Technology, Cleveland, Ohio

A. R. NASIR, M.A. (Panjab) Asst. Professor, Punjab College of Engineering and Technology, Lahore, Pakistan

C. J. OBERIST, M.A. (Columbia) Mathematics Section Chief, U. S. Merchant Marine Academy, Kings Point, N. Y.

R. J. ORAVEC, M.S. (Case) Instructor, Fenn College, Cleveland, Ohio

D. S. PARK, A.B. (Tufts) 507 Beacon Street, Boston, Mass.

R. G. PIECHAN, Student, Central Michigan College of Education, Mount Pleasant, Mich.

M. T. PINEIRO, B.S. (Instituto de la Habana) Blanquizar 112, Luyano, Havana, Cuba.

W. B. PITT, Student, Moorestown Friends' School, N. J.

D. J. PRICE, Ph.D. (London) Lecturer, University of Malaya, Singapore, Malaya

P. H. RENTON, M.S. (Pittsburgh) Dean of Instruction, West Virginia Institute of Technology, Montgomery, W. Va.

W. D. RICE, M.A. (Illinois) Head, Department of Mathematics, Olivet Nazarene College, Kankakee, Ill.

S. L. ROSS, A.M. (Boston U.) Grad. Student, Boston University, Mass.

L. R. SCHAER, Student, University of Buffalo, N. Y.

SISTER MARY DEBORAH, B.A. (Seton Hill) Instructor, Seton Hill College, Greensburg, Pa.

J. L. SLOAN, M.A. (Duke) Asso. Professor, Presbyterian College, Clinton, S. C.

T. F. SMITH, M.A. (Cincinnati) Grad. Student, University of Cincinnati, Ohio

A. K. SNYDER, M.A. (N.Y.U.) Professor, Valley Forge Military Academy and Junior College, Wayne, Pa.

S. P. SPAULDING, B.S. (Harvard) Grad. Student, Boston University, Mass.

M. R. SPIEGEL, Ph.D. (Cornell) Asst. Professor, Rensselaer Polytechnic Institute, Troy, N. Y.

E. T. STAPLEFORD, M.A. (Ohio State) Instructor, Kent State University, Ohio

ANDREW STERRETT, JR., M.S. (Pittsburgh) Acting Instructor, Ohio University, Athens, Ohio

H. C. STOTZ, A.M. (Boston U.) Asso. Professor, U. S. Naval Academy, Annapolis, Md.

P. Y. TANI, B.S. (Stanford) Grad. Student, Stanford University, Calif.

F. H. TAYLOR, B.S. in Ed. (Bucknell) Grad. Assistant, Case Institute of Technology, Cleveland, Ohio

RUTH F. TERRY, M.A. (Duke) Asst. Professor, Florida Southern College, Lakeland, Fla.

J. I. THIGPEN, M.S. (North Carolina S.C.) Chemical Engineer, Hercules Powder Company, Experiment Station, Wilmington, Del.

G. B. THOMAS, JR., Ph.D. (Cornell) Asso. Professor, Massachusetts Institute of Technology, Cambridge, Mass.

EVELYN L. TRENNT, M.A. (Illinois) Instructor, Springfield Junior College, Ill.

L. R. VAN DEVENTER, M.S. (Illinois) Asst. Professor, Eastern Illinois State College, Charleston, Ill.

M. H. VANN, B.S. (Berry C.) Instructor, Reinhardt College, Waleska, Ga.

ELEANOR R. VELEK, B.A. (Carleton) First National Bank, Chicago, Ill.

S. S. WALTERS, Ph.D. (U.C.L.A.) Mathematician, The Rand Corporation, Santa Monica, Calif.

J. A. WEISS, M.A. (Ohio State) Instructor, University of Wyoming, Laramie, Wyo.

E. M. ZAUSTINSKY, A.B. (U.C.L.A.) Grad. Student, University of Southern California, Los Angeles, Calif.

SPRING MEETING OF THE MICHIGAN SECTION

The Spring Meeting of the Michigan Section of the Mathematical Association of America was held at the University of Michigan, Ann Arbor, Michigan, on March 25, 1950, in connection with the Annual Meeting of the Michigan

Academy of Science, Arts, and Letters. Professor L. E. Mehlenbacher, Chairman of the Section, presided at both morning and afternoon sessions as well as at the business meeting held immediately following the luncheon.

A total of one hundred and nineteen persons registered for the sessions including the following eighty-three members of the Association: Bess E. Allen, N. H. Anning, J. W. Baldwin, W. D. Baten, F. A. Beeler, J. E. Bellardo, C. J. Blackall, Harold Blair, Richard Brauer, C. H. Butler, W. H. Cain, C. D. Calhoon, C. R. Carr, R. E. Carr, R. V. Churchill, C. J. Coe, H. B. Coleman, A. H. Copeland, P. C. Cox, J. W. Coy, C. C. Craig, J. W. Crispin, Jr., Wayne Dancer, Violet B. Davis, D. E. Deal, M. L. DeMoss, P. S. Dwyer, P. W. Edmonson, Paul Erdös, K. W. Folley, R. W. Frankel, G. W. Grotts, V. G. Grove, H. H. Hannon, Frank Harary, G. E. Hay, Fritz Herzog, T. H. Hildebrandt, E. E. Ingalls, L. A. Jehn, L. G. Johnson, L. S. Johnston, P. S. Jones, L. M. Kelly, A. E. Lampen, Leo Lapidus, H. D. Larsen, C. C. MacDuffee, G. E. Markle, Elna B. McBride, E. D. McCarthy, L. E. Mehlenbacher, D. C. Morrow, H. W. Nace, A. L. Nelson, J. D. Novak, R. H. Oehmke, F. F. Otis, R. S. Pate, Mary H. Payne, G. Y. Rainich, E. D. Rainville, P. H. Raker, L. L. Rauch, F. A. Reiber, C. C. Richtmeyer, D. D. Rippe, J. P. Roth, E. H. Rothe, L. J. Rouse, K. C. Schraut, Jeanette H. Sherwood, Sister Mary Paula, W. F. Smith, T. H. Southard, J. G. Sowul, R. L. Spencer, E. M. Steinbach, B. M. Stewart, Lois Martin Suprock, P. C. Sweetland, R. M. Thrall, Leonard Tornheim.

The morning session and luncheon were attended by many of the undergraduates who came for the Undergraduate Mathematics Conference which in the afternoon met in a session separate from that of the Michigan Section of the Association. This conference was organized and sponsored by the Michigan Beta Chapter (Wayne University) of Kappa Mu Epsilon in collaboration with the officers of the Michigan Section of the Association. Students were invited from all colleges in Michigan and some in Ohio and Indiana whether or not they had chapters of Kappa Mu Epsilon. Eight undergraduate papers were presented by students from six colleges.

At the business meeting Professor C. C. Richtmeyer reported that the first printing of the guidance pamphlet and chart *A Mathematics Student—To Be or Not to Be?* which his committee had prepared for free distribution to Michigan high schools had been exhausted. A continuing demand had lead the officers to reprint the pamphlet for distribution at a charge of ten cents. Continuing sales have returned a slight profit to the Section. At Professor Richtmeyer's suggestion his committee was discharged, but it was recommended that the officers of the section appoint a new committee to explore other possibilities for improving the relationships between secondary school and collegiate mathematics instruction.

The nominating committee consisting of Professors A. L. Nelson, E. E. Ingalls, and L. J. Rouse reported the names of Professor D. C. Morrow, Wayne University, for 1950–51 Chairman, and Professor P. S. Jones, University of Michigan, for Secretary-Treasurer. They were elected unanimously.

The next annual meeting of the Section was set for March 24, 1951, at Michigan State College in East Lansing, Michigan.

The following papers were presented at the morning and afternoon sessions:

1. *The mathematical determination of "X" stress factors for involute spur gears*, by Mr. L. G. Johnson, Research Laboratories Division, General Motors Corporation.

In determing the bending stress at the fillet of an involute spur gear tooth, it is customary to make a layout in order to determine the so-called "X"factor, or "tooth form factor." Mr. Johnson presented the mathematics involved in such a procedure, together with a method of solving its equations. By so doing, he eliminated the necessity of making time-consuming layouts in any such gear stress determination.

2. *The solution of differential equations by means of an electronic analogue computer*, Mr. H. B. Coleman, Aeronautical Research Center, University of Michigan.

Mr. Coleman presented a general description of the construction of an electronic analogue computer, the operation of the components which perform integration and differentiation as well as the fundamental arithmetic operations, and finally its use in solving several types of first and second order differential equations.

3. *How may collegiate mathematicians assist high school teachers in stimulating their students*, Professor K. C. Schraut, University of Dayton.

Professor Schraut told of a program for encouraging secondary school students to strive for superior academic achievement, especially in mathematics, and for bringing to them an appreciation of the importance of mathematics in our cultural advancement. The major feature has been the organization and promotion of The National Mathematics Honor Society of Secondary Schools.

4. *A vector proof of the Pascal theorem*, Professor G. Y. Rainich, University of Michigan.

Projective geometry is considered as the study of properties invariant under projections of a plane w immersed in a euclidean three-space. It is treated in terms of vectors with a common initial point O of w. A point A is represented by a vector, also denoted by A, of the line OA; a line—by a vector perpendicular to the plane containing that line and O. Vectors differing by a scalar factor represent the same object (point or line of w). The vector product of two vectors representing points represents their join. Incidence of point and line is expressed by vanishing of the inner product of vectors representing them. Vanishing of triple (scalar) product means collinearity (or concurrence). Six points are called Pascalian if the meets of the opposite sides of the hexagon determined by them are collinear. Using the rules of operations on vectors it is proved that the Pascality of $ABCDEF$ is equivalent to the equality of certain two fractions A^* and B^* whose numerators and denominators are each a product of two triple products whose factors are the given points. (Independence of the Pascality of six points on their order is a consequence of structure of the above equality.) Interpreting these triple products in terms of volumes of tetrahedra with common vertex at O we find that A^* gives the cross-ratio of the joins of A with C, D, E, and F, and B^* gives the same for B, so that Pascality of the six points is equivalent to the equality of these cross-ratios. Noticing that if six points are on a circle these cross-ratios are equal because of the equality of the angles between the corresponding joins, we conclude the Pascality of six cocircular points; and considering a conic as the projection of a circle, to the Pascal theorem.

5. *Four squares*, Professor N. H. Anning, University of Michigan.

Drawing his motivation from the figure of Torricelli in which three equilateral triangles based

on the sides of an arbitrary triangle lead to a fourth equilateral triangle, Professor Anning presented a number of properties of four squares related to an arbitrary triangle. His paper will appear in full in the Fall 1950 issue of *The Pentagon.*

6. *The two-area covering problem*, Professor B. M. Stewart, Michigan State College.

This paper will appear in full in this MONTHLY.

7. *Cross-purposes in education*, Professor C. C. MacDuffee, University of Wisconin.

In his hour talk delivered at the invitation of the officers of the section, Professor MacDuffee attacked excessive emphasis on a restricted type of "functionalism" in the design of secondary school programs. He advocated at least two high school curricula in order that superior students may be educated for leadership through a program involving "some degree of difficulty and substance."

He felt that inspiring teachers are essential for that education which opens new vistas and develops "the power to understand, to correlate, and to appreciate" in an atmosphere of intellectual fun. To attract such teachers every possible effort should be made to make the position of teacher an important one in the community.

Professor MacDuffee recommended that each Section of the Association concern itself with educational problems within its borders and support an active committee to this end.

8. *Nicholas Pike and his arithmetic*, Professor E. E. Ingalls, Albion College.

Professor Ingalls told of the life of Nicholas Pike and the times in which he lived. His comments on the recommendations for Pike's *Arithmetic* included an interesting letter from George Washington which Professor Ingalls found quoted in a *History of Newburyport* published in 1854.

9. *Oscillation of a third order non-linear autonomous system*, Professor L. L. Rauch, Department of Aeronautical Engineering, University of Michigan.

This paper was presented by title at the 1949 Summer Meeting of the American Mathematical Society. An abstract appears in the *Bulletin of the American Mathematical Society*, vol. 55, 1949, p. 1062.

10. *The quadric of Lie*, Professor V. G. Grove, Michigan State College.

Let S be a surface, generated by a point P. It is well known that given a line l lying in the tangent plane π of S at P, the reciprocal l' of l with respect to any quadric of Darboux may be defined without using any of the quadrics of Darboux. On l there is set up in a simple manner a family of projectivities between its points. Among these projectivities is one involution, which involution determines a point P_1 on l' in a manner involving the construction of the family of projectivities. Dually there is set up a family of projectivities between the planes through l'. Among these projectivities is one involution, which involution determines dually a plane through l. This plane intersects l' in a point P_2. The locus of the harmonic conjugate of P with respect to P_1, P_2 as l varies in π is the quadric of Lie of S and P.

11. *Unified control-check of the calculation of coeficients in harmonic analysis by the method of least squares*, Dr. Hugo Mandelbaum, Wayne University.

Dr. Mandelbaum's procedure effects a considerable saving in time and effort by a simultaneous calculation of two sets of equations in one operation which also provides for a check on the calculation.

P. S. JONES, *Secretary*

APRIL MEETING OF THE TEXAS SECTION

The annual meeting of the Texas Section of the Mathematical Association of America was held at Abilene, Texas, on April 14–15, 1950, sponsored jointly by Abilene Christian College, Hardin-Simmons University, and McMurry College. Sessions were held Friday afternoon and Saturday morning, H. J. Ettlinger and C. A. Murray presiding. A dinner was given Friday evening at McMurry College at which L. W. Ramsey of Texas Christian University gave an illustrated talk on *A Mathematician's Hobby*, and Dr. A. S. Householder of Oak Ridge Institute of Nuclear Studies, gave an address on *Mathematics and Science.*

Among the seventy-nine persons who registered were the following thirty-two members of the Association: Ina M. Bramblett, H. E. Bray, H. D. Brunk, J. E. Burnam, L. A. Colquitt, J. V. Cooke, H. J. Ettlinger, E. H. Hanson, E. A. Hazelwood, E. R. Heineman, J. J. Henrick, J. M. Hurt, T. M. Jasper, Fay H. Johnson, Hazel L. Mason, Dorothy McCoy, E. D. Mouzon, M. E. Mullings, C. A. Murray, H. C. Parrish, C. B. Rader, L. W. Ramsey, C. R. Sherer, F. W. Sparks, D. W. Starr, W. G. Stokes, Jennie L. Tate, H. E. Taylor, F. E. Ulrich, R. S. Underwood, B. B. Williams, H. E. Woodward.

At the business meeting the following officers were elected for the coming year: Chairman, C. A. Murray, West Texas State College; Vice-Chairman, D. W. Starr, Southern Methodist University; and Secretary-Treasurer, C. R. Sherer, Texas Christian University.

The first six papers were given Friday afternoon, and the remaining papers and the panel discussion were given Saturday morning.

1. *Functions of three variables—a geometric interpretation*, by R. S. Underwood, Texas Technological College.

Given the coordinate system consisting of a U-axis perpendicular to the 3-axes plane at the origin, it can be shown that the locus of the equation $U=f(x, y, z)$ is in general a solid such as, for example, a filled paraboloid. If we represent by Z the special values of U for points on the bounding surface, and are able to find the equation, $Z=F(X, Y)$, of this surface in terms of rectangular coordinates, we may obviously replace the analytic tests for maxima and minima of the function of three variables by the simpler tests concerning two variables.

The transformation is effected as follows: We replace x, y, and z by t, $S+X-t$, and $S-X+t$ respectively, and thus get U in terms of X, S, and t. (For convenience, S is used in place of $Y/\sqrt{3}$. Next we set $\partial u/\partial t=0$, solve for t, and substitute this value of t in the equation $U=G(X, S, t)$, at the same time replacing U by Z.

Exceptionally the locus of $U=f(x, y, z)$ is already a surface. This occurs when $U_y \equiv U_x+U_z$. In such cases the foregoing substitutions yield $U_1=Z=F(X, S)$ directly.

2. *On the relation between the singularities of the two series* $\sum a_n z^n$, $\sum z^n/a_n$, by S. Agmon, The Rice Institute, introduced by the Secretary.

Let $f(z)$ be an analytic function in the whole plane cut along the line $1 \leq x < \infty$, and let its development into Taylor series in the unit-circle be $f(z)=\sum_{n=0}^{\infty} a_n z^n$. Suppose that $a_n \neq 0$, $n=0, 1, \cdots$, and $\lim |a_n|^{1/n}=1$. Put $f^*(z)=\sum_{n=0}^{\infty} z^n/a_n$. Obviously $f^*(z)$ is analytic in the unit circle. Moreover, the boundary of the star of analyticity of $f^*(z)$ is of a very simple character. Thus, denoting by $\rho(\theta)$ the distance from the origin to the first singularity of $f^*(z)$ along the ray

arg $z=\theta$, there exist two numbers a, b, $0\leqq a\leqq\infty$, $0\leqq b\leqq\infty$, such that $\rho(\theta)=\min\,(e^{a\theta}, e^{b}(2\pi-\theta)$ for $0<\theta<2\pi$ and $\rho(0)=1$.

In particular, if both a and b are finite, $f^*(z)$ is analytic in a simply connected region bounded by two spiral arcs and having the boundary for a natural cut.

3. *Some recent developments in the theory of abstract semi-groups*, by J. M. Hurt, University of Texas.

Certain recent developments in the theory of abstract semi-groups such as the work of A.H. Clifford and D. D. Miller on zeroid elements, and of A. H. Clifford on minimal ideals, were discussed. The concept of unitoid element as defined by the author was presented, and a few facts about semi-groups having such elements were established. The power of the new theory was demonstrated by proving several theorems of which the following is typical: A semigroup containing a left-cancellable zeroid element is a group.

4. *Determination of type and properties for a class of Riemann surfaces*, by H. E. Taylor, The Rice Institute.

The construction of a class of open, simply connected Riemann surfaces was described. Each member of the class is topologically equivalent to a semi-infinite cylinder. It can be proved that each member of the class is parabolic and the mapping function has the form $f(z)=\int_0^z f'(t)dt$ and $f'(t)=\prod_{k=1}^{\infty}(1-z/\alpha_k)(1-z/\beta_k)/(1-z/\gamma_k)^2$ with $\alpha_k, \beta_k, \gamma_k$ real, and $\beta_{k+1}<\gamma_{k+1}<\beta_k<\gamma_k<0<\alpha_k<\alpha_k=1$ $(k=1, 2, \cdots)$

5. *Loci of parametric equations on planes of three and four axes*, by Mrs. A. Hays, Abilene Christian College, introduced by the Secretary.

This paper gave a brief report of the type of conic represented by certain parametric equations in three and four variables, and by the use of charts showed how the loci of these equations could be plotted on planes of three and four axes.

6. *The possibilities of weird boundary behavior of analytic functions*, by G. R. MacLane, The Rice Institute, introduced by the Secretary.

7. *The tangent in drawing conic sections in analytic geometry*, by M. E. Mullings, Abilene Christian College.

In a first course in analytic geometry, the quantities a, b, c, d, e, p, and FW are used with the theorem: The tangents at the ends of a latus rectum of a conic section entersect each other at the intersection of the directrix and axis of symmetry, and have slopes numerically equal the eccentricity; to construct geometrically points and tangents in drawing conic sections. This gives eight points and tangents in the ellipse, and six and the asymptotes in the hyperbola.

8. *The effects of the Gilmer-Aiken law on mathematics teachers*, by J. V. Cooke, North Texas State College.

9. *Panel discussion-freshman mathematics*, by E. R. Heineman, and E. A. Hazelwood, Texas Technological College, and C. R. Sherer, Texas Christian University.

The discussion dealt with the following topics: (1) What should we do with the poorly prepared student; (2) What should we do with the gifted student; (3) How can we get the gifted student to take more mathematics.

C. R. SHERER, *Secretary*

APRIL MEETING OF THE OHIO SECTION

The thirty-fourth annual meeting of the Ohio Section of the Mathematical Association of America was held at Denison University, Granville, Ohio, on Saturday, April 22, 1950. Professor E. P. Vance, Chairman of the Section, presided at the morning and afternoon sessions.

One hundred and thirty-nine persons registered attendance, including the following ninety-five members of the Association: J. E. Adney, Jr., C. E. Amos, P. R. Annear, R. P. Bacon, Grace M. Bareis, I. A. Barnett, H. M. Beatty, Theodore Bennett, W. D. Berg, Henry Blumberg, A. P. Boblett, Foster Brooks, O. E. Brown, W. B. Brown, C. D. Calhoon, V. B. Caris, Dorothy I. Carpenter, W. F. Cornell, Wayne Dancer, J. E. Darraugh, R. C. Davis, B. B. Dressler, O. L. Dustheimer, E. R. Epperson, Paul L. Evans, Rudolf Feige, G. R. Glabe, E. L. Godrey, G. Graff, L. J. Green, P. E. Guenther, S. W. Hahn, Marshall Hall, Jr., E. H. Hanson, E. E. Haskins, P. S. Herwitz, F. V. Higgins, L. C. Hill, Clarice S. Hobensack, R. Y. Iwanchuk, S. J. Jasper, E. D. Jenkins, M. L. Johnson, Margaret E. Jones, John Kaiser, Chosaburo Kato, L. C. Knight, Jr., A. C. Ladner, F. C. Leone, H. D. Lipsich, L. L. Lowenstein, W. C. Lowry, R. C. Luippold, H. R. Mathias, Margaret Mauch, S. W. McCuskey, E. J. Mickle, L. H. Miller, C. C. Morris, Max Morris, J. R. Musselman, D. J. Myatt, Helen Olney, P. M. Pepper, C. F. Pinzka, H. S. Pollard, Tibor Rado, S. E. Rasor, P. V. Reichelderfer, R. F. Rinehart, D. L. Robb, S. A. Rowland, Charles Saltzer, W. C. Sangren, K. C. Schraut, Samuel Selby, R. W. Shoemaker, Edward Silverman, Mary Emily Sinclair, G. W. Spenceley, V. C. Stechschulte, R. L. Swain, H. E. Tinnappel, C. W. Topp, W. R. Transue, Bryant Tuckerman, E. P. Vance, W. R. Van Voorhis, R. W. Wagner, R. E. Walters, M. E. White, D. R. Whitney, F. B. Wiley, C. O. Williamson, Alberta Wolfe.

The following officers were elected for the coming year: Chairman, V. C. Stechschulte, Xavier University; Secretary-Treasurer, Foster Brooks, Kent State University; Member of the Executive Committee, D. R. Whitney, Ohio State University; Program Committee: L. C. Knight, Jr., Muskingum College (Chairman), H. R. Mathias, Bowling Green State University, W. R. Transue, Kenyon College.

The following papers were presented:

1. *A different approach to the study of circular functions*, by Professor E. P. Vance, Oberlin College.

A frequent source of confusion in any study of the circular functions is the meaning of sin x, cos x, and so forth, when x is a general real variable. This speaker presented a method for defining these circular functions as functions of a real variable by using the unit circle and the notion of arc length rather than ratios as intuitively given. The development of the whole of elementary trigonometry was outlined by using this method.

2. *An elementary approach to the summation of divergent series*, by Dr. H. D. Lipsich, University of Cincinnati.

Ordinarily a student's first contact with summability occurs when he is confronted with the

Leibniz series, which is shown to be summable by the method of arithmetic means. This paper provides an introduction to the theory of divergent series via the transformation defined by

$$y_M^m = \frac{\sum_{\nu=0}^{m} S_{m-\nu}}{m+1}$$

where m is a fixed non-negative integer. The problems of relative regularity, Tauberian theorems, necessary conditions for summability, and a limiting summability method, are all solved in extremely elementary fashion, and illustrated by means of sequences consisting of zeros and ones, corresponding to generalized series of the Leibniz type. The paper concludes with remarks and results concerning extension of the definition to negative values of m.

3. *A note on Mordell's equation*, by Professor Marshall Hall, Ohio State University.

Mordell's equation $x^3-y^2=k$ has been shown unsolvable in integers for various values of k, by means of criteria depending on quadratic residues. It is shown here that cubic residues may also be used in a similar way. For example, writing $x^3-y^2=77$ in the form $x^3-2=y^2+3\cdot 5^2$ we may show that no solutions exist.

4. *On the circumcircle and the circumhyperbola of a triangle*, by Professor J. R. Musselman, Western Reserve University.

The geometry of the triangle is enriched by considering not only the circumcircle of triangle $A_1A_2A_3$ but also its circumhyperbola—a rectangular hyperbola with asymptotes in a fixed direction. The speaker showed the existence of a line analagous to the Euler line on which lie C, the center of the circumhyperbola; G, the centroid of $A_1A_2A_3$; N^1, the center of the nine-point hyperbola, and H^1, the point of concurrence of the antiparallels to the three altitudes of $A_1A_2A_3$ as to the asymptotes. Various properties of this point H^1 were discussed, including the following theorem "the three lines drawn from any point on the hyperbola parallel to A_iH^1 cut the sides A_jA_k in three collinear points."

5. *Some modern Italian mathematicians*, by Professor Emilio Baiada, Universities of Pisa and Cincinnati (Professor Baiada's address was presented by invitation of the Program Committee).

The speaker discussed the lives and works of mathematicians working in Italy in recent years.

6. *Spherical harmonic, cylindrical harmonic, and elliptic integral solutions of a potential problem*, by Professor R. P. Bacon, Miami University.

The problem of the potential due to a uniform circular lamina is solved in the spherical polar coordinate system and the cylindrical coordinate system. Results are then degenerated into the plane of the lamina, where they are shown to be identical to each other and to a third solution by elliptic integrals. The transformations used for the most part are relations between a Bessel integral and a hypergeometric function and the expression of the complete elliptic integrals of the first and second kind as hypergeometric functions.

7. *On homogeneous linear matrix forms*, by Professor R. W. Wagner, Oberlin College.

This speaker discussed the lack of uniqueness in the multipliers of a linear homogeneous matrix form $f(X)=\sum A_iXB_i$. Of the five theorems in the paper the important ones are (1) that the above form and the form $f'(X)=\sum A_i'XB_i'$ are identically equal if (*I*) $A_i'=\sum a_{ij}A_j$, if (II)$B_i'=\sum b_{ij}B_j$, and if (III) $\sum a_{ri}b_{rj}=\delta_{ij}$; and (2) a partial converse that if the forms are identically

equal (III) is a consequence of (I) and (II) and the linear independence of the two sets of multipliers A_i and B_i.

8. *Rapidly converging expressions for sin x, cos x, sinh x, and cosh x*, by Mr. L. C. Hill, Miami University.

In this MONTHLY, April, 1949, Professors C. L. Seebeck, Jr., and P. M. Hummell, University of Alabama, developed a generalized Taylor's expansion for $f(x)$. Applying this directly to sin x raises some algebraic difficulties which can be avoided by beginning with e^x. The resulting expression for e^x in the form $P_1(x)/P_2(x)$ is a rapidly converging quotient. With this, and allied expressions for e^{-x}, e^{ix}, e^{-ix}, one can obtain equally rapidly converging expressions for sin x, cos x, sinh x, cosh x, in a similar quotient form. It was further found that precisely these expressions can be obtained by the continued fraction development of e^x given, for example, in H. S. Wall's *Continued Fractions*.

9. *A new course for mathematics majors*, by Professor O. L. Dustheimer, University of Toledo.

This speaker outlined a general course, covering certain selected subjects in mathematics, recently given by the author at the University of Toledo.

FOSTER BROOKS, *Secretary*

CALENDAR OF FUTURE MEETINGS

Joint Meeting with American Society for Engineering Education, Michigan State College, East Lansing, June 25–26, 1951.

Thirty-second Summer Meeting, University of Minnesota, Minneapolis, September 3–4, 1951.

The following is a list of the Sections of the Association with dates of future meetings so far as they have been reported to the Secretary.

ALLEGHANY MOUNTAIN, Duquesne University, Pittsburgh, Pennsylvania, May, 1951.

ILLINOIS, University of Illinois, Urbana, May 11–12, 1951.

INDIANA, May 5, 1951.

IOWA, Wartburg College, Waverly, April 20–21, 1951.

KANSAS

KENTUCKY, Eastern Kentucky State College, Richmond, April 28, 1951.

LOUISIANA-MISSISSIPPI, Mississippi State College, State College, February 16–17, 1951.

MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA

METROPOLITAN NEW YORK, Spring, 1951.

MICHIGAN, Michigan State College, East Lansing, March 24, 1951.

MINNESOTA, College of St. Benedict, St. Joseph, April 28, 1951.

MISSOURI, Central College, Fayette, Spring, 1951.

NEBRASKA, University of Nebraska, Lincoln, May 5, 1951.

NORTHERN CALIFORNIA, University of San Francisco, January 27, 1951.

OHIO, Ohio State University, Columbus, April 21, 1951.

OKLAHOMA

PACIFIC NORTHWEST, State College of Washington, Pullman, June 15, 1951.

PHILADELPHIA

ROCKY MOUNTAIN, Colorado State College of Education, Greeley, April, 1951.

SOUTHEASTERN, Vanderbilt University and Peabody College, Nashville, Tennessee, March 16–17, 1951.

SOUTHERN CALIFORNIA, Whittier College, Whittier, March 10, 1951.

SOUTHWESTERN, University of New Mexico, Albuquerque, Spring, 1951.

TEXAS, Southern Methodist University, Dallas, Spring, 1951.

UPPER NEW YORK STATE, Hamilton College, Clinton, May 5, 1951.

WISCONSIN, Carroll College, Waukesha, May, 1951.

GEORGE BANTA PUBLISHING COMPANY, MENASHA, WISCONSIN

THE AMERICAN MATHEMATICAL MONTHLY

THE OFFICIAL JOURNAL OF

THE MATHEMATICAL ASSOCIATION OF AMERICA, INC.

VOLUME 58 NUMBER 2

CONTENTS

FEBRUARY 1951

The AMERICAN MATHEMATICAL MONTHLY

(FOUNDED IN 1894 BY BENJAMIN F. FINKEL)

EDITORIAL CORRESPONDENCE should be addressed to the Editor, C. V. NEWSOM, State Education Building, Albany 1, N. Y.

ADVERTISING CORRESPONDENCE should be addressed to J. F. RANDOLPH, Morey Hall, University of Rochester, Rochester 3, N. Y.

NOTICE OF CHANGE OF ADDRESS by members of the Association as well as correspondence regarding subscriptions to the MONTHLY should be sent to the Secretary-Treasurer, H. M. GEHMAN, University of Buffalo, Buffalo 14, N. Y. Change of address must reach the Secretary-Treasurer about six weeks before the change can become effective.

Entered as second class matter at the post office at Menasha, Wis. Acceptance for mailing at special rate of postage provided for in the Act of February 28, 1925, embodied in Paragraph 4, Section 538, P. L. and R., authorized April 1, 1926.

SUBSCRIPTION PRICE: To Members, $4 a Year, To Others, $5 a Year.

PUBLISHED BY THE ASSOCIATION at Menasha, Wisconsin, and Albany, N. Y. during the months of January, February, March, April, May, June-July, August-September, October, November, December.

THE HISTORY OF CALCULUS*

ARTHUR ROSENTHAL, Purdue University

Everyone knows that Newton and Leibniz are the founders of Calculus. Some may think it suffices to know just this one fact. But it is worthwhile, indeed, to go into more details and to study the history of the development of Calculus, in particular, up to the time of Newton and Leibniz.

In our courses on Calculus we usually begin with differentiation and then come later to integration. This is entirely justified, since differentiation is simpler and easier than integration. On the other hand, the historical development starts with integration; computing areas, volumes, or lengths of arcs were the first problems occurring in the history of Calculus. Such problems were discussed by ancient Greek mathematicians, especially by Archimedes, whose outstanding and penetrating achievements mark the peak of all ancient mathematics and also the very beginning of the theory of integration. The method applied by Archimedes for his proofs was the so-called method of exhaustion, that is, in the case of plane areas, the method of inscribed and circumscribed polygons with an increasing number of edges. This method was first rigorously applied, in the form of a double *reductio ad absurdum*, by the great Greek mathematician Eudoxus at the beginning of the fourth century B.C. He first proved the facts, previously stated by Democritus, that the volume of a pyramid equals one third of the corresponding prism and the volume of a cone equals one third of the corresponding cylinder. The same method was also used by Euclid and then with the greatest success by Archimedes (third century B.C.). It is well known that Archimedes was the first to determine the area and the length of the circle, that is, to give suitable approximate values of π, and moreover to determine the volume and the area of the surface of the sphere and of cylinders and cones. But he went far beyond this [1]; he found the areas of ellipses, of parabolic segments, and also of sectors of a spiral, the volumes of segments of the solids of revolution of the second degree, the centroids of segments of a parabola, of a cone, of a segment of the sphere, of right segments of a paraboloid of revolution and of a spheroid. These were amazing achievements, indeed. Archimedes proved his results in the classical manner, by the method of exhaustion. Sometimes the type of approximation is just the same as we would use. For instance, in order to obtain the volume of a solid of revolution of the second degree, Archimedes approximates the volume by a sum of cylindrical slabs. But the direct evaluation of the limit of such sums was cumbersome. Hence we may ask: what was the method used by Archimedes for finding his results?

There is an indication of his method in the beginning of his book on the quadrature of the parabola. But a full explanation of his procedure was given by him in a work rediscovered as late as 1906. It is his *Method Concerning Mechanical Theorems, dedicated to Eratosthenes*, known as Archimedes' *Method*

* Based on two addresses given at the Mathematics Club of Purdue University, June 2, 1949, and at the meeting of the Indiana Academy of Science, Crawfordsville, Ind., November 4, 1949.

or ἔφοδος [2]. This manuscript was found in Istanbul as a so-called palimpsest. That is to say, in the 10th century A.D. the manuscript of the *Method* was written on this parchment; later, in the 13th century, since nobody there was interested in it any longer or could even understand it, the *Method* was washed off and a religious text of the orthodox church, a so-called euchologion, was written on the parchment. Fortunately most of the Archimedean text could be restored.

The method of Archimedes may be called a mechanical infinitesimal method. We must remember that Archimedes was also the founder of statics, and of hydrostatics too. Now his method of integration consists in an application of the principle of the lever to elementary parts of the figure. As an example, we shall see how Archimedes determines the area of a segment of a parabola; this indeed is the first example given in his ἔφοδος.

Let $\overset{\frown}{AB}$ be the given segment of the parabola, let R be the midpoint of the chord AB, and draw the diameter d (parallel to the axis) through R, intersecting the arc AB in U (cf. Figure 1). We draw the tangent of the parabola at B and the parallel to d through A. So we obtain $\triangle ABC$ and wish to compare the area

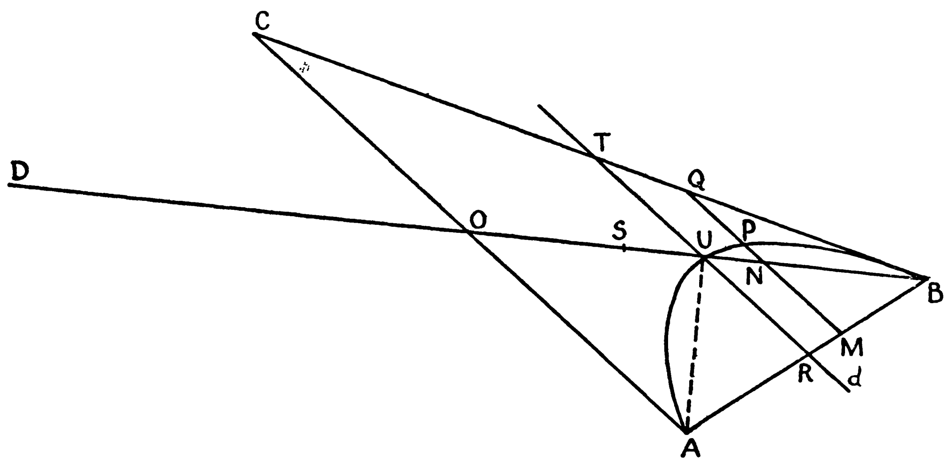


FIG. 1

of the segment $\overset{\frown}{AB}$ with the area of this triangle. Let T be the intersection of the tangent BC with the diameter d. From an elementary property of the parabola we obtain: $UT=UR$. Hence the line BU intersects AC in its midpoint O. The centroid S of the $\triangle ABC$ lies on OB with $OS=\frac{1}{3}OB$. Make $OD=OB$. Now through any point P of the arc AUB draw a parallel to the diameter, $MNPQ$. Archimedes proves as a property of the parabola that $MP:MQ=AM:AB$, hence $MP:MQ=ON:OB=ON:OD$. Therefore $MP\times OD=MQ\times ON$. That is, according to the principle of the lever: If DOB is considered to be the bar of a balance with the fulcrum in O (cf. Figure 2), then the line-segment PM suspended at D is in equilibrium with the line-segment MQ which remains, without

any change, suspended at N. Since this holds for every line parallel to d, it follows that the total segment $\overset{\frown}{AB}$ of the parabola suspended at D is in equilibrium with the $\triangle ABC$ remaining unchanged or, what amounts to the same thing, the segment $\overset{\frown}{AB}$ suspended at D is in equilibrium with the $\triangle ABC$ suspended at its centroid S. Therefore, since $OS=\frac{1}{3}OD$, the segment $\overset{\frown}{AB}=\frac{1}{3}\triangle ABC$. Moreover, since $RU=\frac{1}{2}RT=\frac{1}{4}AC$, we have $\triangle ABC=4\triangle AUB$, and hence also segment $\overset{\frown}{AUB}=\frac{4}{3}\triangle AUB$.

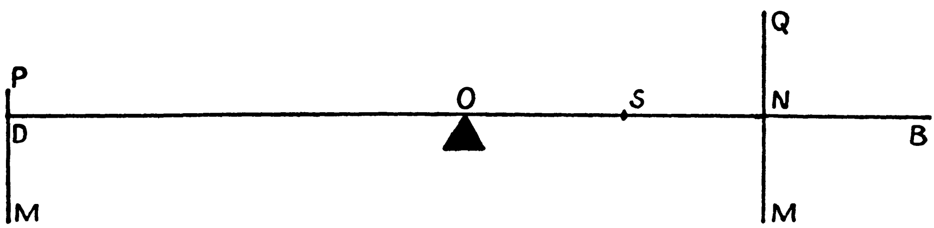


FIG. 2

By this ingenious method of statics, Archimedes found the final result. But he did not regard such reasoning as a proof. Having thus obtained the result, he was then able to give a rigorous formal proof by the method of exhaustion.

One should expect that the wonderful achievements of Archimedes would have become a great stimulus to the further development of Greek mathematics, similar to the great influence of Newton and Leibniz on the mathematical production of succeeding generations. But it is surprising that Archimedes found almost no successors to continue his work. In this connection only one of the subsequent mathematicians is to be mentioned, namely, Dionysodorus who found the volume of the torus. Of course, one has to remember that at the time of Archimedes there lived another outstanding Greek mathematician, Apollonius, about 25 years younger than Archimedes. Apollonius, in a masterly way, completed the Greek theory of conic sections. It is very strange, and I do not understand the reason for it, that soon after Archimedes and Apollonius Greek mathematics declined and that the later development essentially took a different direction. Under the influence of the needs of astronomy, a new branch of mathematics (the roots of which, however, go back also to Archimedes), namely trigonometry, was established and furthermore, much later, the theory of numbers was developed by the work of Diophantus. Original contributions in the direction of Archimedes' work were finally made by one of the latest Greek mathematicians, Pappus (end of the third century A.D.) who stated the important general theorems named after him, in particular, the theorem that the volume of a body of revolution equals the area of the revolving plane figure times the length of the path of the centroid of this area.

When Teutonic tribes, still barbarian at that time, invaded the Roman empire and conquered it, the interest in mathematics almost vanished there; mathematics receded to the Orient, to Byzantium, where at least valuable manu-

scripts were preserved, to Persia, and afterwards to the Arabian countries, where—on the basis of Greek tradition—mathematics flourished in the period about 800–1200 A.D. One of these mathematicians, the Mesopotamian Ibn Al Haitham (about 1000 A.D.), was able to compute the volume of a solid that is generated by rotation of a segment of a parabola about a line perpendicular to its axis.

Under the influence of the Orient, interest in mathematics was slowly awakened in Europe, in particular in the 12th and 13th centuries. As early as the 16th century great discoveries in algebra were made by Italian mathematicians, namely the solution of the algebraic equations of the third and fourth degree. Simultaneously Archimedes' works were studied and understood again.

Then about the beginning of the 17th century the further development of the ideas of Archimedes starts. This was the same great period in which modern science was first established by Galileo. The Flemish engineer Simon Stevin (as early as 1586) and the Italian mathematician Luca Valerio (1604) were the first ones who, by direct passage to the limit, tended to avoid the double *reductio ad absurdum* of the method of exhaustion. Valerio showed directly that the areas under certain curves can be approximated by sums of circumscribed and inscribed rectangles, whose difference can be made arbitrarily small.

Then, in particular, we have to mention the great German astronomer Johannes Kepler who, in 1615, published a book, *Nova stereometria doliorum vinariorum*, on determining the volumes of wine-casks. Somewhat earlier there had been a year of plenty and there was need of barrels for storing the great supply of wine; moreover, Kepler was puzzled by the rules which dealers applied to estimate the approximate contents of a barrel. So he discussed in a popular manner the volumes of various casks and, in particular, asked which cask has the most economic shape. He found that the Austrian barrel was the most economic one. Kepler used the results and methods of Archimedes, but also discussed quite a few new cases. Because of his popular purpose he replaced the rigorous proofs of Archimedes by an intuitive infinitesimal reasoning, in this way stressing the essential points.

Another mathematician of that time had a great influence on further progress; this was the Italian Bonaventura Cavalieri who published in 1635 an important book on the so-called indivisibles, entitled *Geometria indivisibilibus continuorum nova quadam ratione promota*. Indivisibles mean elements of a given dimension which by their motion generate figures of the next higher dimension. Thus a moving point generates a line, a moving line (parallel to a fixed line) generates a plane figure, a moving plane figure (parallel to a fixed plane) generates a solid. Cavalieri speaks, for instance, about "all lines of a plane figure" ("omnes lineae figurae"). Well known is Cavalieri's principle: Two solids (lying between two parallel planes) have the same volume if they intersect each intermediate parallel plane in two equal areas. Cavalieri's views, influenced by late medieval speculations, have somewhat of the spirit of Archimedes' Method, which, however, was not known at that time.

In connection with Cavalieri we must mention also the Swiss Paul Guldin who, besides criticizing Cavalieri, rediscovered Pappus' theorems on bodies of revolution, the Flemish mathematician Gregorius a St. Vincentio who was the first to observe (1647) that the area between a hyperbola and an asymptote behaves like a logarithm, and also the Italian mathematician and physicist Evangelista Torricelli [3] and the French mathematician Gil Persone de Roberval [4]. The important achievements of these last two men will be discussed presently.

About this time another outstanding event occurred in mathematics, the invention of Analytic Geometry by Descartes (1637) and, simultaneously and independently, by Fermat; this invention, of course, had great influence on the development of Calculus. Both Descartes and, in particular, Fermat also made valuable direct contributions to Calculus.

René Descartes, in his *Géométrie*, gave a method of finding the tangents, or rather the normals, to algebraic curves. He draws a circle with center on the x-axis, which cuts the given curve in two points. If these two points coincide, he obtains the normal. Hence the question is reduced to determining double roots of an algebraic equation. Somewhat later, in a letter, Descartes remarked that, instead of circles, intersecting straight lines could also be used for the same purpose.

Fermat's achievements in Calculus were even more important. In fact, he was the greatest mathematician of the first part of the 17th century, not only in general but particularly in the domain of Calculus. Pierre Fermat was a jurist, a councillor of the parliament at Toulouse in southern France. This position left him enough time for intensive mathematical activity. His outstanding work in the theory of numbers is well known. Now, what was *his* method of finding tangents? His procedure was first applied by him to the particular case of determining maxima and minima [5(a)]. He found this method as early as in 1629, communicated it to Descartes in 1638, and had it published in 1642. In order to find the maximum or minimum of an expression, one replaces the unknown A by $A+E$,* and both expressions obtained in this manner are considered approximately equal. One must cancel on both sides all that is possible to cancel. In this way only terms containing E are left. Now divide by E and then drop all terms still containing E. There remains an equation giving that value of A which yields the desired maximum or minimum. That means, if we write $F(A)$ for the given expression, we have to determine A from the equation

$$\left[\frac{F(A+E)-F(A)}{E}\right]_{E=0}=0.$$

This is just our usual method. Of course, the condition is only necessary, but not sufficient for the extreme, and the statement of Fermat yields the result only for polynomials F.

* Fermat always used the letter A for the variable and the letter E for its increment.

Fermat [5(a)] gave at the same time a general method for finding the tangent, in the form of determining the subtangent. Let PT (with T lying on the the x-axis) be the tangent line of the given curve $\mathfrak{C}$ at the point P (cf. Figure 3), let P_1 be a point of $\mathfrak{C}$ in the neighborhood of P, let Q and Q_1 be the projections

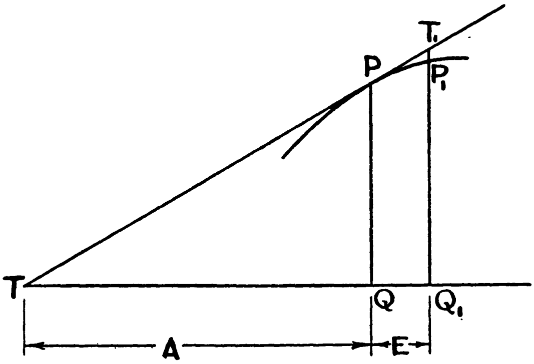


FIG. 3

on the x-axis of P and P_1, respectively, and let T_1 be the point on the tangent line whose projection on the x-axis is Q_1. In order to find the subtangent $A(=TQ)$, whose increment QQ_1 is again designated by E, Fermat uses the similarity of the triangles TQP and TQ_1T_1 and replaces T_1 approximately by P_1. Then he obtains approximately: $A:QP=E:(Q_1P_1-QP)$, that is, in our usual notation if we write the equation of the curve $\mathfrak{C}$ in the form $y=F(x)$,

$$A:F(x) = E:(F(x+E)-F(x));$$

hence

$$A = \frac{F(x)\cdot E}{F(x+E)-F(x)}\cdot$$

Now again one divides the denominator by E and afterwards sets $E=0$.

It should be mentioned here that at this time an entirely different method of constructing tangents of curves, using the parallelogram of velocities, was invented also by Roberval and Torricelli, independently of each other; both of them published it in 1644. On the other hand, somewhat later (before 1659) two Netherlanders Johannes Hudde, for many years mayor of Amsterdam, and René François de Sluse advanced along the road opened up by Descartes and Fermat, giving quite explicit formal rules for finding extremes and subtangents of algebraic curves.

But let us return to Fermat. He had great success also in the theory of integration. He was the first who, by 1636 or earlier, had found and proved the power formula of integration for positive integral exponents n, i.e. a geometrical statement equivalent to the formula which we now write as

$$\int_0^a x^n dx = \frac{a^{n+1}}{n+1}.$$

Roberval also, at the suggestion of Fermat, then found and proved the same theorem. Afterwards Cavalieri discovered it independently,* and he was the first to publish it (1639, 1647); but he proved it explicitly only for the first few cases, including $n=4$, while, as he stated, the general proof which he published was communicated to him by a French mathematician Beaugrand, who quite probably had obtained it from Fermat. At that time Fermat, and then also Torricelli, had already generalized this power formula to rational exponents $n(\neq -1)$. Fermat determined areas under curves which he called "general parabolas" and "general hyperbolas," that is, curves

$$y^m = cx^n, \text{ which, in our notation, leads to } c\int_0^a x^{n/m}dx,$$

and

$$y = \frac{c}{x^m}, \text{ which, in our notation, leads to } c\int_a^{+\infty} \frac{dx}{x^m} \qquad (m > 1).$$

It is remarkable that Fermat in this work does not use subdivisions into equal parts, but subdivisions according to a geometric progression. Other areas were reduced by him to areas under such general parabolas and hyperbolas.

Fermat did interesting work also on the rectification of curves, in a memoir published 1660 [5(b)], where he approximated the arc by segments of tangents, thus using a saw-like figure. At that time various mathematicians obtained rectifications of curves which are now considered classical. In 1645, Torricelli had rectified the logarithmic spiral. The semicubical parabola was rectified independently by the Englishman William Neil (1657), by the Hollander Hendrik van Heuraet, and by Fermat. The rectification of the cycloid was first achieved by the English mathematician and great architect Christopher Wren (1658), and then by Fermat and Roberval, after they had heard of his result. It is noteworthy that Fermat, Neil, van Heuraet, and also Wallis and Huygens, reduced rectifications of curves to the determination of areas of other curves.

Reviewing the achievements of Fermat we see that he was aware of the relation among various problems in differential calculus, and similarly for various problems about definite integrals. But he had not observed the general relation between differentiation and integration.

Another famous French mathematician of that time was Blaise Pascal (a younger friend of Fermat and Roberval), who may be considered a master of integration. Roberval was the first to integrate certain trigonometric functions. Pascal was able to integrate more such trigonometric functions as well as some algebraic functions. As an important means for some of his results Pascal used

* Much later, Blaise Pascal (1654) and Wallis (1656) rediscovered it also.

relations between integrals obtained by interchanging the order of integration in double integrals. Of course, he did this in a geometric form. That is to say, certain volumes were found by means of intersections parallel to one plane and also by intersections parallel to another plane. Since the volume is always the same, Pascal thus obtained a relation between two different integrals, which may be considered to be a kind of integration by parts.

At about the same time important contributions were made by the English mathematician John Wallis, whose book *Arithmetica infinitorum* of 1656 was written in contrast to the geometric work of most of his predecessors. He stressed the notion of the limit. On the other hand, he was very audacious with regard to generalizations and interpolations, but his strong power of intuition kept him on the right track. For instance, he stated the general power formula of integration for any real exponent $n(\neq -1)$.

The notion of limit was carefully considered at that time by the Italian Pietro Mengoli in his book *Geometria speciosa* (1659) [6(a)]. In particular, by modifying the procedure of Luca Valerio (see above), he gave in a precise way a representation of the area under certain special curves as limits of sums of rectangles [6(b)]. Later the same method was also employed by Newton [**11**, book 1, lemma 2].

One should now mention two other mathematicians who, like Wallis and Mengoli, were contemporary with Newton and Leibniz, but began their work earlier, so that they too are to be considered, at least partially, as predecessors of Newton and Leibniz. One of these men is the great Dutch physicist and mathematician Christiaan Huygens who, among other important results, introduced the notion of evolutes and involutes. It is remarkable that Huygens used to great extent the classical methods of Archimedes, and only for differentiation employed Fermat's method.

The second of these two mathematicians is the Scotsman James Gregory, who like Torricelli and Pascal died in the prime of life, when only 37 years old, and whose genius has found its full recognition only recently [**7, 8**]. He did excellent work in integration; for instance, in 1668 he published such a difficult result as the following (written in modern notation):

$$\int_0^a \sec x dx = -\log(\sec a - \tan a)$$

and other trigonometric integrals. Moreover, for example, he obtained Newton's interpolation formula, independently of Newton. But perhaps the most important achievements of Gregory belong to the theory of series.

The first great result in the theory of series is due to the German mathematician Nicolaus Mercator (1668), who found the logarithmic series. For this purpose, Mercator used term by term integration of a geometric series. This method was independently discovered, but not published, by Newton. Subsequent contributions were made by William Lord Brouncker, the first president

of the Royal Society in London. Then the most outstanding results concerning infinite series were obtained by Newton and Gregory, who worked essentially independent of each other, though Gregory was influenced by the knowledge of some of Newton's statements, but not of his methods. Both discovered the binomial series and also many series for trigonometric and inverse trigonometric functions. In particular, Gregory found the series for arctan x; a special case of it is the series $\pi/4=1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\cdots$, which was later found independently by Leibniz. Gregory obtained some more complicated series, *e.g.*, for tan x, sec x, log sec x, using processes of differentiation for determining the coefficients of these series, thus anticipating Brook Taylor by more than forty years. This fact has recently been demonstrated by H. W. Turnbull who studied and interpreted notes in Gregory's handwriting (*cf.* [**7**], pp. 168–176, 350–359). It is obvious that many of the contributions to the theory of series were closely connected with the growth of Calculus.

From all that has been discussed so far we have seen that there certainly was an extensive development of the theory of integration and differentiation in the period immediately before Newton and Leibniz, and that many mathematicans of various nations made great contributions. So we shall ask: What then was missing at that time? One very important point still missing was the general fact that differentiation and integration are inverse processes, that is, the so-called fundamental theorem of integral calculus. It is true that a few mathematicians had already come quite close to this knowledge. First in this connection we must mention Torricelli. In manuscripts left at his death (1647) and published as late as 1919 [cf. **3**], he had obtained the distance $s(t)$ of a moving point by means of the quadrature of the velocity $v(t)$, while by the construction of the tangent of the curve $s(t)$ he could recover $v(t)$. It is very doubtful, however, whether he really conceived the significance of this relation. Secondly, as we have already stated, some of the mathematicians of the time knew that the rectification of one curve $\mathfrak{C}_1$ could be obtained by means of the area under another curve $\mathfrak{C}_2$. Fermat in his memoir [**5**(b)], published in 1660, found the relation between the slopes of these two curves. Moreover, James Gregory, in his book *Geometriae pars universalis* (1668), solved the following problem, which is inverse to the rectification of $\mathfrak{C}_1$, just mentioned: Given a certain curve $\mathfrak{C}_2$, find another curve $\mathfrak{C}_1$ whose length equals the area under $\mathfrak{C}_2$. In the solution of this problem Gregory used the quadrature of an auxiliary curve, which indeed corresponds to obtaining the primitive function of a derivative by means of integration [**9**]. In spite of this, one again may doubt whether, at that time, he perceived in general that differentiation and integration are inverse operations. The first who made this important discovery in full generality was Isaac Barrow, the teacher of Newton at Cambridge University. Barrow first was professor of Greek language at Cambridge, then he was professor of mathematics first in London and then again at Cambridge. In 1669 he resigned his chair to his pupil Newton, whose superior genius he had recognized. Barrow then devoted the rest of his life to theology. His principal mathematical work was his *Lectiones Geo-*

metricae, published in 1670, together with the second edition of his *Lectiones Opticae*. These geometrical lectures [**10**] contained important systematic contributions to the theory of differentiation and integration, almost all in purely geometric form. Here, for the first time, the inverse character of differentiation and integration was explicitly stated and proved.

At this remarkable point of the development of the theory we must again ask the same question as above: What more remained to be done? The answer is: What had to be created was just the *Calculus*, a general symbolic and systematic method of analytic operations, to be performed by strictly formal rules, independent of the geometric meaning. Now it is just this Calculus which was established by Newton and Leibniz, independent of each other and using different types of symbolism. Newton's first discoveries were made about ten years before those of Leibniz; on the other hand, Leibniz' publications preceded those of Newton, and—what is more important—Leibniz' symbolism, the same as that used by all mathematicians at present, is superior to that of Newton.

Isaac Newton, influenced by his teacher Barrow and also by the work of Wallis, started his "method of fluxions" in the years 1665–1666, his most creative period, at the age of 23 years. Some of his early manuscripts were known to friends of his and indications of his method were contained in some of his letters. He wrote his *Methodus fluxionum et serierum infinitarum* in 1670–1671; but it was not published until 1736, nine years after his death. In his profound work *Philosophiae naturalis principia mathematica* (1687) Newton avoided his method of fluxions, except for a few indications [**11**, book 2, lemma 2], and presented his great discoveries in the classical geometric form, though simplified by using the notion of limits. The first publication of both the method and the notation of Newton is found as late as 1693 in Wallis' works, where Wallis included two of Newton's letters to him. Newton himself then published an account of his method, entitled *Tractatus de quadratura curvarum*,* as an appendix to his *Optics* in 1704.

Newton, considering motions, took the time t as the independent variable, called the dependent variable x "fluent" and its velocity "fluxion," and wrote $\dot{x}$ for the fluxion, i.e. for the derivative with respect to t. The higher derivatives were then designated by $\ddot{x}$, $\dddot{x}$, etc. For the increment of the independent variable t Newton used the letter o and called $\dot{x}o$ (i.e., the differential of x) the "moment" of x. In the case of the inverse process, that is, if the variable x is given as a fluxion, he first designated the fluent (i.e., the antiderivative of x) by $\square\, x$ or $\boxed{x}$, later by $\acute{x}$, and used then for iterated integration $\overset{\prime\prime}{x}$, $\overset{\prime\prime\prime}{x}$, etc. It has to be stressed that Newton was the first to use systematically the results of differentiation in order to obtain antiderivatives, and hence to evaluate integrals.

Our notations, now used generally in Calculus, are due to Gottfried Wilhelm Leibniz. He was a universal spirit, extremely versatile and interested in every kind of knowledge and scholarship, perhaps most famous as a philosopher. He

* Most of it was already written in 1676.

started as a jurist, was soon active in diplomacy, then became librarian and historian at Hanover, and later (1700) founded the Berlin Academy of Sciences. In mathematics, his first publication, as a graduate student, concerned combinations and permutations; then he soon became interested in the theory of differences and in constructing a computing machine. At the suggestion of Huygens (1673), Leibniz thoroughly studied the works of previous mathematicians on integration and differentiation. In particular, he was much influenced by the work of Pascal. The story of Leibniz' own discoveries can be traced in all details, since his manuscripts with his dated sketches were found at the Hanover library (edited and published by C. I. Gerhardt in the middle of the 19th century) [**12, 13, 14, 14a**]. Leibniz' new notation was first introduced by him on October 29, 1675 [**12**, pp. 121–127]. On this day, as on preceding days, Leibniz discussed integrations using Cavalieri's "omnes lineae." Here he abbreviated "omnes" or "omnia" to "omn." and applied quite a few formal operations to this symbol. Then he remarked: "It will be useful to write $\int$ for omn., thus $\int l$ for omn. l, that is, the sum of those l's." Hence $\int$ is derived from the first letter of the word *summa*. Later in the same manuscript, he came to the "contrary calculus" and continued thus: "If $\int l \sqcap ya$" ($\sqcap$ is the equal sign of Leibniz), "then let us set $l \sqcap ya/d$. Certainly as $\int$ will increase the dimensions, so d will diminish them. $\int$, however, designates a sum, d a difference." Hence Leibniz first wrote the differential sign d into the denominator of the variable. But two or three weeks later he writes dx, dy, dx/dy, and the integrals $\int ydy$ or $\int ydx$. So he arrived at the notation, now classical.

In 1684, Leibniz first published his differential calculus in a paper (issued in the newly founded *Acta Eruditorum*) with the title *Nova methodus pro maximis, itemque tangentibus*, $\cdots$. Here he deals with differentials, and it is noteworthy that he introduces dx as an arbitrary finite interval and then defines dy by the proportion $dy:dx=$ordinate:subtangent. Then, in 1686, he published also a paper containing his notation of the integral. (The word "integral" was introduced by Jakob Bernoulli, 1690.)

Later Leibniz was accused by friends and followers of Newton of having plagiarized Newton's ideas. This unfortunate and undignified controversy started first in 1699 and became continuously more and more furious, so that Leibniz' last years were filled with bitterness. But considering Leibniz' manuscripts, nobody can doubt at present that Newton and Leibniz founded their Calculus independently.

The invention of Calculus stimulated an immense and energetic further development. On the English side Taylor and Maclaurin, on the continent the eminent Basel mathematicians, first the brothers Bernoulli, then the prodigious Euler, and later the Frenchman D'Alembert and the Italian Lagrange contributed greatly to this development. Moreover, other closely related parts of mathematics soon originated in connection with the Calculus, e.g., the theory of differential equations, the calculus of variations, differential geometry.

While Newton and Leibniz had rather reasonable (although not always con-

sistent) ideas about the fundamentals of the new Calculus, the extremely rapid further development caused the basic concepts to be neglected or to be treated in a very unsatisfactory manner. In particular, Euler is an example of this tendency. A few mathematicians, among them D'Alembert, stressed the necessity of using the notion of the limit as foundation of the Calculus [**15**]. But it was Cauchy in the beginning of the 19th century who, in such a way, developed the Calculus systematically and consistently. So at last Cauchy and his many successors gave a solid basis to the Calculus.

References

1. T. L. Heath, The Works of Archimedes, Cambridge 1897.

2. The Method of Archimedes, edited by T. L. Heath, Cambridge 1912.

3. E. Bortolotti, I progressi del metodo infinitesimale nell' Opera geometrica di Evangelista Torricelli, Periodico di Matematiche (4) 8 (1928), pp. 19–59.

4. Evelyn Walker, A Study of the Traité des Indivisibles of Gilles Persone de Roberval, Teachers College, Columbia University, New York 1932.

5. Œuvres de Fermat, 4 volumes and supplements, Paris 1891–1922. (a) Methodus ad disquirendam maximam et minimam, Œuvres, vol. 1, pp. 133–136; see also pp. 136–179 (French translation: vol. 3, pp. 121–123 and 124–156); (b) De linearum curvarum cum lineis rectis comparatione, Œuvres, vol. 1, pp. 211–254 (French translation: vol. 3, pp. 181–215).

6. A. Agostini, Peridico di Matematiche (4) 5 (1925). (a) La teoria dei limiti in Pietro Mengoli, pp. 18–30; (b) Il concetto d'integrale definito in Pietro Mengoli, pp. 137–146.

7. James Gregory. Tercentenary Memorial Volume, edited by H. W. Turnbull, London 1939.

8. M. Dehn and E. D. Hellinger, Certain mathematical achievements of James Gregory, Am. Math. Monthly 50 (1943), pp. 149–163.

9. A. Prag, On James Gregory's Geometriae pars universalis, in [7], pp. 487–505.

10. J. M. Child, The Geometrical Lectures of Isaac Barrow, Chicago and London, 1916.

11. Sir Isaac Newton's Mathematical Principles of Natural Philosophy and His System of the World. Translated by A. Motte, revised by F. Cajori, University of California Press, Berkeley, Calif., 1946.

12. C. I. Gerhardt, Die Entdeckung der höheren Analysis, Halle 1855.

13. J. M. Child, The Early Mathematical Manuscripts of Leibniz, Chicago & London 1920.

14. D. Mahnke, Neue Einblicke in die Entdeckungsgeschichte der höheren Analysis, Abhandlungen der Preuss. Akad. der Wissenschaften, Phys.-Math. Klasse, 1925, No. 1.

14a. J. E. Hofmann, Die Entwicklungsgeschichte der Leibnizschen Mathematik während seines Aufenthalts in Paris (1672–1676), Munich 1949.

15. As for the long lasting struggle for a rigorous foundation of the Calculus see C. B. Boyer, The Concepts of the Calculus, New York 1939, 2nd printing 1949.

tion (double triangularization) of Burgess' method shows there to be n^3+n multiplications to perform in each case. Thus, in the computation of matrices with random decimal elements, where the work is often done in an entirely unthinking fashion, as if by an automaton, there will be little advantage to the more general method. However, if the computer is alert to advantageous combinations it is clear that there will be more opportunity to take advantage of fortunate combinations using the general method involving both row and column transformations. This is particularly true when some of the elements are integers, or when, in a given column or row, the elements are pre-known factors of one another, or differ by a pre-known amount, as is often the case with actual laboratory data.

SOLUTION OF QUADRATIC EQUATIONS AND TRIANGLES BY MACHINE*

J. P. BALLANTINE, University of Washington

1. Outline. First, the operations on a computing machine are expressed in symbols. These are applied to the solution of quadratic equations. Square roots are considered as solutions of special quadratic equations. A method is then given for finding ϕ from $\cos \phi$, involving 4 square roots. The accuracy can be increased at pleasure by taking more square roots.

Finally, the methods are applied to the solution of triangles. It is shown how every case of a triangle can be solved without reference to tables. The material is almost wholly original with the author.

There are two main practical consequences of our method of solving triangles. Assuming one has available a 10-place computing machine and a 6-place set of tables, then one can obtain a solution to a triangle which is correct to 9 significant places, instead of only 6. If it were required to build a super-machine for the automatic solution of triangles, use of our method would obviate the necessity of building a set of tables into the machine.

For those wishing to solve a triangle in the least time, paragraph 9 is regarded as providing the answer.

2. Machine operations. The computing machine consists of essentially three parts, the keyboard called K, the accumulator called A, and the counter called C. More elaborate machines include devices by which the interplay between K, A, and C may be made more rapid and automatic. These complications in the machine do not change the essential relation between K, A, and C, so that a consideration of the less elaborate machines will suffice.

* This paper was suggested by the brief note by Prof. H. E. Stelson, this MONTHLY, vol. 56, 1949, pp. 94–95, in which he showed how to solve the case of a triangle, given three sides.

There is essentially only one operation, called $T(1)$. As a result of this operation, the number in K is added to the number in A, and the number in C is increased by 1. The operation $T(1)$ is carried out on the older machines by turning a crank, and on the newer machines by touching the plusbar. If the operation $T(1)$ is carried out n times, the notation $T(n)$ is used. $T(1024)$ does not require the crank to be turned 1024 times, for one turn in the 1000-position of the carriage will carry out $T(1000)$.

The counter C counts how many times $T(1)$ is carried out, and the accumulator A accumulates the totals of the readings of K for the various T's.

There are means by which the readings of A, C, and K can be set at any desired number (up to a certain limited number of digits). The operation of making K read n is $K(n)$. Similarly, $A(n)$ or $C(n)$ means that the number n is to be entered in A or C. It is not necessary to introduce a new notation for clearing A, C, or K, for we need only write $A(0)$, $C(0)$, or $K(0)$.

Division is carried out by repeated subtraction. To find y/x, start with the operations $A(0)$, $K(y)$, $T(1)$, as a consequence of which, the reading of A is y. On some machines the operation $A(y)$ can be carried out directly. Then, $C(0)$, $K(x)$. Now start repeating the operation $T(-1)$ as many times as possible, without making the reading of A become negative. On machines having automatic division, the last step is carried out by throwing the division lever. We shall call this operation "Div." The result is the same whether it is done automatically or by hand.

When performing the operation $T(-n)$, the reading of C would be expected to be negative, and hence hard to read. This can be avoided on most machines by setting the machine for division.

The notation $\Delta K(n)$ means to increase the reading of K by n. In most cases, n will be a small number, like 1 or 3, and the operation is carried out by the mere pressing of a key.

Finally, the notation $Q(a, h)$ stands for an operation used in solving quadratic equations. It consists of three parts or sub-operations:

$$Q(a, h) = \Delta K(ah), \qquad T(h), \qquad \Delta K(ah).$$

3. The solution of quadratic equations. In the equation

$$ax^2 + bx + c = 0, \tag{1}$$

it is supposed that $b \geqq 0$. This may require multiplication by -1. Get the machine ready for the solution of equation (1) by the operations:

$$A(c), \qquad K(b), \qquad C(0),$$

and make a mental note of the value of a. As remarked in connection with division, the operation $A(c)$ may require three steps. If c is negative, $A(c)$ is effected by the three steps, $K(-c)$, $T(-1)$, $K(0)$.

With the machine so prepared, the operation $Q(a, h)$ will have the effect of reducing the roots of the equation by h, as in Horner's Method. Since $Q(a, h)$

can be carried out rapidly, one would as soon carry out $Q(a, 1)$ seven times as to carry out $Q(a, 7)$ once. The effect would be the same.

Example 1. Solve the equation $x^2+3x-11=0$.

Solution. Perform the following operations:

$$K(11), \qquad T(-1), \qquad C(0), \qquad K(3),$$

and make a mental note that $a=1$, the coefficient of x^2. With -11 and 3 well to the left in A and K, mark the decimal points, leaving plenty of spaces at the right for decimals.

Either $Q(1, 2)$, or $Q(1, 1)$, $Q(1, 1)$.

More in detail, $Q(1, 1)$ consists of $\Delta K(1)$, $T(1)$, $\Delta K(1)$. After these three steps, K reads 5, and A reads -7. After the second $Q(1, 1)$, K reads $+7$, A reads -1, and C reads 2. The roots of the equation have now been reduced 2 units.

Next, $Q(1, 0.1)$, $Q(1, 0.04)$, $Q(1, 0.00005)$, and so forth.

Read the root, $x_1=2.14005 \cdots$, on C.

$$x_2 = - \ 3 - 2.14005 = -5.14005, \quad \text{or}$$

$$x_2 = -11 \div 2.14005 = -5.140066 \cdots .$$

When h is small, the 3-step operation $Q(a, h)$ can be reduced to its middle step, namely $T(h)$, with only a small resulting error. Suppose the operator proceeded as above and found $x_1=2.14005$ on a 10-place machine equipped with automatic division. Then to avoid the negative reading of A, perform the operation $T(0.00001)$, followed by the operation "Div." The resulting reading of C is 2.140054945, correct to 9 decimal places. It is checked by finding x_2 in two ways.

4. **Extraction of square roots.** To find $\sqrt{N}$, merely solve the quadratic equation $-x^2+0x+N=0$. Note that $a=-1$. Take h negative.

Example 2. Find the square root of 7.

Solution. Prepare the machine for the equation by the operations:

$$A(0), \qquad K(7), \qquad T(1), \qquad K(0), \qquad C(0).$$

Next:

$$Q(-1, -2) = \Delta K(2), \qquad T(-2), \qquad \Delta K(2).$$

If this results in making the reading of C negative, start over, and this time set the machine for division. Then after the operation $Q(-1, -2)$, C will read 2. This is really -2, for we are finding the negative square root of 7.

$$Q(-1, -0.5), \qquad Q(-1, -0.1), \quad \text{or} \quad Q(-1, -0.6), \quad \text{or}$$
$$Q(-1, -0.7), \qquad Q(-1, +0.1).$$

In other words, one must guess how many tenths to take off. One may take one tenth at a time, or guess 5 tenths at once. If this is not enough, take another. If the operator overshoots and takes off 7 tenths, he may put one back. Whichever he does, he has the same result, and K reads 5.2, C reads 2.6, and A reads 0.24. Then follow the operations:

$Q(-1, -0.04)$, $Q(-1, -0.005)$, and so forth.
The result, $2.645 \cdots$, is read on C.

5. Finding cos $\frac{1}{2}\theta$. One practical application of extracting square roots is in finding $\cos \frac{1}{2}\theta$ from $\cos \theta$, by the well known formula:

$$\cos \tfrac{1}{2}\theta = \sqrt{\tfrac{1}{2}(1 + \cos \theta)}.$$

This starts with the operations:

$$A(0), \qquad K(1 + \cos \theta), \qquad T(+0.5), \qquad C(0), \qquad K(0).$$

The 1 of $1+\cos \theta$ is entered in the extreme left place in K. The 0 of $+0.5$ is in the extreme left place in C. The machine is now ready for the extraction of the square root whose value is $\cos \frac{1}{2}\theta$.

In counting the significant figures in $\cos \theta$, initial 9's should not be counted. If $\cos \theta = 0.9999365$, then one has only 3 significant places. It is desirable not to lose significant places in the process of finding $\cos \frac{1}{2}\theta$.

Example 3. If $\cos \theta = 0.98713251$, find $\cos \frac{1}{2}\theta$.

Solution. We shall suppose that there is available an 8-place machine, but as one place is used for the units place, it gives cosines to 7 significant places. In the present example, the initial 9 makes difficulties, as one must go out one more place to have 7 significant places.

$$C(0), \quad K(1.9871325), \quad T(+0.5), \quad K(0.0000001), \quad T(+0.05), \quad K(0), \quad A(0)$$

The machine is now ready to extract the square root of $\frac{1}{2}(1+\cos \theta)$. The machine gives the square root as 0.9967779, but stops there since it is only an 8 place machine. K reads $1.9935 \cdots$, or practically 2. A reads $\cdots 7307 \cdots$. The next two places of the desired square root can be read by inspection. It requires merely dividing $\cdots 7307 \cdots$ by 2, namely, 36 or 37. Therefore, on an 8-place machine, one finds readily that $\cos \frac{1}{2}\theta = 0.996777937$, to the full 7 significant places.

Of course, it would have been easier to use a 10-place machine, but with a 10-place machine, one would expect 9 significant places. The procedure of Example 3 illustrates how one finds $\cos \frac{1}{2}\theta$ to the full $n-1$ significant places on an n-place machine.

6. Finding ϕ from cos ϕ. From Taylor's series, we have

$$\cos \phi = 1 - \frac{\phi^2}{2} + \frac{\phi^4}{24} - \frac{\phi^6}{720} + \cdots \tag{2}$$

$$\cos \tfrac{1}{2}\phi = 1 - \frac{\phi^2}{4\cdot 2} + \frac{\phi^4}{4^2\cdot 24} - \frac{\phi^6}{4^3\cdot 720} + \cdots \tag{3}$$

$$\cos \tfrac{1}{4}\phi = 1 - \frac{\phi^2}{4^2\cdot 2} + \frac{\phi^4}{4^4\cdot 24} - \frac{\phi^6}{4^6\cdot 720} + \cdots . \tag{4}$$

The term in ϕ^4 is readily eliminated between (2) and (3). The term in ϕ^6 may either be dropped, or carried as an approximation to the error. The resulting equation is solved for ϕ^2, and ϕ is found by extracting one square root. The resulting formula is (5) given below. Similarly, working with (2), (3) and (4), the terms in ϕ^4 and ϕ^6 may both be eliminated, and Formula (6) is derived.

$$(5) \qquad \phi = \sqrt{\frac{15 + \cos\phi - 16\cos\frac{1}{2}\phi}{1.5}} + \frac{\phi^5}{2880} + \cdots .$$

$$(6) \qquad \phi = \sqrt{\frac{945 - \cos\phi + 80\cos\frac{1}{2}\phi - 1024\cos\frac{1}{4}\phi}{22.5}} + \frac{\phi^7}{2^7\cdot 8!} + \cdots .$$

When computing with (5) or (6), the final terms are computed only roughly, to see how many significant places are given by the radical. The remainder term in (5) happens to be precisely the same as that of the formula given by Prof. H. E. Stelson.

Example 4. Find the largest angle in a triangle with sides 7, 9, and 14. (Problem taken from H. E. Stelson's paper.)

Solution. By the law of cosines, $\cos\theta = -66/126$.

t	$\cos\theta/t$
1	−0.523809524
2	+0.487950037
4	0.862539865
8	0.9650232808
16	0.99121725185
32	0.99780189713

Since a 10-place machine was used, all cosines are carried to 9 significant places. In (6), take $\phi = \theta/8$. Then

$$\theta = 8(0.2652641622 \cdots) = 2.122113298.$$

With $\phi = 0.26 \cdots$, $\phi^7/2^7.8! = \phi^7/5{,}000{,}000$, which vanishes to 10 decimal places. Had ϕ been taken as $\theta = 2.12 \cdots$, the error would have been much larger.

7. Conversion of angles to degrees. For machine computation, it is more convenient to have all angles in radians rather than in degrees, minutes, and seconds. Machines are not designed to handle addition of numbers in the sexagesimal system. For the computation of sin x, cos x, and tan x from the series, it is best to have x in radians. But suppose it is desired to express an angle in the Babylonian manner. The following procedure is suggested:

Let D, M, and S be, respectively, the number of radians in a degree, in a

minute, and in a second. Then

$$D = 0.01745329252 \cdots$$
$$M = 0.0002908882087 \cdots$$
$$S = 0.000004848136811 \cdots .$$

Example 5. Convert 2.122113298 radians to degrees, minutes, and seconds.

Solution. Since θ is given to 10 places, we suppose a 10-place computing machine. Perform the following operations:

$$A(0), \quad K(\theta = 2.122113298), \quad T(1.000000000), \quad C(0).$$
$$K(D = 1745329252), \quad T(-121.0000000).$$

In other words, carry out the division, so long as the quotient, here 121, remains an integer. Do not clear A, K, or C until directed to.

$$K(M = 2908882087), \quad T(-00035.00000).$$
$$K(S = 4848136811), \quad T(-0000017.288).$$
$$\theta = 121° \; 35' \; 17.288'' \text{ as read from } C.$$

In a similar manner, the angle 121° 35′ 17.288″ can be reconverted to radians, by the use of D, M, and S.

8. Solution of triangles. The procedure for the solution without tables of any triangle is now obvious.

Given the three sides, apply the Law of Cosines, and find each angle from its cosine. Check by the relation $A+B+C=\pi$.

Given two sides and the included angle, a, b, and C, find cos C by the cosine series, and c from the Law of Cosines.

Given the ambiguous case, a, b, and A, one procedure is to start with the Law of Sines. Find sin A by the sine series, sin B by the Law of Sines. Then sin B $=\cos(\frac{1}{2}\pi - B)$, and $\frac{1}{2}\pi - B$ can be found from its cosine, as in Example 4. As the sign of $(\frac{1}{2}\pi - B)$ is in doubt, two values of B result.

Or, if only the value of c is desired, use the Law of Cosines immediately,

$$a^2 = c^2 + b^2 - 2bc \cos A.$$

This is a quadratic equation in c. It is solved as in Example 1.

Example 6. Given $a=12.02$, $b=12.83$, and $A=1.1023$ radians, find c.

Solution. From the Law of Cosines,

$$12.02^2 = c^2 + 12.83^2 - (2)(12.83)(\cos 1.1023)(c).$$

Cos $1.1023 = 0.451545147$, as can be found directly from the cosine series, requiring only 7 terms to give 9 places, or from the more rapidly convergent series sin $(\frac{1}{2}\pi - 1.1023)$.

$$- c^2 + 11.5866485c - 20.1285 = 0.$$
$$c_1 = 2.128067661.$$
$$c_2 = 11.5866485 - c_1 = 9.458580839,$$

or

$$c_2 = 20.1285 \div c_1 = 9.458580838.$$

The case of two angles and one side is handled by the Law of Sines, with the sines of the three angles found by series.

9. Solution of triangles with machine and tables. For the quickest possible method of solving triangles, use both tables and machine. Follow the methods of Section 8, and whenever it is necessary to find a trigonometric function of a given angle, or to find the value of the angle from one of the trigonometric functions, use the tables. The machine will be found helpful in the interpolation.

With the necessary multiplying, squaring, extracting the square root, and other computation done on the machine, every triangle can be solved with 3 applications to the tables. In some cases, this includes the check, in other cases, a fourth reference to the tables will be required for a check. This contrasts with 8 applications to the tables, when the computation is done by logarithms instead of machine.

MATHEMATICAL NOTES

EDITED BY E. F. BECKENBACH, *University of California*

Material for this department should be sent to F. A. Ficken, Institute of Mathematics and Mechanics, New York University, 45 Fourth Ave., New York 3, New York.

ON A CONJECTURE OF KLEE

PAUL ERDÖS, University of Michigan

1. Introduction. Klee[1] denotes by $S_k(m)$ the number of solutions of $\phi(x) = m$, where x has exactly k prime factors which appear to the first power in the factorization of x. Lampek[1] observed that

$$\phi\left(\frac{(n!)^2}{\phi(n!)}\right) = n!.$$

Klee[1] remarks that except for the prime 2, all prime factors of $n!/\phi(n!)$ are multiple. Thus $S_0(n!) > 0$. Klee[1] conjectures that for all n, $S_1(n!) > 0$. Gupta[2] recently proved this conjecture, in fact he proved that $\lim_{n=\infty} S_1(n!) = \infty$. In the present note we prove that $\lim_{n=\infty} S_k(n!) = \infty$ for every k, and state without

[1] This MONTHLY, vol. 56 (1949), pp. 21–26.
[2] *Ibid.*; vol. 57 (1950), pp. 326–329.

proof a few other problems and results.

2. Lemmas. First we prove three lemmas.

Lemma 1. *Let $b|a$, and assume that a/b has the same prime factors as a (that is, all the prime factors of b occur in a with a higher exponent). Then*

$$\phi\left(\frac{a}{b}\right) = \frac{\phi(a)}{b}.$$

This follows immediately from the definition of the ϕ function.

Lemma 2. *The number of primes q, $n<q<2n$, $q\equiv 1$ (mod 6), is greater than $c_1 n/\log n$ for a suitable constant c_1 and sufficiently large n.*

This follows immediately from the prime number theorem for arithmetic progressions (or also from a more elementary result).[3]

Lemma 3. *Let n be sufficiently large. Put*

$$A_{q_1,q_2,\cdots,q_k} = \frac{n!}{\prod (p-1)\cdot(q_1-1)(q_2-1)\cdots(q_k-1)},$$

where p runs through the primes $\leqq n$ and $n<q<2n$, $q\equiv 1$ (mod 6). Then $A_{q_1,\cdots,q_k}$ is an integer, and $p|A_{q_1,\cdots,q_k}$, for $p\leqq n$.

First of all from Lemma 2, for sufficiently large n the number of q's is $>c_1 n/\log n>k$; thus $A_{q_1,\cdots,q_k}$ is defined. Let t be a prime. For $n/2<t\leqq n$, $t|A_{q_1,\cdots,q_k}$ since $t|n!$ while $t\neq p-1$, $q_i-1\not\equiv 0$ (mod t) (since $q_i-1\equiv 0$ (mod 6)). Let next $3<t\leqq n/2$. The denominator of $A_{q_1,\cdots,q_k}$ can be written as

$$2^v\prod \frac{p-1}{2}\prod_{i=1}^{k}\frac{q_i-1}{2},$$

and here all the factors are distinct integers $\leqq n$. But t and $2t$ are never of the form $(q_i-1)/2$ (since $(q_i-1)/2\equiv 0$ (mod 3)). Further not both t and $2t$ can be of the form $(p-1)/2$, since either $2t+1$ or $4t+1$ is a multiple of 3. Thus any $3<t\leqq n/2$ occurs with a higher exponent in $n!$ than in the denominator of $A_{q_1,\cdots,q_k}$. If $t=3$ and $n\geqq 12$, then $3|A_{q_1,\cdots,q_k}$, since $12\neq(p-1)/2$, $12\neq(q_i-1)/2$. Let now $t=2$. The even numbers $6u+2$ are clearly not of the form $p-1$ ($6u+3\equiv 0$ (mod 3)). Thus $n!/\prod(p-1)$ is a multiple of $2^{[(n-2)/6]}$. If $\prod_{i=1}^{k}(q_i-1)$ is a multiple of 2^u we clearly have $2^u<2^k n^k$ and, for sufficiently large n,

$$2^{[(n-2)/6]} > (2n)^k.$$

Thus $2|A_{q_1,\cdots,q_k}$, and Lemma 3 is proved.

3. Theorem. We shall establish the following result.

---

[3] R. Breusch, Math. Zeitschrift, vol. 34 (1932), pp. 505–526; see also P. Erdos, *ibid.*, vol. 39 (1935), ppl 473–491.

THEOREM. *For sufficiently large n, we have*

$$S_k(n!) > c_2 \frac{n^k}{(\log n)^k}.$$

Proof: We have, by Lemmas 1 and 3, with

$$B_{q_1,\cdots,q_k} = \prod_{i=1}^{k} q_i \frac{(n!)^2}{\phi(n!)\prod_{i=1}^{k}(q_i - 1)},$$

$$\phi(B_{q_1,\cdots,q_k}) = \phi\left(\prod_{p\leqq n} p \prod_{i=1}^{k} q_i \frac{n!}{\prod_{p\leqq n}(p - 1)\prod_{i=1}^{k}(q_i - 1)}\right)$$

$$= \phi\left(\prod_{p\leqq n} p \prod_{i=1}^{k} q_i A_{q_1,\cdots,q_k}\right) = n!.$$

It follows from Lemma 2 that there are more than $c_2 n^k/(\log n)^k$ choices for $q_1, \cdots, q_k$; also, by Lemma 3, $B_{q_1,\cdots,q_k}$ contains exactly k prime factors which appear to the first power in the factorization of $B_{q_1,\cdots,q_k}$. This completes the proof of the Theorem.

4. Further questions. One can ask the question how large has n to be in order that $S_k(n!) > 0$. Our proof gives that n has to be greater than $c_3 k \log k$. By a more complicated argument we can show that for a suitable constant c_4, we have $\phi_k| [c_4 k]!| > 0$. It is probable that for every $\epsilon > 0$ and sufficiently large n we have $\phi_k([(1+\epsilon)k]!) > 0$. It is easy to see that $S_n(n!) = 0$ for $n > 2$.

We can also show that $\lim_{n=\infty} S_k(n!)^{1/n} = 1$. On the other hand there exists an absolute constant c_5 so that the number of solutions of $\phi(x) = n!$ is greater than $(n!)^{c_5}$. Previously it was known that there are infinitely many integers m, so that the number of solutions of $\phi(x) = m$ is greater than m^{c_5}. It is an open question whether c_5 can be chosen arbitrarily close to 1.

It seems a difficult question to decide whether $\phi(x) = n!$ is always solvable in squarefree integers x. Similarly it seems difficult to decide whether for sufficiently large n, the equation $\sigma(x) = n!$ is solvable ($\sigma(x)$ denotes the sum of the divisors of x).

If one wants to prove Gupta's[2] result, $S_1(n!) > 0$ for all n, it suffices to remark that for $n \geqq 4$ there always is a prime $q \equiv 1 \pmod 6$ in the interval $(n, 2n)$.[3] Also that for $n \geqq 8$,

$$2^{[(n-2)/6]+2} \Big| \frac{n!}{\prod (p - 1)}$$

(since 8 contains 2 with exponent 3). Further since $q \leqq 2n$, $q \equiv 1 \pmod 6$, if $2^u | (q-1)$ we have $2^u < 2n/3$. Thus if

$$2^{[(n-2)/6]} > n/6,$$

then $S_1(n!) > 0$, and this holds for $n \geqq 14$. For $n < 14$, the relation $S_1(n!) > 0$ can be shown by a short computation. By a slightly longer computation we can show that $S_2(n!) > 0$ for all $n \geqq 2$.

PERFECT SQUARES OF SPECIAL FORM*

VICTOR THÉBAULT, Tennie, Sarthe, France

1. Introduction. This note carries further** the determination of systems of numeration in which there exist pairs of perfect squares having the form

$$aabb = (cc)^2, \qquad bbaa = (dd)^2.$$

2. Necessary and sufficient conditions. It is easy to show† that necessary and sufficient conditions for the above are:

$$(1) \qquad a + b = B + 1, \qquad (2) \qquad 1 \leqq a, b, c < B,$$

$$(3) \qquad c^2 = a(B - 1) + 1, \qquad (4) \qquad d^2 = b(B - 1) + 1,$$

where B, a, b, c are positive integers.

3. Special cases. The form of (3) suggests an examination of the special cases where

$$(5) \qquad c = ma \pm 1,$$

with m an arbitrary positive integer.

From (1), (3) and (5), the following equations result immediately:

$$(6) \qquad B = m(ma \pm 2) + 1,$$

$$(7) \qquad b = (m \pm 1)[(m \mp 1)a \pm 2].$$

Now (4), (6) and (7) combine to give

$$(8) \qquad d^2 = m(m \pm 1)(ma \pm 2)[(m \pm 1)a \pm 2] + 1.$$

This equation is satisfied by $a = 0$, $d = 2m \pm 1$, and by $a = 4$, $d = 4m^2 \pm 2m - 1$. Hence, since the coefficient of a^2 is not a square, there will be infinitely many solutions for each positive integer m (and for many fractional values of m as well).

* Translated (and abridged) from the French by E. P. Starke, Rutgers University.

** See V. Thébault, Mathesis, 1936 (Supplement); This MONTHLY, *Two Classes of Remarkable Perfect Square Pairs*, 1949, pp. 443–448.

† Or see V. Thébault, Mathesis, *loc. cit.*

4. Examples. The following examples sufficiently illustrate the procedure.

(i) Take $m=1$, $c=a+1$. Then $B=a+3$, $b=B+1-a=4$, and $d^2=4B-3$. Since this last is an odd integer, d^2 may be set equal to $(2k+1)^2$, whence $B=k^2+k+1$. This gives the squares

$$\overline{B-3}\ \overline{B-3}\ 4\ 4 = (\overline{B-2}\ \overline{B-2})^2,$$

$$4\ 4\ \overline{B-3}\ \overline{B-3} = (\overline{2k+1}\ \overline{2k+1})^2,$$

to which we have previously called attention.*

(ii) With the values $m=2$ and $c=2a+1$, equation (8) may be written as

$$d^2 - 3(2a+3)^2 = -2,$$

of which the solutions are given by the recurrence relations

$$d_{n+1} = 4d_n - d_{n-1}, \qquad a_{n+1} = 4a_n - a_{n-1} + 3,$$

with $d_0=5$, $a_0=0$; $d_1=19$, $a_1=4$. Thus we have an infinite sequence of pairs of squares of the desired form given by

$$a = a_n, \qquad b = 3a_n + 6, \qquad c = 2a_n + 1, \qquad B = 4a_n + 5, \qquad d = d_n.$$

(iii) With $m=3$, $c=3a-1$, equation (8) becomes

$$(12a-7)^2 - 2d^2 = -1.$$

The solutions of $x^2-2d^2=-1$ are given by

$$x_{n+1} = 6x_n - x_{n-1}, \qquad d_{n+1} = 6d_n - d_{n-1},$$

with $x_0=1$, $d_0=1$; $x_1=7$, $d_1=5$. It is easy to show that a_n, being equal to $(x_n+7)/12$, is an integer for every n for which $n-2$ is a multiple of 4. The desired squares are now given by $n=4k+2$ and

$$a = (x_n+7)/12, \quad b = (2x_n+2)/3, \quad c = (x_n+3)/4, \quad B = (3x_n+1)/4, \quad d = d_n.$$

(iv) For $m=3/2$, $c=(3a-2)/2$, equation (8) may be reduced to

$$16d^2 = 45a^2 - 96a + 64,$$

which in turn is reduced, by substitution of $a=(4x+16)/15$, to

$$x^2 - 5d^2 = -4.$$

The solutions of this last equation may be obtained in the form

$$x_{n+1} = 18x_n - x_{n-1}, \qquad y_{n+1} = 18y_n - y_{n-1},$$

and thus squares of the desired form may be found from

$$a = (4x_n+16)/15, \qquad b = (x_n+1)/3, \qquad c = (2x_n+3)/5,$$
$$B = (3x_n+2)/5, \qquad d = d_n;$$

* Mathesis, 1936, *loc. cit.*

these will be all integers for x_n, d_n in either of the sequences

$$x_0 = 1, \quad d_0 = 1; \quad x_1 = 29, \quad d_1 = 13; \cdots ; \quad \text{with} \quad n = 4k + 2,$$

or

$$x_0 = 4, \quad d_0 = 2; \quad x_1 = 76, \quad d_1 = 34; \cdots ; \quad \text{with} \quad n = 4k + 3.$$

(v) Following a suggestion by M. A. Buquet, we easily verify that the polynomials

$$a = 4k, \qquad k = 8m^3 + 4m^2 - 4m - 1, \qquad d = 2k(2m^2 - 1) - 2m - 1$$

reduce (8) to an identity in m. Therefore a and d, together with

$$b = 4(m^2 - 1)k + 2m + 2, \qquad c = 4mk + 1, \qquad B = 4m^2k + 2m + 1,$$

form squares of the desired form for every integral value of m.

5. Table of values. A table is appended showing a few of the simplest examples obtained by these methods. (We omit those in (i) and those which could be obtained from them upon interchange of a and b, and c and d.)

B	a	b	c	d
81	19	63	39	71
177	45	133	89	153
305	75	231	151	265
313	140	174	209	233
1141	284	858	569	989
2461	616	1846	1231	2131
4261	1064	3198	2129	3691
8611	956	7656	2869	8119

ON A GENERALIZATION OF FEUERBACH'S THEOREM

R. Goormaghtigh, Bruges, Belgium

Using complex coordinates, H. F. Sandham has recently (this Monthly, [1949, 620–622]) given a remarkable generalization of Feuerbach's theorem by showing that *the angle between the nine-point circle of a triangle ABC and the pedal circle of two isogonal conjugate points P and Q is equal to the angle which the segment PQ subtends at the inverse of either P or Q in the circumcircle of ABC.* A very simple proof of this generalization can be derived from properties of the orthopole and Aiyar's theorem.

Let O be the circumcenter of triangle ABC. Since the orthopole of any circumdiameter lies on the pedal circle of any point of that diameter,[1] the intersections, M and M', of the nine-point circle, having center F, and the pedal circle of P and Q, having for center the midpoint ω of PQ, are the orthopoles, with respect to triangle ABC, of OP and OQ. It is known[2] that $\sphericalangle MFM' = 2\sphericalangle POQ$, and therefore $\sphericalangle MF\omega = \sphericalangle POQ$. But, by Aiyar's theorem,[3] if R is

[1] Gallatly, Modern geometry of the triangle, second edition, p. 52.

[2] *Ibid.*, p. 31.

[3] *Ibid.*, p. 79.

the circumradius, $(OP)(OQ)=2R(\omega F)$. Hence, if Q' is the inverse of Q in the circumcircle of ABC, $\omega F/FM=PO/OQ'$. The triangles ωFM and POQ' are thus similar, and the theorem is established.

A CORRECTION

B. H. ARNOLD, Oregon State College, and IVAN NIVEN, University of Oregon

We regret to report that our notes in this MONTHLY [vol. 56, pp. 465–466 (1949) and vol. 57, pp. 246–248 (1950)] contain a basic error. The mappings employed are not continuous. In the first paper the function $g(z)$ for $|z| \leqq 1$ has a discontinuity along the positive real axis. Similarly, in the second paper the function $g(z)$ in equation (8) is discontinuous. No easy remedy seems to be available to avoid this difficulty, which was pointed out to us by J. C. Oxtoby.

CLASSROOM NOTES

EDITED BY C. B. ALLENDOERFER, Haverford College

All material for this department should be sent to C. B. Allendoerfer, Haverford College, Haverford, Penna.

INTEGRATION BY LONG DIVISION

J. P. BALLANTINE, University of Washington

The familiar relation that $\int f(x)\cdot dx=(1/d)f(x)$, used in symbolic operators, can be introduced in elementary calculus for the purpose of integration.

In the process of long division, the first term in the quotient is an approximation to the quotient. The approximation is *usually* found by considering only the first terms of the dividend and divisor. *When dividing by d*, meaning the differential, the approximation is formed by dividing by d assuming that all the first term except its first factor is constant.

The procedure will give every result that can be found by the method of integration by parts. In fact the formula for integration by parts may be obtained by the following division:

$$\begin{array}{rll} d) & u\cdot dv & \quad (uv \\ & \underline{u\cdot dv + v\cdot du} & \\ & \qquad - v\cdot du & \end{array}$$

The quotient, uv, is obtained by dividing $u\cdot dv$ by d, assuming u constant. The remainder, $-v\cdot du$, still remains to be divided by d, with the result: $-\int v\cdot du$.

For example, to divide $x^2\cdot\sin x\cdot dx$ by d, assume that x^2 is constant for the

purpose of obtaining a simple approximation, namely $x^2(1/d)\sin x\cdot dx=-x^2\cos x$. This is taken as the first term in the quotient. The long division then appears as follows:

$$
\begin{array}{rl}
d)\ \ x^2\cdot\sin x\cdot dx & (-x^2\cdot\cos x+2x\cdot\sin x+2\cdot\cos x \\
\underline{x^2\cdot\sin x\cdot dx-2x\cdot\cos x\cdot dx} & \\
+\,2x\cdot\cos x\cdot dx & \\
\underline{2x\cdot\cos x\cdot dx+2\cdot\sin x\cdot dx} & \\
-\,2\cdot\sin x\cdot dx & \\
\underline{-\,2\cdot\sin x\cdot dx} & \\
0 &
\end{array}
$$

When the remainder comes out 0, we know that the division, or rather the integration, is exact. The quotient is the desired integral of $x^2\cdot\sin x$.

In the division of $e^{2x}\cdot\sin 3x\cdot dx$ by d, the remainder never comes out 0, but after two steps the remainder is a constant times the dividend. The fundamental relation between D = dividend, d = divisor and also differential, Q = quotient, and R = remainder, is

$$D=dQ+R.$$

If

$$D=dQ+kD,$$

then

$$(1-k)D=dQ,\quad\text{and}$$

$$D=d\left(\frac{Q}{1-k}\right).$$

The result is $Q/(1-k)$, and the integration is seen to be exact.

The long division for $e^{2x}\cdot\sin 3x\cdot dx$ by d appears as follows:

$$
\begin{array}{rl}
d)\ \ e^{2x}\cdot\sin 3x\cdot dx & (-\frac{1}{3}e^{2x}\cdot\cos 3x+\frac{2}{9}e^{2x}\cdot\sin 3x \\
\underline{e^{2x}\cdot\sin 3x\cdot dx-\frac{2}{3}e^{2x}\cdot\cos 3x\cdot dx} & \\
\frac{2}{3}e^{2x}\cdot\cos 3x\cdot dx & \\
\underline{\frac{2}{3}e^{2x}\cdot\cos 3x\cdot dx+\frac{4}{9}e^{2x}\cdot\sin 3x\cdot dx} & \\
-\,\frac{4}{9}e^{2x}\cdot\sin 3x\cdot dx &
\end{array}
$$

Here $R=-\frac{4}{9}D$, so that $k=-\frac{4}{9}$.

The desired integral is $Q/(1-k)$, namely:

$$\frac{-\frac{1}{3}e^{2x}\cdot\cos 3x+\frac{2}{9}e^{2x}\cdot\sin 3x}{1+\frac{4}{9}}.$$

PARAMETRIC EQUATIONS AND MECHANICAL MANIPULATION OF MATHEMATICAL SYMBOLS

A. D. FLESHLER, Champlain College

Many writers have pointed out the necessity of impressing upon the student the fact that mathematical symbols and formulae are not to be handled mechanically, expecting that an apparent automatic, correct operation will always yield a correct result. Examples illustrating the pitfalls involved in such a mechanical procedure are legion.* Even the better student who knows his integration and trigonometric transformations will perform the following operation:

$$\int_0^{4\pi} \sqrt{1 - \cos x}\, dx = \sqrt{2} \int_0^{4\pi} \sin \frac{x}{2}\, dx = 0(!!).$$

Most students, when the difficulty involved in the above operation is pointed out to them, still remain puzzled why an apparent correct transformation led to the wrong result.

I found that one of the most effective ways of training the student to be on his guard against wrong results arising from such automatic transformations is to ask him to plot a curve given in parametric equations: first by eliminating the parameter and plotting $y=f(x)$, and then plotting directly from the parametric equations $x=h(t)$; $y=g(t)$. The equations, of course, are chosen so that the two methods of plotting lead to two different loci. I gave the class the simple set of equations:

$$x = \sin t; \qquad y = \sin t$$

and I asked them for the locus represented by these equations. The unanimous agreement was that since $x=y$ by eliminating $\sin t$, the locus was a 45 degree line through the origin. I then asked them to plot directly by making a table of values of t, x, y. They quickly realized that they could not get out of the intervals: $-1 \leqq x \leqq 1$; $-1 \leqq y \leqq 1$ and that the parametric equations represented only a finite portion of the line. I gave them innumerable variations of the above set of equations. In each case a mechanical elimination led to $x=y$; while the original set of equations represented different portions of the line.

For example: The equations $x=e^t$; $y=e^t$ represent the portion of the line in the first quadrant; while the equations: $x=\coth(t)$; $y=\coth(t)$ represent the entire line except the interval: $-1<x<1$; $-1<y<1$. Once the students grasped the significance of the above illustrations they were on their guard against such fallacies when the above method was applied to other loci.

Time and again the student had used the following elimination: Given $x=r\cos\theta$; $y=r\sin\theta$. Squaring and adding he obtained $x^2+y^2=r^2$ a circle. When I gave them the following set of equations $x=a(\tanh t)$; $y=a(\operatorname{sech} t)$ a few were

* See F. S. Nowlan: "Objectives in Teaching College Mathematics," this MONTHLY, vol. 57, pp. 73–82, 1950.

tempted to say that the equations represented a full circle, since, following the above procedure we obtain:

$$x^2 + y^2 = a^2(\text{sech } t)^2 + a^2(\tanh t)^2 = a^2$$

The majority, however, realized that y must always be positive; and hence the entire locus must be above the x axis.

The advantage of this method is that once the student grasps the idea he gets real pleasure out of constructing his own illustration and is duly impressed with the importance of a proper interpretation of mathematical symbols and expressions.

AN IMPROVEMENT ON DICKSON'S "BEST METHOD FOR INTEGRAL ROOTS"

J. G. CAMPBELL, Albany, Kentucky

In chapter III, section 20 of L. E. Dickson: "New First Course In the Theory Of Equations," one finds Dickson's "Best Method For Integral Roots." We present the following simplification of the procedure for applying his method. Let $f(x) \equiv a_0x^N + \cdots a_N$ be a polynomial with integral coefficients. We seek its integral roots, r.

THEOREM. *If r is an integral root of $f(x)=0$ and if m is any other integer not a root, then $r-m$ is a divisor of $f(m)$.*

Proof. Consider the factor theorem. If r is a root of $f(x)=0$ then $f(x)=(x-r)Q(x)$. Put $x=m, f(m)=(m-r)Q(m)$. Hence $r-m$ is a divisor of $f(m)$.

Let $(D_1, D_2, \cdots, D_j)$ be the total set of divisors of $f(m)$. Form the subset $(D_1+m, D_2+m, \cdots, D_j+m)$. The root r is contained in this subset, since $(r-m)+m=r$ and $r-m$ is one of the divisors of the first set of numbers.

Now consider the divisors of $f(m_1)$ where m_1 is another integer not a root. Let the total set of divisors be $(d_1, d_2, \cdots, d_k)$. Form the subset $(d_1+m_1, d_2+m_1, \cdots, d_k+m_1)$. The root r is contained in this subset of numbers. Let $(s_1, s_2, \cdots, s_l)$ be the set of numbers common to both subsets. r is contained in this set.

To illustrate his method, Dickson gives the following problem in the above mentioned section of his book.

Example. Find the integral roots of $f(x)=x^3-20x^2+164x-400=0$. The constant term has 30 divisors. There is no negative root and 21 is an upper limit to the roots. Let $m=0$, then $f(0)=-400$. The divisors less than 21 are (1, 2, 4, 8, 16, 5, 10, 20). The subset is the same since any number plus zero is that number. Now let $m_1=1$ and note that $f(m_1)=-255$. The divisors are 3, 5, 17. The subset is $(3+1, 5+1, 17+1)$ or (4, 6, 18). The set common to both subsets is (4). We see that $f(4)=0$ and 4 is the only integral root. In actual practice it is possibly better to consider the subset of the divisors of $f(m_2)$ also, and its intersection with the set common to the first two subsets. 1, -1 and 0 are the suggested values of m (provided they are not roots).

ELEMENTARY PROBLEMS AND SOLUTIONS

EDITED BY HOWARD EVES, Oregon State College

Send all communications concerning Elementary Problems and Solutions to Howard Eves, Mathematics Department, Oregon State College, Corvallis, Oregon. This department welcomes problems believed to be new, and demanding no tools beyond those ordinarily furnished in the first two years of college mathematics. To facilitate their consideration, solutions should be submitted on separate, signed sheets, within three months after publication of problems.

PROBLEMS FOR SOLUTION

E 950. *Proposed by W. R. Ransom, Tufts College*

Show that every positive integral power of $\sqrt{2}-1$ is of the form $\sqrt{m}-\sqrt{m-1}$.

E 951. *Proposed by C. S. Venkataraman, Trichur, South India*

Two circles are drawn touching the sides AB, AC of a triangle ABC at the ends of the base BC and also passing through the midpoint D of BC. If E is the other point of intersection of the circles show that AE is a symmedian of triangle ABC.

E 952. *Proposed by Albert Wilansky, Lehigh University*

Let $P:(r, \theta)$ be given in plane polar coordinates. Define the *p-slope* of the segment P_1P_2 to be $(r_2-r_1)/$(distance from P_1 to P_2). If A, B, C are three collinear points, with B between A and C, show that p-slope $CB \geqq p$-slope $CA \geqq p$-slope BA.

E 953. *Proposed by Joseph Langr, Prague, Czechoslovakia*

Four planes, drawn through a point, parallel to the faces of a tetrahedron, cut the edges of the tetrahedron in 12 points lying on a quadric surface.

E 954. *Proposed by G. W. Walker, Buffalo, N. Y.*

Let T_c be the cth positive integer which is both a triangular number and a square number. Find T_c as a function of c.

E 955. *Proposed by C. L. Dunsmore, University of California, Los Angeles*

Evaluate the nth order determinant $|a_{ij}|$, where $a_{ij}=1/(i+j-1)!$.

SOLUTIONS

Triangle, Circle, and Chords

E 915 [1950, 260]. *Proposed by Victor Thébault, Tennie, Sarthe, France*

The circle passing through the feet L, M, N of the cevians AL, BM, CN which divide the sides $BC=a$, $CA=b$, $AB=c$ of a triangle in the ratios

$$BL/LC = c^n/b^n, \qquad CM/MA = a^n/c^n, \qquad AN/NB = b^n/a^n,$$

intercepts on the lines BC, CA, AB signed segments of lengths x, y, z satisfying

the relation

$$x/a^{n-1} + y/b^{n-1} + z/c^{n-1} = 0.$$

Solution by the Proposer. Let LX, MY, NZ be the chords (or signed lengths x, y, z) intercepted by the circle LMN on the lines BC, CA, AB, and suppose the triangle lettered so that $a \geqq b \geqq c$. Now $AM \cdot AY = AN \cdot AZ$, with similar expressions favoring the other vertices. These relations yield the following equations:

$$bc^n[bc^n + (c^n + a^n)y]/(c^n + a^n)^2 = cb^n[cb^n + (a^n + b^n)z]/(a^n + b^n)^2,$$

$$ca^n[ca^n - (a^n + b^n)z]/(a^n + b^n)^2 = ac^n[ac^n + (b^n + c^n)x]/(b^n + c^n)^2,$$

$$ab^n[ab^n - (b^n + c^n)x]/(b^n + c^n)^2 = ba^n[ba^n - (c^n + a^n)y]/(c^n + a^n)^2.$$

Multiplying these equations, respectively, by a^{2n}, b^{2n}, c^{2n}, then adding and transposing all terms to the left, and finally dividing through by $a^nb^nc^n$, we obtain the required relation.

In $n=1$, in which case AL, BM, CN are the interior angle bisectors of the triangle, the relation reduces to

$$x + y + z = 0, \tag{1}$$

a result given by Lawrence in *The Mathematical Gazette*, 1945, p. 18.

If $n=2$, the relation becomes

$$x/a + y/b + z/c = 0. \tag{2}$$

The segments determined by the circle through the isotomic conjugates of L, M, N, satisfy the relation

$$a^{n+1}x + b^{n+1}y + c^{n+1}z = 0,$$

which reduces to (1) when $n=-1$ and to (2) when $n=-2$.

Also solved by Roger Lessard.

Editorial Note. This problem can lead to an interesting and extended study. Thus, suppose C_n and r_n are the center and radius of the circle LMN. What is the locus of C_n, and what is the expression for r_n in terms of a, b, c, n? Is there an extension of the tetrahedron? Etc.

Orthogonal Spheres

E 918 [1950, 335]. *Proposed by N. A. Court, University of Oklahoma*

Two spheres are orthogonal if they cut a given line in pairs of harmonic points and if the two planes determined by that line and the centers of the two spheres are perpendicular.

Solution by Roger Lessard, Ecole Polytechnique, Montreal. Let A_1, B_1 and A_2, B_2 be the intersections of the given line with the spheres S_1 and S_2 of centers O_1 and O_2 and radii r_1 and r_2. Let O and O' be the midpoints of A_1B_1 and A_2B_2.

Then O_1O and O_2O' are perpendicular to the given line. The harmonic relation gives

$$(OO')^2 = (OA_1)^2 + (O'A_2)^2,$$

whence the perpendicular relation gives

$$\begin{aligned}(O_1O_2)^2 &= (O_1O)^2 + (OO')^2 + (O'O_2)^2 \\ &= (O_1O)^2 + (OA_1)^2 + (O'A_2)^2 + (O'O_2)^2 \\ &= r_1^2 + r_2^2,\end{aligned}$$

and the spheres are orthogonal.

Also solved by the proposer.

An Expression for the Greatest Common Divisor

E 919 [1950, 335]. *Proposed by Leo Moser, University of Manitoba*

Show that the greatest common divisor of a and b is given by

$$(a, b) = \sum_{m=0}^{a-1} \sum_{n=0}^{a-1} (1/a)e^{2\pi i b mn/a}.$$

Solution by F. Bagemihl and W. Seidel, University of Rochester. Let $d = (a, b)$. Then $a = rd$, $b = sd$, where $(r, s) = 1$. Also, $(b/a)m = (s/r)m$, which is an integer for precisely d values of $m \leqq a-1$, namely, $m = 0, r, 2r, \cdots, (d-1)r$. Now if $(b/a)m$ is not an integer

$$\sum_{n=0}^{a-1} e^{2\pi i b mn/a}$$

is a finite geometric series with sum zero. But if $(b/a)m$ is an integer, the same sum is equal to a, since there are then a terms each equal to 1. Hence

$$\sum_{m=0}^{a-1} \sum_{n=0}^{a-1} (1/a)e^{2\pi i b mn/a} = (1/a)da = d.$$

Also solved by P. M. Anselone and V. E. Hoggatt (jointly), D. H. Browne, N. J. Fine, Roger Lessard, A. E. Livingston, H. Ohbayashi, L. B. Rall, L. A. Ringenberg, C. M. Sandwick, Sr., P. J. Schillo, N. T. Seely, Jr., and the proposer.

The proposer pointed out that the expression for (a, b) is given without proof in G. Frobenius, *Vorlesungen über Zahlentheorie*, p. 28. The well known properties of (a, b), such as $(a, b) = (b, a)$ and $(ca, cb) = c(a, b)$, may easily be deduced from it.

A Matrix Identity

E 920 [1950, 335]. *Proposed by T. G. Room, University of Sydney, Australia*

If S and T are any two square matrices of the same order, and the necessary matrices are non-singular, then

$$(I + S)^{-1}(S + T)(I + ST)^{-1}(I + S) = (I - S)(I + TS)^{-1}(S + T)(I - S)^{-1}.$$

Solution by the Proposer. Set

$$A = (S + T)(I + ST)^{-1}(I - S^2),$$

$$B = (I - S^2)(I + TS)^{-1}(S + T).$$

We have to prove that $A = B$. Now we clearly have

$$T(I + ST)^{-1} = (I + TS)^{-1}T, \tag{1}$$

$$T(I + ST)^{-1}S = I - (I + TS)^{-1}, \tag{2}$$

$$(I + ST)^{-1}S = S(I + TS)^{-1}, \tag{3}$$

$$(I + ST)^{-1} = I - S(I + TS)^{-1}T. \tag{4}$$

(The proofs of (1) and (2) are immediate, and (3) and (4) are the same results with S and T interchanged.) Using these four results we now have

$$\begin{aligned} A &= S(I + ST)^{-1} + T(I + ST)^{-1} - S(I + ST)^{-1}S^2 - T(I + ST)^{-1}S^2 \\ &= S - S^2(I + TS)^{-1}T + (I + TS)^{-1}T - S^2(I + TS)^{-1}S \\ &\quad - S + (I + TS)^{-1}S = B. \end{aligned}$$

Also solved by Mohammad Ishaq and Ingram Olkin.

A Multiple-pan Balance

E 921 [1950, 416]. *Proposed by R. T. Hood, Beloit College*

Devise a multiple-pan balance, with minimum number of pans, by which one can weigh any whole number of pounds using weights which are powers of n, n a given positive integer. What other amounts can be weighed with such a balance?

Solution by S. T. Thompson and the Proposer. The required balance has $n-2$ pans on the right side of the fulcrum and one on the left, such that their points of suspension and the fulcrum constitute n equally spaced points. Let the pans be numbered from left to right by $-1, 1, 2, 3, \cdots, n-2$. Suppose p pounds are to be weighed. Let the sequence of digits representing p in the n-ary system be altered by starting from the right and replacing each digit, except 1, by that digit minus n, and at the same time increasing the next digit to the left by 1. Then if the new jth digit is k, put the weight of n^{j-1} pounds in the pan numbered $-k$. This arrangement of weights will balance p/q pounds placed in pan q, $q=1, 2, \cdots, n-2$.

Editorial Note. The above is a generalization of the well known fact that an ordinary two-pan balance can weigh any whole number of pounds by using weights of 1, 3, 3^2, $\cdots$ pounds, if weights are permitted in both pans. The crux of the solution rests on the fact that every positive integer may be represented as a series of multiples by $1, 0, -1, \cdots, -(n-2)$ of the successive powers of n.

A Dissection of a Triangle

E 922 [1950, 416]. *Proposed by Michael Goldberg, Washington, D. C.*

Show that any given triangle can be dissected by straight cuts into four pieces which can be arranged to form two triangles similar to the given triangle.

Solution by Aaron Buchman, Buffalo, New York. Take D, E, F on AB, BC, CA, the sides of any triangle ABC, such that $AD/AB=CE/CB=AF/AC=1/5$. Let G be on AC with $AG=2AF$, and let H be the midpoint of DE. Then the segments DE, DF, GH represent the three cuts. One of the required triangles is the piece BDE. The other three pieces may be rotated in the plane to form the second triangle. The proof is elementary.

Also solved by D. H. Browne, Joseph Langr, Roger Lessard, B. D. Roberts, Joseph Rosenbaum, Harold Shniad, C. W. Trigg, and the proposer.

A great variety of solutions was offered for the problem, but most of them, like the above, led to dissections where the linear ratios of the three triangles in question are 3:4:5. In some of the solutions certain pieces had to be turned over. Remarkable ingenuity was exercised by Rosenbaum, who offered a surprising number of solutions. In addition to many of the 3:4:5 type he found dissections where the linear ratios of the three triangles are, for example,

$$(a_1+a_2)^2-a_i^2:2(a_1+a_2)a_i:(a_1+a_2)^2+a_i^2, \qquad i=1, 2, 3,$$

where a_1, $a_2 \leqq a_1$, a_3 are the sides of the given triangle. He also found solutions applicable to certain limited classes of triangles. In particular, he found a ***three*** piece solution applicable to any nonequilateral triangle. Here the ratios are

$$2a_1a_2:a_1^2-a_2^2:a_1^2+a_2^2, \qquad a_1>a_2.$$

One wonders if there is a three piece solution for the equilateral triangle.

ADVANCED PROBLEMS AND SOLUTIONS

EDITED BY E. P. STARKE, Rutgers University

Send all communications concerning Advanced Problems and Solutions to E. P. Starke, Rutgers University, New Brunswick, New Jersey. All manuscripts should be typewritten, with double spacing and with margins at least one inch wide. Problems containing results believed to be new or extensions of old results are especially sought. Proposers of problems should also enclose any solutions or information that will assist the editor. In general, problems in well known text books or results found in readily accessible sources should not be proposed for this department.

PROBLEMS FOR SOLUTION

4425. *Proposed by P. A. Piza, San Juan, Puerto Rico*

There are infinitely many triangular numbers (numbers of the form $k(k+1)/2$) which are also perfect squares, viz.

$$[(17+12\sqrt{2})^n+(17-12\sqrt{2})^n-2]/32.$$

Find numbers which are at the same time a sum of two squares and a sum of two triangular numbers.

4426. *Proposed by D. J. Newman, New York University*

Let a, b, c, d all have modulus unity. Prove that in the unit circle the polynomial

$$P(z)=az^3+bz^2+cz+d$$

has a maximum modulus not less than $\sqrt{6}$. What is the result for a polynomial of degree greater than three?

4427. *Proposed by Paul Erdös, University of Aberdeen, Scotland*

Let

$$f_n(x)=\prod_{i=1}^{n}(x-x_i), \qquad -1\leqq x_i\leqq 1.$$

Prove that there cannot exist numbers a, b such that

$$|f_n(a)|\geqq 1, \qquad |f_n(b)|\geqq 1, \qquad -1<a<0<b<1.$$

SOLUTIONS

Square Inscribed in Arbitrary Simple Closed Curve

4325 [1949, 39]. *Proposed by Orrin Frink, Pennsylvania State College.*

Show that on every simple closed plane curve there are four points which are the vertices of a square.

Note by A. M. Gleason, Cambridge, Massachusetts. I would like to point out

the inadequacy of the proof printed in the June issue [1950, 423] as a solution of problem 4325. First, it is stated that it is possible to deform a horizontal chord in such a way that it is always horizontal and moves from the first contact to the last contact. While this is easy to see when the curve has only a finite number of maxima and minima, it certainly requires proof that it can be done in general.

Second, the perpendicular bisector of the moving horizontal chord is alleged to have continuous behavior. Since the perpendicular bisector of the curve may have many different intersections with the curve it is easy to give examples in which, say, the first intersection counting from below is not continuous. It may be possible to get around this difficulty by the device of retrograding as before, but this must be combined with the demand for a retrograde motion for XX' and proof is certainly required.

Third, it is assumed that the rhombi constructed for each direction depend continuously on the direction. This is a difficulty which cannot be circumvented by the devices previously employed. Let A, B, C, D be the vertices of a rhombus. A simple closed curve may be easily constructed containing A, B, C, D but not containing any other rhombus near $ABCD$. To do this, simply let A and C lie on cusps pointing to the right while B and D lie on cusps pointing to the left.

Note by J. J. Schaeffer, Instituto de Matematica y Estadistica, Montevideo, Uruguay. In the second part of the proof the greatest difficulties are encountered. This applies particularly to the statement "then rotate m_x continuously through 90° until it reaches the position m_y; now $XX' < YY'$." There seems to be no reason to assume the possibility of returning by continuous rotation to the initial position of the rhombus, and even in the case of fairly simple curves the proof encounters apparently insurmountable obstacles.

For convex figures, however, the idea can be used, as it can be easily proved that convex figures have the property that for any slope there is essentially only one rhombus with a diagonal parallel to this slope. This is the same idea as Emch's, but in order to apply it to general convex curves, which may contain rectilinear segments or angles, the use of continuity should be avoided, and replaced by considerations of limit points of open sets.

Similar comments were received from Arthur Rosenthal, and G. A. Dirac. Further information and discussion of this interesting problem will be welcomed.

A Prime Representing Function

4337 [1949, 186]. *Proposed unsigned.*

If the numbers R_{nk} are defined by

$$\frac{1 - z^2}{\sin \pi z} \prod_{k=2}^{n} \sin (\pi z/k) \equiv \sum_{k=0}^{\infty} R_{nk} z^k,$$

prove that $\underline{\lim}\, R_{nk}^{-1/k}$ is equal to the first prime exceeding n.

Solution by the Proposer. The only possible poles are at the integers, and these are cancelled by zeros of the numerator for all z such that $|z| \leqq n$. As $|z|$ increases beyond n, the numerator will first fail to vanish when z is a prime. The radius of convergence, then, is equal to the distance to the first prime beyond n, and the desired result follows immediately.

Two Feuerbach Point Theorems

4344 [1949, 269]. *Proposed by Victor Thébault, Tennie, Sarthe, France*

(1) If, in a triangle, one of the angles is 120° (or 60°), two of the Feuerbach points are diametrically opposite on the nine-point circle, and conversely. (2) If the triangle is scalene and if the circle through the feet of the interior bisectors (or one interior and two exterior bisectors) passes through one of the vertices, three of the Feuerbach points form an isosceles triangle, and conversely.

Solution by J. W. Clawson, Ursinus College, Collegeville, Pennsylvania. Let ABC be the triangle, and A', B', C' the midpoints of the sides. We shall use the triangle $A'B'C'$ as basic and employ trilinear areal coördinates, taking $B'C' = a$, and so on.

The equation of the nine-point circle of ABC is

$$a^2yz + b^2zx + c^2xy = 0;$$

that of the inscribed circle is

$$a^2yz + b^2zx + c^2xy = [(b-c)^2x + (c-a)^2y + (a-b)^2z](x+y+z);$$

and that of the escribed circle opposite A is

$$a^2yz + b^2zx + c^2xy = [(b-c)^2x + (c+a)^2y + (a+b)^2z](x+y+z).$$

Then the Feuerbach point, F, at which the nine-point circle and the inscribed circle touch has coördinates

$$\frac{a}{b-c},\quad \frac{b}{c-a},\quad \frac{c}{a-b};$$

while F_1, at which the nine-point circle touches the above escribed circle is given by

$$\frac{a}{b-c},\quad \frac{-b}{c+a},\quad \frac{c}{a+b}.$$

Similarly the other two Feuerbach points, F_2, F_3, are respectively

$$\frac{a}{b+c},\quad \frac{b}{c-a},\quad \frac{-c}{a+b};\qquad \frac{-a}{b+c},\quad \frac{b}{c+a},\quad \frac{c}{a-b}.$$

The center of the nine-point circle, N, is

$$a^2(b^2 + c^2 - a^2), \qquad b^2(c^2 + a^2 - b^2), \qquad c^2(a^2 + b^2 - c^2).$$

(1) The condition that F, F_1, N are collinear is easily reduced to $a^2 = b^2 + c^2 - bc$, which means that $A' = A = 60°$. The condition that F_2, F_3, N are collinear is $a^2 = b^2 + c^2 + bc$, which means that $A' = A = 120°$.

(2) The point P at which the interior bisector of angle A meets BC is $b+c$, $c-b$, $b-c$; P', the foot of the external bisector of A, is $b-c$, $-b-c$, $b+c$; with analogous expressions for Q, Q', R, R', the feet of the other bisectors. By substituting in the general equation of a circle

$$a^2yz + b^2zx + c^2xy = (px + qy + rz)(x + y + z),$$

and eliminating p, q, r, one finds the condition that P, Q, R, A be concyclic is

$$a^3 + a^2(b + c) - a(b^2 + bc + c^2) - (b^3 + b^2c + bc^2 + c^3) = 0.$$

The same equation is obtained, somewhat laboriously, as the condition that $F_1F_2 = F_1F_3$. We note that $a = b$ reduces the above equation to $c(b+c)^2 = 0$. Hence if the triangle is not scalene, this condition could not be satisfied for all three of its cases.

In the same way we may deduce that the condition for P, Q', R', A to be concyclic is the same as that for $FF_2 = FF_3$.

Also solved by R. Goormaghtigh, and the proposer.

Editorial Note. Goormaghtigh derives the solution synthetically from the property: when a circumdiameter of a triangle ABC moves about the circumcenter, its orthopole as to ABC moves on the nine-point circle with an angular speed twice that of the circumdiameter. He gives also the following references:

Part (1) has been mentioned before. See Goormaghtigh, *Tôhoku Mathematical Journal*, 1926, p. 107.

For properties of triangles having one angle equal to 60° or 120°, see Neuberg, *Mathesis*, 1895, p. 215, Droz-Farny, Deprez, *Ibid.*, 1897, p. 77, Goormaghtigh, *Ibid.*, 1914, p. 107.

For properties of the special triangles considered in part (2), see Cristeseu, *Gazeta Matematica*, 1902–1903, p. 112, Bastin, *Mathesis*, 1903, p. 32, Neuberg, *Ibid.*, 1903, p. 134.

Properties of these and other special triangles will be found in Neuberg, Bibliographie des triangles spéciaux, *Mémoires de la Société royale des Sciences de Liége*, 1924, Goormaghtigh, Propriétés nouvelles des triangles spéciaux, *Ibid.*, 1925.

Euler's Constant

4353 [1949, 479]. *Proposed by H. F. Sandham, Trinity College, Ireland*

Prove that

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n}\left[\frac{\log n}{\log 2}\right] = \gamma,$$

where $[x]$ denotes the integral part of x, and γ is Euler's constant.

Solution by D. F. Barrow, University of Georgia. From the definition of γ if follows that

$$\gamma = \lim_{n\to\infty}\left(\sum_{s=1}^{2^n} \frac{1}{s} - \log 2^n\right).$$

The expression in parentheses may be written

$$\sum_{s=1}^{2^n} \frac{1}{s} - \log 2 - \log 2^{n-1} = \sum_{s=1}^{2^n} \frac{1}{s} + \sum_{s=1}^{\infty} \frac{(-1)^s}{s} - \log 2^{n-1}.$$

Now, when the terms of the finite series are added to the corresponding terms of the infinite series, the odd terms cancel and the even terms add. Thus we have

$$\sum_{s=1}^{2^{n-1}} \frac{1}{s} - \log 2^{n-1} + \sum_{s=2^n+1}^{\infty} \frac{(-1)^s}{s}.$$

Next apply the same reduction to the first two terms of this expression, obtaining

$$\sum_{s=1}^{2^{n-2}} \frac{1}{s} - \log 2^{n-2} + \sum_{s=2^{n-1}+1}^{\infty} \frac{(-1)^s}{s} + \sum_{s=2^n+1}^{\infty} \frac{(-1)^s}{s}.$$

Let this process be continued until the logarithmic term is used up, and then, in the various series that arise, let terms having a common denominator be added:

$$1 + \sum_{r=1}^{n-1}\left(r \sum_{s=2^r+1}^{2^{r+1}} \frac{(-1)^s}{s}\right) + n \sum_{s=2^n+1}^{\infty} \frac{(-1)^s}{s}.$$

As n grows infinite, we notice that the last term, containing the infinite limit, approaches zero, and this leads to

$$\gamma = 1 + \sum_{s=1}^{\infty} \frac{(-1)^s}{s}\left\{\frac{\log s}{\log 2}\right\},$$

where $\{x\}$ denotes the largest integer less than x. This series becomes the one called for in the problem if we express the initial 1 as the infinite series $1/2+1/4+1/8+\cdots$, and add these terms to those of the infinite summation having like denominators.

Also solved by M. S. Klamkin, Norman Miller, and the proposer.

RECENT PUBLICATIONS

EDITED BY E. P. VANCE, Oberlin College

All books for review should be sent directly to the editor of this department, American Mathematical Monthly, 531 West 116th Street, New York 27, N. Y. and not to any of the other editors or officers of the Association.

University Mathematics. By Joseph Blakley. Glasgow, Blackie & Son, Limited, 1949. 527 pages. 25s.

This is a University textbook for students of science and engineering and is intended to be used in a first-year course in Pure Mathematics. It is essentially a book on analytic geometry and calculus but contains as well two chapters on differential equations including a section on partial differential equations.

The material covered is well integrated and due emphasis is placed on matters of manipulation and drill but not enough attention is paid to questions of theory and rigor. "The limit of the quotient of two functions is equal to the quotient of their limits" (p. 14). In the presentation of partial fractions the statement is made that for every cubic factor ax^3+bx^2+cx+d there will be a term $(Ax^2+Bx+C)/(ax^3+bx^2+cx+d)$ and, further, it is noted that "a quadratic factor or a cubic factor cannot be factorized further" (p. 70). A power series is defined in terms of coefficients which are supposed to be rational (p. 165). Under partial differentiation it is stated that "if $u=f(x, y)$ be a continuous function, then $\partial^2u/\partial x\partial y=\partial^2u/\partial y\partial x$" (p. 210).

By way of explaining the topics under discussion the author works many illustrative problems and there is a set of examples (the only ones given) at the end of each chapter with answers supplied at the back of the book. These examples—more than 500 in number—are mostly taken from London University examination papers and hence would not in general be readily adaptable to use as daily assignments in a regulation American College course.

C. O. OAKLEY

The Mathematics of Great Amateurs. By J. L. Coolidge. Oxford University Press, 1949. 8+211 pages. $6.00.

In his preface the author admits difficulty in defining an amateur in mathematics, and additional difficulty in choosing a limited few from the many who conform to his definition. We would enter upon no debate with him upon his choice in these matters. Intriguing indeed are most of the names which he has chosen—one for each of his sixteen chapters—and, likewise, the topics under each name. They deserve listing here.

Plato starts the list. We have mention of his mathematical training, followed by various topics and ending with his thoughts on commensurability of numbers. Next is Omar Khayyâm, his story and his contribution to cubics. Then comes Leonardo, his background and, naturally, a variety of topics. We then have Dürer with his study of spirals and helices and his lead into descrip-

tive geometry through study of measurements of the human body; Napier, logarithms and various other topics, followed by a word of deserved praise; Pascal, his famous theorem, logic of mathematics and other discussions, including integration; Arnauld who, probably, made the first serious attempt to break away from Euclid, with his logic of mathematics and his new approach to geometry; De Witt, his writings on geometry and on annuities; Hudde, his work on reduction of equations and on equal roots; Brouncker on continued fractions and other studies; L'Hospital, several contributions ending with infinitesimals and conics. Then come Buffon, his search for the infinite, moral arithmetic, probability; Diderot on vibrating strings, involutes, the pendulum; Horner and his famous "method"; and, finally, Bolzano with his study of real functions and his close approach to Cantor's idea of the power of an infinite assemblage.

The very nature of such a work invites—and the author accepts the invitation—the idea of making the treatment replete with the human element. One feels that he is delightfully successful in this. It is refreshing to go back with the author and to rethink the thoughts of those who worked when, mathematically speaking, the "world was young." It is to be hoped that his wish that someone might undertake the task of continuing this type of study and writing, might be granted.

F. B. WILEY

Regular Polytopes. By H. S. M. Coxeter. London, Methuen, 1948; New York, Pitman, 1949. xx+321 pages. $10.00.

A polytope (to give an informal definition) is a figure bounded by line segments, or by portions of planes or hyperplanes; the word is used to describe any member of the sequence *point, segment, polygon, polyhedron, etc.* The study of polytopes in their simplest forms goes back at least as far as the Greeks and their investigations of the properties of polygons and polyhedra. However the middle of the nineteenth century had arrived before Schläfli, one of the first to venture into space of more than three dimensions, discovered the general polytope. His work attracted little attention, and it was not until 1880 that his ideas were rediscovered by Stringham and again later by others. With the publication of this book further rediscoveries are not to be expected.

Professor Coxeter has collected and arranged the scattered material, enriched it with the results of his own research done over a period of twenty-five years, and has presented us with what is, and will probably be for many years, the only organized treatment of the subject. For this reason we should be glad that the work has been done by such a competent scholar and by one who is doubtless more thoroughly acquainted with this field than anyone else.

The arrangement of the material is excellent. The first six chapters are devoted largely to the theory of polyhedra, leading the reader by easy stages from the familiar five Platonic solids, through a mild change in the definition of polyhedron, to the four beautiful Kepler-Poinsot star-polyhedra, which are also

regular. These chapters serve at the same time to introduce methods and concepts which later prove effective in the study of polytopes in higher space. Among these are the use of maps, symmetry groups, honeycombs, truncation, stellating, faceting, and the author's own ingenious *graphical symbolism*. The remaining eight chapters extend the pattern to the hypersolids, and here are encountered the regular polytopes, of which there are sixteen in four-space and three in each of the other higher dimensional spaces.

Nothing of much importance appears to have been omitted. One may regret the exclusion of any discussion of regular skew polyhedra, but apparently there was not room for that. The historical notes, of which there is a rich collection, constitute one of the most fascinating parts of the book. In addition to numerous excellent drawings, there is a set of eight plates furnished by P. S. Donchian, showing various solids and some remarkable models which are solid projections of regular four-dimensional polytopes. Many readers will recall seeing these models at the Chicago Exposition.

The theory of polytopes has attracted the interest of numerous amateur mathematicians, a number of whom have been quite gifted and have made valuable contributions. It is to be hoped that this book will introduce to the subject others who study mathematics for the love of it. From this group there may emerge one of those rare individuals possessing the mysterious power to visualize the hypersolids as easily as most of us visualize the solids of three-space. And there is still work for these amateurs to do. For example, the skew polytopes have not been investigated at all. They will not find it easy, at least after the first few stages, but this will not deter them. One recalls that difficulties have not discouraged amateurs in the study of number theory.

This book belongs in every university, college and high school library, and on the bookshelf of every mathematician. It is a book to read, to refer to and to recommend to students.

H. E. WOLFE

NEW BOOKS RECEIVED

First Course in Probability and Statistics. By J. Neyman. New York, Henry Holt, 1950. 10+350 pp. $3.50.

Practical Mathematics. By C. Palmer and S. Bibb. Part 2, Algebra with Applications, 5th Edition. New York, McGraw-Hill, 1950. 14+252 pp. $2.20.

A Theory of Formal Deducibility. (Notre Dame Mathematical Lectures, No. 6.) By H. Curry. Notre Dame, Indiana, 1950. 10+126 pp.

Theorie des Distributions, Part i. (L'Université de Strasbourg, Vol. 9). By L. Schwartz. Paris, Hermann, 1950. 148 pp.

College Mathematics. By C. Clark. New York, Prentice-Hall, 1950. 6+331 +46 pp. tables. $3.85.

Analytic Geometry. By R. Douglass and S. Zeldin. New York, McGraw-Hill, 1950. 10+216 pp. $2.75.

Contributions to Mathematical Statistics. By R. A. Fisher. New York, Wiley, 1950. 656 pp. $7.50.

A History of Philosophical Systems. Edited by Vergilius Ferm. New York, The Philosophical Library, 1950. xiv+642 pp. $6.00.

Basic Mathematical Analysis. By H. G. Ayre. New York, McGraw-Hill, 1950. xvi+584 pp. $5.00.

Mathematics of Relativity. By G. Y. Rainich. New York, John Wiley and Sons, Inc., 1950. vii+173 pp. $3.50.

Elements of Algebra. By L. C. Peck. New York, McGraw-Hill, 1950. 9+230 pp. $2.75.

Business Mathematics, 3rd Edition. By C. Richtmeyer and J. Foust. New York, McGraw-Hill, 1950. 18+441 pp. $3.50.

Leçons sur les nombres transfinis. By W. Sierpinski. Paris, Gauthier-Villars, 1950. 7+240 pp.

Algebra: Its Big Ideas and Basic Skills. By Aiken Henderson. New York, Harper and Brothers, Publishers, 1950. xv+409 pp.

A Course in the Slide Rule and Logarithms. Revised Edition. By E. J. Hills. Boston, Ginn and Company, 1950. iv+108 pp. $1.40.

Differential Equations. By J. E. Powell and C. P. Wells. Boston, Ginn and Company, 1950. vi+205 pp. $3.00.

Calculus and Analytic Geometry. By C. T. Holmes. New York, McGraw-Hill, 1950. x+416 pp. $4.75.

Fundamentals of the Calculus. By D. E. Richmond. New York, McGraw-Hill, 1950. xi+233 pp. $3.00.

Ordinary Differential Equations. By Michael Golomb and Merrill Shanks. New York, McGraw-Hill, 1950. ix+356 pp. $3.50.

Royal Society of Mathematical Tables I, The Farey Series of Order 1025. By E. H. Neville. New York, Cambridge University Press, 1950. xxvii+2 pp. Decimal Index + 405 pp. $18.50.

Numerical Mathematical Analysis. Second Edition. By J. B. Scarborough. Baltimore, The Johns Hopkins Press, 1950. xviii+511 pp. $6.00.

Bücher der Mathematik und Naturwissenschaften, Differential- und Integralrechnung. By Wilhelm Maak. Wolfenbuttel and Hannover, Wolfenbutteler Verlagsanstalt, 1949. 235 pp.

Riemannian Geometry. By L. P. Eisenhart. Princeton, Princeton University Press, 1949. vii+306 pp.

Vorlesungen Über Analytische Geometrie des Raumes. By Karl Kommerell. Stuttgart, K. F. Koehler Verlag. viii+407 pp. DM 26.

Fourier Series. Translation of Rogosinski's original (by Samml. Goschen, 1930) by Harvey Cohn and F. Steinhardt. New York, Chelsea Publishing Co., 1950. vi+176 pp.

Grundbegriffe Moderner Statistischer Methodik, Erster Teil. By L. von Baranow. Stuttgart, S. Hirzel Verlag, 1950. viii, 112 Seiten mit 16 Abb. DM 6.

Grundbegriffe Moderner Statistischer Methodik, Zweiter Teil. By L. von Baranow. Stuttgart, S. Hirzel Verlag, 1950. 112 Seiten mit 32 Abb.

Theory of Mental Tests. By Harold Gulliksen. New York, John Wiley and Sons, Inc., 1950. xix+486 pp. $6.00.

Theory of the Interior Ballistics of Guns. By J. Corner. New York, John Wiley and Sons, 1950. xiii+443 pp. $8.00.

Mathematical Snapshots. By H. Steinhaus. New York, Oxford University Press, 1950. vi+266 pp. $4.50.

Formulas and Theorems for the Special Functions of Mathematical Physics. By Wilhelm Magnus and Fritz Oberhettinger. New York. Chelsea Publishing Company, 1949. 172 pp.

Mecanique des Milieux Continus et Deformables, Tome I, Parts 1 and 2. By Maurice Roy. (Part I: Thermodynamique et mecanique des milieus continus et deformables. Part II: Equilibre et Mouvement des solides elastiques) Paris, Gauthier-Villars, 1950. xxii+366 pp. $8.36.

Mecanique des Milieux Continus et Deformables, Tome II, Parts 3 and 4. By Maurice Roy. (Part III: Equilibre et Mouvement des Fluides. Part IV: Theorie des Machines) Paris, Gauthier-Villars, 1950. xii+338 pp. $6.87.

Electromagnetic Waves. By F. W. G. White. New York, John Wiley and Sons, 1950. 108 pp. $1.25. (Methuen's Monographs on Physical Subjects).

The Special Theory of Relativity. By Herbert Dingle. (Methuen's Monographs on Physical Subjects). New York, John Wiley and Sons, 1950. 94 pp. $1.25.

Wave Guides. By H. R. L. Lamong. New York, John Wiley and Sons, 1950. 117 pp. $1.50.

Relativity Physics. By W. H. McCrea. New York, John Wiley and Sons, 1950. 87 pp. $1.25.

Bertrand Russell, A Sketch of His Life and Work. By H. W. Leggett. New York, Philosophical Library, 1950. 78 pp. $3.75.

The Philosophy of Mathematics. By E. A. Maziarz. New York, Philosophical Library, 1950. viii+286 pp. $4.00.

Strategy in Poker, Business and War. By John McDonald. New York, W. W. Norton and Company, Inc., 1950. 128 pp. $2.50.

Grundzuge der Galois'schen Theorie (N. Tschebotarow). Translated and revised by H. Schwerdtfeger. Holland, P. Noordhoff, 1950. v+432 pp. f 17.50, geb f 20.

Irrational and Rational Number Theory. By E. Schlytter. Chicago, 1950. Copyright by author. 67 pp. +ii index.

Weltsystem Weltäther und die Relativitätstheorie. By Karl Jellinke. Switzerland, Wepf and Co., 1949. xv+450 pp. SFr. 45.

Verstandliche Elemente der Wellenmechanik, I Teil. By Karl Jellinek. Switzerland, Wepf and Co., 1950. xii+304 pp. SFr. 34.

Ausgleichsrechnung. By V. Happach. Leipzig, Teubner, 1950. 104 pp. $1.50.

CLUBS AND ALLIED ACTIVITIES

EDITED BY L. F. OLLMANN, Hofstra College

Send reports of all activities, such as club reports, special features, topics with references, student papers, and other material of interest to L. F. Ollmann, Hofstra College, Hempstead, New York.

CLUB REPORTS, 1949–50

Pi Mu Epsilon, University of Pennsylvania

The *Pi Mu Epsilon* chapter at the University of Pennsylvania held regular meetings every month. The papers presented were:

Parallel curves, by Prof. C. B. Allendoerfer

Rational and irrational numbers, by Prof. Brinkman

Sylvester's identity, by Herman Zabronsky

P-adic numbers, by Dr. Emil Grosswald

The role of electronic computers in mathematics, by Dr. Herbert Mitchell

Logic of the circuits of electronic digital computers, by Prof. G. W. Patterson.

The activities for the year closed with a dinner at which Prof. J. R. Kline was the featured speaker.

Officers for the year 1950–51 are: Director, Dr. R. D. Anderson; President, Herbert Gurk; Secretary, David Loev; Treasurer, Mildred Goss.

Mathematics Club, Butler University

The *Mathematics Club* of Butler University held regular meetings twice a month during the academic year. At these meetings a paper is presented by a student, faculty member, or guest.

The club also sponsors two outings and a Christmas party each year. One of the outings was a trip to the Johnson's Observatory, while the other two were a Christmas party and a picnic.

Titles and authors of papers presented at the regular meetings are:

The geometric fourth dimension, by Prof. H. E. Crull

Nomography, by W. W. Gorsline

Solids, regular and semi-regular, by Carol Wilson

Isaac Newton, by Eldon Palmer

Applications of mathematics to mechanics, by Kenneth Hallam

Application of mathematics to submarine detection and torpedo firing data, by Dr. H. E. Crull

Geometrical paradoxes, by Miss Jane Uhrhan

The duo-decimal system, by Edwin Britz

Dynamic symmetry in Greek art, by Prof. June Beal

Mathematical paradoxes, by Egon Steinkamp

Several proofs of the Pythagorean theorem, by Frank Rexroth

The binary system, by Ted Souders.

Mathematics Club, University of Kansas

The *Mathematics Club* of the University of Kansas presented the following programs during 1949–50:

Algebraic numbers, by Dr. Robert Schatten
Is the sum of algebraic numbers an algebraic number?, by Robert Fisher
Methods of trisecting any angle, by Dr. G. W. Smith
Unsolved problems in number theory, by Dr. S. Chowla
The proof of the irrationality of pi, by Dr. I. N. Herstein
The planimeter, by Ralph Simmons
A problem in statistics, by Dr. G. B. Price
Schwartz's inequality, by Mr. Kuo-Chih Hsu
Elementary cryptography, by Dr. G. W. Smith
Combinatorial topology, by Calvin Foreman
Reflected triangles, by Arthur Kruse.

The annual picnic was held in the spring. Officers for the year 1950–51 are: President, Robert Newton; Vice-President, Marion Brown; Secretary-Treasurer, Ruth Hurwitz.

Pi Mu Epsilon, Brooklyn College

The following talks were given at open meetings of the *New York Gamma* chapter of *Pi Mu Epsilon* during the academic year 1949–50:

The concept of homotopy, by Prof. Samuel Eilenberg of Columbia University
Topology, by Prof. Norman Steenrod of Princeton University
Simple Simon, miniature mechanical brain, a lecture-demonstration by Edmund C. Berkeley, a consultant mathematician and author of the book "Giant Brains."

Two installations of new members were held during the year. On both occasions, refreshments were served, and a social event followed the initiation ceremonies.

Officers for the Fall term, 1950–51 are: President, Anotole Beck; Vice-President, Lawrence Bennett; Financial Secretary, George Booth; Correspondence Secretary, Alvin Hausner; Director, Prof. Moses Richardson.

Kappa Mu Epsilon, Iowa State Teachers College

The following papers were presented to the *Iowa Alpha* chapter of *Kappa Mu Epsilon*:

Paper constructions, by Lena Abbas
Eisenstein's irreducible criteria, by Don Edwards
Magic squares, by Sam Weigert
The theory and uses of the planimeter, by Don Richardson
Problem of Apollonius, by Eddie Sage
Boolean algebra, by Robert Robinson
Inversions, by Marian Karrys.

The Homecoming Breakfast was held in the Fall at the home of Dr. H. Van Engen, Sponsor of the chapter and National President of *Kappa Mu Epsilon.*

Nine student members attended the convention in Chicago during April, 1950, at which time Robert Allender presented a paper to the gathering.

Officers for 1950–51 are: President, George York; Vice-President, Sam Weigert; Secretary-Treasurer, Gladys Satell; Corresponding Secretary, Prof. George Keppers; Sponsor, Dr. H. Van Engen.

Sigma Phi Mu, Montclair Teachers College

This year's program began with the initiation and welcoming of the Freshmen into *Sigma Phi Mu.* The social program consisted of an annual Christmas party with the *New Jersey Beta* chapter of *Kappa Mu Epsilon*, an interesting and pleasant Probability Night and an annual May picnic.

Papers presented during 1949–50 were:

The use of the mathematics laboratory, by George Kays

The conic sections, by Charles Sensale

So you want to be a mathematics teacher!, by Ralph C. Miller

The functionalism of teaching mathematics in the high school, by Dr. V. S. Mallory

The trisection of an angle by means of various curves, by Audrey Jensen

Interesting facts about e, by William Koellner

Teaching mathematics in the elementary schools, by Lawrence Campbell.

The officers for the coming year are: President, Marie Kanthack; Vice-President, Ruth Maehl; Secretary, Margaret Cotter; Treasurer, Phyllis Friedman.

Kappa Mu Epsilon, Washburn Municipal University

At the monthly meetings of the *Kansas Delta* chapter of *Kappa Mu Epsilon*, the following papers were given:

Theory of numbers, by Dr. S. Chowla, guest speaker

Problems in algebra, a panel discussion, by Hugo Rolfs, Howard Sperry, William Powell, and Kenneth Lake

The trisection of an angle, a panel discussion, by Norman Hoover, Margery Gamble, Edna Metzenthin and Nancy Martin

The mathematical needs of the psychology major, by Robert Hage of the Department of Psychology

On the Hausdorff summability of derived and conjugate derived Fourier series, by Dr. P. E. Eberhart

Zeno's paradoxes, by Kenneth Lake

A few of the interesting topics in algebra, by Betty Jane Moffett

Mobius rings, by Dale Long.

One of the most successful meetings of the year was High School Senior Guest Night. Approximately seventy persons attended the meeting, which consisted of greetings from the President of the University and from the Chairman

of the Department of Mathematics. Four student members of *Kappa Mu Epsilon* gave short talks on mathematics, and the President of the chapter explained *Kappa Mu Epsilon* and its project. We had an enjoyable informal refreshment hour, after which we went to the observatory to look through the telescope. Many notes and calls of appreciation were received from the parents and the seniors from our four surrounding high schools.

Dr. S. Chowla, of India, who is a visiting professor at the University of Kansas, and a former participant in the Institute for Advanced Study, was the guest speaker for the initiation dinner in the fall. Dr. G. Baley Price, an honorary member of the chapter, was also a guest.

Miss Laura Greene, *Kansas Delta* Sponsor, was in charge of the arrangements for the *Kappa Mu Epsilon* luncheon held in connection with the National Council of Teachers of Mathematics meeting in Wichita, Kansas. Dr. O. J. Peterson, *Kansas Beta* chapter of *Kappa Mu Epsilon*, presided at the meeting. Kenneth Lake, spoke on the topic *An elementary discussion of fundamental concepts in modern algebra*. Approximately twenty-five persons attended the luncheon at which Dr. E. H. C. Hildebrandt, honorary member of *New York Alpha*, was a guest.

The officers for 1950–51 are: President, Joe Latas; Vice-President, Dale Long; Secretary, Narra Smith; Treasurer, Margery Gamble; Sponsor, Miss Laura Greene; Corresponding Secretary, Miss Margaret Martinson.

Kappa Mu Epsilon, Central Michigan College of Education

Reports read to the *Michigan Beta* chapter of *Kappa Mu Epsilon* during 1949–50 were:

Mathematics in music, by Margaret King

Mathematics in photography, by Roger Ewing

Congruences, by Darold Comstock

Mechanical brains, by Richard Little.

The paper by Richard Little, *Mechanical brains*, was read by him at the combined meeting, in Ann Arbor, of the Student Section of the Michigan Section of the Mathematical Association of America and the regional chapters of *Kappa Mu Epsilon*.

Dr. Cleon C. Richtmeyer, of *Michigan Beta*, attended the *Kappa Mu Epsilon* Council meetings at Hofstra College during the Christmas recess.

Officers elected for 1950–51 are: President, Darold Comstock; Vice-President, Mary Bolla; Secretary, Doris Brode; Treasurer, Paul Buchholz; Corresponding Secretary and Sponsor, Prof. D. R. Sudborough. However, Prof. H. W. Zeoli will serve for Prof. Sudborough while the latter is on leave during the year.

Kappa Mu Epsilon, Baldwin-Wallace College

The following talks were presented to the *Ohio Gamma* chapter of *Kappa Mu Epsilon* during 1949–50:

Centrifugal force in industry, by Walter H. Craig

The mathematics of sound, by Dr. E. C. Unnewehr

The science and philosophy of mathematics, by Robert Miller and George Smolensky

Exponential functions and their applications to science, by Robert Noblit

The history of mathematics, by Theron Schwegler

Mathematics at NACA, by Miss Betty Hostetler

Quality control, by Dr. Fred Leone.

The annual banquet of *Kappa Mu Epsilon* was held in May at which Dr. Leone gave his paper.

The newly elected officers to serve for 1950–51 are: President, Bruce Glass; Vice-President, Richard Sutton; Secretary, Ruth Oberer; Treasurer, Barbara Ayres; Corresponding Secretary and Sponsor, Dr. Paul Annear.

Kappa Mu Epsilon, Wayne University

During the past year the *Michigan Gamma* chapter of *Kappa Mu Epsilon* took part in many activities to further interest in mathematics. The group has sponsored seven very interesting and informative talks on various aspects of mathematics. The topics and speakers were:

Newton's method of solving equations, by Dr. K. W. Folley

Foundations of mathematics, by Dr. R. Ackoff

The differential analyzer, by Dr. A. W. Jacobson

The calculus of variations, by Dr. Max Coral

Methods of solving equations, by Dr. T. Southard

Topology, by Dr. S. Kaplan

Gibb's phenomena, by Marvin Snyder.

The chapter attended a meeting at the University of Michigan in conjunction with the Michigan Section of the Mathematical Association of America. Faculty papers were presented at the morning session and six student members of *Michigan Gamma* presented papers during the afternoon session. Eight different institutions from Michigan and Ohio were represented with a total student attendance of 79. The conference proved so successful that the *Michigan Beta* chapter of *Kappa Mu Epsilon* at Mt. Pleasant, is sponsoring a similar meeting in the coming spring.

Twenty-seven new members were taken into the group at the annual initiation and banquet, at which Dr. G. Y. Rainich of the University of Michigan gave a talk on *Axioms*.

As a future project, the *Michigan Gamma* chapter hopes to hold joint meetings with the University of Detroit and the University of Michigan.

NEWS AND NOTICES

EDITED BY EDITH R. SCHNECKENBURGER, University of Buffalo

Readers are invited to contribute to the general interest of this department by sending news items to Edith R. Schneckenburger, University of Buffalo, Buffalo 14, New York. Items must be submitted at least two months before publication can take place.

INSTITUTE FOR TEACHERS OF MATHEMATICS

The Association of Teachers of Mathematics in New England is sponsoring an Institute for Teachers of Mathematics which will be held August 23–30, 1951 at Connecticut College, New London, Connecticut. Details of the program will be published in *The Mathematics Teacher*. If you wish a program, please drop a card to C. H. Mergendahl, Newton High School, Newtonville, Massachusetts or Katharine E. O'Brien, Deering High School, Portland, Maine.

NEW GRADUATE PROGRAM IN STATISTICS AT THE UNIVERSITY OF ILLINOIS

The University of Illinois has instituted a graduate program leading to the degree of Master of Arts or to the degree of Doctor of Philosophy in Statistics. Administered by the Department of Mathematics, the new program includes both statistical theory and its application to a specific field of interest. Considerable flexibility is possible in the division of emphasis between theory and its application but in no case will the amount of work in the latter area be less than that for a minor.

Under the new program a student who wishes to concentrate on research in statistical methods will have a choice of areas in which his thesis may be written, depending on his major interest. Each student will have two advisers, one in statistical theory and the other in his area of application, one of whom will be in charge of his thesis.

Beyond the regular Graduate College stipulations, the only special requirement for admission to the program is completion of a thorough course in calculus. For the Ph.D. degree a student, in addition to language and other standard requirements, must show capacity for important original contributions in statistical methods or in application of such methods.

ACKNOWLEDGMENT

The editor of the MONTHLY wishes to make grateful acknowledgment of the services rendered by the following persons who have refereed papers: R. P. Agnew, I. A. Barnett, P. M. Batchelder, Richard Bellman, Garrett Birkhoff, L. M. Blumenthal, C. B. Boyer, R. C. Buck, W. E. Byrne, G. F. Carrier, R. V. Churchill, N. A. Court, Wayne Dancer, Paul Erdös, Tomlinson Fort, V. G. Grove, William Gustin, E. H. C. Hildebrandt, Ralph Hull, B. W. Jones, P. S. Jones, Irving Kaplansky, E. P. Lane, Caroline Lester, V. O. McBrien, N. H. McCoy, A. D. Michal, Oystein Ore, George Pólya, G. B. Price, J. F. Randolph, John Riordan, Arthur Rosenthal, J. B. Rosser, H. N. Shapiro, I. M. Sheffer,

C. E. Springer, A. H. Taub, R. S. Underwood, H. E. Vaughan, P. M. Whitman, R. L. Wilder, H. E. Wolfe.

PERSONAL ITEMS

Professor H. F. Fehr of Teachers College, Columbia University, was the delegate of the Association at the Fifteenth Educational Conference, New York, on October 26–27, 1950.

Dean E. L. Mackie of the University of North Carolina represented the Association at the inauguration of President H. W. Tribble of Wake Forest College on November 28, 1950.

Professor N. H. McCoy of Smith College was the representative of the Association at the inauguration of President Spencer Miller, Jr. of American International College on November 8–9, 1950.

Professor R. F. Schnepp, St. Mary's University of San Antonio, was appointed to represent the Association at the dedication of Science Hall, Incarnate Word College, on November 3, 1950.

Professor C. R. Wylie, Jr. of the University of Utah represented the Association at the inauguration of President L. L. Madsen of Utah State Agricultural College on November 3, 1950.

Dr. R. V. Kadison of the University of Chicago has received an appointment to a National Research Fellowship in Mathematics for the year 1950–51.

Mr. Z. A. Melzak of McGill University was awarded the William Lowell Putnam Scholarship at Harvard University for the Tenth Annual Competition.

Augustana College reports: Associate Professor H. E. Nelson has been promoted to the position of Professor and Head of the Department of Mathematics; Professor W. E. Cederberg has retired with the title of Professor Emeritus.

Hope College announces: Assistant Professor J. E. Folkert has been granted a year's leave of absence for the purpose of doing graduate work at Michigan State College; Assistant Professor Harry Frissel of the Department of Physics has been made a temporary member of the department.

At Mississippi State College: Professor C. R. Pettis, head of the Department of Mathematics, and Associate Professor C. R. Stark have retired; Professor Arthur Ollivier has been named Head of the Department of Mathematics; Dr. L. H. Kanter of the University of Wisconsin has been appointed to an associate professorship.

At Washington Square College, New York University: Professor Wilhelm Magnas of California Institute of Technology has been appointed Research Associate Professor; Associate Professor C. K. Payne has retired.

Northern Illinois State Teachers College announces the following: Mrs. C. E. Hardgrove, formerly an instructor in the Department of Education, Ohio State University, and Dr. W. L. Pickard of the Naval Research Laboratory, Washington, D. C. have been appointed to associate professorships; Assistant Professor R. E. Anderson has been promoted to an associate professorship and has been granted a leave of absence while he is in military service.

Southern Illinois University announces the following appointments: Dr. A. M. Mark of the University of Wisconsin to the position of Assistant Professor of Mathematics and Director of University Statistical Service; Mr. Daniel Orloff, previously graduate student at the University of Chicago, to an instructorship.

Stevens Institute of Technology reports: Professor C. O. Gunther, formerly chairman of the Department of Mathematics, has retired with the title of Professor Emeritus; Assistant Professor M. R. Reeks, who is Chairman of the Department of Mathematics, has been promoted to an associate professorship; Mr. O. J. Karst, formerly of the Department of Physics, has been appointed to an associate professorship.

United States Naval Academy makes the following announcements: Associate Professor E. Hawkins has been promoted to a professorship; Assistant Professor J. F. Paydon has been promoted to an associate professorship; Professor G. R. Clements has retired; Professor J. B. Scarborough has retired with the title of Professor Emeritus; Assistant Professor K. F. McLaughlin has returned after spending two years' leave of absence at the University of Iowa.

At the University of Houston: Professor J. D. Hutchinson has been named Chairman of the Department of Mathematics; Assistant Professor Martin Wright has been promoted to an associate professorship.

The University of Maine reports the promotion of Associate Professor W. S. Lucas to a professorship and the appointment of Mr. G. F. Simmons, formerly teaching assistant at the University of Chicago, to an instructorship.

The University of Oklahoma announces: Dr. Y. W. Chen, formerly at the Institute for Advanced Study, has been appointed to an associate professorship; Assistant Professor A. E. Labarre, Jr., of the University of Idaho has been appointed to an instructorship; Associate Professor B. S. Whitney has been given the title of Director of the Observatory.

The University of Southern California announces the following: Dr. Bernard Sherman of the University of Vermont has been appointed to an assistant professorship; Associate Professor R. S. Phillips is on leave of absence and is at the Institute for Advanced Study.

University of Tennessee reports: Dr. Walter Baum of Eidgenossische Technische Hochschule, Zurich, and Dr. E. G. Kundert of Kantonsschule, Zurich, have been appointed to acting assistant professorships; the contract of the University with the Office of Naval Research has been extended for another year with Professor J. W. Givens and Mr. N. Heerema as investigators.

University of Vermont makes the following announcements: Assistant Professor J. A. Larrivee has been promoted to an associate professorship; Mr. William Baranoff, Mr. J. F. Detlef, Mr. W. T. Fishback, and Mr. J. E. Vollmer have been appointed to instructorships.

At the University of Wisconsin: Graduate Assistant L. E. Fuller has been promoted to an instructorship; Dr. J. B. Kelly is now at the Institute for Advanced Study.

The University of Wyoming announces the following: Assistant Professor V. J. Varineau has been promoted to an associate professorship; Mr. W. G. Brady has been granted a year's leave of absence to study at the University of Pittsburgh.

Wilson Junior College reports: Mr. J. H. Manheim has been appointed to an instructorship replacing Mr. Norman Stein who is now located at Fort Sheridan.

Mr. L. J. Abbeduto, a student at Illinois Institute of Technology, has been recalled to active duty in the Air Force and is located at Selfridge, Michigan.

Mr. D. L. Arenson is now a Research Engineer at Armour Research Foundation, Chicago, Illinois.

Dr. A. V. Baez of the Cornell Aeronautical Laboratory has been appointed Professor of Physics at University of Redlands.

Professor F. R. Bamforth of Ohio State University has been appointed to a professorship at Otterbein College.

Mr. M. Q. Barton, formerly a student at Central College, has received an appointment as graduate assistant at the University of Illinois.

Mr. H. G. H. Bartram, previously a graduate student at Cornell University, has been appointed to an assistant professorship at Coe College.

Associate Professor A. W. Boldyreff of the University of New Mexico has been appointed a member of the staff of Sandia Corporation, Albuquerque, New Mexico.

Mr. S. G. Bourne of Johns Hopkins University is now an instructor at the University of Connecticut and a member of the Institute for Advanced Study.

Dr. F. W. Brown of North American Aviation, Inc. has accepted a position as associate director for research and development, United States Naval Ordnance Test Station, Inyokern, California.

Mr. W. C. Brown, who was a graduate assistant at the University of Oklahoma, is now an actuarial student at Equitable Life Assurance Society, New York.

Dr. K. A. Bush has been appointed to an associate professorship at Champlain College.

Mr. J. A. Carpenter has accepted a position as mathematician with Snow and Schule, Inc., Cambridge, Massachusetts.

Mr. C. R. Carr, previously a teaching fellow at the University of Michigan, has received an appointment at Indiana Technical College.

Professor H. B. Curtis of Lake Forest College has retired with the title of Professor Emeritus.

Mr. D. L. Dunning, graduate assistant at Kent State University, is now Assistant Actuary at Warner-Watson, Inc., Chicago, Illinois.

Lecturer Albert Edrei of the University of Saskatchewan has been promoted to an associate professorship.

Mr. J. V. Finch of the University of Wisconsin has been appointed to an assistant professorship at Beloit College.

Assistant Professor Pearl L. Ford of Western Michigan College of Education has been promoted to an associate professorship.

Mr. R. F. Gabriel of St. Francis College has been appointed to an instructorship at University College, Rutgers University.

Mr. B. T. Goldbeck, Jr., of the University of Wyoming has received an appointment as instructor at Texas Christian University.

Dr. Helmut Grunsky of the University of Tubingen was Visiting Professor at State College of Washington for the first half of the academic year 1950–51.

Professor Marshall Hall, Jr. of Ohio State University has been recalled to active duty in the United States Navy.

Dr. H. J. Hamilton of Los Angeles City College has been appointed to an associate professorship at Pomona College.

Assistant Professor P. W. Healy of Southwestern College, Winfield, Kansas, has been promoted to an associate professorship.

Dr. Sarah B. Hill has been appointed to an assistant professorship at Wheaton College, Norton, Massachusetts.

Professor Aughtum S. Howard of Kentucky Wesleyan College has been appointed Acting Head of the Division of Science and Mathematics for 1950–51.

Dr. J. A. Jenkins has been appointed to an assistant professorship at Johns Hopkins University.

Mr. M. L. Lawson of the University of Oklahoma has been appointed to an associate professorship at Arkansas State College.

Dr. L. H. N. Lee, University of Minnesota, has been appointed to an assistant professorship at the University of Notre Dame.

Dr. Eugene Lukacs, who has been at the United States Naval Ordnance Test Station, Inyokern, California, has joined the staff of the Statistical Engineering Laboratory of the National Bureau of Standards.

Dr. Nathaniel Macon of the University of North Carolina is at the University of Amsterdam during 1950–51.

Mr. Fred Marer has been appointed Chairman of the Mathematics Department of Los Angeles City College.

Associate Professor R. H. Moorman, Tennessee Polytechnic Institute, has been appointed Acting Head of the Department of Mathematics.

Mr. C. H. Murphy, Jr. of the University of Hawaii has accepted a position as mathematician in the Ballistic Research Laboratory, Aberdeen Proving Ground, Maryland.

Assistant Professor P. B. Norman of Wagner College has been promoted to an associate professorship.

Mr. J. M. Patterson of Wayne University has been appointed to an instructorship at New York University.

Dr. C. J. Pipes of the University of Oklahoma has been appointed to an assistant professorship at Southern Methodist University.

Professor D. W. Pugsley, head of the Department of Mathematics of Berea

College, was on sabbatical leave during the first semester of 1950–51 while studying at Columbia University.

Assistant Professor A. O. Qualley of Drake University has been appointed Superintendent, Farnhamville Independent School, Iowa.

Mr. C. H. Reid, Jr., who has been teaching at Sargent Consolidated School, Monte Vista, Colorado, has been appointed to an instructorship at Fort Lewis Agricultural and Mechanical College.

Assistant Professor G. L. Tiller of Utica College, Syracuse University, has been appointed to an associate professorship at Southwestern at Memphis.

Dr. S. E. Urner has been appointed Professor and Chairman of the Department of Mathematics of the Los Angeles State College of Applied Arts and Sciences.

Professor C. E. VanHorn of Fisk University has been appointed Professor and Head of the Department of Mathematics of Wartburg College.

Professor Oswald Veblen has retired from active service in the Institute for Advanced Study.

Dr. J. E. Wilkins, Jr., formerly a mathematician at American Optical Company, Buffalo, New York, has accepted a position with the Nuclear Development Associates, Inc., New York City.

Mr. Herbert Wolf, previously a graduate student at the University of North Carolina, has been appointed to an assistant professorship at Winthrop College.

Dr. C. S. Yih of the University of British Columbia has received an appointment as associate professor at Colorado Agricultural and Mechanical College.

Mr. J. A. Zilber of Columbia University has been appointed to an instructorship at Johns Hopkins University.

Professor W. A. Ballou of the Department of Physics of Morehead State College died on November 20, 1950.

Professor M. M. Culver of the University of Pittsburgh died on November 2, 1950.

Dean W. E. Duckering of the University of Alaska died on October 7, 1950.

Professor R. O. Hutchinson, head of the Department of Mathematics of Tennessee Polytechnic Institute, died on October 22, 1950.

Sister M. Bertrand of Marywood College died on November 18, 1950.

THE MATHEMATICAL ASSOCIATION OF AMERICA

Official Reports and Communications

NEW MEMBERS

Professor H. M. Gehman, Secretary-Treasurer, announces that the following ninety persons have been elected to membership by the Board of Governors on applications duly certified.

A. H. ALBERT, M.S. (Michigan) Instructor, Kansas State Teachers College, Emporia, Kans.

J. S. AREND, S.B. (M.I.T.) Part-time Instructor, University of Colorado, Boulder, Colo.

H. H. BERRY, B.S. (Kentucky) 859 S. Broadway, Lexington, Ky.

A. H. BLUE, Ph.D. (Iowa) Professor, Culver-Stockton College, Canton, Mo.

S. R. BODNER, B.C.E. (Poly. Inst. of Brooklyn) Instructor, Polytechnic Institute of Brooklyn, N. Y.

C. W. BOSTICK, B.S. (M.I.T.) Grad. Student, University of Illinois, Urbana, Ill.

F. E. BOTHWELL, Ph.D. (M.I.T.) Asso. Professor, Northwestern University, Evanston, Ill.

HOLMES BOYNTON, Ed.D. (Columbia) Professor, Northern Michigan College of Education, Marquette, Mich.

T. F. BRIDGLAND, JR., B.S. (Florida) Grad. Assistant, University of Florida, Gainesville, Fla.

W. J. BURGESS, Faculty Member, Morris College, Sumter, S. C.

R. A. BURROWS, Student, Albion College, Mich.

VIRGINIA CARLOCK, A.B. (LaGrange C.) Teaching Fellow, Alabama Polytechnic Institute, Auburn, Ala.

W. E. CARSON, B.S. (Poly. Inst. of Brooklyn) Grad. Student and Part-time Instructor, Polytechnic Institute of Brooklyn, N. Y.

L. E. CARVILLE, M.A. (Boston U.) Mathematician, Army Map Service, Washington, D. C.

ELSIE T. CHURCH, M.A. (Kentucky) Instructor, University of Kentucky, Lexington, Ky.

C. R. CLARK, A.M. (Michigan) Research Assistant, Sandia Corporation, Albuquerque, N. M.

D. R. CLUTTERHAM, M.S. (Arizona) Grad. Assistant, University of Illinois, Urbana, Ill.

W. J. CODY, JR., Student, Elmhurst College, Ill.

E. L. DOLNEY, M.S. (Notre Dame) Instructor, University of Alaska, College, Alaska

ANDERSON DUGGAR, JR., Student, University of Detroit, Mich.

META M. EWING, M.S. (Michigan S. C.) Teacher, Bay City Junior College, Mich.

J. E. FAULKNER, B.S. (Utah S. C.) Grad. Assistant, Kansas State College, Manhattan, Kans.

D. H. FIRL, B.A. (Gustavus Adolphus C.) Grad. Assistant, Kansas State College, Manhattan, Kans.

R. J. FLANAGAN, B.S. (New Mexico) Mathematical Analyst, Sandia Corporation, Albuquerque, N. M.

J. R. FOOTE, Ph.D. (M.I.T.) Asst. Professor, Iowa State College, Ames, Iowa.

R. S. FOUCH, M.S. (Chicago) Asst. Professor, Arizona State College, Tempe, Ariz.

E. A. FRANZ, M.S. (Iowa) Instructor, Culver-Stockton College, Canton, Mo.

W. T. GLACKEN, Student, Purdue University, Lafayette, Ind.

V. D. GOKHALE, Ph.D. (Chicago) Professor, Atlanta University, Ga.

M. A. GOLUB, M.S. (New Brunswick) Instructor, University of Missouri, Columbia, Mo.

R. E. GRAVES, Ph.D. (Minnesota) Asst. Professor, University of Minnesota, Minneapolis, Minn.

W. R. GREANEY, Kingsbridge P. O., New York, N. Y.

E. V. GREER, M.A. (Illinois) Professor, Bethany-Peniel College, Bethany, Okla.

J. P. GWIN, Student, University of Tulsa, Okla.

SEYMOUR HABER, B.A. (Yeshiva C.) Part-time Instructor, Syracuse University, N.Y.

D. J. HANNAN, B.A. (Loras C.) Teaching Fellow, University of Detroit, Mich.

H. K. HARTMAN, B.S. (Bucknell) Claim Set-

tlement Agent, Department of Public Assistance, Harrisburg, Pa.

D. M. HOUSER, B.S. (Ottawa U.) Grad. Assistant, Kansas State College, Manhattan, Kans.

ERNEST IKENBERRY, Ph.D. (Louisiana State U.) Asst. Professor, Alabama Polytechnic Institute, Auburn, Ala.

R. G. INGLE, A.B. (Bethany-Peniel C.) Meteorologist, Sandia Corporation, Albuquerque, N. M.

R. T. JOHN, M.S. (Notre Dame) Clergyman and Instructor, Crosier Seminary, Onamia, Minn.

C. W. JORDAN, JR., B.A. (Williams C.) Asst. Professor, Williams College, Williamstown, Mass.

ABRAHAM KARRASS, Student, Brooklyn College, N. Y.

W. M. KOGER, Student, Southern Methodist University, Dallas, Tex.

E. R. LANCASTER, M.A. (Wayne) Instructor, University of Detroit, Mich.

G. N. LANDES, B.A. (Wooster) Mathematical Analyst, Sandia Corporation, Albuquerque, N. M.

E. V. LEWIS, Ph.D. (M.I.T.) Asst. Professor, University of Delaware, Newark, Del.

J. A. LEWIS, Ph.D. (Brown) Research Physicist, Corning Glass Works, N. Y.

G. O. LOSEY, Student, University of Michigan, Ann Arbor, Mich.

R. W. MALLEY, B.S. (Southeastern Louisiana C.) Mathematical Analyst, Sandia Corporation, Albuquerque, N. M.

J. H. MANHEIM, M.S. (Illinois) Instructor, Wilson Junior College, Chicago, Ill.

J. A. MANSOUR, B.S. (Detroit) Teaching Fellow, University of Detroit, Mich.

W. J. McCLINTOCK, M.A. (Michigan) Teaching Fellow, University of Detroit, Mich.

GARNER McCROSSEN, M.A. (Wyoming) Part-time Instructor, University of Colorado, Boulder, Colo.

J. E. McLAUGHLIN, Ph.D. (C.I.T.) Instructor, University of Michigan, Ann Arbor, Mich.

S. A. McLEOD, M.A. (North Carolina) Dean, Lander College, Greenwood, S. C.

G. W. MEDLIN, M.A. (North Carolina) Part-time Instructor, University of North Carolina, Chapel Hill, N. C.

E. P. MERKES, M.S. (DePaul) Lecturer, DePaul University, Chicago, Ill.

J. J. MILLER, B.S. in M.E. (New Mexico) Supervisor, Applied Mathematics Section, Sandia Corporation, Albuquerque, N. M.

MILTON MORRISON, M.A. (Columbia) Instructor, Stevens Institute of Technology, Hoboken, N. J.

J. H. MOSS, JR., M.S. (N.Y.U.) Asst. Professor, Alabama Polytechnic Institute, Auburn, Ala.

T. P. PALMER, A.M. (Harvard) Asso. Professor, Rose Polytechnic Institute, Terre Haute, Ind.

P. I. PETERS, M.A. (Ohio State) Asso. Professor, Union College, Barbourville, Ky.

R. B. PINSON, M.A. (Texas) Asso. Professor, Stephen F. Austin State College, Nacogdoches, Tex.

C. G. PITNER, Professor, Harding College, Searcy, Ark.

ETHEL H. RAYBOULD, M.A. (Queensland) Senior Lecturer, University of Queensland, Brisbane, Australia.

J. W. REED, B.S. (New Mexico) Supervisor, Meteorological Section, Sandia Corporation, Albuquerque, N. M.

D. F. RIDDLE, A.B. (Illinois) Grad. Student, University of Michigan, Ann Arbor, Mich.

R. A. SARTOR, Student, University of Detroit Mich.

JOSEPHINE SAVARO, Librarian, University of Scranton, Pa.

W. R. SEUGLING, M.A. (U.C.L.A.) Research Assistant, University of California at Los Angeles, Calif.

W. T. SHARP, M.A. (Toronto) Asst. Professor, University of Maryland, College Park, Md.

SISTER IRENE, M.A. (Catholic U.) Teacher, Annhurst College, Putnam, Conn.

SISTER M. CONSTANTIA, M.A. (Notre Dame) Head of Mathematics Department, Central Catholic High School, Toledo, Ohio.

D. O. SNOW, M.A. (Acadia) Asso. Professor, Acadia University, Wolfville, N. S.

SANDERS STONE, Student, Wilmington College, Undergraduate Center, Dayton, Ohio.

ANN A. TAYLOR, B.S. (Georgia) Teaching Fellow, Alabama Polytechnic Institute, Auburn, Ala.

P. M. TREUENFELS, M.S. (N.Y.U.) Mathematician, Ballistics Research Laboratory, Aberdeen Proving Ground, Md.

L. M. TULLOCH, A.M. (Brown) Asso. Professor, Southwest Texas State College, San Marcos, Tex.
G. E. UHRICH, Ph.D. (Washington) Asst. Professor, Montana State College, Bozeman, Mont.
C. M. VICK, Supervisor, Mathematical Analyst Section, Sandia Corporation, Albuquerque, N. M.
D. T. WALKER, B.S. (Wofford C.) Grad. Student, University of Georgia, Athens, Ga.
W. H. WARNER, B.S. (Carnegie) Teaching Assistant, Carnegie Institute of Technology, Pittsburgh, Pa.
S. E. WARSCHAWSKI, Ph.D. (Basel, Switzerland) Professor, University of Minnesota, Minneapolis, Minn.
W. J. WAYNE, Student, Butler University, Indianapolis, Ind.
G. P. WEEG, M.S. (Oklahoma A & M) Instructor, St. Ambrose College, Davenport, Iowa.
SONIA S. WOHL, M.S. (Columbia) Asst. Librarian, Rockefeller Institute for Medical Research, New York, N. Y.
F. A. YOKE, Student, San Diego State College, Calif.
W. M. ZARING, A.B. (Kentucky Wesleyan) Grad. Student, University of Kentucky, Lexington, Ky.
A. D. ZIEBUR, Ph.D. (Wisconsin) Asst. Professor, Oklahoma Agricultural and Mechanical College, Stillwater, Okla.

FEBRUARY MEETING OF THE LOUISIANA-MISSISSIPPI SECTION

The February meeting of the Louisiana-Mississippi Section of the Mathematical Association of America was held at Centenary College, Shreveport, Louisiana, February 24–25, 1950. Professor G. R. Trott, Chairman of the Section, presided.

Ninety-three persons attended the meeting, including the following thirty-nine members of the Association: T. A. Bickerstaff, L. Virginia Carlton, Margaret Davis, H. T. Donohoe, W. L. Duren, Virginia I. Felder, L. R. Ford, L. M. Garrison, M. E. Gillis, W. C. Griffith, B. H. Gundlach, J. A. Hardin, S. L. Hull, H. T. Karnes, C. G. Killen, Z. L. Loflin, A. C. Maddox, J. W. McClimans, Betty McKnight, R. A. Miller, B. E. Mitchell, S. B. Murray, Arthur Ollivier, R. L. O'Quinn, C. L. Perry, P. K. Rees, D. P. Richardson, F. A. Rickey, H. F. Schroeder, V. B. Temple, W. B. Temple, J. F. Thompson, B. B. Townsend, G. R. Trott, M. P. Tullier, Jr., A. D. Wallace, Eleanor B. Walters, M. C. Wicht, T. F. Wilbanks.

The following officers were elected for the coming year: Chairman, H. F. Schroeder, Louisiana Polytechnic Institute; Louisiana Vice-Chairman, T. F. Wilbanks, Southwestern Louisiana Institute; Mississippi Vice-Chairman, S. B. Murray, Mississippi State College; Secretary-Treasurer, F. A. Rickey, Louisiana State University.

By the invitation of the Executive Committee, Professor L. R. Ford delivered two addresses, one at the Friday evening dinner on the subject *The Superior Student*, the other at the Saturday morning session on *Some Remarkable Theorems about Areas.*

The following short papers were presented:

1. *Haversine-saversine trigonometry*, by B. E. Mitchell, Millsaps College.

The trigonometry of the haversine-saversine combination is analogous to that of the sine-cosine pair. The saver sine of an angle is defined as the haversine of the supplement of the angle. The angle being θ, the symbols employed are hav θ and sav θ. The fundamental relation (identity) connecting them, hav $\theta+$sav $\theta=1$, is indicative of the simplicity of their relations. Their definitions

as lines, their graphs and goniometry, calculus, development into series, and their use in solving both plane and spherical triangles are discussed.

2. *Extensional invariance*, by A. D. Wallace, Tulane University.

Any completely regular Hausdorff space has a homeomorph X densely contained in a compact Hausdorff space X' so that (denoting closure in X' by $'$) if A, B are closed in X then $(A\cap B)' = A'\cap B'$ provided that either (i) X is normal, or (ii) X is locally compact and one of A, B is compact. This is merely the Tychonoff imbedding (as improved by Cech and Stone) together with the additional result about closure. Case (i) can be deduced from some results of Wallman but case (ii) cannot, since Wallman's compaction is not Hausdorff unless X is normal. This result is applied to prove the extensional invariance of certain properties of X. For example unicoherence, if X has property S.

3. *Some variations in the content of an analytic geometry course*, by L. Virginia Carlton, Northwestern State College.

In an effort to make the concepts of plane analytic geometry more fruitful, the author of this paper proposes two variations—the teaching of polar coordinates simultaneously with rectangular coordinates, and greater use of determinants. Based on the recent study of trigonometry, the polar approach is not only clearer in itself, but by contrast clarifies the advantages of the rectangular system. Three of the many uses of determinants are discussed, namely the derivation of the formula for the distance from a point to a line, the determination of the turning point of a quadratic function, and the characteristic determinant in eliminating the xy-term from the general second-degree equation.

4. *On metric extensors*, by B. B. Townsend, Louisiana State University.

"Metric" extensors, $g_{\alpha a \beta b}$ and $g^{\alpha a \beta b}$, are obtained from the Riemannian metric tensor g_{ab} and its contravariant analogue g^{ab}. These extensors enjoy many of the analogous properties in extensor theory as do the corresponding tensors in tensor theory. In particular, full range contractions on upper and lower doublet indices yield the product of two Kronecker deltas. This property makes possible the use of the well known technique of raising and lowering indices.

5. *On simplification of the equation of a quadric surface*, by S. L. Hull, University of Arkansas.

This paper deals with a method of removing the cross-product terms from the equation of an ellipsoid, without using rotation formulas. The equation is written in the form of a symmetric matrix. A parameter λ is subtracted from each element of the main diagonal of its determinant, and the resulting cubic equation is solved. The validity of this introduction of λ is established by a consideration of the gradient on an axis of the ellipsoid. The three roots of this cubic equation are the coefficients in the new equation having no cross product terms. A numerical example is solved to illustrate the method.

6. *Laplace transform and boundary value problems*, by Margaret LaSalle, Louisiana State University, introduced by F. A. Rickey.

The object of the theory as developed by H. S. Carslaw and J. C. Jaeger is to show the use of the Laplace transform in the solution of the partial differential equation $v_2 = v_{11}$, where $v = f(x, t)$.

By use of the Laplace transform, this is reduced to a total differential equation, which is solved by usual methods. The Fourier-Mellin inversion theorem and contour integration are used in determining the particular solution satisfying the initial conditions of the problem.

7. *Two versions of spectral theory*, by W. L. Duren, Tulane University.

The proof of the equivalence of two definitions of the spectrum of a linear transformation is presented in this paper. The classical definition asserts that λ is in the spectrum of the transforma-

tion T in an algebra of linear transformations if and only if $T-\lambda I$ has no inverse in $\mathfrak{A}$. A modern form of spectral theory arises from an isomorphism of $\mathfrak{A}$ into the algebra of all continuous complex valued functions on some compact Hausdorff space. The image of T is a function whose range is called the spectrum of T in the second definition.

8. *The testing program*, by H. T. Karnes, Louisiana State University.

This was the second in a series of annual reports of the testing program as given by the departments of mathematics in the colleges of Louisiana and Mississippi to entering Freshmen. The report was for the year 1949–50. There was an indication that the 1949–50 class was somewhat better than the 1948–49 class. This could have been due to uncontrolled factors. In view of this, it was decided to administer the test for at least one more year.

F. A. RICKEY, *Secretary*

APRIL MEETING OF THE IOWA SECTION

The Iowa Section of the Mathematical Association of America held its 37th annual meeting at the State University of Iowa, Iowa City, Iowa, on Friday and Saturday, April 21, 22, 1950. The Chairman, Professor B. E. Gillam of Drake University, presided.

One hundred and four persons attended the meeting, including the following forty-eight members of the Association: E. W. Anderson, W. S. Bicknell, E. W. Chittenden, B. B. Clark, Byron Cosby, Jr., A. T. Craig, Marian E. Daniells, W. M. Davis, R. M. Deming, Rev. L. E. Ernsdorff, A. M. Feyerherm, L. R. Ford, R. W. Gardner, R. E. Gaskell, B. E. Gillam, Cornelius Gouwens, Clara L. Hancock, F. S. Harper, Gertrude A. Herr, J. J. L. Hinrichsen, Helen P. Hoffman, D. L. Holl, G. E. Kaldenberg, Dora E. Kearney, L. A. Knowler, O. C. Kreider, R. J. Lambert, C. E. Langenhop, Ta Chung-Heng Li, Mary Beth Lieberknecht, F. W. Lott, Jr., R. B. McClenon, J. V. McKelvey, Martha M. McKelvey, C. G. Maple, W. H. Marlow, E. E. Moots, E. N. Oberg, H. V. Price, R. W. Raymond, Fred Robertson, Hazel M. Rothlisberger, M. F. Smiley, F. M. Stein, O. R. Taylor, H. P. Thielman, Henry Van Engen, Roscoe Woods,

Professor Van Engen reported for the committee to formulate by-laws for the section. The recommendations of the committee were adopted by an unanimous vote.

The following members were elected to hold office for the ensuing year: Chairman, Professor D. L. Holl, Iowa State College; Vice-Chairman, Rev. L. E. Ernsdorff, Loras College; and Secretary-Treasurer, Professor Fred Robertson, Iowa State College.

The following papers were presented at the meeting:

1. *Early history of the Iowa section*, by Professor R. B. McClenon, Grinnell College.

Professor McClenon gave a short account of the organization of the Iowa Section and of its first ten regular meetings. The Section held regular fall and spring meetings for a good many years, but in recent years it has met annually with the Iowa Academy of Science. The Section has shown a very gratifying growth in numbers and in strength with the passing years.

2. *Bounds for the derivatives of the solution of the Neumann problem*, by Professor C. G. Maple, Iowa State College.

The hypercircle method in function space introduced by Pfager and Synge is used to obtain bounds at a point for the derivatives of the solution of the Neumann problem. It is assumed that the hypercircle in function space on which the solution vector is located has already been determined. Then a free Green's vector is defined, and bounds fixed for the scalar product of the solution vector and the Green's vector. This same scalar product is also expressed in terms of certain calculable integrals and the value of a derivative of the solution function at an interior point. A combination of these two results gives bounds for the derivative of the solution at any point interior to the domain of definition of the problem. Only first and second order derivatives are considered.

3. *Note on Levinson's existence theorem for forced periodic solutions of a second order differential equation*, by C. E. Langenhop, Iowa State College.

Levinson (N. Levinson, *Journal of Math. and Phys.*, vol. 22, 1943, p. 41) has proved the existence of a periodic solution of the equation

$$\ddot{x} + f(x, \dot{x})\dot{x} + g(x) = e(t)$$

when $e(t)$ is periodic, and $f(x, \dot{x})$ and $g(x)$ satisfy certain rather general conditions, among which is $\lim_{x\to\infty}[g(x)/G(x)]=0$, where $G(x)=\int_0^x g(x)dx$. By altering Levinson's proof slightly at a critical point, the existence can be shown when this condition is replaced by $\lim_{x\to\infty} \sup [g(x)/x]$. These two conditions cover all possible $g(x)$ satisfying the other requirement on $g(x)$, i.e. $\lim_{x\to\pm\infty} g(x) = \pm\infty$. It does not appear to be possible to treat both cases the same way throughout.

4. *Potential flow into circumferential openings in drain tubes*, by Don Kirkham, Iowa State College, introduced by the Secretary.

A theoretical analysis of the effect of the spaces between drain tube units as used in the artificial drainage of soil is given. The problem is one of potential flow; therefore, the results are applicable to heat flow, etc. The basic problem solved is that for axially symmetric flow from an external cylindrical boundary at constant potential to a series of equal, equally-spaced openings at a lower potential, all located axially on, and comprising a part of, the otherwise impervious drain tube. The radii of the open sections and impermeable sections of the drain tube are equal. The basic problem is extended to obtain the solution to the practical problem of the seepage of groundwater into drain tubes beneath a horizontal watertable. The exact solution of the basic problem is obtained, but it is not suitable for numerical work. Accordingly, approximate solutions of specified uncertainty are derived and are utilized for tabulation of numerical results. As an example, the analysis shows, in the case of 6 inch diameter drain tubes, having 1 foot long impermeable sections and buried 4 feet deep in uniformly permeable soil, that increasing the openings from 1/32 inch width to 1/4 inch width will increase the flow 36 per cent; while embedding the tubes in gravel, to make the 1/32 inch openings of effectively infinite width, will increase the flow 180 per cent.

5. *Elastic deflection of a split ring*, by E. W. Anderson, Iowa State College.

The investigation is concerned with the stresses and deflections developed in split rings due to the action of circumferentially applied radial loads which are distributed symmetrically with respect to a diametral axis through the gap. Assuming that a condition of plane stress exists, biharmonic Airy's stress functions in Fourier series form are used to satisfy equilibrium and boundary conditions.

6. *On functional equations, generalized logarithms, and commutative functions*, by H. P. Thielman, Iowa State College.

Necessary and sufficient conditions on the rational, integral function $p(x, y)$ were given in order that the functional equation $F[p(x, y)] = F(x)+F(y)$ might have continuous, monotone solu-

tions. The functional equations $f(xy)=y^\alpha f(x)+x^\alpha f(y)$, and $g(xy)=\alpha g(x)\cdot g(y)+G(x)+g(y)$ were used to define generalized logarithms. These functional equations and their solutions were used to construct one-parameter continuous groups of commutative functions. Two functions $f(x)$ and $g(x)$ were defined to be commutative with each other if $f[g(x)]=g[f(x)]$. A particular example of the one parameter groups considered was $f(\gamma, x)=(\gamma x^\alpha+1-\gamma)^{1/\alpha}$ where γ is the parameter. Here $[\gamma, f(\beta, x)]=f(\gamma\beta, x)$. The identity of the group was given as $f(1, x)=x$.

7. *Use of complex variable in plane rigid motion*, by Professor L. R. Ford, Illinois Institute of Technology.

This was an invited address. By the use of complex variables, Professor Ford discussed some of the motions and some of the properties of the configurations thus generated by a point in a smooth plane sliding over a smooth stationary plane.

8. *Bernoulli numbers*, by Professor R. B. McClenon, Grinnell College.

This paper by Professor McClenon was concerned with the origin of these numbers in the *Ars Conjectandi* of James Bernoulli. Later work, by Euler especially, was referred to, and the usefulness of the Bernoulli numbers in many expansions was shown in the case of trigonometric and other functions.

9. *Measures for predicting calculus achievement*, by O. C. Kreider, Iowa State College.

Seven hundred eighty-nine algebra students chose curriculums requiring calculus; only 552 finished first-quarter calculus at the Iowa State College. Biserial r was used to test measures of mortality for the students who did not persevere through calculus. Correlation coefficients, regression equations, F and t tests were used to test measures for predicting scholastic achievement in calculus.

Mortality can be forecast reasonably well from the first-quarter all-college quality point average. Prediction of a calculus mark can be done reasonably well by using the algebra achievement test score, algebra course mark, and all-college average. All these measures were taken at the end of the students' first quarter in college.

10. *Note concerning the coefficients of the moments of the Bernoulli distribution*, by Dorothy DeWitt, Iowa State Teachers College, introduced by the Secretary.

In this note it is proved that the coefficients of the Bernoulli or binomial distribution are related to the differences of zero. A simple method for determining the coefficients of the rth moment from those of the $(r-1)$st moment is given.

11. *Note on a theorem of Miquel*, by Roscoe Woods, State University of Iowa.

This note is concerned with the envelope of a side, say DE, of the Miquel triangle DEF associated with the Miquel point M. The locus of M is also considered when DE is revolved about a fixed point in the plane. Conditions that the locus M be a circumconic of the reference triangle are considered in detail.

12. *On the stochastic independence of a ratio and its denominator*, by R. V. Hogg, State University of Iowa, introduced by the Secretary.

The purpose of this paper is to obtain a necessary and sufficient condition that the random variables y and x/y be stochastically independent. If $M(t_1, t_2)\equiv E[\exp(t_1x-t_2y)]$ be the moment generating function of the two dimensional random variable (x, y), this condition is, under appropriate restriction, that

$$\frac{\partial^k M(0,0)}{\partial t_2^k}\cdot\frac{\partial^k M(0,t_2)}{\partial t_1^k} \equiv \frac{\partial^k M(0,0)}{\partial t_1^k}\cdot\frac{\partial^k M(0,t_2)}{\partial t_2^k}$$

for $k=0, 1, 2, \cdots$.

13. *Some properties of quotients of additive set functions*, by E. W. Chittenden, State University of Iowa.

This paper was read by title.

14. *On the transcendence of e^x*, by Ta Chung-Heng Li, Drake University.

Let $A(z)$, $B(z)$, $\alpha(z)$, $\beta(z)$, and $H(z)$ denote integral functions with rational Taylor's coefficients. For given $A(z)$ and $B(z)$ there exist "uniquely" $\alpha(z)$ and $\beta(z)$ such that

$$A(z)\alpha(z) + B(x)\beta(x) \equiv H(x)$$

where $H(z)$ denotes the H. C. F. of the given functions. If $B(z)$ is "irreductible," and $A(z)$ has one zero in common with $B(z)$, then $A(z)$ is divisible by $B(z)$. By virtue of this fact, we can easily show that e^x is transcendental for every algebraic value of x.

15. *Constant rank matrices*, by H. D. Mills, Iowa State College, introduced by the Secretary.

The matrix polynomial

$$A(t) = A_0 + A_1 + \cdots + A_k t^k$$

where the elements of A_i are in a field F, and t is a scalar variable in a subfield F^1 of the algebraic closure of F, is said to be of constant rank r over F^1 if $A(t)$ is exactly of rank r for every t in F. If F^1 contains more than $k(r+1)$ distinct elements, a necessary and sufficient condition for $A(t)$ is to be of constant rank r over F^1 is that A_0 be of rank r and $D_1(t)$ the first invariant factor of $A(t)$ have no zero in F^1.

If $A_s=0$, $s=2, 3, \cdots$, and $A(t)$ is of constant rank r, $A(t)$ can be put in the canonical form,

$$PA(t)Q = \begin{pmatrix} I + B_1t & B_2t \\ B_3t & 0 \end{pmatrix},$$

where P, Q and B_i are constant, I and B_1 are $r\times r$ matrices, and

$$B, B_1^m B_2 = 0 \qquad m = 0, 1, 2, \cdots$$

with certain other conditions, the last equation is shown to be necessary and sufficient for $A(t)$ to be of constant rank r.

16. *Panel discussion: The teaching of college algebra and trigonometry* by J. V. McKelvey, Iowa State College (Chairman), Henry Van Engen, Iowa State Teachers College, H. V. Price, University High School, Iowa City, Iowa, Viola Smith, High School, Maquoketa, Iowa, and L. W. Swanson, Coe College.

This discussion included such topics as the solution of word problems, intelligent reading of text material, accurate use of mathematical symbols, mathematics courses for liberal arts and engineering students, algebraic and trigonometric identities, the solution of simultaneous linear equations, and testing of entering freshmen. A brief question and answer period closed the discussion.

The Iowa Academy of Science offers a \$50 prize for the best paper presented at its annual meeting. Each section submits the non-invited paper considered best on its program, and the prize winning paper is selected from this group.

As the Iowa Section meets jointly with the Iowa Academy of Science, the papers presented at this meeting are eligible to compete for the prize. The committee selected the paper by Professor E. W. Anderson on *Elastic deflection of a split ring* as the entry from Mathematics.

FRED ROBERTSON, *Secretary*

APRIL MEETING OF THE ROCKY MOUNTAIN SECTION

The thirty-third annual meeting of the Rocky Mountain Section of the Mathematical Association of America was held at the University of Denver, Denver, Colorado, April 28 and 29, 1950. There were three sessions with Professor A. J. Lewis presiding at each.

The meeting was attended by approximately one hundred persons including the following fifty-three members of the Association: R. V. Anderson, C. F. Barr, D. L. Barrick, W. G. Brady, W. E. Briggs, J. R. Britton, R. L. Calvert, R. C. Campbell, F. M. Carpenter, F. L. Celauro, Nancy V. H. Cheney, A. G. Clark, G. S. Cook, G. A. Culpepper, L. C. Dawson, David DeVol, J. R. Everett, O. J. Falkenstern, A. B. Farnell, F. N. Fisch, Katherine C. Garland, H. T. Guard, R. R. Gutzman, Leota C. Hayward, I. L. Hebel, LeRoy Holubar, Burrowes Hunt, C. A. Hutchinson, B. W. Jones, Claribel Kendall, A. J. Lewis, C. C. MacDuffee, J. C. McKenzie, W. K. Nelson, Greta Neubauer, K. L. Noble, D. O. Patterson, H. C. Peterson, A. W. Recht, L. W. Rutland, Jr., Nathan Schwid, S. R. Smith, W. N. Smith, L. C. Snively, M. E. Sperline, K. H. Stahl, J. M. Staley, P. O. Steen, J. F. Stockman, E. P. Tovani, V. J. Varineau, W. W. Varner, Lillie C. Walters.

At the business meeting, the following officers were elected for the coming year: Chairman, Professor D. O. Patterson, Colorado State College of Education; Vice-Chairman, Professor F. L. Celauro; Secretary-Treasurer, Professor J. R. Britton, University of Colorado.

The following program of papers was presented:

1. *A note on operators*, by Mr. H. C. Peterson, University of Denver.

2. *A nonlinear differential equation of heat conduction type*, by Professor Nathan Schwid, University of Wyoming.

The solution of the differential equation for the flow of heat in one direction, when the thermal conductivity K and the specific heat C each is of the form $\alpha+\beta u$, where u is the temperature and the ratio β/α is small, was considered for a semi-infinite and for a finite bar. With suitable boundary conditions a solution can be obtained if the ratio K/C depends upon the temperature.

3. *Some properties of Fibonacci sequences*, by Mr. David DeVol, University of Colorado.

Defining Fibonacci sequences by the property $u_{n+1}=u_n+u_{n-1}$, several relations between the terms are easily obtained by the manipulation of two-by-two matrices whose elements are terms of the sequence. The speaker concluded by pointing out a geometric connection between the Fibonacci sequences and the sequences of polygonal numbers.

4. *Determination of a class of solvable biquartic equations*, by Professor L. C. Dawson, Colorado A and M College.

Solvable biquartic equations of the form

$$x^8 + ax^6 + bx^5 + cx^4 + dx^3 + ex^2 + fx + g = 0$$

may be formed by assigning certain real values to the coefficients. We impose the condition that the general biquartic be expressed as a difference of two squares, thus reducing the given biquartic to two quartics each solvable by known methods. This procedure yields two necessary conditions: $a^2-4c=0$ and $(a+b)^2-4(c+d+e+f+g)=0$, whereby the coefficients may be chosen so that the biquartic decomposes into a pair of quartics. A similar procedure is applicable to the determination of a class of solvable bicubics.

5. *A problem in the theory of runs*, by Professor A. G. Clark, Colorado A and M College.

In a set of independent trials of an event where the probability of a specified outcome is constant, an asymptotic expression was obtained for P_n, the probability that a run of given length will result for the first time with the nth trial. With this definition of n, $E(n)=\sum_{j=1}^{n}P_j=1/2$. Furthermore, $E[n-E(n)]^2$ was determined as a measure of the lack of stability of n. Attention was focussed on the extent to which the solution by elementary methods of this problem in the classical theory of probability makes use of subject matter pertinent to nearly every course offered in the usual undergraduate curriculum in mathematics.

6. *A note on the calculation of residues*, by Professor C. A. Hutchinson, University of Colorado.

The expression for the residue of an analytic function at a pole of order n is obtained as a determinant of order $n-1$. In an illustrative example, the determinant is evaluated by means of a second-order linear difference equation.

7. *Linear equations without determinants*, by Professor C. C. MacDuffee, University of Wisconsin.

8. *Cross-purposes in education*, by Professor C. C. MacDuffee.

This was an evening address at which Professor MacDuffee was the guest speaker.

9. *Progress in mathematics by the U.S.S.R. since World War II*, by Mr. R. J. Howerton, Regis College.

Since January 1948, all Russian scientific journals have been published in the languages of Russia only. A survey was made of the titles and authors of the papers appearing in the six leading mathematical journals of the U.S.S.R. for 1948–49. Four of these were carried back through 1947 and one, *Akademiya Nauk, S.S.S.R., Doklady*, was carried back through 1946 since it carried the greatest number of papers for 1947–48. A classification was then made of the papers into six general categories of mathematics. The following conclusions were drawn: (1) There was a general increase of activity in 1949 over 1948, the greatest increase being shown by topology and group theory; (2) The most profitable journal for an American (Russian reading) would be *Akademiya Nauk, S.S.S.R., Doklady*, unless he were in the field of applied mathematics, in which case *Prikladnaia Matematika i Mechanika* would be the most fruitful; (3) Due to the difficulty caused by transliteration from the Latin alphabet to the Russian and back again, no conclusive evidence was obtained to show an increase in the number of Germanic names among the authors of papers; (4) The work of the Russians seems to be of the highest quality and would do credit to any American Journal. (The same results were obtained by Mr. Paul W. Howerton in the field of organic chemistry. See *Russian literature in the field of organic chemistry*, Journal of Chemical Education, April, 1949); (5) Several writers have turned out a large volume of work, the most prolific being N. G. Chebotarev, with ten papers in two years; (6) There is no evidence of any political slant to any of the papers read.

10. *Problems in the training of teachers of mathematics*, Professor A. W. Recht, University of Denver.

After the program of papers, a joint meeting was held with the Mathematics Section, Eastern Division, Colorado Education Association. The discussion was concerned with the formation of the Colorado Council of Teachers of Mathematics.

J. R. BRITTON, *Secretary*

MAY MEETING OF THE INDIANA SECTION

The twenty-seventh annual meeting of the Indiana Section of the Mathematical Association of America was held at Wabash College, Crawfordsville, Indiana, on Saturday, May 6, 1950. Two sessions were held at which Professor Ralph Hull of Purdue University, Chairman of the Section, presided.

There were sixty-two in attendance including the following thirty-six members of the Association: Juna L. Beal, L. G. Black, Stanley Bolks, C. F. Brumfiel, G. E. Carscallen, W. W. Chambers, T. E. Cheatham, H. E. Crull, M. W. DeJonge, V. E. Dietrich, P. D. Edwards, W. R. Fuller, E. L. Godfrey, Michael Golomb, S. H. Gould, G. H. Graves, J. R. Hadley, N. R. Hughes, Ralph Hull, M. W. Keller, E. L. Klinger, R. A. Lufburrow, R. B. Merrill, P. T. Mielke, P. M. Nastocoff, C. C. Oursler, P. W. Overman, Philip Peak, J. C. Polley, Arthur Rosenthal, M. E. Shanks, Jane A. Uhrhan, R. O. Virts, J. L. Wilson, Florence A. Wirsching, W. D. Wood.

The following officers were elected: Chairman, H. E. Crull, Butler University; Vice-Chairman, M. W. Keller, Purdue University; Secretary, J. C. Polley, Wabash College.

On the matter of awarding Association medals as prizes in high school mathematics contests the chairman was authorized to appoint a committee with power to act. The committee was instructed to investigate the possibility of making such awards in connection with the Indiana State Mathematics Contest and the Indiana Science Talent Search.

The annual meeting of 1951 will be held on Saturday, May 5, the place of meeting to be announced later.

The following papers were presented:

1. *Mathematics for engineers*, by Professor M. E. Shanks, Purdue University.

Of two significant trends in mathematics for freshmen, terminal courses designed solely to fill the cultural gap, and a unified non-compartmentalized course in algebra, trigonometry, and analytic geometry, in part cultural but chiefly motivated by a need for bringing so called advanced ideas down into the undergraduate program, the latter was emphasized. In the author's opinion the need of the modern engineer for the advanced ideas, for pure mathematics, is essential, and once the engineer recognizes that the less traditional course could clearly increase his mathematical "power" he would welcome the change.

2. *A proof of the existence of a real zero for a polynomial of odd degree with real coefficients which is not dependent on continuity*, by Professor J. C. Polley, Wabash College.

This proof of the existence of a real zero for the polynomial $P(x)=\sum_{i=0}^{n}a_ix^{n-i}$, where the a_i are real, and $a_0>0$, is based on the theorems proving the existence: (1) of a number A such that, for all $x>A$, $P(x)>0$; (2) of a number B such that, for all $x<B$, $P(x)<0$; (3) of a number $d>0$ such that, for all x for which $0<x\leqq d$ and $-d\leqq x<0$, $|p(x)|=\sum_{i=0}^{n-1}a_ix^{n-i}<D$, where D is any positive number, however small. The proof consists in showing that at $x=c$, the least x, such that $P(x)>0$ for all greater x, $P(x)$ can be neither positive nor negative, whence, being defined, it must vanish.

3. *The crystallographic groups*, by Mr. C. L. Hassell, Purdue University, introduced by Professor Ralph Hull.

The 32 classes of crystallographic groups were described in terms of their representation as subgroups of the full orthogonal groups of three dimensions. It was pointed out that eleven of them are subgroups of the proper orthogonal groups, others are obtained from those of the first kind by adjoining the central reflection, and the rest are obtained from those of the first kind in another way. Illustrations were given for many of the classes, and it was mentioned that apparently no crystal substance is known for one of the classes, or at least this was the case up to 1938.

4. *On the content of the first course in mathematical statistics*, by Professor C. F. Kossack, Purdue University, introduced by Professor Ralph Hull.

As a first course in mathematical statistics is commonly taught the mathematics is superimposed upon a standard course in statistical methods, each derivation being treated separately and somewhat isolated from the others, whence the student fails to appreciate the theoretical basis of statistical methods. A course should be developed stressing the logical basis of the methods and the inductive approach to problems, introducing the elements of probability as needed and approaching statistical theory from at least a semi-axiomatic basis. The main ideas of hypotheses, testing, etc., should be stressed, and the student's material restricted to the simpler illustrations which he can handle mathematically.

5. *Linear graphs and the economics of transportation*, by Professor Tjalling C. Koopmans, Director of Research of the Cowles Commission, University of Chicago.

This was an invited address. The theory of linear graphs was applied to finding the most economical routing plan for transportation equipment (ships, say) in carrying out a given program consisting of constant monthly cargo flows between each pair (i, j) of n ports. If, for $i, j=1, \cdots, n$, y_{ij} are cargo flows in shiploads per month, x_{ij} flows of empty ships, t_{ij} and s_{ij} average times for loading and empty movement respectively, b_i the monthly surplus of ships, and Z the amount of shipping required for a routing plan y_{ij}, x_{ij}, then $b_i=\sum_j y_{ji}-\sum_j y_{ij}$ and $b_i=\sum_j x_{ij}-\sum_j x_{ji}$ where $x_{ij}\geqq 0$ (thus defining a convex polyhedral set). Hence $Z=\sum_{ij}t_{ij}y_{ij}+\sum_{ij}s_{ij}x_{ij}=Y+X$, say, where X is a linear function of x_{ij} which has at most one minimum anywhere in the set, but possibly at more than one point.

With any point x_{ij} minimizing X exists a non-negative number set p_{ij} representing nominal prices for a unit of transportation services, and a set p_i representing evaluations of the location of a ship under efficient routing, such that $p_{ij}-p_i+p_j-t_{ij}\leqq 0$ and $-p_i+p_j-s_{ij}\leqq 0$ for all (i, j), the former being zero for $y_{ij}>0$, and the latter zero for $x_{ij}>0$. The p_{ij} and p_i are the same for all programs permitting an efficient routing plan for which the linear graph G, consisting of all routes (i, j) such that $x_{ij}>0$, is the same.

Since the p_{ij} are reflected in freight rates in a perfectly competitive market, the analysis applies in developing a system of rates for a regulated transportation system so as to induce social efficiency in the choice of industrial locations by private entrepreneurs.

J. C. POLLEY, *Secretary*

ANNUAL MEETING OF THE MINNESOTA SECTION

The annual meeting of the Minnesota Section of the Mathematical Association of America was held at Macalester College in St. Paul, Minnesota, on Saturday, May 6, 1950. A session was held in the forenoon followed by a joint luncheon and afternoon session with the Minnesota Council of Teachers of Mathematics. Professors E. J. Camp, W. L. Hart and Kenneth May (Chairman of the Section) presided at the respective sessions. Professor J. R. Mayor of the University of Wisconsin was the principal speaker at the luncheon.

One hundred and thirty persons attended the meeting, including the following sixty-eight members of the Association: H. M. Anderson, F. J. Arena, J. E. Bearman, E. J. Berger, K. H. Bracewell, L. E. Bush, W. H. Bussey, R. H. Cameron, E. J. Camp, H. D. Colson, W. F. Crum, K. E. Dubbert, Helen Engebretson, I. C. Fischer, Walter Fleming, G. C. Francis, C. B. Germain, H. W. Godderz, Ruby M. Grimes, K. L. Hankerson, Charles Hatfield, J. Stanley Hill, J. E. Hafstrom, Hildegarde Horeni, G. H. Jaeger, D. A. Johnson, Reverend W. C. Kalinowski, G. K. Kalisch, Karlis Kaufmanis, P. G. Kirmser, L. S. Laws, R. L. Lokensgaard, W. S. Loud, Walter Lyche, H. B. MacDougal, W. R. McEwen, R. B. McHugh, C. R. McIntosh, K. W. McVoy, S. L. Mason, Kenneth May, J. R. Mayor, Sigurd Mundljeld, M. J. Norris, E. P. Northrop, F. R. Ohnsorg, Margaret Owchar, G. C. Priester, Mrs. Ruth Scholten, Dorothy V. Schrader, S. C. Simonson, Sister Ada Marie, Sister M. Joanne, Sister Mary Leontius, Sister Mary Seraphim, F. C. Smith, R. C. Staley, Irwin Stoner, A. G. Swanson, F. J. Taylor, Takashi Terami, Matilda B. Thompson, Marian W. Thornton, H. L. Turrittin, E. C. Varnum, O. E. Walder, K. W. Wegner, Irene L. Wente.

The nominating committee, consisting of Professor M. J. Norris, Chairman, Professor H. L. Turrittin, and Professor Kenneth Wegner, presented the following slate of nominees for the coming year: Chairman, J. M. H. Olmsted, University of Minnesota; Secretary, L. E. Bush, College of St. Thomas; Executive Committee, Kenneth May, Carleton College; Walter Fleming, Mankato State Teachers College; H. M. Anderson, Gustavus Adolphus College. These were duly elected. The following resolutions were passed: (1) That the term of office of all appointed committees shall end at the annual meeting; (2) That the committees on the high school mathematics contest and on cooperation with the Minnesota Council of Teachers of Mathematics be continued for another year.

By invitation of the Executive Committee, Dean E. P. Northrop of the University of Chicago delivered an address at the morning session on *The Role of Mathematics in General Education.*

The following seven short papers were presented:

1. *On two circular-hyperbolic functions and their application to differential equations*, Professor F. J. Arena, North Dakota State College.

Professor Arena discussed the properties of the functions

$$c(x) = \sum_0^\infty \frac{x^k}{(2b)!}, \qquad s(x) = \sum_0^\infty \frac{x^k}{(2b+1)!},$$

and showed how the use of these functions simplifies the solution of the linear differential equation of the second order with constant coefficients.

2. *An extension of Schwarz's lemma*, Professor Irwin Stoner, University of Minnesota.

In order to prove Riemann's mapping theorem, H. A. Schwarz formulated the lemma which states that if $f(z)$ is analytic, regular for $|z|<1$, if $|f(z)|<1$ for $|z|<1$, and if $f(0)=0$, then $|f(z)| \leqq |z|$ for $|z|<1$. By omitting the condition $f(0)=0$, Carathéodory and Koebe were able to extend this lemma. If $r<1$, Carathéodory showed that the values of $f(z)$ are in the circle whose center is at the origin and whose radius is $[r+|f(0)|]/[1+r|f(0)|]$, and Koebe showed that the values of $f(z)$ are in a circle whose center is at the point $f(0)$ and whose radius is $r[1+|f(0)|^2]/[1-r|f(0)|]$.

The author obtained a sharper result than the two given above. He showed that the values of $f(z)$ are in the circle whose center is at the point $|f(0)|(1-r^2)/[1-r^2|f(0)|^2]$ and whose radius is $r[1+|f(0)|^2]/[1-r^2|f(0)|^2]$. This circle is entirely contained in the intersection of the above two circles.

3. *A perpetual calendar*, Professor Charles Hatfield, Jr., University of Minnesota.

A simple circular perpetual calendar based upon Gauss' related congruence was presented and demonstrated.

4. *On a continuous function with derivative nowhere*, Professor M. J. Norris, College of St. Thomas.

An example of a function of the type under discussion given by van der Waerden is modified in a way suggested by the use of binary notation rather than decimal notation. The main interest of the paper is in the discussion of differentiability, which can be made almost intuitive.

5. *In defense of the factoring method of solving quadratic equations*, Professor K. W. Wegner, Carleton College.

A defense provoked by the proposal of W. R. Ransom in the *Mathematics Teacher* (March, 1948) that the factoring method be dropped from elementary algebra courses.

6. *A modification of normal curve grading techniques as applied to achievement examinations*, Mr. J. Stanley Hill, Minnesota Mutual Life Insurance Company.

The paper presupposes general acceptance of the belief that normal curve grading techniques have value in certain situations. Frequency distributions of achievement test scores are often markedly skewed in the direction of the lower grades. When usual normal curve grading techniques are applied to such distributions a paucity of A's is produced together with a redundant number of failures. These difficulties are ameliorated by determining separate measures of dispersion for the scores lying above the mean on the one hand and the scores lying below the mean on the other hand. The measures used are identical in definition with the standard deviation except that summation for both numerator and denominator is performed only on variates lying above the mean in one case and on variates lying below the mean in the other case. A "short method" for calculating these measures of dispersion, analogous to the short method for calculating the standard deviation, has been developed. As a practical matter the division between the upper and lower groups for summation is taken at the class limit nearest the mean. The arithmetic work is not in-

creased materially over that required for the usual standard deviation technique; no additional columns on the work sheet are necessary.

7. *Circular nomograms*, Mr. E. C. Varnum, Barber-Colman Company.

Although there are many nomograms in the technical literature, very few are based on the closed, symmetric form of a circle which embodies eye appeal as well as inclusion of all scales in a finite region. During the last year, the author constructed fifteen circular nomograms of which one was published in *Industrial Quality Control* (Jan., 1950), and nine have been printed by Barber-Colman Company for use by its inspection, engineering and sales personnel. Copies of three circular nomograms were distributed to the audience, and the method of constructing the new type of nomogram was outlined.

L. E. BUSH, *Secretary*

THE MAY MEETING OF THE NEBRASKA SECTION

The twenty-sixth annual meeting of the Nebraska Section of the Mathematical Association of America was held at the Nebraska Wesleyan University in Lincoln, on Saturday, May 6, 1950. Professor C. B. Gass presided at the two sessions.

Thirty-eight persons attended the meetings including the following twenty-two members of the Association: M. A. Basoco, H. W. Becker, A. K. Bettinger, Jessie W. Boyce, C. C. Camp, F. Marion Clarke, H. M. Cox, Morris Dansky, H. W. Doss, Jr., J. M. Earl, C. B. Gass, C. F. Gayton, Jr., Edwin Halfar, L. M. Larsen, W. G. Leavitt, W. T. Lenser, E. J. Lowry, C. R. Perisho, H. L. Rice, Lulu L. Runge, R. G. Sanger, and C. J. Tegels.

Officers elected at the meeting were: Chairman, Morris Dansky, Creighton University; Vice-Chairman, C. B. Gass, Nebraska Wesleyan; Secretary, Lulu L. Runge, University of Nebraska. The annual meeting in 1951 is to be held on Saturday, May 5, at the University of Nebraska.

Professor R. G. Sanger of Kansas State College was the guest speaker, and presented a paper at both the morning and afternoon sessions.

The following papers were presented:

1. *Some simple properties of periodic mappings*, by Professor Edwin Halfar, University of Nebraska.

Considered was a class of commutative mappings (f) on an arbitrary point set S into S. A mapping was said to be periodic at a point x if for some $n, f^n(x)=x$ where f^n means n repetitions of f. The principal results were that if $fg=gf$, f periodic at x, $fg(X)\subset X$ for each $X\subset S$, and g is 1-1, then there exist integers p and q such that $f^p(x)=g^q(x)$; if S is a complete lattice ordered by "$\leqq$," then for each p and f monotonic increasing, the set of points $E[f^p(x)=x]$ is a complete lattice ordered by "$\leqq$" and is non-empty.

2. *Multimeter disturbance factors*, by H. W. Becker, Electronic Radio-television Institute, Omaha.

The speaker demonstrated two corollaries of Thevenin's. When an ammeter of internal impedance Z_{AM} is introduced into a circuit of series impedance Z, then the undisturbed current $=$(measured current) $(1+Z_{AM}/Z)$. When a voltmeter of internal impedance Z_{VM} is put across a branch having impedance Z', in series with a network having impedance Z, then the undisturbed voltage$=$(measured voltage) $[1+(Z||Z')/Z_{VM}]$, where $Z||Z'=ZZ'/(Z+Z')$.

3. *Comments on elliptic integrals*, by Professor R. G. Sanger, Kansas State College.

In the early part of this century elliptic integrals and elliptic functions were assiduously studied. Since then, courses in elliptic integrals and the associated functions have gradually disappeared from the offerings of departments of mathematics. An attempt was made in this paper to describe elliptic integrals, their properties and associated functions, and to indicate why interest in them has rapidly dwindled.

4. *A locus of navigational interest*, by Professor O. C. Collins, University of Nebraska, introduced by the Secretary.

The locus considered was that of a navigator who sees three fixed markers, and so manoeuvres his vessel that the middle marker remains angularly equidistant from the other two. On a spherical earth, the corresponding locus is that of ships so placed that two fixed stars are sighted at equal bearings to the east and to the west of north. An extension to space navigations was considered.

5. *On the elastic theory of a certain toroid*, by T. I. Gilroy, Creighton University, introduced by the Secretary.

6. *Supervised study laboratory at the University of Nebraska*, by S. R. Barnett, Extension division, University of Nebraska, introduced by the Secretary.

7. *The development of the calculus of variations*, by Professor R. G. Sanger, Kansas State College.

The development of the calculus of variations from the historical standpoint was considered in this paper. The problems which inspired this subject and the results obtained by workers in this field before 1950 were mentioned. Then, the need for and the problem of determining sufficient conditions for problems in this field were considered. Some results which have been obtained in the present century, and the present trends in the development of this subject, were indicated.

8. *Developments of the Nebraska mathematics test*, by Professor H. M. Cox, Bureau of Instructional Research, University of Nebraska.

The speaker reviewed the studies of entrance examinations in mathematics at the University of Nebraska. Beginning in 1939 with a revision of a short algebra test for students registering in the College of Engineering, the program was extended to all entering freshmen and sophomore students before 1942. The 1942 edition of the Nebraska Mathematics Test was planned to integrate high school mathematics, and it provided an array of questions of increasing difficulty for students of various preparations.

The edition of the Test now in use represents a third pattern of measurement, and includes an effort to partially offset the factor of "forgetting." Charts were distributed showing a study of performance of students in mathematics courses in their second semester of enrollment in college.

9. *Use of elementary structures of modern algebra for traditional questions in elementary theory of equations*, by F. Marion Clarke, University of Nebraska.

It was shown how the abstract definitions of group, ring, integral domain, and field can be used by the undergraduate to give logical strength to an analysis of the number and kinds of solutions that exist for an equation in one variable, and for m linear equations in n variables over a given coefficient domain. Examples from the domains of weaker structure give emphasis to the meaning of techniques which have become traditional habits for the domains of real or complex numbers.

10. *Abstract algebra for elementary teachers*, by Professor W. G. Leavitt, University of Nebraska.

This speaker considered the question as to whether or not some knowledge of modern algebra would be useful to teachers of elementary mathematics. A few brief comments were made, first, on the broader aspects of the question, after which were mentioned a number of possible specific applications of a knowledge of abstract theory to elementary instruction.

11. *Note on elementary functions*, by W. T. Lenser, University of Nebraska.

The author noted the parallelism which exists between certain theorems on elementary functions with those on algebraic numbers, even to the details of proof.

LULU L. RUNGE, *Secretary*

CALENDAR OF FUTURE MEETINGS

Joint meeting with American Society for Engineering Education, Michigan State College, East Lansing, June 25–26, 1951.

Thirty-second Summer Meeting, University of Minnesota, Minneapolis, September 3–4, 1951.

Thirty-fifth Annual Meeting, Brown University, Providence, Rhode Island, December 29, 1951.

The following is a list of the Sections of the Association with dates of future meetings so far as they have been reported to the Associate Secretary.

ALLEGHENY MOUNTAIN, Duquesne University, Pittsburgh, Pennsylvania, May, 1951.

ILLINOIS, University of Illinois, Urbana, May 11–12, 1951.

INDIANA, May 5, 1951.

IOWA, Wartburg College, Waverly, April 20–21, 1951.

KANSAS, University of Kansas, Lawrence, April 7, 1951.

KENTUCKY, Eastern Kentucky State College, Richmond, April 28, 1951.

LOUSIANA-MISSISSIPPI, Mississippi State College, State College, February 16–17, 1951.

MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA.

METROPOLITAN NEW YORK, Manhattan College, April 7, 1951.

MICHIGAN, Michigan State College, East Lansing, March 24, 1951.

MINNESOTA, College of St. Benedict, St. Joseph, April 28, 1951.

MISSOURI, Central College, Fayette, April 6, 1951.

NEBRASKA, University of Nebraska, Lincoln, May 5, 1951.

NORTHERN CALIFORNIA

OHIO, Ohio State University, Columbus, April 21, 1951.

OKLAHOMA

PACIFIC NORTHWEST, State College of Washington, Pullman, June 15, 1951.

PHILADELPHIA, University of Pennsylvania, Philadelphia, November 24, 1951.

ROCKY MOUNTAIN, Colorado State College of Education, Greeley, April 20–21, 1951.

SOUTHEASTERN, Vanderbilt University and Peabody College, Nashville, Tennessee, March 16–17, 1951.

SOUTHERN CALIFORNIA, Whittier College, Whittier, March 10, 1951.

SOUTHWESTERN, University of New Mexico, Albuquerque, March 23–24, 1951.

TEXAS, Southern Methodist University, Dallas, Spring, 1951.

UPPER NEW YORK STATE, Hamilton College, Clinton, May 5, 1951.

WISCONSIN, Carroll College, Waukesha, May 12, 1951.

GEORGE BANTA PUBLISHING COMPANY, MENASHA, WISCONSIN

THE AMERICAN MATHEMATICAL MONTHLY

THE OFFICIAL JOURNAL OF

THE MATHEMATICAL ASSOCIATION OF AMERICA, INC.

VOLUME 58 NUMBER 3

CONTENTS

MARCH 1951

The AMERICAN MATHEMATICAL MONTHLY

(FOUNDED IN 1894 BY BENJAMIN F. FINKEL)

EDITORIAL CORRESPONDENCE should be addressed to the Editor, C. V. NEWSOM, State Education Building, Albany 1, N. Y.

ADVERTISING CORRESPONDENCE should be addressed to J. F. RANDOLPH, Morey Hall, University of Rochester, Rochester 3, N. Y.

NOTICE OF CHANGE OF ADDRESS by members of the Association as well as correspondence regarding subscriptions to the MONTHLY should be sent to the Secretary-Treasurer, H. M. GEHMAN, University of Buffalo, Buffalo 14, N. Y. Change of address must reach the Secretary-Treasurer about six weeks before the change can become effective.

THIS IS THE OFFICIAL JOURNAL OF THE
MATHEMATICAL ASSOCIATION OF AMERICA, INC.

(Devoted to the Interests of Collegiate Mathematics)

Entered as second class matter at the post office at Menasha, Wis. Acceptance for mailing at special rate of postage provided for in the Act of February 28, 1925, embodied in Paragraph 4, Section 538, P. L. and R., authorized April 1, 1926.

SUBSCRIPTION PRICE: To Members, \$4 a Year, To Others, \$5 a Year.

PUBLISHED BY THE ASSOCIATION at Menasha, Wisconsin, and Albany, N. Y. during the months of January, February, March, April, May, June-July, August-September, October, November, December.

GEOMETRICAL SOLUTION OF SPHERICAL TRIANGLES

J. M. THOMAS, Duke University

1. Introduction. If three parts of a spherical triangle are given arcs, it at once follows from the formulas of spherical trigonometry that the other three parts can be constructed in a plane by ruler and compass. Actual constructions, moreover, have been given. Book [**1**] cited in the references at the end of this paper specifies, for example, certain plane triangles which contain the unknown parts or their functions as elements, so that the construction is effectively indicated, although the emphasis is upon the computation. The books [**2**], [**3**] employ descriptive geometry for constructing the unknown parts. A reading of Dutton's construction [**3**; 189, 190] led to the constructions to be given here.

With applications primarily in view the present paper (i) employs, except in §4, coördinates on the sphere, rather than spherical triangles; (ii) effectively replaces the three circles used by Dutton by a single circle; and (iii) gives a construction for finding the latitude and longitude of a point from which the altitudes of two stars have been simultaneously observed. The effect of (i) is to make the construction immediately and uniformly applicable in whatever quadrants the data of an astronomical problem lie; that of (ii) is to confine all points involved in the construction to the area of one circle and all measurement of arcs to the circumference of that circle. Among the advantages which accrue from (ii) is the possibility of designing an instrument to perform the operations mechanically. In §4 constructions for spherical triangles as such are deduced from the constructions developed in terms of coördinates.

It is to be noted that the treatment given here presupposes only certain almost intuitive results from spherical geometry and none from plane or spherical trigonometry. A class in solid geometry, for example, can easily be taught to find the distance between two points on the earth from their geographical coördinates. Furthermore, it is possible—and, if the construction is to be frequently used, highly desirable—to dispense with the oblique projection usually employed for representing a spherical triangle and to rely entirely on the representation in Figure 1, discussed in the next section. The reader versed in spherical trigonometry may, of course, prefer at the outset to follow the steps in Figure 4 on an auxiliary three-dimensional diagram.

It should further be remarked that the law of cosines, which effectively implies all the formulas of spherical trigonometry but is irrelevant to the present discussion, can be easily deduced in all generality from Figure 1. Conditions for the consistency of data can likewise be obtained as inequalities among arcs.

2. Representation on the plane. Figure 1 shows all that is essential for the construction. The positive sense for all angles is clockwise. For the point 0 on the earth's surface the circle shown represents the meridian on the celestial sphere and its opposite. This circle will be called the *fundamental circle.* Point 1 is (that is, represents) the north pole of the celestial sphere. Point 2 is the intersection of the meridian and the equator; it is 270° from 1. The zenith of 0 is 3.

The north and south points of the horizon are respectively 4, 5. The arc 2-3 is the latitude. A point on the celestial sphere is represented by its orthographic

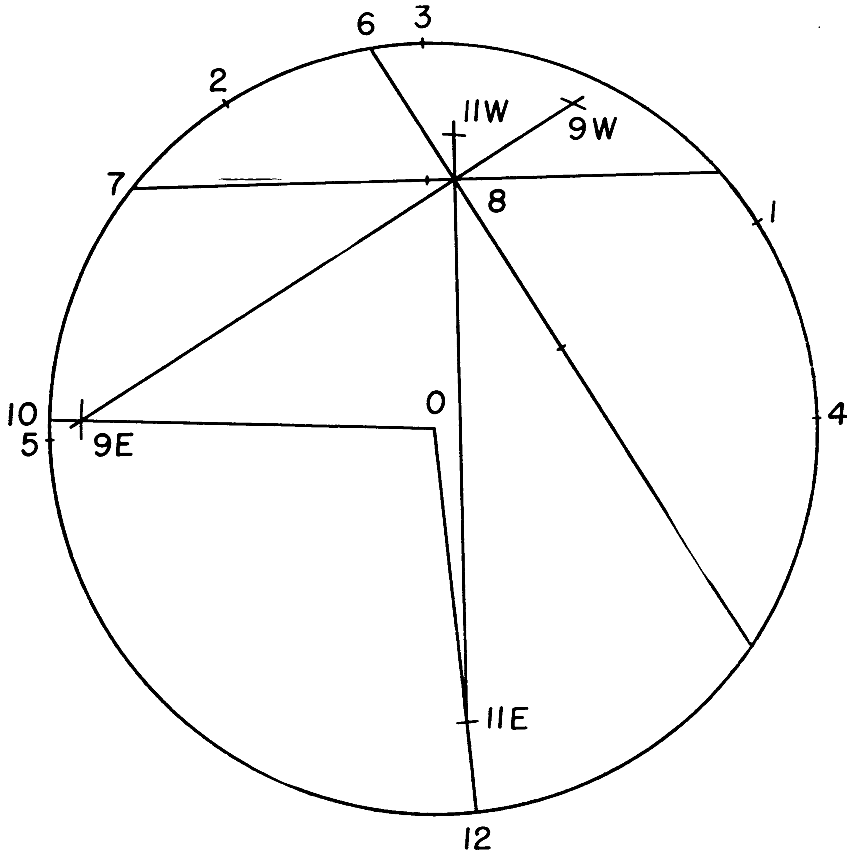


FIGURE 1

1	north pole	2–3	lat
2	equator	2–6	dec
3	zenith	2–10	LHA
0–9	radius diurnal circle	5–7	alt
0–11	radius altitude circle	4–12	az

projection 8 on the plane of the paper, the point of sight being to the east. To simplify language, we exclude from consideration the points on the fundamental circle; how to handle such points is obvious. Each point 8 interior to the circle represents exactly two points on the celestial sphere, one to the east (towards the reader) and one to the west. Arc 2-6 is the declination (north in the figure). The chord 8-6 is parallel to or coincides with 0-2 and is the projection of the diurnal circle. The representation of a given star (with fixed declination) moves with simple harmonic motion of period 24 sidereal hours along 8-6. Erect a perpendicular to 6-8 at 8. A circle with center at 0 and radius equal to that of

the diurnal circle (half the full chord 6-8) cuts the perpendicular in two points, one of them nearer 1 than the other. The one farther from 1 is numbered 9E and belongs to the point E in the east whose representation is 8. The one nearer to 1 is numbered 9W and belongs to the point W in the west. Let the radius 0-9E cut the fundamental circle in 10. Then the LHA of E is arc 2-10. The chord 8-7 parallel to or coinciding with 4-5 is the projection of the altitude circle. The altitude is arc 5-7. A circle with center 0 and radius equal to that of the altitude circle cuts the perpendicular to 7-8 at 8 in two points 11E, 11W, the one nearer to 3 being associated with the west. Let 0-11E cut the circle in 12. The azimuth of the point in the east represented by 8 is arc 4-12.

The proof of the foregoing is as follows. Let the diurnal circle be rotated about its projection 6-8 into the plane of the paper so that its western half is on the same side of 6-8 as is 1. Then translate the circle so that its center falls on 0, retaining only the points of intersection 9E, 9W with the perpendicular. The altitude circle is treated similarly.

3. Application of the representation. Although the construction just given is truly ruler-and-compass, it is convenient in applications to employ a transparent circular protractor which has the same size as the fundamental circle and is graduated clockwise from 0° to 360°. This permits the data and answers to be expressed numerically.

For the present suppose that the latitude and longitude are given. The fundamental construction problem is this: given one of the sets of coördinates

(dec, GHA), (alt, az), (dec, alt)

for a point on the celestial sphere, find either or both of the others. The pair (dec, alt) must, of course, be accompanied by a designation east or west in order that the point represented be unique.

Any one of the phases of the problem is immediately solved by reference to Figure 1. If GHA is to be read or set, the protractor is placed so that the reading at 2 is the longitude (the value between 0° and 360°) of the point whose zenith is 3.

Star charts for a given time and place can readily be constructed from the Nautical Almanac. The representative points 8 for the stars in the west can be constructed exactly as described. It is convenient to have those in the east on a separate chart and, since they are to be viewed from the west, to reflect the whole of Figure 1 in a vertical line by moving 1 to the left and turning the protractor over so that angles are measured counterclockwise. To construct subsequently a chart for the stars in the south, construct the point corresponding to 11 by measuring the azimuth from the lowest point of the circle and then project on the horizontal chord as before. In general, if the azimuth of the observer (that is, the azimuth of the point of sight for projection) is B, shift clockwise through $90° - B$ the point 4 from which the azimuth is measured in Figure 1; that is, rotate each horizontal projection 11 through B-90° into 11′ and determine the new representation 8′ by projecting 11′ orthogonally upon the unaltered trace 7-8 of the altitude circle.

4. Solution of spherical triangles. When a spherical triangle is proposed for solution, the problem will be called *lateral* or *angular* according as at least two sides or at least two angles are given. If a problem is angular, the corresponding problem for the polar triangle is lateral. Hence we limit ourselves to the discussion of lateral problems.

Given (a, b, c) to find A we may take $a=90°-\text{alt}$, $b=90°-\text{dec}$, $c=90°-\text{lat}$ and determine A as the GHA on the assumption that the longitude is 0 and that the star is in the west. It is convenient, however, to rearrange matters for direct reading with a semicircular protractor as has been done in Figure 2. Signs being ignored, a can be measured from 3 and b, c from 1. It is also convenient to measure A from 1: this can be done by using a perpendicular to 0-9 rather than 0-9

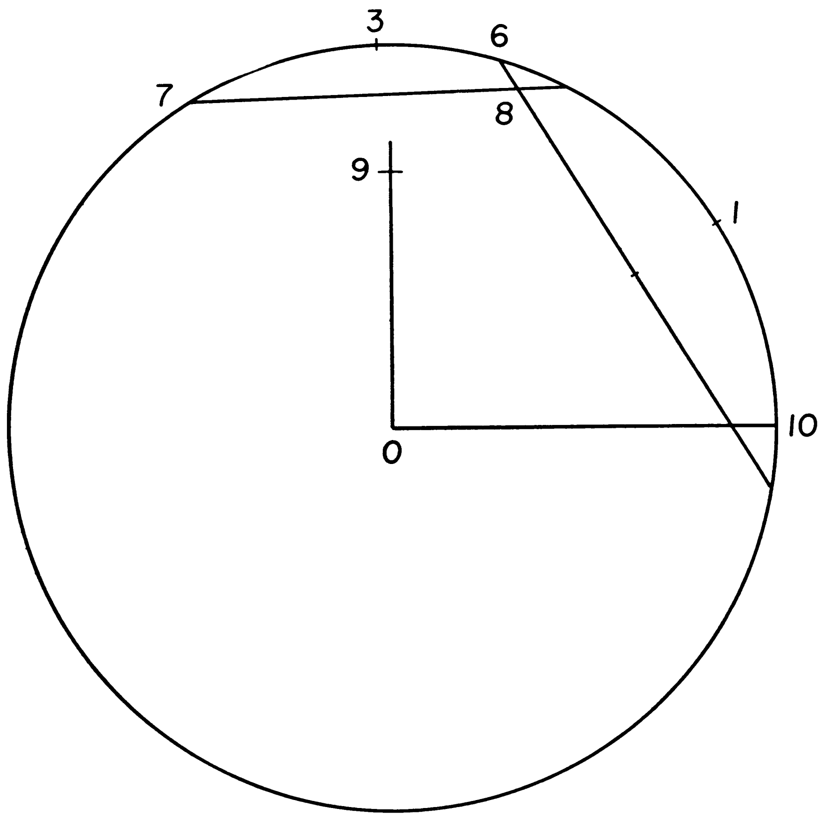


FIGURE 2

$3\text{–}7=a, \quad 1\text{–}6=b, \quad 1\text{–}3=c, \quad 1\text{–}10=A.$

itself. It is not necessary to invoke the rule for choosing between 9E and 9W: if 9E is used, a numerically equal negative A is found. The reason behind this is that the construction is analytically equivalent to the equation of the law of

cosines in which A appears only through its cosine. Figure 2 is constructed for $a=30°$, $b=42°$, $c=60°$, whence $A=32°$, roughly.

Given (A, b, c) to find a, we use exactly the same figure, drawing 7-8 last.

If (a, b, A) are given, the construction is illustrated in Figure 3, in which it is found convenient to place the pole 1 at the top since the zenith is unknown. Points 6, 8, 9 are found much as before, and the rest of the problem is essentially to give chord 7-8 the proper direction. Point 13 is found by striking from 9E, 9W arcs of radius equal to half the chord of an arc of length $2a$ in the fun-

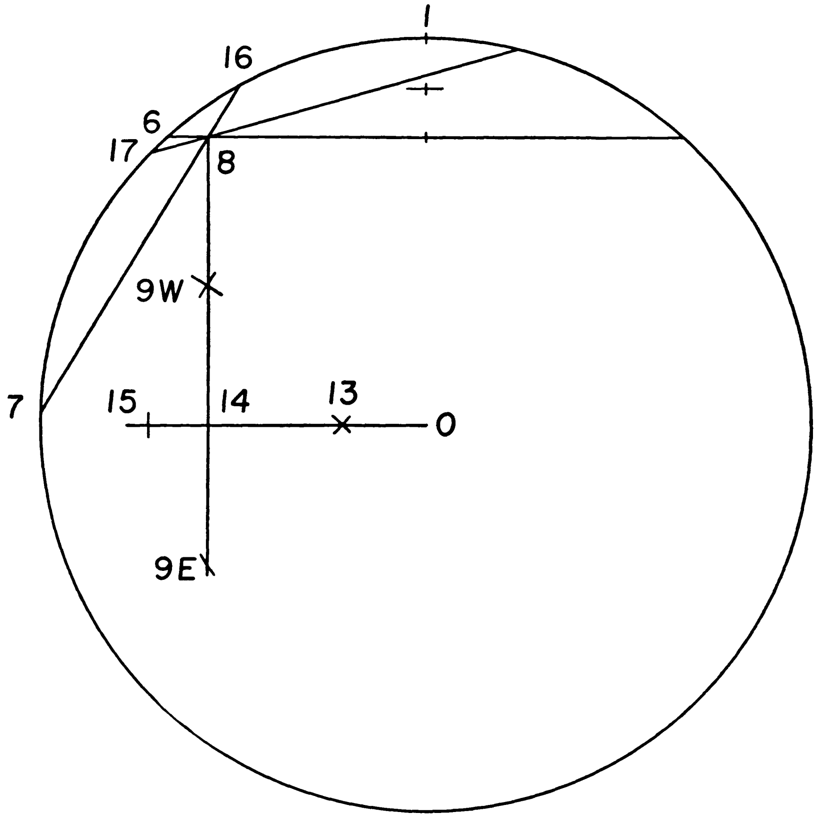


FIGURE 3

damental circle: the radius was found in Figure 3 by marking arc a on either side of 1. Point 15 is obtained by striking an arc of the radius 13-9W with 13 as center. Points 16, 17 are found by striking arcs of radius 14-15 with 8 as center. The chord or chords 7-8 are obtained by joining 8-16 and 8-17. Since c must be positive, point 6 must lie in the arc less than 180° subtended by any chord which is to give a solution.

The situations which can arise with arbitrary data are quite varied. If each of the given parts is between 0° and 180°, it is convenient to divide the situa-

tions into five mutually exclusive and exhaustive types illustrated by the following numerical data:

	a	b	A	
(4.1)	30°	60°	45°	No solution.
(4.2)	30°	45°	45°	One solution in which $B = 90°$.
(4.3)	30°	20°	45°	One solution in which $B \neq 90°$.
(4.4)	30°	40°	45°	Two solutions.
(4.5)	90°	90°	90°	Infinitely many solutions.

In passing, it should be remarked that type (4.5) is usually overlooked in the analysis of the ambiguous case given in textbooks, possibly because the reasoning in them follows too closely the pattern for the plane, where a circle of radius a cannot coincide with the geodesic which supports side c.

5. The location problem. The so-called astronomical triangle has for vertices the three points North Pole, Zenith, Star. Each of these vertices is the pole of a system of polar coördinates on the sphere. Among the coördinates of the vertices themselves in these systems, three are replaced by their complements and with two of the angles are given the names

$$\text{(lat, dec, LHA, alt, az)}. \tag{5.1}$$

In addition, the polar system with pole at the north pole is connected with a fixed polar system, in which the angular coördinate is GHA, by the equation

$$\text{GHA} = \text{LHA} + \text{long}. \tag{5.2}$$

It is necessary in problems such as the one we are about to discuss to consider simultaneously several triangles of points analogous to the astronomical triangle. We shall employ the same names as appear in (5.1) for the analogous coördinates and to avoid confusion shall write $(\text{alt, az})_{ABC}$ to mean "the coördinates analogous to alt, az when the north pole, zenith and star are replaced by points A, B, C, respectively."

Let now P be the north pole, Z the zenith of a fixed point on the earth, not at either pole, and S, T two stars whose GHA's do not differ by 0° or 180° and neither of which coincides with P or Z. Suppose further that the notation is adjusted so that star S is to the eastward in the sense that

$$\text{GHA}T - \text{GHA}S \equiv D \pmod{360°}, \tag{5.3}$$

where $0° < D < 180°$. Set

$$(\text{long, lat, dec, GHA, alt})_{PZS} = (x, y, d_1, G_1, a_1),$$
$$(\text{dec, GHA, alt})_{PZT} = (d_2, G_2, a_2).$$

We suppose that the observer measures a_1, a_2 simultaneously at GCT t, that he notes whether the orientation STZ is positive (clockwise as viewed

from outside the celestial sphere) or negative and whether S is to the east or west of the meridian, and that he determines (d_1, G_1), (d_2, G_2) for time t from the Nautical Almanac. Our problem is then to find (x, y) by construction.

The construction consists of three changes of coördinates of the nature described in §3. They are described in tabular form below.

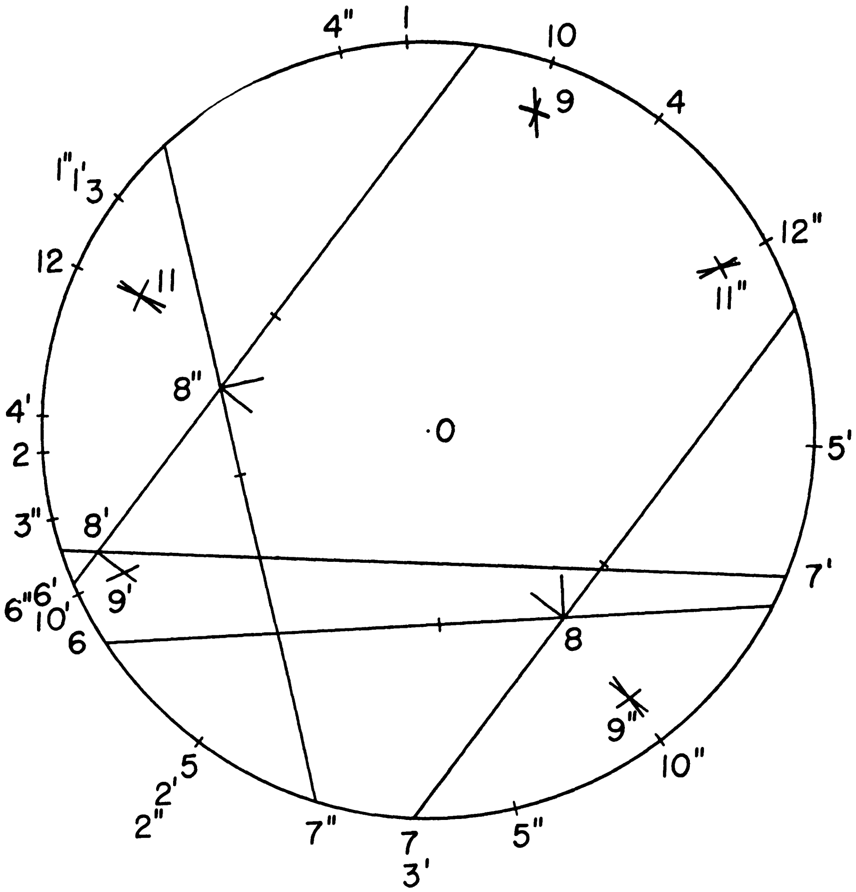


FIGURE 4

	Given	Find
(i)	$(\text{lat, dec, LHA})_{PST} = (d_1, d_2, D)$	$(\text{alt, az})_{PST} = (p, q)$
(ii)	$(\text{lat, dec, alt})_{STZ} = (p, a_1, a_2)$	$(\text{LHA})_{STZ} = u.$
(iii)	$(\text{lat, dec, LHA})_{SPZ} = (d_1, a_1, q + u)$	$(\text{alt, az})_{SPZ} = (y, v).$

The longitude is given by

$$x = \text{GHA}S + v.$$

The data for constructing Figure 4 and the coördinates determined for Z are shown in the table below.

	PST	STZ	SPZ	
	$+$	$+$	$-$	
	dec	alt	GHA	D
S	40°	30°	236°	
T	−30°	20°	348°	112°
Z	30°		310°	

The notation in Figure 4 is the same as that in Figure 1, the two extra stages being distinguished by accents. Portions of some of the lines have been omitted for convenience. The order in which the points are obtained is as follows:

1	2	3	4	5	6	10	9	8	7	11	12
1′	2′	3′	4′	5′	6′	7′	8′	9′	10′		
1″	2″	3″	4″	5″	6″	10″	9″	8″	7″	11″	12″

The position of the unknown point Z is (50°E, 30°N).

References

1. J. D. H. Donnay, Spherical Trigonometry after the Cesàro method, New York, 1945.
2. J. F. Dowsett, Advanced Constructive Geometry, London, 1927.
3. B. Dutton, Navigation and Nautical Astronomy, 7th edition, United States Naval Institute, Annapolis, Md., 1942.

ON THE THEOREM OF FRULLANI

F. G. TRICOMI, University of Turin and California Institute of Technology

1. Introduction. Among the papers left by the late Professor Harry Bateman, there is much material on definite integrals which is now being used for the construction of a table of integrals.* In this material, a little known theorem of an Italian mathematician of the first half of the nineteenth century, the theorem of Frullani [1] plays quite an extensive role. This theorem makes possible in many cases the evaluation of an improper integral of the type

$$\int_0^\infty [f(ax) - f(bx)]\frac{dx}{x} \tag{1}$$

or of integrals of more general types, even when the two integrals of which (1) is the difference, do not exist separately.

* This work is sponsored by the Office of Naval Research.

Frullani's formula,

$$\int_0^{\infty} [f(ax) - f(bx)]\frac{dx}{x} = f(0)\log\frac{b}{a}, \tag{2}$$

and the more general result,

$$\int_0^{\infty} [f(ax) - f(bx)]\frac{dx}{x} = [f(0) - f(\infty)]\log\frac{b}{a}, \tag{3}$$

have been investigated recently [2] with a view of obtaining the least restrictive conditions on f which will ensure the validity of the previous results.

The purpose of this paper is something else. Precisely, I generalize the formulas (2) and (3) in two directions, first, by using two, in some degree arbitrary, functions $p_1(x)$ and $p_2(x)$ in place of the linear functions ax and bx, and, second, by allowing the constants a and b (which are real in Frullani's work) to assume complex values. The list of references collected by Bateman was very helpful to me.

2. The generalization. Let us consider the integral,

$$I_{\delta\omega} = \int_{\delta}^{\omega}\left\{f[p_1(x)]\frac{p_1'(x)}{p_1(x)} - f[p_2(x)]\frac{p_2'(x)}{p_2(x)}\right\}dx \qquad (0 < \delta < \omega), \tag{4}$$

where for the sake of simplicity we shall impose the unnecessarily restrictive conditions that $f(x)$ is continuous, and that $p_1(x)$ and $p_2(x)$ are positive and continuously differentiable, for $x > 0$.

With $p_1(x) = y_1$, $p_2(x) = y_2$,

$$I_{\delta\omega} = \int_{p_1(\delta)}^{p_1(\omega)} f(y_1)\frac{dy_1}{y_1} - \int_{p_2(\delta)}^{p_2(\omega)} f(y_2)\frac{dy_2}{y_2} = \int_{p_1(\delta)}^{p_2(\delta)} - \int_{p_1(\omega)}^{p_2(\omega)} f(y)\frac{dy}{y} \tag{5}$$

and hence

$$\int_0^{\infty}\left\{f[p_1(x)]\frac{p_1'(x)}{p_1(x)} - f[p_2(x)]\frac{p_2'(x)}{p_2(x)}\right\}dx = \lim_{\delta\to 0,\,\omega\to\infty} I_{\delta\omega} = L_0 - L_{\infty}, \tag{6}$$

where

$$L_0 = \lim_{\delta\to 0}\int_{p_1(\delta)}^{p_2(\delta)} f(y)\frac{dy}{y}, \qquad L_{\infty} = \lim_{\omega\to\infty}\int_{p_1(\omega)}^{p_2(\omega)} f(y)\frac{dy}{y}. \tag{7}$$

Thus we see that the integral (6) can be evaluated whenever we are able to evaluate both limits (7).

Now, by the first mean value theorem of integral calculus,

$$\int_{p_1(\xi)}^{p_2(\xi)} f(y)\frac{dy}{y} = f(p)\log\frac{p_2(\xi)}{p_1(\xi)}$$

where p is a suitable value between $p_1(\xi)$ and $p_2(\xi)$, and so whenever

$$\lim_{x\to+0} f(x) = f(0), \qquad \lim_{x\to+0} \frac{p_2(x)}{p_1(x)} = \lambda_0 \neq 0; \tag{8}$$
$$\lim_{x\to+\infty} f(x) = f(\infty), \qquad \lim_{x\to+\infty} \frac{p_2(x)}{p_1(x)} = \lambda_1 \neq 0$$

exist, the two limits (7) and (8) also exist, and

$$L_0 = f(0)\log\lambda_0, \qquad L_\infty = f(\infty)\log\lambda_1.$$

Hence in this case

$$\int_0^\infty \left\{ f[p_1(x)]\frac{p_1'(x)}{p_1(x)} - f[p_2(x)]\frac{p_2'(x)}{p_2(x)} \right\} dx = f(0)\log\lambda_0 - f(\infty)\log\lambda_1. \tag{9}$$

If $p_1(x) = ax$, $p_2(x) = bx$ and $a, b > 0$, this reduces to (3). A formula equivalent to the particular case $p_1(x) = x$ of (9) was given by Lerch [**3**].

If only the first two limits in (8) exist, and in addition

$$\lim_{x\to+\infty} p_1(x) = \lim_{x\to+\infty} p_2(x) = +\infty \tag{10}$$

and the infinite integral

$$\int_h^\infty f(y)\frac{dy}{y}, \qquad (h > 0), \tag{11}$$

exists, then clearly $L_\infty = 0$, and we have

$$\int_0^\infty \left\{ f[p_1(x)]\frac{p_1'(x)}{p_1(x)} - f[p_2(x)]\frac{p_2'(x)}{p_2(x)} \right\} dx = f(0)\log\lambda_0, \tag{12}$$

which reduces to (2) when $p_1(x) = ax$, $p_2(x) = bx$ and $a, b > 0$.

It is worth noting that there is a relation between the present matter and the theory of the Cauchy principal value of an integral. In (5), we change y_2 into $-y_2$, and define $f(y)$ for negative y by the relation $f(-y) = f(y)$. Then

$$I_{0\omega} = \lim_{\delta\to 0}\int_{-p_2(\omega)}^{-p_2(\delta)} f(y)\frac{dy}{y} + \int_{p_1(\delta)}^{p_1(\omega)} f(y)\frac{dy}{y}$$

so that under the assumptions

$$\lim_{x\to+0} p_1(x) = \lim_{x\to+0} p_2(x) = 0, \quad \lim_{x\to+0}\frac{p_2(x)}{p_1(x)} = \lambda_0 \neq 0, \quad \lim_{x\to+0} f(x) = f(0) \tag{13}$$

it follows that

$$I_{0\omega} = \int_{-p_2(\omega)}^{*\, p_1(\omega)} f(y) \frac{dy}{y} + f(0) \log \lambda_0, \tag{14}$$

where the asterisk denotes the principal value of the integral. A similar formula is valid for $I_{\delta\infty}$ and, consequently, even for $I_{\delta\omega}$.

3. The case of a periodic function. Another case of interest arises when only the first two limits in (8) exist, and instead of the convergence of (11) we assume that $f(y)$ is an integrable periodic function with period p.

As is well known, in this case the corresponding indefinite integral

$$F(x) = \int_0^x f(y)dy$$

is the sum of a (continuous) periodic function $\phi(x)$ and the linear function Mx, where M is the mean value of $f(x)$ over the period, *i.e.*,

$$M = \frac{1}{p}\int_0^p f(y)dy.$$

Next, if we consider another function

$$\Phi(x) = \int_h^x f(y)\frac{dy}{y}, \qquad (h > 0),$$

we have, by integration by parts,

$$\begin{aligned}\Phi(x) &= \left[\frac{1}{y}F(y)\right]_h^x + \int_h^x F(y)\frac{dy}{y^2} \\ &= \left[M + \frac{1}{y}\phi(y)\right]_h^x + M\int_h^x y\frac{dy}{y^2} + \int_h^x \phi(y)\frac{dy}{y^2} \\ &= \frac{1}{x}\phi(x) - \frac{1}{h}\phi(h) + M\log x - M\log h \\ &\quad + \int_h^\infty \phi(y)\frac{dy}{y^2} - \int_x^\infty \phi(y)\frac{dy}{y^2}\cdot\end{aligned}$$

Consequently, if the constant H is defined as

$$H = -\frac{1}{h}\phi(h) - M\log h + \int_h^\infty \phi(y)\frac{dy}{y^2}$$

and we observe that, by means of the substitution $y = xt$,

$$\int_x^\infty \phi(y)\frac{dy}{y^2} = \frac{1}{x}\int_1^\infty \phi(xt)\frac{dt}{t^2} = O\left(\frac{1}{x}\right), \tag{15}$$

we may write

$$\Phi(x) = M \log x + H + O(1/x).$$

In view of this formula (which is interesting in itself) we have

$$\int_{p_1(\omega)}^{p_2(\omega)} f(y) \frac{dy}{y} = \Phi[p_2(\omega)] - \Phi[p_1(\omega)]$$
$$= M \log \frac{p_2(\omega)}{p_1(\omega)} + O\left(\frac{1}{p_1(\omega)}\right) + O\left(\frac{1}{p_2(\omega)}\right).$$

Therefore, if in addition to conditions (10) we suppose that

$$\lim_{x\to+\infty} \frac{p_2(x)}{p_1(x)} = \lambda_1 \neq 0, \tag{16}$$

we can write the desired formula

$$\int_0^\infty \left\{ f[p_1(x)] \frac{p_1'(x)}{p_1(x)} - f[p_2(x)] \frac{p_2'(x)}{p_2(x)} \right\} dx = f(0) \log \lambda_0 - M \log \lambda_1. \tag{17}$$

In the particular case $p_1(x)=ax$, $p_2(x)=bx$, a formula similar to (17) was found by Schlömilch [**4**] but Schlömilch's method (expansion in a Fourier series) introduces unnecessary restrictions on the function f.

With regard to generality, it should be noted further that in the preceding proof the only essential property of $\phi(x)$ is that the integral (15) vanishes as $x\to+\infty$. Consequently the proof is not only valid for periodic functions but for all integrable functions f for which a constant M exists such that we can write

$$F(x) = \int_0^x f(y)dy = Mx + o(x),$$

i.e., for all the functions f with a finite mean value on the whole positive half axis:

$$\lim_{x\to+\infty} \frac{1}{x}\int_0^x f(y)dy = M.$$

4. Extension into the complex field. For sake of simplicity, in this last section we shall limit ourselves to the case $p_1(x)=ax$, $p_2(x)=bx$. It is well known, that by means of the substitutions,

$$x = e^\xi, \qquad a = e^\alpha, \qquad b = e^\beta, \qquad f(e^\xi) = F(\xi), \tag{18}$$

the equation (3) can be written in the form,

$$\int_{-\infty}^{\infty} [F(\xi+\alpha) - F(\xi+\beta)]d\xi = (\beta-\alpha)[F(-\infty) - F(\infty)], \tag{19}$$

where $F(-\infty)$ and $F(\infty)$ represent the limit values of $F(\xi)$ as $\xi\to\mp\infty$, and α

and β are two arbitrary (positive, negative or zero) real constants.

In this form, if F is an analytic function of ξ, Frullani's theorem can easily be extended to the complex field, *i.e.*, to the case of two arbitrary complex numbers α and β, where it will be assumed that $\Im\alpha \leqq \Im\beta$. Moreover we shall suppose: (i) that in the strip $\Im\alpha \leqq \Im\xi \leqq \Im\beta$ there lies a finite number of singular points of $F(\xi)$, and none are on the boundary of the strip; (ii) that, as $\Re\xi \to \pm\infty$, $F(\xi)$ approaches the limit values $F(\infty)$ and $F(-\infty)$, respectively, uniformly in the strip.

Under this hypothesis we have

$$(20) \qquad \int_{-\infty}^{\infty} [F(\xi + \alpha) - F(\xi + \beta)]d\xi = (\beta - \alpha)[F(-\infty) - F(\infty)] + 2\pi i S,$$

where the path of integration is the real axis and S is the sum of the residues of F in the strip (without considering the point at infinity).

This formula is easily demonstrated by applying the theorem of residues to the integral of $F(\xi)$ taken around the parallelogram with vertices $\alpha-\omega_1$, $\alpha+\omega_2$, $\beta+\omega_2$, $\beta-\omega_1$, (ω_1, ω_2 positive) and passing to the limit as ω_1 and ω_2 approach infinity. This demonstration was given by G. H. Hardy and H. W. Carjel in a solution to a problem proposed by E. E. Elliot in the *Educational Times* [5].

However, it does not seem to have been observed that even the Cauchy-Elliot equation (3) can be extended to the complex field by a transformation of equation (20) by means of the inverses of the substitutions (18). But here we must bear in mind the fact that these inverse substitutions are generally multiple-valued. They become single-valued, however, if we assume that the imaginary parts of ξ, α, β are in $(-\pi, \pi)$, and this does not impose restrictions on the three complex numbers x, a, b, other than that they are different from zero. Moreover, it should be noted that a line parallel to the real axis in the ξ-plane corresponds to a half-line issuing from the origin in the x-plane.

Thus we obtain the formula,

$$(21) \qquad \int_{0}^{\infty} [f(ax) - f(bx)]\frac{dx}{x} = (\log b - \log a)[f(0) - f(\infty)] + 2\pi i S^*,$$

where:

(i) the imaginary parts of both logarithms lie in the range $(-\pi, \pi)$;

(ii) if γ is the sector $\arg a \leqq \arg x \leqq \arg b$, then S^* represents the sum of the residues of $f(x)$ inside γ, each divided by the affix of the corresponding singular point;

(iii) $f(0)$ and $f(\infty)$ represent respectively the limit values of $f(x)$ as $|x| \to 0$ and $|x| \to \infty$ in γ with the condition that $f(x)$ approaches these limits uniformly with respect to $\arg x$.

For the proof, it is sufficient to observe that if the analytic function $F(\xi)$ has a singular point with residue R at $\xi = \xi_0$, the function $f(x) = F(\log x)$ has a

singular point with residue $x_0 R$ at $x=x_0=e^{\xi_0}$. In fact, we have

$$\frac{R}{\xi-\xi_0}=\frac{R}{\log x-\log x_0}=\frac{R}{\log\left(1+\dfrac{x-x_0}{x_0}\right)}=\frac{R}{\dfrac{x-x_0}{x_0}-\dfrac{1}{2}\left(\dfrac{x-x_0}{x_0}\right)^2+\cdots}$$

$$=\frac{Rx_0}{x-x_0}+\cdots.$$

Finally, it should be noted that if some simple poles of $f(x)$ lie on the boundary of γ, formula (21) remains valid providing the integral on the left is interpreted as a Cauchy principal integral and the residues at the corresponding poles are halved in the sum S^*.

For example, in the case

$$f(x)=e^{-x},\qquad a=a_1+ia_2,\qquad b=b_1+ib_2,\qquad (a_1>0,\ b_1>0)$$

we have

$$f(0)=1,\qquad f(\infty)=0,\qquad S^*=0,$$

and (21) gives us, by separating the real and imaginary parts, the interesting formulas,

$$\text{(22')}\qquad \int_0^\infty \left[e^{-a_1x}\cos(a_2x)-e^{-b_1x}\cos(b_2x)\right]\frac{dx}{x}=\frac{1}{2}\log\frac{b_1^2+b_2^2}{a_1^2+a_2^2},$$

$$\text{(22'')}\qquad \int_0^\infty \left[e^{-a_1x}\sin(a_2x)-e^{-b_2x}\sin(b_2x)\right]\frac{dx}{x}=\arctan\frac{a_2}{a_1}-\arctan\frac{b_2}{b_1},$$

where both arctangents are taken between $-\pi/2$ and $\pi/2$ (since the real parts of a and b are positive). This is a generalization of a known result of Poisson [6].

References

1. Mem. Società Ital. d. Scienze, vol. 20, 1828, p. 448. See also T. J. Bromwich, Infinite Series (London, Macmillan & Co., 1908) p. 432.

2. Iyengar, J. Indian Math. Soc., vol. 4, 1940, 145–150; Proc. Cambridge Phil. Soc., vol. 37, 1941, 9–18. More complete results in this direction were obtained by Professor A. Ostrowski of Basel who reported on his (as yet unpublished) work to the Unione Matem. Italiana (Pisa, September 1948) and to the University of California (Los Angeles, May, 1949).

3. M. Lerch, Abhand. Akad. d. Wiss. Prag vol. 1^2, 1891 pp. 123–131. See also G. H. Hardy, Messenger of Math. (2) vol. 34, 1905 pp., 11–18 and 102.

4. O. Schlömilch, Archiv. Math. u. Physik (Grunert) (1) 5 (1866) pp. 152–155.

5. Educational Times v. 72 (1900) pp. 89–90, Question 14164.

6. S. D. Poisson, Jour. École Polyt. vol. 9, 1813, p. 215 (see pp. 220–221). The formula of Poisson is (22'') when $b_2=0$.

MATHEMATICS FOR THE MILLION, OR FOR THE FEW?*

WILLIAM BETZ, Rochester, New York

It is now thirteen years since the appearance of a much-debated volume entitled "Mathematics for the Million." That title was decidedly misleading. In reality, the book presupposed a mathematical maturity ordinarily achieved only by "the few." However, it served to focus attention on the real question at issue; namely, what kind of mathematical education should the schools provide not merely for a gifted minority, but also for the average citizen?

Ever since the organization of the International Commission on the Teaching of Mathematics, at the Fourth International Congress of Mathematicians, in Rome, Italy, in 1908, published reports on the mathematical curricula of the participating nations have gradually become available. In the United States, summaries of these documents have appeared in such publications as the National Report of 1923 and the Fourth Yearbook of the National Council of Teachers of Mathematics. They made it quite clear that in the majority of the reporting nations the schools provided one type of education for the masses and a radically different kind of education for the "classes." That kind of program is possible only in a dual system of education consisting of elementary schools for the many and secondary schools for the few.

However, in the United States that arrangement has long been considered undemocratic. Instead, the prevailing policy is that of admitting "all the children of all the people" to every kind of public school. Accordingly, our schools have become a sort of educational laboratory on a huge scale. And it may well be that both our successes and our failures in the field of mass education hold valuable lessons for the other nations. In the light of this background, let us briefly examine what has been done here, or is being proposed, by way of answering the question embodied in the title of this paper.

Three principal approaches have been tried or suggested in American secondary schools in their attempts to provide functioning types of mathematical training for all American youth.

The first is anchored on the popular doctrine that "life situations" should be the primary basis of all curricula. But since mathematics is a system of ideas and processes, whereas life situations are incurably unsystematic, this approach has failed completely wherever it has been tried. It has always resulted merely in a sort of chaotic "mathematics without mathematics" and it ignored fundamental aspects of the problem we are considering.

The second approach hopes to find dependable answers in the recommendations of authoritative committees and in the techniques of curriculum workshops and laboratories. However, the thousands of mathematical syllabi now crowding the shelves of our curriculum morgues have merely dramatized a hopeless confusion of objectives. And even the reports of national committees

* A report submitted to Section VII of the International Congress of Mathematicians, at Harvard University, September 1, 1950.

have regularly been attacked by leading educators as mere reflections of unsupported private opinion.

There remains a third approach, often explored partially, but never with anything like scientific completeness or thoroughness. It is that of making a really dependable, full-length study of the role of mathematics in the modern world, from both a practical and a cultural standpoint.

To accomplish that, we should have to look carefully at many aspects of the question. How and where do we need mathematics in our homes, our vocations and professions, as wage earners, as citizens, as readers, as students, and so on? We should have to find out how mathematics is related to the industrial, commercial, and engineering enterprises of the nation. And in what way do mathematical modes of thinking, attitudes, and types of appreciation permeate the very warp and woof of modern civilization? To obtain the information we need, it would be necessary to analyze the work of many groups of people, such as businessmen, builders, contractors, mechanics, machinists, designers, draftsmen, economists, scientists, statisticians, accountants, navigators, and pilots. When such studies become available, we are sure to discover the utterly untenable character of the idea that only "the few" need genuine mathematical training. On the contrary, it will be found that mathematics is a subject of global significance, and that a citizen of the modern world simply cannot afford to be ignorant of a basic knowledge of mathematics, which must surely involve a little more than grocery store arithmetic and a few rules of mensuration.

It is believed that only on the basis of such objective findings will it be possible to obtain a convincing and enduring foundation for the mathematical curricula of the modern school. It will then be our major task to build a two-track program, or even a three-track program, for the divergent needs of those seeking an immediate life preparation and of those who aim at foundational studies in our colleges and universities.

This brief report is submitted in the ardent hope that it may enlist the active concern of another International Congress of Mathematicians pertaining to this "unfinished business," namely, our common problem of providing more adequate curricula both for the many and the few. Is it too much to expect that a work which was begun so auspiciously in 1908, at the instance of our own David Eugene Smith, be resumed and eventually brought to more complete consummation? It is therefore urged that Section VII of this Congress transmit to the International Mathematical Union an expression of its profound interest in this problem, together with the definite suggestion that proper steps be taken forthwith to initiate its solution.

AN ACKNOWLEDGMENT

Theorem 1 of my paper *Some applications of matrices in the theory of equations* (this MONTHLY 57 (1950), 156) is equivalent to an earlier result of W. V. Parker and M. M. Flood (this MONTHLY 42 (1935), 164 and 43 (1936), 562).

C. C. MACDUFFEE

ART AND MATHEMATICS: A BRIEF GUIDE TO SOURCE MATERIALS

W. L. SCHAAF, Brooklyn College

Sometime ago the writer asked a person well-known in the field of modern art for some material which would show the relation of geometry to art. Somewhat to his amazement came the crisp reply that there is no relation between art and geometry. Upon delving into the literature, it became apparent that there are two schools of thought about the matter, and that considerable expository material, in scattered form and not always readily accessible, nevertheless does exist.

Not only for the immediate question of the relation between art and mathematics, but also for the education of teachers of mathematics, the matter would seem to deserve consideration. It is generally conceded that the well-qualified teacher must be competent in each of three areas: (1) he must understand the ways of youth, and the nature of learning; (2) he must be well versed in mathematics; and (3) he must have attained sound bearings in some of the other domains of human achievement—at least in the physical sciences, the social disciplines, the crafts and technologies, and the fine arts. If this thesis is accepted, then clearly what is demanded is not merely a professionally trained person, but a truly cultured individual.

To be sure, much of a teacher's experience and attainment in this third category are secured in the course of time, as his interest, opportunities and resources are likely to dictate. Frequently, however, the resources are not as convenient as could be desired; the material is not always at hand, or it is scattered, or adequate guides to source materials are lacking. This is perhaps more true of material dealing with mathematics and art than of material dealing with mathematics and science, or mathematics and technology. Hence it is to be hoped that the following list of reference materials on art and mathematics, despite its incompleteness and other shortcomings, may be helpful to mathematics teachers in service, to prospective teachers of mathematics (and of art), to students of the subject, and to lay readers as well.

1. GENERAL

Ahern, Lorella. Art in geometry. *Mathematics Teacher*, 1939, 32: 156–162.

Birkhoff, G. D. Aesthetic Measure. Cambridge, Harvard University Press, 1933; 226 pp.

Polygonal forms; ornaments and tilings; vases; harmony and melody in music; aesthetic theories.

Birkhoff, G. D. Polygonal Forms. *National Council of Teachers of Mathematics*, Sixth Yearbook, 1931, pp. 165–195.

Bragdon, Claude. Art and arithmetic. *Theatre Arts Magazine*, 1924, 8: 505–514.

Colman, Samuel, and Coan, C. A. Proportional Form. New York, G. P. Putnam's Sons, 1920; 265 pp.

Mathematical forms in Nature, in the human figure, in art and in architecture.

Court, N. A. Mathematics and esthetics. *School Science and Mathematics*, 1930, 30: 31–32.

Craven, T. J. Art and relativity. *Dial*, 1921, 70: 535–539.

Davis, R. C. An evaluation and test of Birkhoff's aesthetic measure formula. *Journal of General Psychology*, 1936, 15: 231–240.

Dubisch, Roy. A mathematical approach to aesthetics. *School Science and Mathematics*, 1941, 41: 718–723.

Emch, Arnold. Mathematical principles of aesthetic forms. *Monist*, 1900, 11: 50–64.

Evans, Griffith. Form and appreciation. *National Council of Teachers of Mathematics, Eleventh Yearbook*, 1936, pp. 245–258.

Ghyka, Matila C. Essai sur le Rythme. Paris, Gallimard, 1938; 186 pp.

Ghyka, Matila C. Esthêtique des Proportions dans la Nature et dans les Arts. Paris, Gallimard, 1927; 452 pp.

An unusual collection of material dealing with the relationship of form in Nature and in art to mathematics.

Ghyka, Matila C. The Geometry of Art and Life. New York, Sheed & Ward, 1946; 174 pp.

A stimulating and very readable presentation.

Ivins, W. M. Art and Geometry. Cambridge, Harvard University Press, 1946; 135 pp.

Somewhat philosophical discussion of Greek art, space perception, perspective, and the history of geometric ideas and the development of art; a scholarly and suggestive exposition.

Levy, Hyman. Modern Science. New York, Alfred Knopf, 1939.

Chapt. 29, Geometrical Thinking and Feeling, discusses briefly geometric, dynamic and artistic forms.

Lietzmann, Walther. Lustiges und Merkwürdiges von Zahlen und Formen. Breslau, F. Hirt, 4th edit., 1930; 307 pp.

Discussion of parquet designs, pp. 239–244; geometry and painting, pp. 292–298.

Lietzmann, Walther. Mathematik und bildende Kunst. Breslau, F. Hirt, 1931; 150 pp.

Chapters on architecture, ornament in utensils, perspective, bas relief, stereographs, the human figure, and the composition of paintings; excellent illustrations; well worth perusing, even if unfamiliar with German.

Lindberg, Viola G. Mathematics and art. *Bulletin, Kansas Assoc. of Teachers of Mathematics*, 1937, 11: 10–11.

Mathematics—or Modern Art? *Science Illustrated*, January, 1948, pp. 8–10.

Interesting photographs; no text.

Moritz, R. E. On the beauty of geometrical forms. *Scripta Mathematica*, 1936, 4: 25–36.

Ogden, R. M. Naive geometry in the psychology of art. *American Journal of Psychology*, 1937, 49: 198–216.

Raymond, G. L. The Genesis of Art Form. New York, G. P. Putnam's Sons, 1908.

Read, Herbert. The Anatomy of Art. New York, Dodd, Mead & Co., 1932.

Richter, Irma A. Rhythmic Form in Art. London, John Lane, 1932; 127 pp.

Exposition of compositions based on principles of perspective and dynamic symmetry as exemplified in the works of the great masters.

Schillinger, Joseph. The Mathematical Basis of the Arts. New York, Philosophical Library, 1948; 696 pp.

A ponderous tome, presenting a unique point of view; discusses continuity, periodicity, permutations, involution, ratio, rotation, symmetry and other concepts as applied to design, music, and other arts.

Smith, D. E. Esthetics and mathematics. *Mathematics Teacher*, 1927, 20: 419–428.

Somervell, Edith L. A Rhythmic Approach to Mathematics. London, George Philip & Sons, 1906; 67 pp.

An unusual little pamphlet describing designs and patterns based upon the parabola, the circle, the curve of pursuit, and the spiral of Archimedes. Original edition no longer available, but reprints are obtainable from Miss L. E. Christman, 1217 Elmdale Avenue, Chicago, Ill.

Weideman, C. C., and Giles, M. A. Integration of geometric form and art ideas. *Proceedings, National Education Association*, 1937, pp. 207 ff.

Winsey, A. R., and Sands, L. B. Contributions of mathematics in the development of art. *School Science and Mathematics*, 1942, 42: 845–852.

Wolff, Georg. Mathematics and the arts. *National Council of Teachers of Mathematics, Eleventh Yearbook*, 1936, pp. 226–244.

Wolff, Georg. Mathematik und Malerei. Leipzig, Teubner, 1925; 85 pp.

Perspective in painting; applications of mathematics to the composition of paintings and portraits; good illustrations; brief bibliography.

Wulff-Parchim, L. Dürer als Mathematiker. 1928.

2. ORNAMENT AND DESIGN

Barvalle, H. v. Dynamic beauty of geometrical forms. Scripta Mathematica, 1946, 12: 294–297.

Four plates, designs based on the cardioid; no text.

Barvalle, H. v. Geometry at the Junior High School Grades and the Waldorf School Plan. Garden City, N. Y., Adelphi College, 1948; 32 pp.

Contains many suggestive ideas relating to geometry and design.

Barvalle, H. v. Transformation of curves by inversion. Scripta Mathematica, 1948, 14: 113–125; 266–272.

Baravalle, H. v. The number *e*—the base of natural logarithms. *Mathematics*

Teacher, 1945, 38: 350–355.

Interesting designs based on logarithmic spirals.

Barlow, Bronson. Mathematics of Design; for Use in Arts, Letters and Science. Chicago, Ill., The Author, 1929. (Out of print.)

Blake, E. M. Method for the creation of geometric designs based on structure. *Journal American Ceramic Society*, 1945, 28: 156–167.

Boyd, Rutherford. Mathematical ideas in design. *Scripta Mathematica*, 1948, 14: 128–135.

Boyd, Rutherford. Mathematical Themes in Ornament. Files of Scripta Mathematica.

Scores of miscellaneous plates; no text, but many of the subjects are unusually interesting.

Bragdon, Claude. Ornament from magic squares. *Architectural Record*, 1926, 60: 506–516.

Bragdon, Claude. More ornament from magic squares. *Architectural Record*, 1927, 62: 473–480.

Bragdon, Claude. Ornament from the Platonic solids. *Architectural Record*, 1928, 63: 504–512.

Bragdon, Claude. Projective Ornament. Rochester, N. Y., 1915.

Carroll, John. Pattern Drawing and Design; the application of geometrical drawing to the construction of ornament and the planning of patterns. London, Burns & Oates, Ltd., 1902; 112 pp.

Cartlidge, S. J. Original Design: Geometric Pattern for Beginners. New York, E. P. Dutton & Co., 1923.

Collatz, L. Gleichungen geometricher Ornamente. *Zeitschrift für mathematischen und naturwissenschaftlichen Unterricht*, 1933, 64: 165–169.

Crane, Walter. The Bases of Design. London, G. Bell & Sons, 1902; 1925.

Crane, Walter. Line and Form. London, G. Bell & Sons, 1901; Second edit., 1914; 287 pp.

Diefenbach, L. Geometrische Ornamentik. Berlin, Max Spielmeyer, n.d.

A collection of geometric designs.

El-Milick, M. Eléments d'Algèbre Ornementale. Paris, Dunod, 1936.

A unique volume; numerous ornamental curves representing specific algebraic equations; see files of Scripta Mathematica for typical examples.

Ewer, A. B. Geometric Design. Denver, Smith-Brooks Co., 1935; 28 pp.

Pamphlet of designs, with text; out of print; possibly obtainable from the author, through the publisher.

Hagan, N. Designing from geometric forms. *School Arts Magazine*, 1927, 27: 209–211.

Havkin, Nathan Ben-Zion. Ornamography; the principles of geometrical ornament and its use in decorative art. Jerusalem, R. Mass, 1945; 32 p.+192 plates.

Hornung, C. P. Handbook of Designs and Devices. Revised edition. New York, Dover Publications, 1946; 218 pp.

Excellent collection of over 1800 geometric elements; bibliography.

Hurst, C. Geometric designs on Mimbres bowls. *American Antiquarian*, 1940, 6: 107–114.

Jenks, A. E. Geometric designs on Mimbres bowls. *Art and Archaeology*, 1932, 33: 137–139+.

Koller, E. L. Space Filling; prepared especially for home study. Scranton, Pa., 1936; 52 pp.

An exercise book in decorative design.

Mathematical Designs. *News Week*, 1937, v. 9, p. 31.

Mechanical Toy which Draws Geometrical Designs. *Scientific American*, 1907, v. 96, p. 496.

Meyer, F. S. Handbook of Ornament—Geometrical and Floral. Pelham, N. Y., Bridgman, Publishers, 1928; 64 pp.

McGiffert, James. The economy of symmetry. *National Mathematics Magazine*, 1938, 12: 210–212.

Müller, Edith. Gruppentheoretische und strukturanalytische Untersuchungen der maurischen Ornamente aus der Alhambra in Granada. Rüschlikon, Buchdruckerei Baublatt A.-G., 1944; 128 pp.

Perry, L. D., and Buck, R. O. Compass curve in design. *Industrial Education Magazine*, 1928, 29: 319–321.

Schudeisky, Albrecht. Geometrisches Zeichnen. Leipzig, Teubner, 1919; 99 pp.

Over 200 illustrations of decorative designs and geometrical drawings.

Tennyson, J. Anna. Window transparencies. *National Council of Teachers of Mathematics, Eighteenth Yearbook*, 1945, pp. 86–87.

Tessin, L. D. Developing the geometric design. *School Arts Magazine*, 1926, 25: 432–435.

Wornum, R. N. Analysis of Ornament. London, 1896.

Young, J. W. Symmetry. *National Council of Teachers of Mathematics, Fifth Yearbook*, 1930, pp. 145–148.

3. GEOMETRIC PATTERNS; REPEATING DESIGNS

Ahrens, Walther. Mathematische Unterhaltungen und Spiele. Leipzig, Teubner, 1921; 2 vols.

Discussion of space-filling regular polygons, Vol. 1, Chapt. 5, p. 129 ff.

Ballard, P. B., and Hamilton, E. R. Fundamental Geometry Tiles. London, Hodder and Stoughton, 1941.

Bradley, A. D. The Geometry of Repeating Design and Geometry of Design for High Schools. New York, Teachers College, Columbia University, Contribution to Education No. 549, 1933; 131 pp.

A comprehensive and scholarly monograph; extensive bibliography; excellent source book.

Dawson, T. R. Ornamental squares and triangles. *Mathematical Gazette*, 1946, 30: 19–21.

Day, L. F. Pattern Design. London, B. T. Botsford; New York, Scribner's Sons, 1903.

Analysis of certain geometric patterns.

Fauré, P. Blanc et Noir: La Décoration Géométrique. Paris, Andre, Daly fils et Cie., n.d.

Designs based upon the subdivision of squares; designs based upon concentric circles.

Günther, S. Vermischte Untersuchungen zur Geschichte der mathematischen Wissenschaften. Leipzig, Teubner, 1876.

Historical account of the development of star polygons and star polyhedrons, Chapt. I, pp. 1–92.

Jones, Owen. Designs for Mosaic and Tessalated Pavements. London, 1842.

Jones, Owen. The Grammar of Ornament. London, Bernard Quaritch, 1856; 1868; reprinted, 1910.

Larkin, N. J. An Essay on a Species of Mosaic Pavement Formed of Right-angled Triangles of Different Colours. London, J. Taylor, 1818.

MacMahon, P. A. New Mathematical Pastimes. Cambridge University Press, 1921; 116 pp.

Unusual patterns formed from polygons, usually triangles and squares, by the ingenious use of colored compartments, or by various transformations.

MacMahon, P. A. Pythagoras's theorem as a repeating pattern. *Nature*, 1922, 109: 479; 579.

Morgan, Thomas. Romano-British Mosaic Pavements. London, 1886.

Pelikan, A. G. Space-filling element in design. *Industrial Arts Magazine*, 1926, 15: 92–93.

Richmond, C. A. Repeating designs in surfaces of negative curvature. *American Mathematical Monthly*, 1937, 44: 33–35.

Schorer. Tapetenmuster und der Satz des Pythagoras. *Zeitschrift für mathematischen und naturwissenschaftlichen Unterricht*, 1929, 60: 434–439.

Shaw, Henry, Specimens of Tile Pavements. London, 1858.

Shorter, S. A. The mathematical theory of the sateen arrangement. *Mathematical Gazette*, 1920–21, v. 10, pp. 92–97.

A sateen arrangement is a "certain regular method of arranging points on squared paper, each point being placed at the center of a square."

Speiser, Andreas. Theorie der Gruppen von endlicher Ordnung. Berlin, J. Springer, 1927.

Material on the geometry of ornament and design; bibliography.

Sykes, Mabel. A Source Book of Problems for Geometry, based upon Industrial

Design and Architectural Ornament. Boston, Allyn & Bacon, 1912.

Pp. 1–149 treat of tile designs; parquet floor designs; repeating patterns of all sorts; designs based on star polygons, rosettes, regular polygons, etc.; bibliography; excellent source book.

White, Gleeson. Practical Designing. London, 1894.

Chapters on designs for tiled floors, linoleums, and print fabrics.

Wyatt, M. D. Specimens of Geometric Mosaics of the Middle Ages. London, 1848; 26 pp.

4. PERSPECTIVE; PAINTING; GEOMETRIC DRAWING

Baravalle, H. v. Geometric drawing. *National Council of Teachers of Mathematics, Eighteenth Yearbook*, 1945, pp. 64–81.

Bartel, K. Malerische Perspektive. Grundsätze. Geschichtlicher Überblick, Ästhetik, v. I. (Edited by W. Haack.) Leipzig, 1934.

Becker, Hugo. Geometrisches Zeichnen . . . neubearbeitet von Prof. J. Vonderlinn. Berlin, Leipzig, W. de Gruyter & Co., 1920, 1923; 136 pp.

Cornelius, H. Elementargesetze der bildenden Kunst. Leipzig, Teubner, 1921.

Doehlemann, K. Grundzüge der Perspektive nebst Anwendungen; 3 Auflage. Leipzig, Teubner, 1928.

Gardner, Helen. Understanding the Arts. New York, Harcourt Brace, 1932.

The division of areas in paintings is discussed on pp. 214–232.

Jackson, A. W. Art via geometry. *Pencil Points*, 1935, v. 16, supp., p. 13.

Lubschez, B. J. Perspective. New York, Van Nostrand, 1927.

Morris, I. H. Geometrical Drawing for Art Students. New York, Longmans Green & Co., 1926, 1941.

Njorling, E. R. Perspective Made Easy. New York, Macmillan, 1939; 203 pp.

Pressland, A. J. Geometrical Drawing. London, 1892.

Schudeisky, A. Geometrisches Zeichnen. Leipzig, 1919.

Spanton, J. H. Geometrical Drawing and Design. London, Macmillan, 1930; 262 pp.

Spanton, J. H. Science and Art Drawing; Complete Geometrical Course. London, 1895.

Timerding, H. E. Zeichnerische Geometrie. Leipzig, Akademische Verlagsgesellschaft, 1928.

Wolff, Georg. Linearzeichenunterricht und Kunsterziehung. *Zeitschrift für Mathematischen und Naturwissenschaftlichen Unterricht*, 1916, 47: 253 ff.

Wolff, Georg. Neue Perspektiven für die Geschichte der Perpsektive. *Zeitschrift für Mathematischen und Naturwissenschaftlichen Unterricht*, 1915, 46: 263 ff.

5. ARCHITECTURE

Bond, Francis. Gothic Architecture in England. London, B. T. Botsford, 1905.

Boyd, Rutherford. The controls over design. *Architecture*, June 1935, pp. 311–318.

Bragdon, Claude. The Beautiful Necessity. New York, Alfred Knopf, 1922; 111 pp.

Bragdon, Claude. The Frozen Fountain: being Essays on Architecture and the Art of Design in Space. New York, Alfred Knopf, 1932; 125 pp.

Craver, Mary. The mathematical foundations of architecture. *Mathematics Teacher*, 1939, 32: 147–155.

Durach, F. Mittelalteriche Bauhütten und Geometrie. Stuttgart o.J., Julius Hoffmann, 1929.

Geilen, V. Mathematik und Baukunst als Grundlagen abendländischer Kultur. Wiedergeburt der Mathematik aus dem Geiste Kants. Braunschweig, 1921.

Hewlett, R. S. Significance of Architectural Composition in the Teaching of Secondary Mathematics. Unpublished master's thesis, Teachers College, Columbia University, 1914.

Lund, F. M. Ad Quadratum; a Study of the Geometric Bases of Classic and Medieval Religious Architecture. 2 vol.; 1921.

Micheli, Geneviève L. Le decor géométrique dans la sculpture de l'Aisne et de l'Oise au XI^e siècle. Paris, J. Haumont, 1939; 112 p.

Phillips, E. C. Some applications of mathematics to architecture: Gothic tracery curves. *American Mathematical Monthly*, 1926, 33: 361–368.

Salit, Charles. The geometric form as an architectural matrix. *Mathematics Teacher*, 1946, 39: 113–116.

Sharpe, Edmund. Decorated Windows: A Series of Illustrations. London, 1849.

Shedd, C. R. Mathematical arches. *Scripta Mathematica*, 1945, 11: 362–363.

Sleight, Norma. Arches through the ages. *School Science and Mathematics*, 1945, 45: 21–25.

Smith, D. E. Mathematica Gothica. Paris, Presses Universitaires de France, 1925; 30 pp.

Sykes, Mabel. Source Book of Problems for Geometry. Boston, Allyn and Bacon, 1912.

Although long since out of print, this well-known book contains many examples of Gothic arches and Gothic traceries.

6. DYNAMIC SYMMETRY

Armfield, M. Dynamic symmetry and its practical value. *International Studio*, 1921, v. 74, supp., pp. 76–85.

Bowes, Julian. Dynamic symmetry. *Scripta Mathematica*, 1933, 1: 236–244; 309–314.

Bragdon, Claude. Dissertation on dynamic symmetry. *Architectural Record*, 1924, 56: 305–315.

Bragdon, Claude. Old Lamps for New. New York, Alfred Knopf, 1925.

Chapter 9: observations on dynamic symmetry.

Bragdon, Claude. Regulating lines. *Architecture*, 1931, 64: 329–334.

Bush, F. Anent pictorialism and dynamic symmetry. *American Photography*, 1934, 28: 414–420.

Caskey, L. D. Geometry of Greek Vases; Attic vases in the Museum of Fine Arts analyzed according to the principles of proportion discovered by Jay Hambidge. Boston, Museum of Fine Arts, 1922; 235 pp.

Dynamic Symmetry. *Scientific American Monthly*, 1921, 4: 23–28.

Frank, H. Dynamic nonsense. *American Photography*, 1935, 29: 742–746.

Gardner, R. W. Standardized proportion of the Greek vase and ornament. *Journal of the American Ceramic Society*, 1926, 9: 426–436.

Gugle, Marie. Dynamic symmetry. *Journal of the American Ceramic Society*, 1926, 9: 157–161.

Gugle, Marie. Dynamic symmetry. *National Council of Teachers of Mathematics Third Yearbook*, 1928, pp. 57–64.

Hambidge, Jay. Dynamic Symmetry: The Greek Vase. Yale University Press, 1920; 161 pp.

Hambidge, Jay. Dynamic Symmetry in Composition: As Used by the Artists. Yale University Press, 1923, 1948; 88 pp.

Hambidge, Jay. Dynamic symmetry: the Greek vase. *American Journal of Archaeology*, 1921, 25: 18–36.

Hambidge, Jay. The Elements of Dynamic Symmetry. New York, Brentanos', 1926; Yale University Press, 1948; 133 pp.

Hambidge, Jay. Greek design. *Journal of American Institute of Architects*, 1920, 8: 288–295.

Hambidge, Jay. The Parthenon and other Greek Temples and their Dynamic Symmetry. Yale University Press, 1924.

Hambidge, Jay. Practical Applications of Dynamic Symmetry. Yale University Press, 1932.

Hambidge, Jay. Symmetry and proportion in Greek art. *American Architecture*, 1919, 116: 597–605.

Hoadley, E. C. Dynamic symmetry and art education. *School Arts Magazine*, 1925, 24: 304–309.

Hogan, G. F. Dynamic symmetry for photographers. *American Photography*, 1935, 29: 487–480.

Kimball, R. Dynamic symmetry in design. *Industrial Arts Magazine*. 1924, 13: 222–224; 423.

Kimball, R. Dynamic symmetry in nature. *Industrial Arts Magazine*, 1925, 14: 68–69.

Southwick, A. A. Dynamic symmetry. *American Architecture*, 1922, 121: 54–57.

Swartout, E. Greek proportions theoretically and otherwise. *American Architecture*, 1921, 120: 379–383.

Van Dyck, A. Dynamic symmetry in radio design. *Proceedings, Institute of Radio Engineers*, 1932, 20: 1481–1511.

Warington, A., and Welo, S. Dynamic symmetry in advertising layout. *Inland Printer*, 1937, 98: 69–72.

Warington, A., and Welo, S. Uses of dynamic symmetry. *Inland Printer*, 1936, 97: 52–54.

7. THE GOLDEN SECTION

Barvalle, H. v. The geometry of the pentagon and the Golden Section. *Mathematics Teacher*, 1948, 41: 22–31.

Baravalle, H. v. Die Geometrie des Pentagrammes und der goldene Schnitt. Stuttgart, Waldorf Verlag, 1932.

Cavallero, Vincenzo. Nuove ricerche sulla genesi della sezione aurea. *Bollettino di Matematica*, 1926, No. 2–3.

Engelhardt, Rudolph. Der goldene Schnitt im Buchgewerbe; eine buchgewerbliche Harmonielehre für Buchdrucker und Buchgewerbler. Leipzig, J. Mäser, 1922; 282 pp.

Golden section discussed from the printer's viewpoint; over 330 figures, many plates.

Funck-Hellet, Ch. Les oeuvres peintes de la renaissance italienne et le nombre d'or. Paris, Librairie Le François, 1932; 55 pp.

Gaudet, C. Golden measure and Greek art. *Spectator*, 1920, 124: 235–236.

Ghyka, Matila C. Le nombre d'or; rites et rythmes pythagoriciens dans le développement de la civilisation occidentale. Paris, Gallimard, 1931.

A sequel to "Esthétique des Proportions dans la Nature et dans les Arts." Discussion of number symbolism; Pythagorean philosophy of numbers; rhythm in art; the Golden Section.

Golden mean in art and nature. *Literary Digest*, 1928, 98: 22–23.

Göringer, A. Der goldene Schnitt (göttliche Proportion) und seine Beziehung zum menschlichen Körper, usw. München, 1893.

Graesser, R. F. The golden section. *The Pentagon*, 1943–44, v. 3, pp. 7–19.

Hagge, K. Besondere Dreiecke, die mit dem goldenen Schnitt in Beziehung stehen. *Zeitschrift für Mathematischen und Naturwissenschaftlichen Unterricht*, 1910, 41: 21 ff.

Hagge, K. Zum goldenen Schnitt. *Zeitschrift für Mathematischen und Naturwissenschaftlichen Unterricht*, 1905, 36: 498 ff.

Hagen, F. W. Der goldene Schnitt in seiner Anwendung. Leipzig, Engelmann, 1857.

Kalbe, O. Der goldene Schnitt. Hannover, Ost, 1890.

Matthias, J. Die Regel vom goldenen Schnitt im Kunstgewerbe. Leipzig, Hässel, 1886.

Niebel, W. Beitrag zur ästhetischen Bedeutung des goldenen Schnittes. *Zeitschrift für Mathematischen und Naturwissenschaftlichen Unterricht*, 1929, 60: 249 ff.

Pfeifer, F. X. Der goldene Schnitt und dessen Erscheinungsform in Mathematik, Naturwissenschaft und Kunst. Augsburg, Huttler, 1885.

Timerding, H. E. Der goldene Schnitt. Leipzig, Berlin, Teubner, 1919, 1929; 57 pp.

Wittstein, Theodor. Der goldene Schnitt. Hannover, Hahn, 1874.
Zeising, A. Der goldene Schnitt. (Posth.) Leipzig, Engelmann, 1884.
Zeising, A. Neue Lehren von den Proportionen des menschlichen Körpers aus einem bisher unbekannt Gebliebenen und die ganze Natur und Kunst durchdringenden Grundgesetze. Leipzig, 1854.

MATHEMATICAL NOTES

EDITED BY E. F. BECKENBACH, *University of California, Los Angeles*

Material for this department should be sent to F. A. Ficken, Institute of Mathematics and Mechanics, New York University, 45 Fourth Ave., New York 3, New York.

ON THE QUADRATIC RECIPROCITY LAW

B. W. BREWER, Oregon State College

1. Introduction. Kronecker [2] obtained the quadratic reciprocity law for rational primes p, q by considering the decomposition of the prime p into prime ideal factors in the quadratic extension $R(\eta)$ of the rational number field R generated by a root η of the period equation $x^2+x+[1-(-1|q)q]/4=0$. However, Kummer [see **5**, p. 78] had shown that the decomposition of p in $R(\eta)$ parallels in every respect the decomposition of $x^2+x+[1-(-1|q)q]/4$ modulo p; and Gauss [**1**], Lebesgue [**3**], and Pellet [**4**] obtained the reciprocity law by using the theory of higher congruences to discuss directly the decomposition of $x^2+x+[1-(-1|q)q]/4$ modulo p. The theory of Galois (finite) fields as distilled from the older theory of higher congruences is discussed in many recent texts on modern algebra and number theory; and though various well-known proofs of the reciprocity law, including that of Kronecker mentioned above, appear in these texts, none are based on this theory. A proof based on Galois field theory is both simple and direct, and we give such a proof here, borrowing a little from each of the above mentioned proofs of Gauss, Lebesgue, and Pellet.

2. Proof. Let p and q be distinct primes, q odd and $q=4k\pm1$. If p belongs to the exponent e modulo q, the Galois field $GF[p^e]$ contains a primitive qth root of unity ρ since the multiplication group of $GF[p^e]$ is cyclic of order p^e-1. The $(q-1)/2$-nomial periods of ρ are $\eta=\sum_a\rho^a$, $\eta'=\sum_b\rho^b$, where a ranges over the quadratic residues and b ranges over the quadratic non-residues modulo q.

Now η and η' are roots of a quadratic equation in $GF[p]$; to obtain the coefficients of this period equation, we need to find $\eta+\eta'$ and $\eta\eta'$. The roots of $x^q-1=0$ are 1 and the primitive qth roots of unity. Hence $\eta+\eta'=-1$. To find $\eta\eta'=\sum_{a,b}\rho^{a+b}$, we need to determine the number of distinct representations (mod q) of each of the residues $r=0, 1, \cdots, q-1$ in the form $a+b$. We note that the number of distinct representations of a given non-zero residue r in

the form $a+b$ is equal to the number of distinct representations of 1 in this form. The residue 0 admits of no representation in the form $a+b$ if $(-1|q)=1$, and exactly $(q-1)/2$ distinct representations if $(-1|q)=-1$.

Hence if $(-1|q)=1$, the number of distinct representations of each non-zero residue r in the form $a+b$ is exactly $(q-1)/4$ and we have

$$\eta\eta' = \frac{q-1}{4}\cdot \sum_{(r\neq 0)} {}_r\, \rho^r = \frac{1-q}{4}.$$

But if $(-1|q)=-1$, the number of distinct representations of each non-zero residue r in the form $a+b$ is exactly $(q-3)/4$ and we have

$$\eta\eta' = \frac{q-1}{2} + \frac{q-3}{4}\cdot \sum_{(r\neq 0)} {}_r\, \rho^r = \frac{1+q}{4}.$$

Therefore the quadratic equation in $GF[p]$ having η and η' as roots is $x^2+x-(-1|q)k=0$.

Incidental to the above argument is the fact that $(-1|q)=1$ implies $q\equiv 1 \pmod 4$, and $(-1|q)=-1$ implies $q\equiv 3 \pmod 4$. Hence this gives

$$(-1 \mid q) = (-1)^{(q-1)/2},$$

the first supplement to the reciprocity law.

We observe that $\eta\neq\eta'$, since $\eta=\eta'$ with $\eta+\eta'=-1$ and $\eta\eta'=-(-1|q)k$ imply $4(-1|q)k=-1$ in $GF[p]$, and this in turn implies $q\equiv 0$ modulo p, a contradiction.

If now $(p|q)=1$, then

$$\eta^p = \left[\sum_a \rho^a\right]^p = \sum_a \rho^{pa} = \sum_a \rho^a = \eta,$$

and η is an element of $GF[p]$ since the roots of $x^p-x=0$ are the p elements of $GF[p]$. But if $(p|q)=-1$, then

$$\eta^p = \left[\sum_a \rho^a\right]^p = \sum_a \rho^{pa} = \sum_b \rho^b = \eta',$$

and η is not an element of $GF[p]$ since $\eta\neq\eta'$ and hence not a root of $x^p-x=0$. Thus $x^2+x-(-1|q)k$ splits into linear factors in $GF[p]$ if and only if $(p|q)=1$.

On the other hand, the discriminant of $x^2+x-(-1|q)k$ is $q(-1|q)$, and hence if p is odd, $x^2+x-(-1|q)k$ splits into linear factors in $GF[p]$ if and only if $(q(-1|q)|p)=1$. This with the above result gives

$$(p \mid q) = (q(-1)^{(q-1)/2} \mid p),$$

the Gaussian form of the reciprocity law.

Finally, if $p=2$, $x^2+x-(-1|q)k$ splits into linear factors in $GF[p]$ if and only if $k\equiv 0 \pmod 2$, and hence if and only if q is of the form $8n\pm 1$. But q is

of the form $8n \pm 1$ if and only if $(-1)^{(q^2-1)/8} = 1$, and this with the above result gives

$$(2 \mid q) = (-1)^{(q^2-1)/8},$$

the second supplement to the reciprocity law.

References

1. K. F. Gauss, Werke, vol. 2, pp. 233–235.
2. L. Kronecker, Über die Potenzreste gewisser complexer Zahlen, Berlin Akad. Monatsber., 1880, pp. 404–407.
3. V. A. Lebesgue, Note sur les congruences, Comptes Rendus, vol. 51 (1860), pp. 9–13.
4. A. E. Pellet, Sur les fonctions réduites suivant un module premier, Bulletin de la Société Mathématique de France, vol. 17 (1889), pp. 156–167.
5. H. Weyl, Algebraic Theory of Numbers, Princeton, 1940.

A SYMBOLIC FORM OF THE CLASSICAL COMPLEX INVERSION FORMULA FOR A LAPLACE TRANSFORM

D. V. Widder, Harvard University

In this journal [1] the author showed that an inversion formula, due to E. Phragmén, for the Laplace transform

$$F(s) = \int_0^\infty e^{-st}\Phi(t)dt \tag{1}$$

is symbolically equivalent to $\{1/\Gamma(D)\}\ F(e^{-x})$, where D stands for differentiation with respect to x and $\Gamma(s)$ is the Gamma-function. In an earlier note [2] the same result was established for the Post-Widder inversion of (1). However, it seems not to have been pointed out that this same symbolic operator $1/\Gamma(D)$ also yields the classical [3] complex inversion

$$\Phi(t) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{st}F(s)ds, \qquad 0 < t < \infty. \tag{2}$$

Here the path of integration is the vertical line $\sigma = c$ in the plane of the complex variable $s = \sigma + i\tau$. The constant c must be greater than σ_c, the abscissa of convergence for the integral (1); and the Cauchy principal value for the integral (2) may be needed.

For completeness let us sketch heuristically how one may arrive at the above operator. We remind the reader that $e^{aD}f(x)$ is interpreted as $f(x+a)$ in view of the MacLaurin expansion

$$e^{aD} = \sum_{n=0}^{\infty} \frac{a^n D^n}{n!},$$

$$e^{aD}f(x) = \sum_{n=0}^{\infty} \frac{a^n}{n!} f^{(n)}(x) = f(x + a). \tag{3}$$

Now if we make an exponential change of variable, equation (1) becomes

$$(4) \qquad f(x) = \int_{-\infty}^{\infty} G(x-t)\phi(t)dt = \int_{-\infty}^{\infty} G(t)\phi(x-t)dt,$$

where

$$G(x) = e^{-e^{-x}}, \qquad f(x) = F(e^{-x}), \qquad \phi(x) = \Phi(e^{x})e^{x}.$$

From the definition of the Gamma-function we have

$$\Gamma(D) = \int_0^{\infty} t^{D-1}e^{-t}dt = \int_{-\infty}^{\infty} e^{-tD}G(t)dt.$$

By use of (3) and (4) we consequently obtain

$$\Gamma(D)\phi(x) = \int_{-\infty}^{\infty} \phi(x-t)G(t)dt = f(x),$$

and a formal solution of this equation yields the desired symbolic inversion formula

$$(5) \qquad \phi(x) = \frac{1}{\Gamma(D)} f(x).$$

The appropriate representation of $1/\Gamma(s)$ for the present purposes is one due to Laplace [**4**],

$$\frac{1}{\Gamma(s)} = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{z}z^{-s}dz, \qquad c > 0,\ \sigma > 0.$$

Replacing s by D and again using (3) we obtain

$$\frac{1}{\Gamma(D)} f(x) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{z}f(x - \log z)dz.$$

In terms of the original functions equation (5) becomes

$$\Phi(e^{x})e^{x} = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{z}F(ze^{-x})dz,$$

or, if $e^{x} = t$,

$$\Phi(t)t = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{z}F\left(\frac{z}{t}\right)dz \qquad \frac{c}{t} > \sigma_c.$$

This equation is equivalent to (2) if $z = ts$. The path of integration becomes $\sigma = tc$, but by Cauchy's theorem a single vertical line may be used for all t.

References

1. D. V. Widder, A symbolic form of an inversion formula for a Laplace transform, this Monthly, vol. 55 (1948), pp. 489–491.

2. D. V. Widder, The inversion of a generalized Laplace transform, Proceedings of the National Academy of Sciences, vol. 33 (1947), pp. 295–297.
3. D. V. Widder, The Laplace Transform, Princeton, N. J., 1941, p. 66.
4. E. T. Copson, Theory of Functions of a Complex Variable, Oxford, 1935, p. 231.

A SIMPLE EVALUATION OF AN IMPROPER INTEGRAL

WACLAW KOZAKIEWICZ, University of Montreal.

The purpose of this note is to give a simple proof of the relation

$$\int_0^{+\infty} \frac{\sin x}{x}\, dx = \frac{\pi}{2}. \tag{1}$$

Our proof of (1) is based on the following theorem due to Riemann: *If $f(x)$ is Riemann integrable in the interval $a \leqq x \leqq b$, then*

$$\lim_{k\to\infty} \int_a^b f(x) \sin kx dx = 0. \tag{2}$$

To proceed with our proof, we may notice that

$$\int_0^{\pi} \frac{\sin (n + \frac{1}{2})x}{2 \sin \frac{x}{2}}\, dx = \frac{\pi}{2}, \qquad (n = 0, 1, 2, \cdots). \tag{3}$$

The relation (3) is obtained by integrating both sides of the identity

$$\tfrac{1}{2} + \cos x + \cos 2x + \cdots + \cos nx = \frac{\sin (n + \frac{1}{2})x}{2 \sin \frac{x}{2}}$$

from 0 to π.

Let

$$\phi(x) = \frac{1}{x} - \frac{1}{2 \sin \frac{x}{2}} = \frac{2 \sin \frac{x}{2} - x}{2x \sin \frac{x}{2}}, \qquad 0 < x \leqq \pi. \tag{4}$$

The fünction $\phi(x)$ is continuous for $0<x\leqq\pi$. By applying twice L'Hospital's rule to (4), we see that

$$\lim_{x\to 0} \phi(x) = 0.$$

Consequently, if we put $\phi(0)=0$, $\phi(x)$ will be continuous for $0\leqq x\leqq\pi$. Therefore $\phi(x)$ certainly satisfies the condition of the Riemann theorem; then the rela-

tion (2), with $k=n+\frac{1}{2}$, gives

$$\lim_{n\to\infty}\int_0^{\pi}\left\{\frac{1}{x}-\frac{1}{2\sin\frac{x}{2}}\right\}\sin\left(n+\tfrac{1}{2}\right)x\,dx = 0.$$

Taking into account (3), we see that

$$\lim_{n\to\infty}\int_0^{\pi}\frac{\sin\left(n+\frac{1}{2}\right)x}{x}\,dx = \frac{\pi}{2},$$

or, making the substitution $u=(n+\frac{1}{2})x$,

$$\lim_{n\to\infty}\int_0^{(n+1/2)\pi}\frac{\sin u}{u}\,du = \frac{\pi}{2}\cdot \tag{5}$$

Since

$$\int_0^{+\infty}\frac{\sin u}{u}\,du$$

is convergent, (5) yields the relation (1).

CLASSROOM NOTES

EDITED BY C. B. ALLENDOERFER, Haverford College

All material for this department should be sent to C. B. Allendoerfer, Haverford College, Haverford, Pa.

FUNDAMENTALS IN THE TEACHING OF UNDERGRADUATE MATHEMATICS*

MOSES RICHARDSON, Brooklyn College

1. Introduction. What I have to say today is neither new nor startling. It has all been said before many times both by others and by myself.** I undertook to say it again because, judging by my own behavior in the classroom, frequent repetitious prodding is needed to keep one on the best path whenever the best path does not coincide with the easiest. Teaching mathematics well at the undergraduate level is not an easy task. I hasten to add that what I mean by teaching "well" may not be what others mean by it; and I do not insist that those who disagree with me are wrong by definition. In fact, I believe that diversity in points of view is healthy and should be encouraged by allowing individual instructors maximum freedom rather than impose a stifling uniformity.

---

* Presented before Section 7 of the International Congress of Mathematicians at Cambridge, Mass., September 1, 1950.

** Cf. bibliography at the end of the paper.

In passing, let me mention one troublesome feature of undergraduate teaching which can be largely overcome, namely the ever present necessity for compromise due to the fact that it is frequently impossible, at that level, to tell the whole truth. A good cure for this is to try to tell nothing but the truth, even at the cost of restricting generality rather than to use faulty arguments to bolster general truths. If a theorem has to be used which is too hard to prove, then one should assume it or give an informal heuristic discussion rather than pass off an incorrect proof as correct.

In general, *I deplore the tendency to claim that mathematics stimulates clear thinking and then to teach it as meaningless memorized routine techniques having little to do with thought.* This tendency is unfortunately aided and abetted by many textbooks. It is admittedly difficult to compel oneself to think carefully in an elementary class which one teaches perennially, or when writing an elementary textbook. For example, how many textbooks on analytic geometry even mention the fact that a converse proposition is necessary in establishing an equation of a locus such as an ellipse† or hyperbola? Converses are traditionally stressed in secondary school classes in plane geometry and frequently forgotten thereafter. Or, how many calculus texts refrain from stating without qualification the half-truth that

$$\int \frac{dx}{x} = \ln x + C? \tag{1}$$

This is not, to my mind, mere pedantry. Evil consequences may attend any lie, no matter how small. For example, (1) causes the loss of a whole family of curves in one of the simplest examples in a first course in differential equations. From the family of curves $xy=c(c\neq 0)$, we get the differential equation $dx/x+dy/y=0$. Integrating, (1) yields $\ln x+\ln y=\ln k(k>0)$, or $xy=k(k>0)$. Where have the curves $xy=k(k<0)$ gone? The correct formula‡

$$\int \frac{dx}{x} = \ln |x| + C \tag{2}$$

yields, of course, all the curves $|xy|=k$ or $xy=\pm k(k>0)$. Few students, in my experience, can locate the lost curves because the incomplete formula (1) is so strongly entrenched in our textbooks.

Time does not permit further examples. The point I wish to make is that *many ills can be traced to excessive formalism in instruction.*

2. Proper emphasis in the teaching of undergraduate mathematics. Every course in mathematics touches to some extent each of the following three

† For a correct treatment, see, for example, Osgood and Graustein, Plane and Solid Analytic Geometry, Macmillan 1930.

‡ Cf. R. Courant, Differential and Integral Calculus, vol. 1 (Nordemann 1937), translated by E. J. McShane, for the derivation.

aspects of the subject:

T. Techniques;
A. Applied problems;
F. Fundamentals.

Under techniques I include the formal algorithmic aspects of mathematics. Under fundamentals I include fundamental concepts, careful reasoning including reasonable justification of the techniques, logical structure, historical evolution of ideas, the relevance of mathematics to other subjects and its cultural significance including justification of its position at the base of the Tree of Knowledge.*

In the practice of teaching, T is too often overstressed to the detriment of A, which runs a poor second, and the virtual exclusion of F which receives little but lip-service. In my opinion, these three aspects should be equally stressed for science or mathematics majors, while for majors in other divisions the order should be exactly reversed.

In recent years, progress has been made in creating terminal courses for non-science and non-mathematics majors which stress F and A while limiting T to bare essentials. These relatively new courses differ in details but share, I think, the principal general objectives.

However, today I wish to concern myself with science and mathematics majors, including prospective teachers of mathematics at secondary school and junior college levels. For this class of students, the problem of successfully teaching F without diminishing the desired attention to T and A remains a difficult but important one. Too many in this class emerge from undergraduate instruction equipped with a collection of heterogeneous techniques, a smattering of applications, and little or no grasp of fundamentals. One consequence of this lack is the student's inability to teach or defend his subject intelligently. Truly, mathematics is a study often pursued but seldom overtaken. I am convinced that many intelligent colleagues in other departments who wish to remove mathematics from the list of prescribed courses have misunderstood the value of the subject because it is so often presented as senseless memorized drill rather than with emphasis on its reasonableness and relevance to the real world. They have themselves been exposed to coaching courses for stereotyped examinations rather than to genuine courses in mathematical thinking. Needless to say, unrationalized teaching of formal techniques is also a source of many technical errors, such as the "Universal Cancellation Law" and the "Universal Distributive Law." The first of these says that a symbol may be crossed out in both places if it appears twice in the same expression, as in, $\sin x/\sin y = x/y$, $\sin x = \sin y$ implies $x=y$, etc. Familiar examples of the latter are $\log (x+y) = \log x + \log y$, $\sin (x+y) = \sin x + \sin y$, $\sqrt{x+y} = \sqrt{x} + \sqrt{y}$, $D(f\cdot g) = Df \cdot Dg$, etc.

* This justification is intended here not merely in the sense of mathematics as the science of space and quantity but also in the sense of mathematics as the totality of hypothetico-deductive sciences and their concrete interpretations.

3. Reasons why fundamentals are understressed. The lack of stress on fundamentals is to my mind the outstanding weakness of much contemporary teaching of undergraduates both in mathematics and in natural science. The liberating effect of scientific rationalism which was felt so strongly in preceding centuries seems now to be lost in the authoritarian dicta of the instructor and the undigested, regurgitated rote responses of the student. To what causes can we attribute this sorry state of affairs?

(a) A few teachers seem to feel that fundamentals should not be taught to most undergraduates. They act as if they believed that it is some how improper for young students to think but entirely proper for them to learn tricks of calculation blindly. Those who hold this extreme view are possibly authoritarians by nature and their number is, fortunately, not great.

(b) Some teachers believe that fundamentals should be taught to undergraduates but that it cannot be done successfully. These are inclined to render superficial lip-service to fundamentals and hurry on to formal techniques with which the student is apt to have easier success.

(c) It is admittedly more difficult to teach F than either T or A. F demands of the teacher resources of patience, eloquence, and energy far greater than those needed for T or A. Being a drill-master and stressing memorized mastery of routine algorithms makes the teacher's life easier but harder to justify.

(d) It is difficult to test and, especially, to grade results of time spent on F.

(e) T and A leave little time free for F. In fact, neglect of F is often due to the teacher's anxiety to "cover" ground in T and A. However, ground can be "covered" as with an opaque blanket as a result of this anxiety. I am convinced that even a routine technique will be remembered longer and applied better if it is understood than if it is merely memorized.

4. Suggested remedies. I have no panacea for these conditions. However, the following suggestions are offered as partial remedies.

(a) Adequate time should be allowed in every course for discussion of fundamentals. The student will learn best whatever the teacher stresses. If the teacher indicates by his attitude that he doesn't expect the student to learn something or that he doesn't consider it important, the average student can scarcely be expected to grasp it.

(b) Effort should be made to include some questions on fundamentals on examinations. It is hardly necessary to labor this obvious point.

(c) Supplementary reading is a device which is not sufficiently used in mathematics courses although it is traditional in other departments. I have personally known able students who never looked at a book other than the textbook for the course they were taking. On the other hand, I have received gratifying expressions of thanks from others for guiding them to some of the good literature of mathematics of whose existence they had been unaware. In fact, several mathematics majors have told me that they derived as seniors considerable enjoyment and enlightenment from some of the books written for

non-science freshmen. This phenomenon is in itself evidence supporting my contention that fundamentals are understressed.

(d) Special courses in fundamentals at an elementary level may be of value especially for prospective teachers of secondary and junior college mathematics who are not ordinarily exposed to extensive graduate study in mathematics. I would consider such a course of greater value to this class of students than one more technical course in advanced mathematics.

5. Conclusion. *The main need, as I see it, is for teachers to be so convinced of the value of fundamentals that they will push themselves along that path even though it is not the path of least effort.*

Mathematics is more than a collection of tricks for calculating and solving problems. Its ideas constitute a main stream in the development of human thought. It is characterized not by sly devices which mysteriously seem to work but rather by persistent and complete honesty in its explicit recognition of underlying assumptions and in its adherence to logical proof of its theorems and precision in its use of terms. How few students come away from their undergraduate studies with a firm grasp of these ideals! Only the more gifted students will develop really independent insight and creative imagination; but the average student *can* understand in large measure the standards of critical thinking. The realization that theories depend on their underlying hypotheses and that various assumptions may be equally tenable is in itself a long step toward maturity and tolerance. I would measure the real success of undergraduate teaching of mathematics at least as much by the extent to which these fundamental ideas are instilled in the student as by the efficiency with which he can perform routine techniques such as Horner's method, complicated formal integrations and the like.

Bibliography

1. C. B. Allendoerfer, Mathematics for liberal arts students, this MONTHLY 54 (1947) pp. 573–578.
2. E. P. Northrop, Mathematics in a liberal education, this MONTHLY 52 (1945) pp. 132–135.
3. F. S. Nowlan, Objectives in the teaching of college mathematics, this MONTHLY 57 (1950) pp. 73–82.
4. O. Ore, Mathematics for students of the humanities, this MONTHLY 51 (1944) pp. 453–458.
5. M. Richardson, On the teaching of elementary mathematics, this MONTHLY 49 (1942) pp. 498–505.
6. M. Richardson, The place of mathematics in a liberal education, Nat. Math. Magazine 19 (1945) pp. 349–358.
7. 15th Yearbook of the National Council of Teachers of Mathematics, Bureau of Publications, Teachers' College, Columbia University 1940.

REMARKS ON NATURAL NUMBERS

M. O. GONZALEZ, University of Havana and J. D. MANCILL, University of Alabama

The authors' somewhat unorthodox yet rational approach to the system of natural numbers (see, *On the System of Natural Numbers*, this MONTHLY, vol. 57, 1950, pp. 104–112) has aroused certain criticisms (see, Rosser, *Problems on*

given to decide for any two distinct elements of the set which is the preceding. However, it is probably not sufficiently clear from our statement of Postulate 4 that it is to apply only to ordered sets as previously discussed and it would be better to have the fourth property of linear order explicitly stated in it.

Definition 5 should read: "Elements A and B, A preceding B, of an ordered set are said . . . ."

In the first sentence of Section 11 the product ab is defined with the stipulation $b>1$. Thus $a\cdot 0$ and $a\cdot 1$ require further definitions. We considered the sum of sets as a set operation or function on two or more arguments. This should have been pointed out in Section 10. It is true that multiplication of natural numbers can be defined in terms of sets independently of addition in such a way as to cover all cases if the use of the null set is allowed. In this case it is not necessary to decide whether a sum can have fewer than two summands. However, we decided against this procedure in our elementary treatment.

In the proof of the principle of complete induction on pages 107 and 108, the element K has an immediate predecessor J since by hypothesis K cannot be the first element of the set. This is clearly implied by the phraseology used in the proof.

The theorem on top of page 108 follows from the induction on the elements of a set Y whose elements are the subsets $X', \cdots, X'', X''', \cdots, X$ of the original set (for examples of similar proofs, see, Hobson, Loc. cit., pp. 4–8). Since this proof involves a particularly defined notation, perhaps the following indirect proof is more cogent. Let X be a given finite set, not a unit set. Let x_1 denote its first element (Definition 6) and remove it. If the *residual* set X_1, which is finite, is not a unit set, let x_2 denote its first element. By removing the element x_2 from X_1, we obtain a finite residual set X_2. Proceeding in this manner, if we do not reach a unit set X_u, we will have found a subset $(x_1, x_2, \cdots)$ of X which has no last term, which contradicts the hypothesis that X is finite. Since a unit set can be made empty by Postulate 2 above, the theorem is proved.

A SIMPLE ENDPOINT MINIMUM

FRANK HAWTHORNE, Hofstra College

C. O. Oakley (this MONTHLY, vol. 54, p. 407) has discussed endpoint maxima and minima with examples. A particularly simple example which may be used to illustrate this point is:

Find that point on the circle $x^2+y^2=1$ which is nearest to (2, 0).

The unsuspecting student sets up this problem using x as independent variable and obtains, for the square of the distance from (x, y) to (2, 0),

$$L^2 = (2-x)^2 + 1 - x^2 = 5 - 4x.$$

Differentiating and equating the result to zero he obtains the disconcerting expression $-4=0$. This certainly does not lead to the obvious solution (1, 0).

If, however, the student uses the central angle as independent variable this difficulty is avoided.

ELEMENTARY PROBLEMS AND SOLUTIONS

EDITED BY HOWARD EVES, Oregon State College

Send all communications concerning Elementary Problems and Solutions to Howard Eves, Mathematics Department, Oregon State College, Corvallis, Oregon. This department welcomes problems believed to be new, and demanding no tools beyond those ordinarily furnished in the first two years of college mathematics. To facilitate their consideration, solutions should be submitted on separate, signed sheets, within three months after publication of problems.

PROBLEMS FOR SOLUTIONS

E 956. *Proposed by H. F. Sandham, Trinity College, Ireland*

Let a_0 and b_0 be positive numbers and define a_{n+1} and b_{n+1} as the arithmetic and harmonic means, respectively, of a_n and b_n. Show that

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n = (a_0b_0)^{1/2}.$$

E 957. *Proposed by C. O. Oakley, Haverford College*

A *palindrome* is an integer that reads the same forwards as backwards as, for example, 2332. By definition the ten digits 0, 1, 2, $\cdots$, 9 will be considered palindromes. How many palindromes are there of at most n digits?

E 958. *Proposed by Herbert Scarf, Temple University*

If $a_n = 1/n!$ show that

$$\begin{vmatrix} a_1 & a_0 & 0 & 0 & \cdots & 0 \\ a_2 & a_1 & a_0 & 0 & \cdots & 0 \\ a_3 & a_2 & a_1 & a_0 & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdots & \cdot \\ a_n & \cdot & \cdot & \cdot & \cdots & a_1 \end{vmatrix} = a_n.$$

E 959. *Proposed by C. S. Ogilvy, Columbia University*

Let C_0 be a square, C_{n+1} the convex polygon obtained from C_n by measuring off one fourth of the length of every side of C_n and cutting off the corners, and $C = \lim_{n\to\infty} C_n$. Prove that C consists of four parabolic arcs.

E 960. *Proposed by Victor Thébault, Tennie, Sarthe, France*

Let AB be a diameter of a circle (O) and let C and D be the midpoints of the radii OA, OB. Let the circles $A(O)$, $B(O)$ cut (O) in A_1, B_1 on the same side of AB. On the other side of AB draw the semicircles $C(O)$ and $D(O)$. Consider the sequence of circles (O_1), (O_2), (O_3), $\cdots$, (O_n), $\cdots$ tangent to the arcs OA_1, OB_1 of $A(O)$ and $B(O)$ and, in turn, to (O), (O_1), (O_2), $\cdots$, (O_{n-1}), $\cdots$. Also consider the sequence of circles (O_1'), (O_2'), (O_3'), $\cdots$, (O_n'), $\cdots$ tangent to the semicircles $C(O)$ and $D(O)$ and, in turn, to (O), (O_1'), (O_2'), $\cdots$, (O_{n-1}'), $\cdots$. Show that the sum of the lengths of the circumferences of all the circles (O_n) and (O_n') is equal to the length of the circumference of (O).

SOLUTIONS

Possible Number of Edges for a Polyhedron

E 923 [1950, 416]. *Proposed by R. C. Buck, University of Wisconsin*

Show that no polyhedron in three-space can have exactly seven edges, while any other integer greater than five is admissible.

Solution by E. P. Starke, Rutgers University. Let $n>3$. A simple polyhedron of $2n$ edges is a pyramid with an n-gon for its base. If the n-gon is folded along a diagonal so that it lies in two planes before its vertices are joined to a point outside the two planes, the resulting polyhedron has $2n+1$ edges.

Each vertex of a polyhedron is also the vertex of a polyhedral angle with at least three edges, and each edge of the polyhedron serves as an edge of two polyhedral angles. A polyhedron with four vertices is a tetrahedron and has six edges. Every other polyhedron has five or more vertices, hence at least (5) (3) edges to its polyhedral angles, and hence at least $(5)(3)/2$ edges. Thus seven edges is impossible.

Also solved by Norman Anning, F. Bagemihl, Alan Berndt, Ray Jurgensen, H. C. Kranzer, Roger Lessard, Norman Miller, L. A. Ringenberg, Harold Shniad, C. W. Trigg, and the proposer.

Most of the above solutions assumed the polyhedron to be simply-connected, and hence subject to the Euler-Descartes formula: $v+f=e+2$. Bagemihl pointed out that this case was solved by Euler in *Elementa doctrinae solidorum*, Novi comm. acad. sc. imp. Petropolitanae 4 (1752), pp. 109–140.

Anning located the problem as No. 7 of Challenge Exercise 63 in H. G. Forder's *School Geometry*, p. 285. On p. 260 of his *Higher Course Geometry*, Forder gives a solution to the problem.

The proposer noted that for the corresponding problem in n-space no value of $e<n(n+2)/2$ is admissible except $n(n+1)/2$.

An Interesting Limit

E 924 [1950, 416]. *Proposed by D. J. Newman, New York University*

Find $\lim_{n\to\infty} n \sin (2\pi en!)$.

I. *Solution by F. Bagemihl and W. Seidel, University of Rochester.* Using the familiar series for e and $\sin x$ and the periodicity of the latter, we have

$$\begin{aligned} n \sin (2\pi en!) &= n \sin \left[2\pi n! \sum_{k=0}^{\infty} 1/k! \right] \\ &= n \sin \left[2\pi \sum_{k=1}^{\infty} 1/(n+1)(n+2)\cdots(n+k) \right] \\ &= 2\pi n/(n+1) + O(1/n). \end{aligned}$$

Hence

$$\lim_{n\to\infty} n \sin (2\pi e n!) = 2\pi.$$

II. *Solution by C. H. Murphy, Chevy Chase, Md., and Eugene Usdin, Purdue University.* As above

$$n \sin (2\pi e n!) = n \sin (2\pi R_n),$$

where

$$R_n = \sum_{k=1}^{\infty} 1/(n+1)(n+2) \cdots (n+k).$$

Now

$$1/(n+1) < R_n < \sum_{k=1}^{\infty} 1/(n+1)^k = 1/n.$$

Therefore

$$\lim_{n\to\infty} R_n = 0 \quad \text{and} \quad \lim_{n\to\infty} nR_n = 1,$$

whence

$$\begin{aligned}\lim_{n\to\infty} n \sin (2\pi e n!) &= \lim_{n\to\infty} n \sin (2\pi R_n) \\ &= \lim_{n\to\infty} 2\pi n R_n [\sin (2\pi R_n)/2\pi R_n] \\ &= 2\pi.\end{aligned}$$

Also solved by P. M. Anselone, S. D. Berkowitz, N. J. Fine, P. G. Kirmser, H. C. Kranzer, Norman Miller, L. A. Ringenberg, J. W. Ross, Harold Shniad, O. E. Stanaitis, R. W. Wild, A. W. Wortham, and the proposer.

M. Golomb noted that this limit also proves that e is irrational. For if e was rational then $n!e$ would be integral for all sufficiently large n, and consequently we would have

$$\lim_{n\to\infty} n \sin (2\pi e n!) = 0.$$

Weighted Dice

E 925 [1950, 416]. *Proposed by J. B. Kelly, University of Wisconsin*

Is it possible so to weight a pair of dice that the probability of occurrence of every sum from 2 to 12 shall be the same?

I. *Solution by Leo Moser and J. H. Wahab, University of North Carolina.* Let p_i and q_i denote the probabilities of an i appearing on the first and second die, respectively. Let s_j denote the probability of the occurrence of the sum j. Now assume $s_2 = s_{12}$. Then

$$p_1q_1 = s_2 = s_{12} = p_6q_6,$$

whence

$$(p_1 - p_6)(q_1 - q_6) \leqq 0,$$

so that

$$s_2 + s_{12} = p_1q_1 + p_6q_6 \leqq p_1q_6 + p_6q_1 \leqq s_7.$$

Thus no loading of the dice can yield equiprobable sums.

II. *Solution by J. V. Finch and P. R. Halmos, University of Chicago.* Define p_i and q_i as above. If $P(x) = \sum_{i=0}^{5} p_{i+1}x^i$ and $Q(x) = \sum_{i=0}^{5} q_{i+1}x^i$, then the requirement that all sums (between 2 and 12) are equally probable is equivalent to the identity $P(x)Q(x) = (1/11)\sum_{i=0}^{10} x^i$. The zeros of the polynomial on the right are all complex (they are, in fact, the complex 11th roots of unity) whereas either of the factors on the left (having real coefficients and odd degree) has at least one real root. It follows that no such factoring is possible and therefore that no loading of the dice can yield equiprobable sums.

Also solved by G. M. Dillon, N. J. Fine, Daniel Finkel, Ray Jurgensen, H. C. Kranzer, Roger Lessard, Bart Park, Harold Shniad, P. B. Wood, A. W. Wortham, and the proposer.

Four Terms in Arithmetic Progression

E 926 [1950, 483]. *Proposed by H. L. Lee, University of Tennessee*

Show that there exist arithmetic progressions with integral terms such that the sum of the squares of four consecutive terms is a 2^kth power of an integer, k a non-negative integer.

I. *Solution by S. T. Thompson, Tacoma, Washington.* We establish a slightly more general result. Let $a-d$, a, $a+d$, $a+2d$ be four consecutive terms of an arithmetic progression. If we take

$$a = 6^{n-1}, \qquad d = 2a, \qquad n > 0,$$

we find that

$$(a - d)^2 + a^2 + (a + d)^2 + (a + 2d)^2 = 36a^2 = 36^n.$$

The given problem is settled, for $k>0$, by taking $n=2^{k-1}$; it is trivial for $k=0$.

II. *Solution by K. Subbarao, M. R. College, Vizianagram, India.* Let $a-3d$ be the first of the four consecutive terms of the arithmetic progression with common difference $2d$. Then

$$(a - 3d)^2 + (a - d)^2 + (a + d)^2 + (a + 3d)^2 = (2a)^2 + 5(2d)^2 = x^2 + 5y^2,$$

where x and y are even. Setting

$$x = p^2 - 5q^2, \qquad y = 2pq, \qquad p,\ q \text{ even},$$

we have

$$x^2 + 5y^2 = (p^2 + 5q^2)^2.$$

Again, setting

$$p = m^2 - 5n^2, \qquad q = 2mn, \qquad m, n \text{ even},$$

we have

$$x^2 + 5y^2 = (p^2 + 5q^2)^4.$$

Repetition of this process leads to the desired result.

Following are some theorems related to the given problem.

1. The Diophantine equation

$$x_1^2 + x_2^2 + \cdots + x_{2p+1}^2 = (2p + 1)z^n,$$

where $x_1, x_2, \cdots, x_{2p+1}$ are in arithmetic progression and $3p(p+1)$ is non-square, has infinitely many solutions.

2. The Diophantine equation

$$x_1^2 + x_2^2 + \cdots + x_{2p}^2 = 2pz^n,$$

where $x_1, x_2, \cdots, x_{2p}$ are in arithmetic progression and $3(4p^2-1)$ is non-square, has infinitely many solutions.

3. The Diophantine equation

$$x_1^2 + x_2^2 + \cdots + x_m^2 = z^{2n+1},$$

where $x_1, x_2, \cdots, x_m$ are in arithmetic progression, has infinitely many solutions.

4. The Diophantine equation

$$x_1^3 + x_2^3 + \cdots + x_m^3 = z^n,$$

where $x_1, x_2, \cdots, x_m$ are in arithmetic progression, has infinitely many solutions for n of the form $3s+1$ or $3s+2$.

Also solved by D. H. Browne, C. A. Johnson and S. J. Pagano (jointly), M. S. Klamkin, Roger Lessard, W. V. Parker, R. E. Wild, and the proposer.

An Unsolved Problem

E 534 [1942, 475, 1943, 261, 1950, 557]. *Proposed by D. H. Browne, Buffalo, N. Y.*

Show that 4, 5, 7 are the only values of n for which $n!+1$ is a perfect square.

Comment by H. D. Grossman, New York, N. Y. The solution that we offered for this problem [1950, 557] contains an invalidating oversight. We cannot conclude, as stated in the third paragraph, that 6, 10, 14, $\cdots$ divide s. Be it resolved that E 534 is still "An Unsolved Problem."

ADVANCED PROBLEMS AND SOLUTIONS

EDITED BY E. P. STARKE, Rutgers University

Send all communications concerning Advanced Problems and Solutions to E. P. Starke, Rutgers University, New Brunswick, New Jersey. All manuscript should be typewritten, with double spacing and margins at least one inch wide. Problems containing results believed to be new or extensions of old results are especially sought. Proposers of problems should also enclose any solutions or information that will assist the editor. In general, problems in well known textbooks or results in readily accessible sources should not be proposed for this department.

PROBLEMS FOR SOLUTION

4428. *Proposed by C. D. Olds, San Jose State College, California*

If p is any prime number >3, prove that

$$_{kp-1}C_{p-1} \equiv {}_{jp-1}C_{p-1} \pmod{p^3},$$

where j, k are any integers and $_nC_r$ is the number of combinations of n different things taken r together.

4429. *Proposed by Ky Fan, University of Notre Dame*

Let H be a non-negative (positive semi-definite) Hermitian transformation in the n-dimensional unitary space. If the eigenvalues $\lambda_i (1 \leqq i \leqq n)$ of H are arranged in ascending order: $0 \leqq \lambda_1 \leqq \lambda_2 \leqq \cdots \leqq \lambda_n$, show that, for any positive integer $k \leqq n$, the product $\lambda_1\lambda_2 \cdots \lambda_k$ of the first k eigenvalues is equal to the minimum of the expression

$$\prod_{i=1}^{k} (Hx_i, x_i),$$

when k orthonormal vectors $x_1, x_2, \cdots, x_k$ vary in space. Here (Hx_i, x_i) denotes the inner product of the vector Hx_i with x_i. (This result is related to the Proposer's paper, *On a theorem of Weyl concerning eigenvalues of linear transformations*, I, Proceedings of the National Academy of Sciences, U. S. A., v. 35 (1949), pp. 652–655.)

4430. *Proposed by Ky Fan, University of Notre Dame.*

Let H', H'' be two n-rowed non-negative Hermitian matrices, and let $H = \alpha H' + \beta H''$, where α, β are two non-negative numbers with $\alpha + \beta = 1$. Let the eigenvalues of H', H'' and H be denoted by λ_i', λ_i'' and $\lambda_i (1 \leqq i \leqq n)$ respectively and so arranged that

$$\lambda_i' \leqq \lambda_{i+1}', \qquad \lambda_i'' \leqq \lambda_{i+1}'', \qquad \lambda_i \leqq \lambda_{i+1}.$$

Prove that for any positive integer $k \leqq n$:

$$(1) \qquad \lambda_1\lambda_2 \cdots \lambda_k \geqq (\lambda_1'\lambda_2' \cdots \lambda_k')^\alpha (\lambda_1''\lambda_2'' \cdots \lambda_k'')^\beta,$$

in particular:

$$\text{Det } H \geqq (\text{Det } H')^{\alpha}(\text{Det } H'')^{\beta}. \tag{2}$$

4431. *Proposed by M. S. Klamkin, Brooklyn Polytechnic Institute, New York*

If $S_n = \sum_1^n 1/r$, prove:

$$\sum_{n=1}^{\infty} \frac{S_n}{n^2} = 2\sum_{n=1}^{\infty} \frac{1}{n^3} = 2\sum_{n=1}^{\infty} \frac{S_n}{(n+1)^2}.$$

4432. *Proposed by Victor Thébault, Tennie, Sarthe, France*

With the midpoints of the sides of a triangle as centers circles are described passing through the feet of the corresponding altitudes. Show that the circle orthogonal to these three circles is tangent to the nine-point circle of the given triangle at a point such that its distance from the foot of one altitude is equal to the sum of the distances from the feet of the other two altitudes.

SOLUTIONS

Professor Banach's Matches

4348 [1949, 343]. *Proposed by D. A. Darling, Rutgers University*

This problem was brought from Poland by Professor H. Steinhaus. It appears that Professor Banach was accustomed to carrying a box of matches in each of two coat pockets. To light his pipe, he would take a match from either box at random. The boxes contained originally n matches each. Banach's question is: when first a box is opened and found empty, what is the expected number of matches left in the other box?

Solution by W. D. Berg, Kenyon College, Gambier, Ohio. The probability of drawing a particular box exactly n times in $n+x$ ($x=0, 1, \cdots, n$) trials and drawing it again on the $(n+x+1)$th trial is $\binom{n+x}{n}(\frac{1}{2})^{n+x+1}$. Therefore when first a box is opened and found empty the probability that $n-x$ matches remain in the other box is

$$2\binom{n+x}{n}(\tfrac{1}{2})^{n+x+1} = \binom{n+x}{n}(\tfrac{1}{2})^{n+x}.$$

Since this situation must obtain for one and only one value of x, we note that

$$\sum_{x=0}^{n}\binom{n+x}{n}(\tfrac{1}{2})^{n+x} = 1. \tag{1}$$

The expected number E of matches left in the other box is therefore

$$E = \sum_{x=0}^{n}(n-x)\binom{n+x}{n}(\tfrac{1}{2})^{n+x},$$

which, with the use of (1) may be simplified as follows.

$$
\begin{aligned}
E &= (2n+1)\sum_{x=0}^{n}\binom{n+x}{n}\left(\tfrac{1}{2}\right)^{n+x} - \sum_{x=0}^{n}(n+1+x)\binom{n+x}{n}\left(\tfrac{1}{2}\right)^{n+x} \\
&= (2n+1) - 2(n+1)\sum_{x=0}^{n}\binom{n+1+x}{n+1}\left(\tfrac{1}{2}\right)^{n+1+x} \\
&= (2n+1) - 2(n+1)\left\{1 - \binom{2n+2}{n+1}\left(\tfrac{1}{2}\right)^{2n+2}\right\} \\
&= \frac{(2n+1)!}{n!n!}\left(\tfrac{1}{2}\right)^{2n} - 1.
\end{aligned}
$$

Also solved by Colin Blyth, Herbert Carus, J. A. Clarkson, E. H. Cutler, Jacques Dutka, Solomon Leader, Frederick Mosteller, Bart Park, C. F. Pinzka, Lois B. Whitman, and the proposer. Bernard Friedman, R. E. Greenwood, Fritz Herzog, J. G. Millar, Leo Moser, and Frederick Mosteller found the answer $\binom{2n}{n}n/2^{2n-1}$ for the allied problem: What is the expected number of matches remaining when one of the boxes is first emptied.

Editorial Note. Several solvers point out that E is asymptotically equivalent to $2\sqrt{n/\pi}$. Millar observes that E is one less than the coefficient of x^n in the expansion of $(1-x)^{3/2}$. Five readers sent in replies based on the unwarranted assumption that all applicable permutations of $n+1$ A's and $n+1$ B's (where A, B represent a drawing from one box or the other) are equally likely.

Dutka notes that the problem is quoted and solved in Feller, *An Introduction to Probability Theory and Its Applications*, New York, 1950, pp. 108–109, 176–177. Mosteller points out its similarity to that of average sample size in truncated single sampling in industrial sampling inspection. See, for example, Statistical Research Group of Columbia University, *Sampling Inspection*, McGraw-Hill, 1948. In industrial inspection plans the n's would ordinarily be unequal.

An Infinite Continued Ratio

4349 [1949, 344]. *Proposed by H. F. Sandham, Trinity College, Dublin, Ireland*

Prove that

$$\frac{2}{1}\Big/\frac{5}{4}\Big/\frac{8}{7}\Big/\frac{11}{10}\cdots = \sqrt{3}.$$

I. *Solution by J. B. Kelly, University of Wisconsin.* From the well known identity

$$\frac{\sin \pi z}{\pi z} = \prod_{n=1}^{\infty}\left(1 - \frac{z^2}{n^2}\right)$$

one obtains

$$\frac{z}{w}\prod_{n=1}^{\infty}\frac{(n-z)(n+z)}{(n-w)(n+w)}=\frac{\sin \pi z}{\sin \pi w}.$$

In particular, the substitution $2z=a$, $2w=1-a$ gives a form which reduces to

$$\frac{a}{1-a}\bigg/\frac{1+a}{2-a}\bigg/\frac{2+a}{3-a}\bigg/\frac{3+a}{4-a}\cdots = \tan\frac{\pi a}{2}.$$

The proposed formula is the special case where $a=2/3$.

II. *Solution by S. T. Parker, Kansas State College.* The problem may be generalized to provide similar developments for other radicals. The fraction

$$F(a)=\frac{2\cdot 4\cdot 6\cdots 2a}{1\cdot 3\cdot 5\cdots(2a-1)}=\frac{2^{2a}a!a!}{(2a)!}$$

diverges as the integer a increases without limit, but the quotient

$$\frac{F(na)}{F(a)}=\frac{2^{2a(n-1)}(na)!(na)!(2a)!}{a!a!(2na)!} \tag{1}$$

converges to the limit $\sqrt{n}$, as is seen at once from Stirling's formula. For n odd, if each factor in $F(a)$ is multiplied by n, (1) is equivalent to $F(na)$ with all multiples of n removed. It can then be easily expanded as a continued ratio $r_0/r_1/r_2\cdots$, where

$$r_k=\frac{(kn+2)(kn+4)\cdots(kn+n-1)}{(kn+1)(kn+3)\cdots(kn+n-2)}.$$

For example, the present problem and

$$\sqrt{5}=\frac{2\cdot 4}{1\cdot 3}\bigg/\frac{7\cdot 9}{6\cdot 8}\bigg/\frac{12\cdot 14}{11\cdot 13}\bigg/\frac{17\cdot 19}{16\cdot 18}\cdots,$$

$$\sqrt{7}=\frac{2\cdot 4\cdot 6}{1\cdot 3\cdot 5}\bigg/\frac{9\cdot 11\cdot 13}{8\cdot 10\cdot 12}\bigg/\frac{16\cdot 18\cdot 20}{15\cdot 17\cdot 19}\cdots.$$

If n is even, however, every multiple of n occurs in the numerator of $F(na)$. Therefore (1) is equivalent to $F(na)$ with each odd multiple of n repeated and each even multiple removed. The corresponding continued ratio is then $r_0/r_1/r_2\cdots$, where

$$r_k=\frac{(kn+2)(kn+4)\cdots(kn+n)}{(kn+1)(kn+3)\cdots(kn+n-1)} \qquad \text{if } k \text{ is even,}$$

$$r_k=\frac{(kn+1)(kn+3)\cdots(kn+n-1)}{(kn)(kn+2)\cdots(kn+n-2)} \qquad \text{if } k \text{ is odd.}$$

For example,

$$\sqrt{2} = \frac{2}{1}\Big/\frac{3}{2}\Big/\frac{6}{5}\Big/\frac{7}{6}\Big/\frac{10}{9}\cdots,$$

$$\sqrt{4} = \frac{2\cdot 4}{1\cdot 3}\Big/\frac{5\cdot 7}{4\cdot 6}\Big/\frac{10\cdot 12}{9\cdot 11}\Big/\frac{13\cdot 15}{12\cdot 14}\cdots,$$

$$\sqrt{6} = \frac{2\cdot 4\cdot 6}{1\cdot 3\cdot 5}\Big/\frac{7\cdot 9\cdot 11}{6\cdot 8\cdot 10}\Big/\frac{14\cdot 16\cdot 18}{13\cdot 15\cdot 17}\cdots.$$

Also solved by Michael Aissen, Vern Hoggatt, L. S. Kennison, M. S. Klamkin, Roger Lessard, and the proposer.

Approximation by a Linear Function

4343 [1949, 269]. *Proposed by C. D. Olds, San Jose State College, California*

Find an approximation to $(x^2+y^2)^{1/2}$ by a linear function $\alpha x+\beta y$ where $ax\leqq y\leqq bx$, and $0\leqq a\leqq b$, such that the maximum of the absolute value of the relative error

$$[(x^2+y^2)^{1/2}-(\alpha x+\beta y)]/(x^2+y^2)^{1/2}$$

shall be as small as possible.

Solution by Roger Lessard, École Polytechnique, Montreal, Canada. In the plane XOY, the true value is the distance from $P(x, y)$ to the origin, and the approximate value is $(\alpha^2+\beta^2)^{1/2}$ times the distance from P to the line $\Delta{:}\,\alpha x+\beta y=0$.

For a fixed true value (that is, for all points on an arc of a circle with center at the origin) the approximate value will be a maximum for the point whose distance to Δ is maximum, that is, the point P_c on the perpendicular to Δ passing through the origin, and will be a minimum for the point nearest Δ. As the interval $ax\leqq y\leqq bx$ corresponds to the arc intercepted by the two lines $y=ax$ and $y=bx$, the best condition will be obtained when both the following requirements are met:

(I) The two extreme points, P_a and P_b are equally distant from Δ, that is, Δ is the exterior bisector of the angle formed by $y=ax$ and $y=bx$. Therefore

$$\frac{a+b}{1-ab}=\frac{2\beta/\alpha}{1-\beta^2/\alpha^2}. \tag{1}$$

(II) The greatest negative relative error is numerically equal to the greatest positive relative error, that is, the sum of the maximum and minimum approximate values equals twice the true value:

$$\sqrt{\alpha^2+\beta^2}+\frac{\alpha+\beta a}{\sqrt{1+a^2}}=2. \tag{2}$$

For given values of a, b, it is easy to determine α and β from (1) and (2).

Also solved by the Proposer in his paper, *The Best Polynomial Approximation of Functions*, this MONTHLY [1950, pp. 617–621].

Set of Minimum Points

4351 [1949, 414]. *Proposed by Albert Wilansky, Lehigh University, Bethlehem, Pa.*

Let $f(x, y)$ be continuous for all (x, y). On each circle with center at the origin f assumes a minimum at certain points. Is the set of all such points throughout the plane connected?

Solution by William Gustin, Indiana University. Let C_r be the circle of radius r centered at the origin. Any continuous real-valued function f assumes a minimum value on C_r, and we denote by S_f the set of all such circlewise minimum points of f as r ranges over $r \geqq 0$. We shall prove that this circlewise minimum set S_f of any continuous function f is a closed set which intersects every circle C_r and, conversely, that any closed set S, not necessarily connected, which intersects every circle C_r is the circlewise minimum set S_f of some continuous function f.

For any continuous function f the set S_f by definition intersects every C_r. To show that S_f is closed, let $p_n \to p$ with p_n in S_f and let q be any point on the same circle C_r as p. We are to demonstrate that $f(p) \leqq f(q)$. To this end choose points q_n such that $q_n \to q$ and such that q_n lies on the same circle as p_n. Since p_n is a circlewise minimum point of f, we see that $f(p_n) \leqq f(q_n)$; and since f is continuous, $f(p_n) \to f(p)$ and $f(q_n) \to f(q)$; whence we conclude that $f(p) \leqq f(q)$: so S_f is closed.

Now consider a closed set S which intersects every circle C_r. Define $f(p)$ to be the distance from the point p to the set S. Then f is a continuous function vanishing on S and positive elsewhere. Since S intersects every C_r the minimum of f on C_r is 0 and is assumed at precisely those points of C_r common to S. This proves that $S_f = S$.

A closed set which intersects every circle C_r can be obtained by adding to an arbitrary closed set a ray issuing from the origin. Although such a set may be very disconnected it will of course possess at least one non-degenerate component. However, it is possible to construct a totally disconnected closed set S intersecting every circle C_r, as follows. Let $\theta \to r$ map the Cantor ternary discontinuum D based on the closed interval $0 \leqq \theta \leqq 2\pi$ continuously onto the closed interval $0 \leqq r \leqq \infty$ so that 2π and only 2π maps into ∞. The plane set S of polar coördinate graph points (r, θ) of this mapping $\theta \to r$ for θ in D, omitting $\theta = 2\pi$, is then closed, totally disconnected, and intersects every C_r.

Also solved by R. P. Agnew, H. D. Block, Fritz Herzog, J. B. Kelly, Norman Miller, and M. H. Protter.

RECENT PUBLICATIONS

EDITED BY E. P. VANCE, Oberlin College

All books for review should be sent directly to the editor of this department, American Mathematical Monthly, 531 West 116 Street, New York 27, N. Y. and not to any of the other editors or officers of the Association.

Introduction to the Theory of Probability and Statistics. By N. Arley and K. R. Buch. Wiley and Sons, New York, 1950. 11+236 pages. $4.00.

Even to those familiar with the many applications of mathematical statistics it is a source of surprise and some pleasure to see the constantly expanding field of studies strongly influenced either by the fundamental ideas of modern probability or by the more recent developments of statistical theory. Physics with a long history of interest in the theory of least squares has awakened rather recently to the possibilities inherent in statistics, although until now no text was adequately designed with this particular group in mind. For physicists, electrical engineers, or astronomers interested in stochastic processes or in problems of experimentation the authors have written an outstanding text, although it is desirable to supplement it by a standard work such as Hoel's *Introduction to Mathematical Statistics.*

Briefly some striking features are the simplified axiomatic treatment of probability (based on the modern definition of Kolmogoroff), the extensive discussion of probability functions, the emphasis on the theory of estimation, the use of matrices in the theory of errors, and the many references to stochastic processes in physics. Topics meagerly discussed or completely omitted are the theory of testing hypotheses, the power function, confidence limits, the design and cost of experimentation, quality control, time series, regression theory, analysis of variance, the χ^2 test, sequential analysis, and computational methods. One wonders whether a student finishing such a text can actually complete a statistical analysis of any kind. The difficulty is that too much space has been given to the theory of probability and the real spirit and character of statistics has been lost. It is for this reason that we recommend supplementation by Hoel. It is unfortunate that the authors follow a particular school of statistics and thus overlook much of the work in statistics completed since 1930.

While certain inadequacies have been noted, on the whole we have been so pleased with this work that we shall use it as a text in a course for physicists and engineers at Hughes Aircraft Company.

L. A. AROIAN

College Algebra. By E. B. Miller and R. M. Thrall. Ronald Press Company, New York, 1950. 493 pages. $3.75.

This textbook for first-year college students is pitched at an intermediate level in that it avoids both the "complexity of the too advanced text" and the "sterility of the over simplified presentation." In choice of subject matter, the authors have adhered rather closely to that common to most of the texts now in

use in this country. The distinctive features of the book therefore lie in the organization of the material and in the manner of its presentation. On both of these scores, the work under review can be recommended.

Organization of the material is facilitated by dividing the book into three parts: Introductory Topics; Functions of Real Variables, Solution of Equations; and Functions of Integers. The two introductory chapters include a brief discussion of the number system of algebra and a review of high school algebra.

The second part centers around the concept of a function and the solution of its associated equation. A desirable point is the inclusion of the range of the independent variable in the definition of function. Graphical representation is effectively used as a tool in studying the linear and quadratic cases and explicit solutions of the cubic and quartic are obtained. For roots of equations of the nth degree, Horner's and the linear approximation methods are explained but there is no graphical treatment of multiple roots. Exponential functions, logarithms and inequalities are discussed with clarity and accuracy. Decomposition into partial fractions is explained in detail but proofs are not given and there is no use of the euclidean algorithm technique although this has previously been used in finding greatest common divisors. (For this theoretically and computationally effective method, see G. Birkhoff and S. MacLane, *A Survey of Modern Algebra*, Macmillan, 1941, pp. 103–105.)

The treatments of sequences, series and mathematical induction are clear and are at about the usual level of rigor. Matrices and determinants are first mentioned in Chapter 4 and are treated more fully in Chapter 17. Here a determinant is defined by a recursion formula involving expansion in terms of the elements in the first row as a number associated with a matrix. The reviewer agrees fully with the authors that the matrix concept should precede that of determinant but is of the opinion that greater use of matrices is required for the distinction to be fully appreciated by the student. Explicitly, the augmented matrix could have been used in an improved discussion of the solution of linear equations by the method of successive elimination (unfortunately called "the addition and subtraction method").

The problems appear well chosen and are sufficiently numerous. Tables, an index and answers to the odd numbered problems are included. The format and printing are excellent. A few misleading statements were noted but no outright errors were found.

WALLACE GIVENS

Analytic Geometry. By L. M. Kells and H. C. Stotz. New York, Prentice-Hall Inc. 1949. 8+280 pages. $2.85.

When this book arrived the reviewer had every intention of procrastinating until vacation time. However, who can resist a peek into a new book? The clear legible type and numerous careful drawings give the book an attractive format. In short, this text looks interesting. One's attention is caught by the diagram on page 150 where the navigational aid LORAN is discussed as an application

of the hyperbola. Further perusal reveals other interesting discussions and applications of analytic geometry to supplement the usual text book applications. The subject matter is that of a conventional course in analytic geometry which does not make use of the calculus. This has been supplemented by material on trigonometric and logarithmic functions, followed by curve fitting as well as an introduction to solid analytical geometry. The material on plane analytics is carefully presented. An interesting discussion of loci of the forms $y=|x|$ and $\sqrt{(x-4)^2+y^2}=|x-2|+4$ is given, and the distinction between $x^2+y^2=a^2$ and $y=\sqrt{a^2-x^2}$ is emphasized.

The first chapter includes a section on vector addition and deft use is made of simple vector arithmetic throughout the text. One may wonder why the vector interpretation of complex numbers is omitted, especially since the section on polar coördinates is unusually well developed. Perhaps the answer lies in the title of the text itself.

In using this text one may wish to spend two or three days on Appendix I before beginning the text proper. Formulae for the solution of quadratic equations, logarithms, and trigonometry are discussed here in a concise form. The authors should be commended for having the courage to admit that tan 90° and tan 270° do not exist, rather than using the symbol ∞ which is so glibly misinterpreted by the pupil.

The discussion of translation of axes early in the course is, in the opinion of this reviewer, a desirable innovation, especially as the concept is well integrated into the remaining material and used in subsequent derivations.

In view of the careful treatment given the distance from a line to a point as a signed quantity and the discussion of equations of the type $\sqrt{(x-4)^2+y^2}=|x-2|+4$, one wonders if focus directrix definitions of the conics, which ignore these distinctions may not trouble a thoughtful student.

A more serious difficulty presents itself if one wishes to discuss the important material of Chapter 11 on the use of logarithmic and semilogarithmic papers. The reviewer feels that the majority of his students would require considerable additional explanation before anything approaching proficiency in the use of these papers could be attained.

After discussing the conic sections and other algebraic curves, the authors have commendably written a section on the sketching of trigonometric and exponential functions including composition of ordinates and damped vibration. Students of engineering, physics, and applied mathematics who study this book should welcome the opportunity to become familiar with these important functions.

This reviewer believes it would be a pleasure either to teach or to study from this text. Certainly *Analytic Geometry* by Kells and Stotz deserves the careful consideration of anyone wishing a text in analytics which does not introduce methods of the calculus.

R. V. ANDREE

NEW BOOKS RECEIVED

Vektoranalysis. By R. Gans. Leipzig, Teubner, 1950. 119 pp. $1.40.

Plane Trigonometry. By E. R. Heineman. Alternate Edition. New York, McGraw-Hill, 1950. 14+184 pp. (Plus 75 pages of tables.) $2.50.

Evariste Galois. (Beihefte zur Zeitschrift-Elemente der Mathematik: Vol. 7). By L. Kollros. Basel, Birkhauser, 1949. 24 pp. Fr. 3.50.

Jakob Steiner, (Beihefte zur Zeitschrift-Elemente der Mathematik: Vol. 2). By L. Kollros. Basel, Birkhauser, 1947. 24 pp. Fr. 3.50.

Vector and Tensor Analysis. (International Series in Pure and Applied Mathematics). By H. Lass. New York, McGraw-Hill, 1950. 12+347 pp. $4.50.

Einführung in die Höhere Analysis. By E. Lindelöf and E. Ullrich. Leizig, Teubner, 1950. 9+526 pp. *3*3.55.

Die Mathematischen Hilfsmittel des Physikers. (Die Gundlehren der Mathematischen Wissenschaften in Einzeldarstellungen, Band IV). by E. Madelung. Berlin, Springer-Verlag, 1950. 18+531 pp. DMark 49.70.

Niels Henrik Abel. (Beihefte zur Zeitschrift-Elemente der Mathematik: Vol. 8). By O. Ore. Basel, Birkhauser, 1950. 23 pp. Fr. 3.50.

Jost Burgi und die Logarithmen. (Beihefte zur Zeitschrift-Elemente der Mathematik: Vol. 5). By El Voellmy. Basel, Birkhauser, 1948. 24 pp. Fr. 3.50.

The Human Use of Human Beings. By N. Wiener. Boston, Houghton Mifflin, 1950. 241 pp. $3.00.

Ludwig Schlafli. (Beihefte zur Zeitschrift-Elemente der Mathematik: vol. 4). By J. J. Burckhardt. Basel, Birkhauser, 1948. 23 pp. Fr. 3.50.

Colloque de Geometrie Algebrique. (Centre Belge de Recherches Mathematiques.). Paris, Masson, 1950. 195 pp. 1.400 francs.

Calcul Vectoriel et Calcul Tensoriel. (Oevres de Henri Poincare: Vol. 5). By A. Delachet. Paris, Gauthier-Villars, 1950. 8+552 pp. 7,200 fr.

Johann and Jakob Bernoulli. (Beihefte zur Zeitschrift-Elemente der Mathematik: Vol. 6). By J. O. Fleckenstein. Basel, Birkhauser, 1949. 24 pp. Fr. 3.50.

Leonhard Euler. (Beihefte zur Zeitschrift-Elemente der Mathematik: Vol. 3). By R. Fueter. Basel, Birkhauser, 1948. 24 pp. Fr. 3.50.

Statistical Inference in Dynamic Economic Models. By T. C. Koopmans. New York, John Wiley and Sons, 1950. xiv+438 pp. $6.00.

An Introduction to the Theory of Statistics. By G. U. Yule and M. G. Kendall. Revised Edition. New York, Hafner Publishing Co., 1950. xxiv+701 pp. $7.00.

Statistical Decision Functions. By Abraham Wald. New York, John Wiley and Sons, Inc., 1950. ix+179 pp. $5.00.

Resṕonse of Physical Systems. By J. D. Trimmer. New York, John Wiley and Sons, Inc., 1950. ix+268 pp. $5.00.

CLUBS AND ALLIED ACTIVITIES

EDITED BY L. F. OLLMANN, Hofstra College

Send reports of all activities, such as club reports, special features, topics with references, student papers, and other material of interest to L. F. Ollmann, Hofstra College, Hempstead, New York.

OLD MAN DELTA

Tune: self-evident

(Dedicated to the students of the Calculus)

Freshmen all work at their mathematics,
Freshmen all work while instructors yawn,
Racking their brains for the right solution,
Staying up nights till the break of dawn.
Look it up, and write it down,
Never get time to go down town,
Oh my back! Oh my head!
I'll be doin' this till I'm dead.
Let me go 'way from these sines and cosines,
Let me go 'way from infinite sums,
Show me that stuff known as long division,
Let me do math on my fingers and thumbs.

Chorus:

Old Man Delta, that Old Man Delta,
He's not a nothin', he's just a somethin'
That keeps on shrinkin', he keeps on shrinkin' away.
He don't buy 'taters, he don't buy cotton,
And what he's good for, I've quite forgotten,
But Old Man Delta, he keeps on shrinkin' away.
You and me, we sweat and strain,
Eyeballs all bloodshot and racked with pain,
Change that sign! Raise that power!
It's three A.M. by the Lib'ry tower!
I gets weary, this life's disgustin',
I'm tired of crammin', and feared of bustin',
But Old Man Delta, he just keeps shrinkin' away.

—G. B. ROBISON, Cornell University

CLUB REPORTS, 1949–50

Mathematics Section, Chamberlin Science Club, Beloit College

The following papers were presented at the monthly meetings of the *Mathematics Section* of the *Chamberlin Science Club* of Beloit College during the academic year 1949–50:

Some interesting problems from the MONTHLY, by Dr. R. C. Huffer

The life of Leibnitz, by Richard Bening

Magic squares, by Shirley Aten

Diophantine equations, by William Quelch

Digital roots and *The casting out of nines*, by Donald Walsh

Prime numbers, by Margery Jenkins

Topology by Margery Jenkins

Topology and the four-color problem, by Donald Runge

Fibonacci's progression, by Rodney T. Hood

Ruler and compass construction, by Arthur Ahlgrim

Geometric exercises in paper folding, by Betty Clayton.

All sections of the *Chamberlin Science Club* were guests of the Beloit College *Sigma Xi* chapter for an address by Dr. C. M. Huffer of the University of Wisconsin on *The Schmidt Telescope of Mt. Palomar*, and the showing of the film *The Story of Palomar*.

Officers for the year 1949–50 were: President, William Quelch; Program Chairman, Marjorie Schueppert; Secretary, Betty Clayton; Advisers, R. C. Huffer and R. T. Hood.

Pi Mu Epsilon, Montana State University

Montana Alpha chapter of *Pi Mu Epsilon* started the current academic year by celebrating the tenth anniversary of the awarding of Pi Mu entrance prizes. These are given annually to the three Freshmen from Montana High Schools who place highest on an examination in mathematics. This year's awards went to Jo Anne Grundstrom, J. Hollis McCrea, Jr., and Robert J. McRae.

Papers presented at meetings during the year were:

Number theory, by Dr. H. Chatland

Fun with mathematics, by Dr. Chatland (a joint meeting with the Mathematics Club)

Thermo-dynamics, by Dr. Lory of the Chemistry Department

Philosophy of mathematics, by Dr. T. G. Ostrom

Accomplishments made in physics during the first half of the 20th century, by Dr. D. R. Jeppesen of the Physics Department

Master's thesis, by Daniel Coffey

Pursuit curve characteristics—mathematics as an aid to air force gunnery, by John Peterson.

Initiation of new members took place at the annual banquet. The members attended the annual *Math-Chem Club* picnic in May.

The following officers were elected for 1950–51: Director, Merton Robertson; Vice-Director, Evan Rempel; Secretary, Marybelle Fry; Treasurer, Dr. T. G. Ostrom.

Pi Mu Epsilon, St. Louis University

The *Missouri Gamma* chapter of *Pi Mu Epsilon*, during the academic year 1949–50, held three regular meetings, including business meetings and elections, besides the annual initiation banquet. The programs consisted of lectures and demonstrations on the following topics:

Seismic ray paths, by Carl Kisslinger

Tabulating machines, by Dr. John D. Elder and S. A. Michalski, Director of Tabulating department

Topological concepts, by Roman Gawkowski and Robert Pohrer

Life with statistics, by Prof. G. W. Snedecor, Professor of Statistics, Iowa State College.

A drive is currently being conducted by the chapter to raise a ten thousand dollar endowment fund, the income of which will be used to support the lecture held at the annual spring meeting and banquet. This lecture, which was formerly sponsored by the Student Conclave, will be known as the James E. Case Mathematics Lecture in honor of the Reverend James E. Case, S.J., who was head of the Mathematics Department until his death last August.

The fourth Annual Prize Essay Contest, opened to undergraduate students only, was won by Miss Evelyn Murrill. The title of her paper was *Pierre Fermat, His Life and Works*. Ninety new members were inducted into the chapter.

Officers for 1949–50 were: Director, Edward Thirkhill; Vice-Director, John Hagan; Secretary-Treasurer, Virginia Herre; Faculty Advisor and Permanent Secretary, Dr. Francis Regan.

Carleton Mathematics Club, Carleton College

Included among the talks presented at the monthly meetings of the *Mathematics Club* of Carleton College by members of the faculties of Carleton and the University of Minnesota, students, and businessmen, were:

Application of group theory in wallpaper design, by Prof. J. M. H. Olmsted

Indeterminant equations, by Prof. C. Gingrich

Calculating machines, by Clinton Karr

Group theory, by Prof. K. May

Theory of numbers and fundamental operations, by Prof. Graves

Techniques of operators, by A. Galambos

Fourier series, by Kirk McVoy.

Refreshments were served after several of the meetings, and a banquet preceded the December talk.

Officers for 1949 were: President, Peggy Brown; Vice-President, Shirley Mills; Secretary, Jane Hauser. Officers for 1950 are: President, Lyman Van-Slyke: Vice-President, James Pierce; Secretary, Irene Daniells; Faculty Advisor, Mr. Winston Crum.

Mathematics Club, Eastern Illinois State College

Papers presented to the *Eastern Mathematics Club* during 1949–50 were:

Approximations of pi, by Don Fraembs

Problems of antiquity, by Kenneth Wilson, Claude Towne, and Don Fraembs

Pythagorean triples, by Prof. L. A. Ringenberg

Fallacies and paradoxes, by Duane Bruce, Jim Cody, Jim Wynn, Lloyd Coad, and Don Fraembs

Automorphic numbers, by Prof. L. A. Ringenberg

Sources of inspiration during a professional teaching career, by Prof. Emeritus E. H. Taylor.

Newly elected officers are: President, Larry Leathers; Vice-President, George Swinford; Secretary-Treasurer, Cora Coombes.

Mathematics Club, Stanford University

The following activities of the *Mathematics Club* of Stanford University took place during the year 1949–50:

Picture writing, by Prof. George Polya

Algebraic numbers, by Al Novikoff

Critical points, by Prof. Max Shiffman

Paradoxes of logic, by William Firey

The ham-sandwich problem, by Prof. A. W. Tucker

Difference equations, by Prof. Herman Rubin

Map coloring, by Edgar Smith

The sums of squares, by Martin Vitousek

Some simple generalizations of Rolle's theorem, by Prof. Charles Loewner

Almost periodic functions, by Bent Fuglede.

The annual picnic was held in May.

The officers for the year 1949–50 were: President, Bent Fuglede; Secretary, William Perry; Treasurer, Martin Vitousek.

Kappa Mu Epsilon, Albion College

During the academic year 1949–50, the *Michigan Alpha* chapter of *Kappa Mu Epsilon* held eight meetings, including two initiations and a picnic. The following papers were presented:

A new method for evaluating determinants, by Prof. P. Cox

Carl Friedrich Gauss, by Edwin Kehe

Life of Abel, by Deane Floria

Repeating decimals, by Philip McKean

Some uses of mathematics in chemistry, by David Harmer

Solutions of the quadratic equation, by Raymond Gillespie

Tesselations, by Prof. H. D. Larsen.

The following officers were elected for 1950–51: President, William Fryer; Vice-President, Robert Warren; Secretary, Eva Krohns; Faculty Sponsor, Prof. H. D. Larsen.

Pi Mu Epsilon, University of Richmond

The *Virginia Alpha* chapter of *Pi Mu Epsilon* held three business meetings, two program meetings, and the Annual Banquet during the year 1949–50. The papers heard were:

Mathematical oddities, by Dr. E. R. Sleight

Development of our number system, by H. V. Caldwell and P. H. Dalle Mura

The origin and development of systems of notation and numeration, by Dr. R. E. Gaines.

NEWS AND NOTICES

EDITED BY EDITH R. SCHNECKENBURGER, University of Buffalo

Readers are invited to contribute to the general interest of this department by sending news items to Edith R. Schneckenburger, University of Buffalo, Buffalo 14, New York. Items must be submitted at least two months before publication can take place.

APPOINTMENTS AVAILABLE IN THE DEPARTMENT OF DEFENSE

The Armed Forces Security Agency has an urgent and immediate need for graduates or postgraduates with a major in mathematics or statistics. These positions are in the research field of mathematics. Annual salaries range from $3100 to $5400, depending upon individual education and experience. All positions are permanent in so far as this Agency is concerned and are located in the Metropolitan area, Washington, D. C.

For further information write to Miss Gertrude E. Kirtland, Head, Employee Utilization, Armed Forces Security Agency, Department of Defense, Washington, D. C.

MATHEMATICS INSTITUTE AT LOUISIANA STATE UNIVERSITY

The Second Annual Mathematics Institute of Louisiana State University will be held on June 24–30, 1951. There will be study groups for both elementary and high school teachers on arithmetic and high school mathematics. In addition there will be study groups on enrichment materials and the slow learner. Leaders for these groups are outstanding specialists from various parts of the country. Also there are to be general sessions. Rooms and meals will be available on the campus at reasonable rates.

For further information write to Professor H. T. Karnes, Department of Mathematics, Louisiana State University, Baton Rouge 3, Louisiana.

MATHEMATICS INSTITUTE AT THE UNIVERSITY OF HOUSTON

The University of Houston is sponsoring its first Mathematics Institute to be held on July 30 to August 2, 1951. There will be sessions of interest to all

teachers of mathematics led by nationally known people in the field of mathematics teaching. Also, there will be sectional meetings for teachers of each level. Rooms and meals will be available on the campus.

For further information write to Professor Esther F. Gibney, University of Houston, Houston 4, Texas.

SUMMER COURSES

The University of Southern California announces that the following advanced courses in mathematics will be offered during its 1951 Summer Session, June 25 to August 4: Professor Sherman, theory of probability and statistics I; differential equations; Professor Snapper, theory of equations and determinants, advanced calculus; Professor Hyers, advanced calculus, vector analysis; Professor Whiteman, history of mathematics I, matrix theory; Professor Dugundji, introduction to theory of complex variables; Professor Gelbart, functions of a real variable, advanced topics in analysis; Professor White and staff, dissertation.

PERSONAL ITEMS

Professor Marston Morse of the Institute for Advanced Study has been appointed a member of the Board of the National Science Foundation.

Professor H. B. Curry of Pennsylvania State College, Dr. H. W. Kuhn of Princeton University, and Professor S. S. Wilks of Princeton University, have been awarded Fulbright research scholarships. Professor Samuel Eilenberg, Columbia University, and Professor Norbert Wiener, Massachusetts Institute of Technology, have been appointed Fulbright visiting lecturers, for 1950–51.

Dr. M. A. Rosenlicht of the University of Chicago has been appointed to a National Research Fellowship in Mathematics for the academic year 1950–51.

Professor E. P. Wigner of Princeton University has received the Franklin Medal from the Franklin Institute.

Assistant Professor Edith R. Schneckenburger of the University of Buffalo was the representative of the Association at the Conference on Improving the Effectiveness of College Faculties which was held in Chicago, Illinois, on December 7–9, 1950. This Conference was sponsored by the United States Office of Education and the American Council on Education.

Catholic University announces the following appointments to instructorships: Mr. W. W. Boone of Rutgers University, Mr. Joseph Fennell, formerly A.E.C. fellow at Princeton University, and Mr. J. F. Hannan, previously research assistant at the University of North Carolina.

At the College of the Holy Cross: Assistant Professor V. O. McBrien has been appointed to an associate professorship; Mr. E. J. McGillicuddy has been appointed to an instructorship; Reverend W. F. Burns is on leave for further study at Fordham University.

Gonzaga University reports the following: Mr. J. J. Murray, formerly a

graduate student at St. Louis University, has been appointed to an assistant professorship; Mr. K. W. Charlton, graduate student at Washington State College, has been appointed to an instructorship; Mr. H. R. Clark, previously a student at Gonzaga University, has been appointed to a teaching fellowship; Mr. D. M. Clarke, who has been an instructor at the University, is now engaged in graduate study at Northwestern University.

At Indiana University: Assistant Professor David Gilbarg has been promoted to an associate professorship; Teaching Fellow D. H. Porter has been promoted to an instructorship; Dr. R. E. MacKenzie of Princeton University has been appointed to an instructorship.

The Department of Mathematics of Iowa State College announces the following: Mr. Buchanan Cargal and Mr. Dean Benson, formerly in charge of Mathematics at Sioux Falls College, have been appointed to instructorships; Associate Professor R. E. Gaskell served as consulting engineer for the Physical Research Unit of Boeing Airplane Company, Seattle, Washington, during the summer of 1950; Dr. B. R. Seth has returned to Hindu College, New Delhi, India; Instructor R. F. Deniston has resigned to accept an industrial position in Detroit, Michigan.

The Department of Statistics of Iowa State College reports: Associate Professor T. A. Bancroft has been appointed Head of the Department of Statistics and Director of the Statistical Laboratory; Dr. S. L. Isaacson, who was a Naval Research Assistant at Columbia University, has been appointed Assistant Professor of Statistics; Mr. Franklin Graybill and Mr. David Huntsberger have been appointed graduate assistants; Dr. Garnet McCreary, previously a graduate student in the Department, has been appointed Assistant Professor of Mathematics at the University of Manitoba.

Lehigh University makes the following announcements: Dr. Samuel Goldberg of Cornell University and Mr. Samuel Schecter of Syracuse University have been appointed to instructorships; Mr. R. A. C. Lane, previously student at Indiana University, Mr. T. A. Schwegler, student at Baldwin-Wallace College, and Mr. R. J. Smurthwaite, formerly student at the University of Buffalo, have been appointed to graduate assistantships; Instructor Michael Tikson has been appointed to a graduate assistantship at Ohio State University.

Ohio State University announces the following: Dr. A. P. Calderon has been appointed Visiting Associate Professor for 1950–51; Dr. I. N. Herstein of the University of Kansas has been appointed Visiting Lecturer; Dr. D. J. Lewis has been appointed to an instructorship; Instructor H. G. Harp has been appointed Chairman of the Department of Mathematics of Ohio Northern University.

Ohio University announces the following appointments to instructorships: Mr. A. L. Fisch, University of Wisconsin, Mr. Huleent Hunzeker of Iowa State College, and Mr. Andrew Sterrett of the University of Pittsburgh.

Purdue University reports the following: Professor Jacob Korevaar of Technical High School, Delft, Holland, who was Visiting Lecturer at the University during the year 1949–50, has been appointed Visiting Assistant Professor for

the year 1950–51; Dr. P. E. Irick has been promoted to an assistant professorship; Instructor G. C. Sipple has retired; Dr. C. E. Goldman has resigned to accept a research position at Wright Field, Dayton, Ohio; Instructor R. W. Peck has returned to a position at Jefferson High School, Lafayette, Indiana.

Tulane University announces the following: Dr. D. R. Morrison of the University of Wisconsin has been appointed to an assistant professorship; Associate Professor J. L. Kelley of the University of California at Berkeley has been appointed Visiting Associate Professor for the year 1950–51; Mr. R. S. Novosad and Mr. C. E. Capel have been appointed to instructorships; Dr. Judson Sanderson, University of Illinois, has been appointed to an assistant professorship in Sophie Newcomb College; Mr. J. W. Keesee, who has completed work for the Ph.D. degree at the University, has been appointed to an assistant professorship at the University of Arkansas.

At the University of California, Berkeley: Lecturers Evelyn Fix, R. M. Lakness, T. K. Pan, and Elizabeth L. Scott have been promoted to instructorships; Dr. P. L. Chambre, Michael Fell, Hewitt Kenyon, Antares Parvulescu, Pao Ming Pu, Maurice Sion, and F. B. Thompson have been appointed Lecturers; Assistant Professor A. R. Williams has retired with the title of Assistant Professor Emeritus; Assistant Professors R. C. James and L. H. Swinford are on sabbatical leave; Associate Professor J. L. Kelley and Assistant Professor E. L. Lehmann are on special leave.

Dr. Bess E. Allen of Wayne University has been promoted to an assistant professorship.

Mr. Nils Blomquist of the Royal Institute of Technology, Stockholm, Sweden, has been appointed to an instructorship at Boston University.

Dr. F. E. Bothwell of the University of Chicago has been appointed to an associate professorship in the Department of Electrical Engineering of Northwestern University.

Assistant Professor J. L. Botsford of San Jose State College has been appointed Associate Professor and Acting Head of the Department of Mathematics of the University of Idaho.

Mr. C. F. Briggs of the University of Michigan has been appointed to an assistant professorship at Emory University.

Dr. Eleazer Bromberg of the Reeves Instrument Corporation has accepted a position as head of the Mechanics Branch, Office of Naval Research, Washington, D. C.

Assistant Professor A. C. Burdette of the University of California at Davis has been promoted to an associate professorship.

Associate Professor A. J. Cook of the University of Alberta has been promoted to the position of Professor and Director of Student Advisory Services.

Dr. J. B. Crabtree has been appointed to an assistant professorship at the University of New Hampshire.

Mr. E. J. Delate has accepted a position as statistician in the Quality Control Department of "Cel-O'Seal," E. I. DuPont de Nemours and Company, Buf-

falo, New York.

Dr. Allen Devanatz, previously graduate student at Harvard University, has been appointed to an instructorship at Illinois Institute of Technology.

Mr. W. C. Foreman of the University of Kansas has been appointed to a professorship at Baker University.

Mr. A. M. Gleason of Harvard University has been promoted to an assistant professorship; he is now on military leave.

Assistant Professor W. J. Harrington of Pennsylvania State College has been promoted to an associate professorship.

Dr. V. C. Harris has been appointed to an assistant professorship at San Diego State College.

Associate Professor D. M. Krabill, Bowling Green State University, has been promoted to a professorship.

Instructor W. R. Mann of the University of North Carolina has been promoted to an assistant professorship.

Associate Professor C. G. Maple of Iowa State College has been recalled to active duty with the United States Navy.

Mrs. Margaret Marshall has been appointed to an instructorship at West Virginia University.

Mr. K. A. McMurtrie has a position as physicist at Aberdeen Proving Ground, Maryland.

Professor J. S. Miller of Dillard University will be a member of the staff of the University of Alberta during the 1951 Summer Session.

Mr. Z. I. Mosesson, who has been Senior Actuarial Assistant, has been promoted to the position of Chief Actuarial Assistant in the Prudential Insurance Company of America.

Professor F. D. Murnaghan has a position at the Technological Institute of Aeronautics, State of São Paulo, Brazil.

Dr. H. A. Palmer has been appointed Head of the Physical Science Department of Midwestern University.

Mr. L. B. Rall, formerly a student at the College of Puget Sound, is now a graduate assistant at Oregon State College.

Dr. R. W. Rempfer of the Farrand Optical Company has been appointed to a professorship at Antioch College.

Dr. L. G. Riggs of Northwestern University has been appointed to an assistant professorship at San Diego State College.

Miss Marcia C. Saile of West Virginia University has accepted a position as instructor at Bay City Junior College.

Mr. D. H. Shaftman has accepted a position as junior mathematician at the Argonne National Laboratory.

Lieutenant W. K. Spears, formerly a graduate student at Purdue University, is now in active service in the United States Army.

Mr. E. A. Stavinoha, formerly a graduate fellow of Oklahoma Agricultural and Mechanical College, is teaching at Brownsville High School, Texas.

Assistant Professor R. L. Swain of Ohio State University has been appointed to a professorship at State Teachers College, New Paltz, New York.

Mr. P. Y. Tani, previously a graduate student at Stanford University, is now a research physicist in the Aero Research Group, Minneapolis-Honeywell Regulator Company, Minnesota.

Mr. G. J. Trammell, Jr., a graduate student at Tulane University, has been appointed to an instructorship at Louisiana State University.

Mr. G. E. Uhrich of the University of Washington has been appointed to an assistant professorship at Montana State College.

Dr. M. C. Waddell, previously a graduate student at Johns Hopkins University, has been appointed to an assistant professorship at Western Reserve University.

Mr. Harold Weintraub, previously a teaching fellow at Harvard University, has been appointed to an instructorship at Tufts College.

Mr. J. C. Wilson, formerly graduate fellow at Oklahoma Agricultural and Mechanical College, is a trainee for the Wolverine Insurance Company, Lansing, Michigan.

Dr. Alfred Hume, who was Professor of Mathematics and Chancellor Emeritus of the University of Mississippi, died December 25, 1950 at the age of eighty-four years. He was a charter member of the Association.

Associate Professor S. L. Mason of the University of North Dakota died on December 12, 1950. He had been a member of the Association for sixteen years.

Sister Edward Joseph of St. Mary's College, Notre Dame, Indiana, died on November 8, 1950.

THE MATHEMATICAL ASSOCIATION OF AMERICA

Official Reports and Communications

THIRTY-FOURTH ANNUAL MEETING OF THE ASSOCIATION

The thirty-fourth annual meeting of the Mathematical Association of America was held at the University of Florida, Gainesville, Florida, on Saturday, December 30, 1950, in conjunction with the annual meeting of the American Mathematical Society and the Christmas meeting of the National Council of Teachers of Mathematics. A total of three hundred and seventy-one persons were registered, including the following two hundred and thirty-seven members of the Association:

C. R. Adams, C. B. Allendoerfer, Beulah M. Armstrong, J. D. Armstrong, H. G. Ayre, W. L. Ayres, E. A. Bailey, Aaron Bakst, R. H. Bardell, Helen P. Beard, W. S. Beckwith, E. G. Begle, Barbara B. Betts, R. G. Blake, Laura Blakeley, L. M. Blumenthal, F. C. Bolser, T. A. Botts, S. G.

Bourne, G. F. Bradfield, J. E. Brown, K. E. Brown, M. C. Brown, C. F. Brumfiel, Emalou Brumfield, C. C. Camp, Richard Crowley Campbell, Dorothy I. Carpenter, C. L. Carroll, Jr., A. B. Carson, H. W. Charlesworth, Sarvadaman Chowla, R. S. Christian, Elsie T. Church, C. E. Clark, Reginald Cobb, Nathaniel Coburn, J. B. Coleman, W. J. Conner, G. M. Conwell, Lennie P. Copeland, F. E. Cothran, J. C. Currie, J. F. Daly, P. H. Daus, A. H. Diamond, Annie L. Dicks, V. E. Dietrich, L. L. Dines, B. F. Dostal, W. L. Duren, J. C. Eaves, P. D. Edwards, David Ellis, D. H. Erkiletian, H. F. Fehr, W. E. Ferguson, G. B. Findley, C. H. Fischer, Edward Fleisher, E. E. Floyd, C. W. Foard, Tomlinson Fort, J. S. Frame, Orrin Frink, Jr., J. W. Gaddum, W. A. Gager, A. E. Gault, H. M. Gehman, J. W. Givens, A. M. Gleason, H. E. Goheen, D. B. Goodner, J. B. Gregg, E. H. Hadlock, B. F. Hadnot, P. R. Halmos, P. C. Hammer, Frank Harary, F. S. Harper, B. T. Harris, E. A. Hedberg, Marguerite Z. Hedberg, C. E. Heilman, Edwin Hewitt, E. H. C. Hildebrandt, R. G. Hill, A. J. Hoffman, O. H. Hoke, A. S. Householder, J. V. Howell, J. A. Hratz, C. W. Huff, G. B. Huff, Ralph Hull, M. Gweneth Humphreys, Elaine Hundertmark, W. R. Hutcherson, Ernest Ikenberry, S. L. Jamison, B. W. Jones, P. S. Jones, H. T. Karnes, J. L. Kelley, L. M. Kelly, A. J. Kempner, Evelyn M. Kennedy, J. R. F. Kent, A. J. Killebrew, M. S. Klamkin, J. R. Kline, J. C. Knipp, R. H. Knox, Jr., R. J. Koch, F. W. Kokomoor, Jacob Korevaar, Ruth O. Lane, G. B. Lang, R. E. Langer, W. I. Layton, Nathan Lazar, R. J. Levit, F. A. Lewis, Kenneth Lewis, Z. L. Loflin, Elizabeth C. Lukacs, C. C. MacDuffee, H. M. MacNeille, H. F. MacNeish, A. C. Maddox, R. H. Mason, J. R. Mayor, S. W. McInnis, E. J. McShane, H. A. Meyer, Herman Meyer, D. D. Miller, Nellie P. Miser, W. L. Miser, Benjamin Ernest Mitchell, Josephine M. Mitchell, C. N. Moore, J. A. Morales, L. J. Mordell, D. R. Morrison, B. H. Mount, Jr., W. B. Moye, C. V. Newsom, T. A. Newton, O. M. Nikodym, M. M. Ohmer, A. J. Owens, W. V. Parker, P. B. Patterson, W. D. Peeples, Jr., Sallie E. Pence, F. W. Perkins, I. E. Perlin, C. G. Phipps, T. J. Pignani, Z. M. Pirenian, F. M. Pulliam, K. S. Purdie, J. F. Randolph, W. W. Rankin, L. T. Ratner, C. B. Read, L. M. Reagan, C. J. Rees, Dorothy L. Rees, Mina S. Rees, P. K. Rees, B. P. Reinsch, J. H. Roberts, J. M. Robertson, F. Virginia Rhode, J. B. Rosser, W. A. Rutledge, Arthur Saastad, R. G. Sanger, J. W. Sawyer, M. A. Scheier, Edith R. Schneckenburger, Carol S. Scott, Esther Seiden, W. T. Sharp, C. N. Shuster, T. M. Simpson, C. B. Smith, D. W. Snader, W. S. Snyder, Andrew Sobczyk, D. E. South, T. H. Southard, W. H. Spragens, Jr., E. T. Stapleford, Marion E. Stark, H. R. Stevens, Ruth W. Stokes, P. M. Swingle, Gabor Szego, H. E. Taylor, J. M. Thomas, S. L. Thompson, G. L. Tiller, John Todd, C. B. Tompkins, Marian M. Torrey, H. C. Trimble, C. A. Truesdell, A. W. Tucker, C. M. Tyler, Jr., Gilbert Ulmer, Henry Van Engen, John von Neumann, T. L. Wade, Jr., Earl Walden, D. T. Walker, R. J. Walker, A. D. Wallace, F. A. Wallace, J. L. Walsh, L. E. Ward, Jr., Susie L. Ward, K. W. Wegner, Marie J. Weiss, J. G. Wendel, G. T. Whyburn, W. M. Whyburn, M. C. Wicht, W. L. Williams, R. L. Wilson, W. H. Wilson, G. N. Wollan, J. W. Wright, J. W. Young, J. H. Zant.

Sessions of the Association were held on Saturday morning and afternoon in the Chemistry Auditorium, Leigh Hall of the University of Florida. President R. E. Langer presided at the morning session and at the business meeting and Professor L. M. Blumenthal presided at the afternoon session. The Program Committee for the meeting consisted of Ralph Hull, Chairman, L. M. Blumenthal, and Marie J. Weiss.

FIRST SESSION OF THE ASSOCIATION

"Network Topology," by Professor A. W. Tucker, Princeton University.

"A Class of Iterative Methods for Solving Equations," by Dr. A. S. Householder, Oak Ridge National Laboratory.

"Sampling Methods in Census Work," by Dr. J. F. Daly, Bureau of the Census.

"The Difference Equation in Pure and Applied Mathematics," by Professor Tomlinson Fort, University of Georgia.

SECOND SESSION OF THE ASSOCIATION

"Partial Orderings and Moore-Smith Convergence," by Professor E. J. McShane, University of Virginia.

"Characterization Problems in Elementary Geometry," by Professor L. M. Kelly, Michigan State College.

MEETING OF THE BOARD OF GOVERNORS

The Board met on Friday afternoon in Florida Union. Eighteen members of the Board were present. Among the more important items of business transacted were the following:

The Board voted to approve the appointment by President Langer of the following Nominating Committee for 1951: W. L. Ayres, Chairman, V. G. Grove, and R. D. James.

The Board voted that hereafter the representatives of the Association on the American Council on Education shall be the President, the Editor, and the Secretary-Treasurer, *ex officiis*.

The Board voted that the representatives of the Association on the Policy Committee for Mathematics shall be the President and the Secretary-Treasurer, *ex officiis*, together with a third representative to be elected by the Board. The elected member is Dr. C. V. Newsom, whose term expires at the end of 1951.

Mrs. Jewell H. Bushey of Hunter College was elected by the Board to the office of Second Vice-President for 1951–1952.

The Board voted to approve the proposal that the National Council of Teachers of Mathematics be admitted to membership on the Policy Committee for Mathematics with one voting member.

The final report of the Committee on a Guidance Pamphlet was accepted with an expression of thanks by the Board to the members of the Committee.

ANNUAL BUSINESS MEETING OF THE ASSOCIATION

The annual business meeting of the Association was held on Saturday, December 30, 1950, at 2:00 P.M. in the Chemistry Auditorium of Leigh Hall of the University of Florida.

The Secretary announced the results of the balloting for officers, in which 1185 votes were cast. Saunders MacLane of the University of Chicago was elected President for the two-year term 1951–1952. H. S. M. Coxeter of the University of Toronto and B. W. Jones of the University of Colorado were elected Governors for the three-year term 1951–1953.

AWARD OF THE CHAUVENET PRIZE

Announcement was made of the award of the Chauvenet Prize for the period 1947–1949 to Professor Mark Kac of Cornell University for his paper, "Random Walk and the Theory of Brownian Motion," published in this MONTHLY, Vol-

ume 54 (1947), pp. 369–391. The Committee on the 1950 Chauvenet Prize consisted of W. B. Carver, Chairman, R. E. Gilman, and D. L. Holl.

MEETINGS OF OTHER ORGANIZATIONS

The sessions of the American Mathematical Society began on the afternoon of Wednesday, December 27, and continued through Friday afternoon. The Josiah Willard Gibbs Lecture was delivered by Professor G. E. Uhlenbeck of the University of Michigan on "Some Basic Problems of Statistical Mechanics." Invited addresses were given by Professor M. H. Heins of Brown University on "Regularity of Growth of Subharmonic Functions," and by Professor L. J. Mordell of the University of Pennsylvania and Cambridge University on "The Product of n Homogeneous Linear Forms." Professor J. L. Walsh of Harvard University delivered the retiring Presidential address on "The Location of Critical Points."

The National Council of Teachers of Mathematics held sessions on Thursday and Friday, December 28 and 29. Addresses at general sessions of the Council were given by Professor E. H. C. Hildebrandt of Northwestern University on "What is Good Mathematics Teaching?" and by Professor H. F. Fehr of Columbia University on "The Psychology of Learning Applied to Classroom Teaching of Mathematics." At the banquet of the National Council held on Thursday evening Professor C. C. MacDuffee of the University of Wisconsin spoke on "Defeatism in Education."

ARRANGEMENTS, ENTERTAINMENT AND RECREATION

The Committee on Arrangements for the Meeting consisted of C. G. Phipps, Chairman, R. G. Blake, W. A. Gager, H. M. Gehman, W. R. Hutcherson, F. W. Kokomoor, H. A. Meyer, W. M. Whyburn.

Registration headquarters was in the Bryan Lounge of Florida Union of the University of Florida. Rooms in the University dormitories and in Florida Union were available from noon Tuesday until Sunday morning. Meals were served in the University Cafeteria next to the Florida Union.

A reception for members of the mathematical organizations was held on Wednesday evening in the Social Room of Florida Union.

On Wednesday afternoon a sight-seeing trip was taken by bus to Silver Springs. On Thursday an all-day trip was taken by bus to Daytona Beach, Marineland Ocean Studios, St. Augustine and Penney Farms Memorial Community.

A joint dinner for members of the mathematical organizations and their guests was held at 6:30 P.M. on Friday in the East Wing of the University Cafeteria. Dean T. M. Simpson of the University of Florida acted as toastmaster. The visitors were welcomed by President J. Hillis Miller of the University of Florida. Professor R. E. Langer responded to the address of welcome. Dean Simpson gave a brief history of the University of Florida and of its Mathematics Department. In conclusion, President J. L. Walsh of the American Mathematical Society presented a silver plate to Professor J. R. Kline in recog-

nition of his ten years of service as Secretary of the Society, from which position Professor Kline retired at the end of 1950. Professor Kline then spoke about some of his experiences, particularly in connection with the successful completion of the International Congress of Mathematicians.

A motion prepared by Professor P. H. Daus was adopted by those present expressing deep gratitude to the University of Florida and its members for the hospitality which was enjoyed during the week spent on the University Campus. In particular, a special debt of gratitude is due to the local Committee on Arrangements.

H. M. GEHMAN, *Secretary-Treasurer*

MAY MEETING OF THE ILLINOIS SECTION

The twenty-ninth annual meeting of the Illinois Section of the Mathematical Association of America was held at Southern Illinois University, Carbondale, Illinois, on Friday afternoon and Saturday forenoon, May 12–13, 1950. Professor M. G. Moore, Chairman of the Section, presided at all meetings.

There were thirty-six in attendance, including the following twenty-two members of the Association: A. H. Black, B. K. Brown, S. S. Cairns, A. E. Gault, Dilla Hall, E. C. Kiefer, Harry Levy, C. W. Mathews, C. T. McCormick, W. C. McDaniel, A. W. McGaughey, Karl Menger, B. E. Meserve, E. B. Miller, G. E. Moore, M. G. Moore, Ruth B. Rasmussen, L. D. Rodabaugh, R. C. Stephens, A. F. Svoboda, Alice Kelsey Wright, and R. K. Zeigler.

At the business meeting Saturday morning the following officers were elected for the coming year: Chairman, W. C. McDaniel, Southern Illinois University; Vice-Chairman, S. S. Cairns, University of Illinois, Urbana; and Secretary, E. C. Kiefer, Millikin University. The annual meeting for 1951 will be held at the University of Illinois in Urbana on May 11–12.

Resolutions, presented by a committee composed of A. W. McGaughey, R. K. Zeigler and C. T. McCormick, were presented and approved thanking the administration and members of the mathematics department of Southern Illinois University for their hospitality and efforts in making the meetings a success.

A committee composed of L. R. Ford, M. Anice Seybold, and C. T. McCormick, was appointed last year to report this year on the formation of an official organization for the Illinois Section. C. T. McCormick presented by-laws recommended by the committee, and these were adopted after discussion and some minor changes. There was no further business.

The following program was presented:

1. *Teaching functional thinking*, by Professor C. T. McCormick, Illinois Normal University.

Professor McCormick analyzed functional thinking, and stated that the development of functional thinking should be a primary objective of mathematical instruction. A historical survey of the meaning of the terms 'variable' and 'function' was given, and the significance of the historical development in teaching functional thinking was indicated. Several examples were used to show the advantages in defining a function in terms of the variables involved when the

function concept is being introduced to beginners. It was pointed out that order and correspondence are essential characteristics of a functional relationship. The importance of the tabular and graphical presentation in teaching functional thinking was emphasized throughout the paper.

2. *The development of mathematical statistics*, by Professor W. G. Madow, University of Illinois, introduced by Professor S. S. Cairns.

The main objectives of this paper were to review briefly the development of mathematical statistics and to discuss the implications of that development with respect to the training of statisticians. After summarizing the changes in the formulation of the logic of statistical inference due to R. A. Fisher, J. Neyman, and A. Wald, it was pointed out that though each of these changes affected the most elementary practical applications of statistical inference, the textbooks lag considerably and are essentially all now out of date in this subject. The development in the areas of analysis of variance and covariance, design of experiments, multi-variant analysis, quality control, sequential inference, survey techniques, theory of games, etc., were also mentioned, and the many branches of mathematics needed for their study discussed. The lessening importance of the derivation of distributions of statistical functions was stressed. It was recommended that the studen-interested in statistical theory be advised to take, during his first two years, a one-semester course in statistical methods such as that embodied in S. S. Wilks' *Elementary Statistical Analysis* (no calculus is really needed for this book), build up his mathematics to the point where as a junior or senior he can take courses on the level of A. M. Mood's *Introduction to Mathematical Statistics*, and then be in a position to decide whether to do graduate work in mathematical statistics. Certain applied statistics courses might also be taken. Since many good books are now appearing, the two named should not be taken to be the only two good available books. It was recommended that the student study many fields of mathematics since so many different areas of mathematics are already useful, and more areas are becoming useful from time to time. Finally it was emphasized that the time had passed when a person who had had six or twelve hours of statistics could be called a statistician any more than a similar amount of training would suffice for, say, a mathematician, economist, or psychologist.

3. *Some curves associated with the cuspidal cubic*, by Professor A. H. Black, Southern Illinois University.

From a point Q on a cuspidal cubic curve there is just one tangent line to the cubic. Call the point of contact T. As point Q describes the cubic, any point P on the line QT will describe a curve. The type and properties of the curve depend on the location of point P on the line QT. Several cases were considered.

4. *The aeracom and its uses*, by Dr. E. L. Buell, Northwestern University.

The aeracom is an electrical analog computer recently installed and partially developed at the Aerial Measurements Laboratory, located in the Technological Institute of Northwestern University. The machine is owned by the Bureau of Aeronautics, Navy Department. Its historical development, present elements, and theory of operation were described briefly, and an example of its application to the solution of an unclassified engineering design problem was presented. The duties of a mathematician in such a computing organization, and the need for further theoretical developments, were indicated.

5. *Mathematics in banking*, by Mr. Melvin Lockhard, Vice-President, First National Bank, Cobden, Illinois, introduced by Professor E. C. Kiefer.

This address was delivered at the dinner meeting.

6. *The determination of a quadric surface from its equation in general form*, by Dr. Ruth B. Rasmussen, Chicago City College, Wilson Branch.

If the equation (1) $ax^2+by^2+cz^2+2fyz+2hxy+21x+2my+2nz+d=0(c\neq 0)$ is solved by the quadratic formula, we have $z=-(gx-fy-n)/c\pm\sqrt{R/c}$ where $R=x^2(g^2-ac)+2xy(fg-ch)+y^2(f^2-bc)+2x(gn-cl)+2y(fn-cm)+(n^2-cd)$. Let $z=z_1+z_2$, where $z_1=-(gx+fy+n)/c$, and $z_2=\pm\sqrt{R/c}$, or (2) $c^2z_2=R$. Then since $z=z_1+z_2$, the required surface is obtained by adding the z_1 of the plane $z=z_1$ to the z_2 of the surface whose equation is given by (2) in the following manner. For a fixed value of z, say $z=k_1$, where $0\leqq k_1\leqq z_2$, let the cylinder projecting the curve of section of the plane $z=k_1$, and the surface (2) onto the z-plane be extended to intersect the plane $z=z_1$. On the projecting cylinder thus obtained lay off lengths equal to k_1, above and below the plane $z=z_1$. The locus of all such points will be two curves of section on the required surface. In every case the surfaces whose equations are given by (1) and (2) will be the same kind. The surface may also be identified by examining its coefficients. A surface can usually be identified more quickly by examining equation (2) than by applying the traditional methods to equation (1).

7. *The euclidian division algorithm*, by Professor B. E. Meserve, University of Illinois.

The greatest common divisor, d, of two integers m, and n may be computed using simply an array of two rows. Two more rows may then be added to the array to obtain A and B in the expression $d=Bm+An$. In the ring of polynomials in a single variable x there is a second synthetic procedure. Given $f=ax^m+a_1x^{m+1}+\cdots+a_m$, $a\neq 0$ and $g=bx^n+b_1x^{n-1}+\cdots b_n$, b—0, $n\leqq m$, the operation $f\otimes g=ag-bx^{n-m}f$ may be used to construct a new array containing the coefficients of the greatest common divisor of f and g. This array may be used in the determination of multiple roots and Sturm functions.

8. *The teaching of calculus from the engineering college point of view*, by Professor Karl Menger, Illinois Institute of Technology.

There are two principal schools of thought concerning the teaching of calculus to engineering students; one stresses the mechanical handling of the formulae, the other, the rigorous foundation of the results. The former group, mainly represented by older engineers, occasionally resorts to statements which, to the mathematician, are unmotivated if not untenable. The latter group, consisting of younger mathematicians, presents concepts without appeal to the practical man, and deductions whose use an engineering student can hardly appreciate. The present paper is devoted to a third point of view which is developed in the author's lecture notes "Introduction to Calculus"; to stress the *understanding* of the main facts of calculus. Three examples illustrate this plan. (1) Even to the beginner it should be obvious that, by virtue of the very meaning of the symbols, we have $\int_a^b f(x)dx=\int_a^b f(z)dz$. Engineering students often consider this equality as an astounding truth and sometimes even doubt its correctness. (2) We stress the inverse character of the derivative and the definite integral which too frequently is buried in a sea of formulae or a mass of deductions. (3) It is a strange inconsistency that engineers who deal exclusively with measurable finite quantities, interpret the formula

$$df=\frac{\partial f}{\partial x}dx+\frac{\partial f}{\partial y}dy$$

as expressing the effect on the function f of infinitely small changes of x and y. We emphasize really operational inequalities, e.g., in case that $\Delta x>0>\Delta y$, we have

$$\Delta x\cdot\text{Min}\cdot\frac{\partial f}{\partial x}+\Delta y\cdot\text{Max}\cdot\frac{\partial f}{\partial y}\leqq f(x+\Delta x,y+\Delta y)-f(x,y)\leqq\Delta x\cdot\text{Max}\cdot\frac{\partial f}{\partial x}+\Delta y\cdot\text{Min}\cdot\frac{\partial f}{\partial y},$$

where the maxima and minima refer to the rectangle with the corners (x, y) and $(x+\Delta x, y+\Delta y)$. If it is difficult to compute the precise maxima and minima they may be replaced by slightly larger and smaller numbers, respectively.

E. C. KIEFER, *Secretary*

JUNE MEETING OF THE PACIFIC NORTHWEST SECTION

The fourth annual meeting of the Pacific Northwest Section of the Mathematical Association of America was held at the University of Washington, Seattle, Washington, on June 16, 1950, in conjunction with the four-hundred and sixtieth meeting of the American Mathematical Society. Professor R. M. Winger, Chairman of the Section, invited Professors Moursund, Griffin and Merrill to share the duties of presiding at the afternoon session.

Some fifty persons were in attendance including the following thirty-seven members of the Association: R. W. Ball, J. P. Ballantine, R. A. Beaumont, J. L. Brenner, D. G. Chapman, Harold Chatland, Paul Civin, C. M. Cramlet, D. B. Dekker, Howard Eves, J. R. Fleming, K. S. Ghent, F. L. Griffin, S. G. Hacker, Mary E. Haller, Edwin Hewitt, J. W. Hurst, R. D. James, S. A. Jennings, L. G. Jones, J. M. Kingston, M. S. Knebelman, L. H. McFarlan, A. S. Merrill, M. R. Moore, A. F. Moursund, D. C. Murdoch, Ivan Niven, Gloria Oliver, T. S. Peterson, Louise J. Rosenbaum, R. A. Rosenbaum, W. M. Stone, N. Y. Tang, Sylvia Vopni, L. B. Williams, R. M. Winger.

A business meeting was held in the evening at which the following officers were elected: Chairman, Professor A. F. Moursund, University of Oregon; Vice-Chairman, Professor R. D. James, University of British Columbia; Secretary-Treasurer, Professor J. M. Kingston, University of Washington.

It was decided to hold the next meeting of the section at the State College of Washington on June 15, 1951, and to make it, if possible, a joint meeting with the Society as in 1950. The Committee on Improvement in Qualifications of Mathematics Teachers in Secondary Education in the Pacific Northwest was discharged, having transmitted its findings to superintendents of education throughout the United States. The Committee on a Problem Book in Mathematics was complimented on its accomplishments and asked to continue its work. At the time of the latter committee chairman's report at the afternoon session, mimeographed copies of the present stage of the book (containing 77 problems) were distributed.

The afternoon session consisted of the following five thirty-minute papers and an invited hour address:

1. *Lucas' matrix*, by Professor J. L. Brenner, State College of Washington.

The nth power of the matrix

$$\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \text{ is } \begin{pmatrix} u_{n-1} & u_n \\ u_n & u_{n-1} \end{pmatrix},$$

where u_n is Fibonacci's number. More generally, the nth power of

$$\begin{pmatrix} a-b & -ab \\ 1 & 0 \end{pmatrix} \text{ is } \begin{pmatrix} u_{n-1} & -abu_n \\ u_n & -abu_{n-1} \end{pmatrix},$$

where $u_n=(a^n-b^n)/(a-b)$ is Lucas' number. From these facts it is easy to deduce a part of the general theory of these numbers.

The sequences $u_n = A_1u_{n-1} \cdots A_ru_{n-r}$ have properties some of which are quickly obtained from the study of a matrix of dimension r which generalizes the matrices above.

2. *Some remarks on the teaching of mathematical statistics to undergraduates*, by Professor S. A. Jennings, University of British Columbia.

This paper is a report on two years' experience of the author with a course in mathematical statistics at the University of British Columbia. The main point brought out is that the usual emphasis in calculus courses, directed as they are towards engineering and physics students, leaves students totally unprepared for many of the concepts and analytical tools that are important in the field of mathematical statistics. Various examples are given to illustrate this point.

3. *Dissections*, by Professor Howard Eves, Oregon State College.

This was an invited hour address. The paper furnished a survey of dissection theory from the early uses made of it in classical Greek geometry up to the recent decomposition paradoxes of modern point set theory. The concepts and elementary properties of congruency by addition and congruency by subtraction were first introduced and illustrated, and then the fundamental theorem of polygonal dissection was established. Some corollaries were considered, and then the analogous questions in three space were examined. This latter was based on the work of Dehn, Schatunowsky, Kagan, Lennes, and Suss. Next followed a selected catalogue of twenty dissection curiosities. Here was summarized the meager known results on the minimum dissection problem. Consideration was given to some of the unusual dissections of Dudeney, Travers, Escott, Goldberg, and Wheeler. Also considered were some of the problems arising from the recent work on squaring the square, chess board dissections, ham sandwich problems, tesselations in the euclidean and hyperbolic planes, and a theorem due to Moser. Many unsolved problems were mentioned. The paper concluded with a discussion of some of the paradoxes which arise when dissection theory is extended to general point sets. Such paradoxes are those of von Neuman, Sierpinski-Mazurkiewicz, Ruziewicz, Hausdorff, and Banach-Tarski.

4. *Elementary properties of convex functions*, by Professor R. D. James, University of British Columbia.

The purpose of this note is to point out that in an elementary calculus course, convex functions are easier to deal with than continuous functions. It is a simple matter to show that a bounded convex function has left and right-hand derivatives everywhere and that the derivatives are monotone increasing functions. Moreover, the Riemann integral of a bounded monotone increasing function is a convex function of its upper limit.

5. *Report of the problem book committee of the P. N. W. section of the Association*, by Professor R. A. Rosenbaum, Reed College.

The chairman of the committee set up last year by this section of the Association reported on the progress made during the year. A preliminary list of problems, most of them with short discussions, was mimeographed and distributed to interested mathematicians. The purpose of the problem set (to interest good undergraduate students to proceed beyond their everyday class work) was reiterated, and the cooperation of those present was solicited in an effort to extend and improve the set of problems.

6. *Euclid's algorithm in quadratic fields*, by Professor Harold Chatland, Montana State University.

Let m be a square-free rational integer. The field $R(m^{1/2})$ is said to be euclidean if for integers α, $B \neq 0 \in R(m^{1/2})$ there exists an integer $\gamma \in R(m^{1/2})$ such that

$$|N(\alpha - B\gamma)| < |N(B)|.$$

Many investigators have obtained results for various values of m. One may find an account of the literature in a paper by H. Chatland, *On the euclidean algorithm in quadratic number fields*, Bull. Am. Math. Soc., vol. 55, No. 10, pp. 948–953. A very significant result was recently obtained by H. Davenport who showed the non-existence of Euclid's Algorithm for $m > 2^{14}$. This result may be found in his paper, *Indefinite binary quadratic forms, and euclid's algorithm in real quadratic fields*, which is soon to appear in the *Proc. London Math. Soc.* As a result of all previous investigations and those mentioned above, the existence or non-existence of Euclid's algorithm in $R(m^{1/2})$ was then decided for all values of m except $m = 193, 241, 313, 337, 457, 601$. The algorithm was shown not to exist for any one of these six values by K. Inkeri, *Über den euklidischen Algorithmus in quadratischen Zahlkörpern*, Ann. Acad. Sci. Fenn., No. 41, 1947, pp. 1–35, and independently by Chatland and Davenport, *Euclid's algorithm in real quadratic fields* to appear in the *Canadian Journal of Mathematics*.

Thus as a result of all investigations it has been established that Euclid's Algorithm holds in $R(m^{1/2})$ if $m = -11, -7, -3, -2, -1, 2, 3, 5, 6, 7, 11, 13, 17, 19, 21, 29, 33, 37, 41, 57, 73, 97$, and in no other case.

J. M. KINGSTON, *Secretary*

CALENDAR OF FUTURE MEETINGS

Joint meeting with American Society for Engineering Education, Michigan State College, East Lansing, June 25–26, 1951.

Thirty-second Summer Meeting, University of Minnesota, Minneapolis, September 3–4, 1951.

Thirty-fifth Annual Meeting, Brown University, Providence, Rhode Island, December 29, 1951.

The following is a list of the Sections of the Association with dates of future meetings so far as they have been reported to the Associate Secretary.

ALLEGHENY MOUNTAIN, Duquesne University, Pittsburgh, Pennsylvania, May, 1951.

ILLINOIS, University of Illinois, Urbana, May 11–12, 1951.

INDIANA, May 5, 1951.

IOWA, Wartburg College, Waverly, April 20–21, 1951.

KANSAS, University of Kansas, Lawrence, April 7, 1951.

KENTUCKY, Eastern Kentucky State College, Richmond, April 28, 1951.

LOUISIANA-MISSISSIPPI

MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA, United States Naval Academy, Annapolis, Maryland, April 28, 1951.

METROPOLITAN NEW YORK, Manhattan College, April 7, 1951.

MICHIGAN, Michigan State College, East Lansing, March 24, 1951.

MINNESOTA, College of St. Benedict, St. Joseph, April 28, 1951.

MISSOURI, Central College, Fayette, April 6, 1951.

NEBRASKA, University of Nebraska, Lincoln, April 21, 1951.

NORTHERN CALIFORNIA

OHIO, Ohio State University, Columbus, April 21, 1951.

OKLAHOMA

PACIFIC NORTHWEST, State College of Washington, Pullman, June 15, 1951.

PHILADELPHIA, University of Pennsylvania, Philadelphia, November 24, 1951.

ROCKY MOUNTAIN, Colorado State College of Education, Greeley, April 20–21, 1951.

SOUTHEASTERN, Vanderbilt University and Peabody College, Nashville, Tennessee, March 16–17, 1951.

SOUTHERN CALIFORNIA

SOUTHWESTERN, University of New Mexico, Albuquerque, March 23–24, 1951.

TEXAS, Southern Methodist University, Dallas, April 27–28, 1951.

UPPER NEW YORK STATE, Hamilton College, Clinton, May 5, 1951.

WISCONSIN, Carroll College, Waukesha, May 12, 1951.

GEORGE BANTA PUBLISHING COMPANY, MENASHA, WISCONSIN

THE AMERICAN MATHEMATICAL MONTHLY

THE OFFICIAL JOURNAL OF

THE MATHEMATICAL ASSOCIATION OF AMERICA, INC.

VOLUME 58 NUMBER 4

CONTENTS

APRIL 1951

The AMERICAN MATHEMATICAL MONTHLY

(FOUNDED IN 1894 BY BENJAMIN F. FINKEL)

EDITORIAL CORRESPONDENCE should be addressed to the Editor, C. V. NEWSOM, State Education Building, Albany 1, N. Y.

ADVERTISING CORRESPONDENCE should be addressed to J. F. RANDOLPH, Morey Hall, University of Rochester, Rochester 3, N. Y.

NOTICE OF CHANGE OF ADDRESS by members of the Association as well as correspondence regarding subscriptions to the MONTHLY should be sent to the Secretary-Treasurer, H. M. GEHMAN, University of Buffalo, Buffalo 14, N. Y. Change of address must reach the Secretary-Treasurer about six weeks before the change can become effective.

THIS IS THE OFFICIAL JOURNAL OF THE
MATHEMATICAL ASSOCIATION OF AMERICA, INC.

(Devoted to the Interests of Collegiate Mathematics)

Entered as second class matter at the post office at Menasha, Wis. Acceptance for mailing at special rate of postage provided for in the Act of February 28, 1925, embodied in Paragraph 4, Section 538, P. L. and R., authorized April 1, 1926.

SUBSCRIPTION PRICE: To Members, $4 a Year, To Others, $5 a Year.

PUBLISHED BY THE ASSOCIATION at Menasha, Wisconsin, and Albany, N. Y. during the months of January, February, March, April, May, June-July, August-September, October, November, December.

THE FOREMOST TEXTBOOK OF MODERN TIMES*

C. B. BOYER, Brooklyn College

The most influential mathematics textbook of ancient times (or, for that matter, of all times) is easily named. The *Elements* of Euclid, appearing in over a thousand editions [1], has set the pattern for the teaching of elementary geometry ever since it was composed more than two and a quarter millenia ago. The medieval textbook which most strongly influenced mathematical development is not so easily selected; but a good case can be made out for *Al jabr wa'l muquabala* of Al-Khowarizmi [2], just about half as old as the *Elements*. From this Arabic work algebra took its name and, to a great extent, its origin. Is it possible to indicate a modern textbook of comparable influence and prestige? Some would mention the *Gēomētrie* of Descartes or the *Principia* of Newton or the *Disquisitiones* of Gauss; but in pedagogical significance these classics fell far short of a work less well known. The *Gēomētrie* was not strictly a textbook, and hence most mathematicians learned their analytic geometry from the works of other authors, such as Schooten, De Witt, Sluze, and Lahire. The *Principia*, the greatest of all works in the field of science, affected the course of pure mathematics only indirectly; few readers understood the elements of the calculus which it contained, and the effective teachers of the differential calculus were Leibniz, L'Hospital, and the Bernoullis. The *Disquisitiones*, a work of great profundity, was too specialized to make its influence widely felt except among ardent number-theorists. It is perhaps significant that none of these three modern works appeared in what may be considered the greatest age of textbooks in recent times. The eighteenth century often is characterized as a prosy age in the history of mathematics, for it contributed no single discovery which captured the imagination as had analytic geometry and the calculus. And yet the century was of capital importance in the consolidation of earlier work, a task which was facilitated by the appearance of outstanding textbook writers. At the opening of the century one finds the texts of L'Hospital dominating the fields of analytical conics and the calculus; at the close there were the textbooks of Lacroix which covered the whole elementary field and which appeared in dozens of editions [3], not to mention the Legendre *Euclid* of the same time. But over these well known textbooks there towers another, a work which appeared in the very middle of the great textbook age and to which virtually all later writers admitted indebtedness. This was the *Introductio in analysin infinitorum* of Euler, published in two volumes in 1748 [4]. Here in effect Euler accomplished for analysis what Euclid and Al-Khowarizmi had done for synthetic geometry and elementary algebra, respectively. The function concept and infinite processes had arisen by the seventeenth century, yet it was Euler's *Introductio* which fashioned these into the third member of the mathematical triumvirate comprising geometry, algebra, and analysis. From the point

* Presented at the International Congress of Mathematicians, Cambridge, Mass., Sept. 1, 1950.

of view of leading textbooks, then, one might refer (with, of course, some oversimplification) to geometry as ancient, algebra as medieval, and analysis as modern.

Euler was not the first to use the word "analysis," or even to incorporate it into the title of a book; but he did give the word a new emphasis. Plato's analysis had reference to the logical order of steps in geometrical reasoning, and the analytic art of Viète was akin to our algebra; but the analysis of Euler comes close to the modern orthodox discipline, the study of functions by means of infinite processes, especially through infinite series. It is in this newer sense that Euler, especially after 1748, used the word; and it is for this reason that he has been referred to as "analysis incarnate" [5]. The word, analysis, took on a new lease of life. Euler himself used it in the titles of dozens of his published papers; and soon others were publishing books on "analytic optics" and "analytical mechanics," on "analytical trigonometry" and "analytic geometry." Euler avoided the phrase "analytic geometry," probably to obviate confusion with the older Platonic usage; yet the second volume of the *Introductio* has been referred to, not inappropriately, as "the first text on analytic geometry" [6], and this appeared more than a century after the publication of *La géométrie*! It is the earliest systematic graphical study of functions, not only of a single variable, but of two as well; and Euler's analysis is transcendental as well as algebraic. The notion of "elementary function" stems largely from the *Introductio*, but the book contains also such higher curves as $y=x^{\sqrt{2}}$ and $y^x=x^y$. Polar coordinates had appeared in at least half a dozen earlier works, including Newton's *Method of fluxions*; but the clarity and generality of Euler's treatment in the *Introductio* were such that most subsequent writers traced the use of polar coordinates back to this book. Here, in fact, the spiral of Archimedes appears in its dual form, probably for the first time. The use of parametric equations, implicit even in the work of Descartes, was first systemized in the *Introductio*; and here also one finds formalized for the first time the equations for the transformation of coordinates for two and three dimensions, the latter in a form still referred to as "Euler's equations." The *Introductio* was the first textbook to recognize the five proper general quadric surfaces as members of a single family, a century and a half after Kepler had done the same for the conics, and the names proposed were very similar to those now adopted [7]. In this same book Euler also did for plane quartic curves what Newton had done for cubics—he ordered them according to genus and species.

The word "first" is a hazardous one to use in the history of mathematics, and yet it has been applied freely to Euler's *Introductio.* The cases already cited by no means exhaust the respects in which this textbook was first. It contains the earliest algorithmic treatment of logarithms as exponents and of the trigonometric functions as numerical ratios. It was the first textbook to list systematically the multiple-angle formulas, calling attention to the periodicities of the functions; and it included the first general analytic treatment of these as infinite products, as well as their expansion into infinite series. The well-

known "Euler identities," relating the trigonometric functions to imaginary exponentials, are also found here. The first volume contains as well an exposition of continued fractions and some excellent work of the zeta function and number theory [8].

In scope alone the *Introductio* ranks among the greatest of textbooks, for it is doubtful that any other essentially didactic work includes as large a portion of original material which survives in the college courses of today. Yet the book is outstanding also for its pedagogical lucidity. The immortal Gauss, a man not given to exaggerated expressions of flattery, held that "The study of Euler's works will remain the best school for the various fields of mathematics, and nothing can replace it" [9]. Written as it was more than two hundred years ago (a letter from Euler to D'Alembert indicates that it was completed by 1745), the *Introductio in analysin infinitorum* nevertheless can be read with comparative ease by the modern student—unlike the *Géométrie*, the *Principia*, or the *Disquisitiones*. Not only is the viewpoint quite similar to that of today; even the terminology and notation are almost modern—or perhaps, as Struik has well written, "we should better say that our notation is almost Euler!" [10]. Under the circumstances one should expect that a textbook exhibiting the qualities of the *Introductio* would boast an impressive list of editions and translations; but the facts belie this. The work was not reprinted until the time of the author's death, thirty-five years later; and, including reprintings and incomplete translations, the fewer than a dozen editions are about equally distributed among the three languages Latin, French, and German [11]. No English translation has appeared, and a partial Russian translation apparently has not been published. However, that the worth of a textbook is not necessarily measured by the number of its editions is conclusively evidenced by the *Introductio*, the influence of which was unusually pervasive. Almost without exception the authors of the ubiquitous compendia of the second half of the eighteenth century refer to Euler as the source of their analysis. The *Introductio* became, in a sense, the prototype of modern textbooks. Is not imitation the sincerest form of flattery?

References

1. An excellent English edition is *The thirteen books of the Elements*, translated from the text of Heiberg with introduction and commentary by T. L. Heath, 3 vols., Cambridge, 1908.

2. See *Robert of Chester's Latin translation of the Algebra of Al-Khowarizmi*, with an introduction, critical notes, and an English version by L. C. Karpinski, New York, 1915.

3. In 1848, for example, there appeared at Paris the 20th edition of his *Traité élémentaire d'arithmétique* and the 16th edition of his *Éléments de géométrie*; and ten years later his *Éléments d'algèbre* appeared in a 20th edition.

4. Published at Lausanne.

5. See, e.g., E. T. Bell, *Men of mathematics* (New York, 1937), chapter 9.

6. D. J. Struik, *A concise history of mathematics* (2 vols., New York, 1948), II, 169.

7. His names were elliptoides, elliptico-hyperbolicae, hyperbolico-hyperbolicae, elliptico-parabolicae, and parabolico-hyperbolicae.

8. An extensive account of this work is found in the preface, by Andreas Speiser, to series 1, volume 9, of Euler's *Opera omnia* (Geneva, 1945).

9. *Ibid.*, p. viii. Speiser adds to the words of Gauss: "The *Introductio* in this connection may stand in first place."

10. Struik, *op. cit.*, II, 174.

11. Latin editions appeared in 1748, 1783, 1797, and in volumes 9 and 10 of Euler's *Opera*, series 1; French editions were published in 1785, 1796, and 1835; German editions appeared in 1788, 1835, and 1885. For bibliographic details on Euler's multitudinous works see Gustav Eneström, "Verzeichnis der Schriften Leonhard Eulers," *Jahresbericht der Deutschen Mathematiker-Vereinigung, Ergänzungsband* IV, Leipzig, 1910.. Cf. also P. H. Fuss, *Correspondance mathématique et physique de quelques célèbres géomètres du XVIII^{ème} siècle. Précédée d'une notice sur les travaux de Léonard Euler* (2 vols., St. Pétersbourg, 1843); and Felix Müller, "Über bahnbrechende Arbeiten Leonhard Eulers aus der reinen Mathematik," *Abhandlungen zur Geschichte der mathematischen Wissenschaften*, XXV (1907), 61–116.

PHYSICAL FAMILIES IN THE GRAVITATIONAL FIELD OF FORCE

EDWARD KASNER AND JOHN DE CICCO, Columbia University and De Paul University

1. Introduction. The important physical families of curves connected with an arbitrary positional field of force are: (a) dynamical trajectories, the paths of motion of a particle; (b) general catenaries, the curves formed from the resulting equilibrium when an inextensible flexible string is suspended in any arbitrary field of force; (c) brachistochrones, the curves in a general conservative field of force along which the time of the constrained motion between any two points is least; (d) velocity curves, the curves in an arbitrary field of force, to any one of which there corresponds a speed v_0 such that a particle, starting from any lineal element of the curve with the speed v_0, describes a trajectory osculating the curve.

These four systems of physical curves may all be obtained as special cases of the following general problem [1]: To find all curves along which a constrained motion is possible such that the pressure P, normal to the curve, is proportional to the normal component N of the force vector. We shall term any such family a *physical system* S_k. Thus along any curve of a system S_k, we have $P=kN$, where $k(\neq -1)$ is the constant proportionality.

The important special cases of physical interest of a system S_k are (a) dynamical trajectories, $k=0$, (b) general catenaries, $k=1$; (c) generalized brachistochrones, $k=-2$: (d) velocity curves, $k=\infty$. These four families are systems S_0, S_1, S_{-2}, S_∞.

In our previous work [2], we have discussed systems S_k of ∞^3 curves in any given arbitrary field of force. It is our purpose to investigate the system S_k in the gravitational field of force where the force vector is identically constant. The integration of a system S_k can be obtained in terms of a single quadrature. If k is a rational, then the only systems S_k which may be expressible

in terms of elementary functions are those for which k is of the form $k=(1-n)/n$, or of the form $k=(1-2n)/(1+2n)$, where n is an integer, positive, negative, or zero. If it be required that a system S_k be expressible in terms of elementary functions and that k be an integer, then the only solutions are (a) dynamical trajectories S_0, (b) catenaries S_1, (c) brachistochrones S_{-2}, (d) velocity curves S_∞, and (e) another system S_{-3} of circles.

The development of the present article will be from elementary principles.

2. Preliminary formulas. The time t is measured in seconds and the magnitude of the force vector F is measured in dynes. The distance s and the mass m may be measured in centimeters and grams, or in feet and pounds, respectively. The force exerted by m grams or m pounds is mg dynes where g is 980 or 32 according as the first or second system of units is used [3].

For the two systems of units discussed above, Newton's second law of motion is $F=ma$, where F and a are the force and acceleration vectors, and m is the mass.

Let a particle of mass m traverse a curve C. At a time t_0, let its position on C be (x_0, y_0), its speed be v_0, and the inclination of its direction of motion be θ_0. At the time t_0, the horizontal and vertical components of its velocity vector are $(v_0 \cos \theta_0, v_0 \sin \theta_0)$.

The normal component a_n and the tangential component a_t of the acceleration vector a of the particle are

$$(1) \qquad a_n = \frac{v^2}{r}, \qquad a_t = \frac{dv}{dt} = v\frac{dv}{ds},$$

where v is the speed, s is the length of arc, and r is the radius of curvature.

The rectangular components of the force vector F in a gravitational field of force, acting on a particle of mass m, are

$$(2) \qquad \phi(x, y) = 0, \qquad \psi(x, y) = -mg.$$

If the particle is constrained to move along a given curve C, the normal component N and the tangential component T of the force vector are

$$(3) \qquad N = \psi\frac{dx}{ds} - \phi\frac{dy}{ds} = -mg\frac{dx}{ds}, \qquad T = \phi\frac{dx}{ds} + \psi\frac{dy}{ds} = -mg\frac{dy}{ds}.$$

By (1) and (3), the pressure P of the particle along this curve is given by the formula

$$(4) \qquad P = \frac{mv^2}{r} - N = m\left(\frac{v^2}{r} + g\frac{dx}{ds}\right).$$

Also by (1) and (3), the speed v of the particle obeys the condition $vdv/ds = -gdy/ds$. Integrating this, the result is the energy equation

$$\frac{v^2}{2} = h - gy, \tag{5}$$

where the energy constant has the value

$$h = \frac{v_0^2}{2} + gy_0. \tag{6}$$

3. The systems S_k. A system S_k, where $k \neq -1$, in an arbitrary field of force is defined to consist of curves along each of which a constrained motion is possible such that the pressure $P = mv^2/r - N$ is proportional to the normal component N of the force vector F. Thus $P = kN$ where $k \neq -1$.

The equations of motion of a system S_k in a general field of force are

$$mv^2 = (k + 1)rN, \qquad mv\frac{dv}{ds} = T. \tag{7}$$

For each fixed k, a system S_k is a three parameter family of curves. Thus S_k consists of ∞^3 curves.

By (3) and (4), the equations of motion of a system S_k of ∞^3 curves in the gravitational field of force are

$$v^2 = -(k + 1)gr\frac{dx}{ds}, \qquad v\frac{dv}{ds} = -g\frac{dy}{ds}. \tag{8}$$

By use of Euler's equation in the Calculus of Variations, it can be shown that these are the extremals of the variation problem

$$\int v^{k+1}ds = \text{min.}, \quad \text{or} \quad \int (h - gy)^{(k+1)/2}ds = \text{min.} \tag{9}$$

We consider the *system S_∞* of velocity curves. In an arbitrary field of force, a curve C is a velocity curve corresponding to the speed v_0 if any dynamical trajectory, starting from any lineal element of C with the speed v_0, is initially osculated by C. It is found that the system of velocity curves is the limiting case of a system S_k as k becomes infinite. For this reason, we denote a velocity system by the symbol S_∞.

In a general field of force, the differential equation of a velocity system S_∞ is

$$Nr = mv_0^2. \tag{10}$$

A system S_∞ consists of ∞^3 curves.

In a gravitational field of force, the system S_∞ of velocity curves is given by the differential equation

$$gr\frac{dx}{ds} = -v_0^2. \tag{11}$$

It can be shown by Euler's equation in the Calculus of Variations that these are the extremals of the variation problem

$$\int e^{-gy/v_0^2}ds = \min. \tag{12}$$

4. The integration of the velocity system S_∞. Let θ denote the inclination of the tangent line to a curve. Equation (11) can be written in the form

$$gdx = -v_0^2 d\theta, \tag{13}$$

where $dy = \tan\theta dx$.

The velocity system S_∞ in the gravitational field of force is

$$x = x_0 - \frac{v_0^2}{g}(\theta - \theta_0), \qquad y = y_0 + \frac{v_0^2}{g}\log\frac{\cos\theta}{\cos\theta_0}, \tag{14}$$

where θ is the inclination of the tangent line. The explicit form of any velocity curve is

$$y = y_0 + \frac{v_0^2}{g}\log\left[\cos\frac{g}{v_0^2}(x - x_0) + \tan\theta_0 \sin\frac{g}{v_0^2}(x - x_0)\right]. \tag{15}$$

5. The integration of the system S_k where $k \neq -1$ and $k \neq \infty$. By (5) and (8), we find

$$v^2 = 2(h - gy) = -(k+1)gr\frac{dx}{ds}\cdot \tag{16}$$

Thus it is necessary to solve the equations

$$dx = \frac{2(gy - h)}{(k+1)g}d\theta, \qquad dy = \tan\theta dx. \tag{17}$$

From (17), it is deduced that

$$\frac{-gdy}{h - gy} = \frac{2}{k+1}\tan\theta d\theta. \tag{18}$$

Using (6), it is found that

$$y = y_0 + \frac{v_0^2}{2g}\left[1 - \left(\frac{\cos\theta_0}{\cos\theta}\right)^{2/(k+1)}\right]. \tag{19}$$

Also

$$(20) \qquad dx = \cot\theta dy = -\frac{v_0^2}{g(k+1)}(\cos\theta_0)^{2/(k+1)}(\sec\theta)^{2/(k+1)}d\theta.$$

Thus

$$(21) \qquad x = x_0 - \frac{v_0^2}{g(k+1)}(\cos\theta_0)^{2/(k+1)}\int_{\theta_0}^{\theta}(\sec\theta)^{2/(k+1)}d\theta.$$

Any system S_k where $k \neq -1$ and $k \neq \infty$ in the gravitational field of force consists of the ∞^3 curves

$$(22) \qquad \begin{aligned} x &= x_0 - \frac{v_0^2}{g(k+1)}(\cos\theta_0)^{2/(k+1)}\int_{\theta_0}^{\theta}(\sec\theta)^{2/(k+1)}d\theta, \\ y &= y_0 + \frac{v_0^2}{2g}\left[1 - \left(\frac{\cos\theta_0}{\cos\theta}\right)^{2/(k+1)}\right], \end{aligned}$$

where θ is the inclination of the tangent line. In explicit form, this family of ∞^3 curves is given by the Tchebycheff integral

$$(23) \qquad x = x_0 + \int_{y_0}^{y}\left[\sec^2\theta_0\left\{1 - \frac{2g}{v_0^2}(y - y_0)\right\}^{k+1} - 1\right]^{-1/2} dy.$$

By a theorem of Tchebycheff [4] this integral (23) is expressible in terms of elementary functions only in the case where $1/(1+k)$, or $(1-k)/2(1+k)$, is an integer. Hence $1/(1+k) = n$, or $(1-k)/2(1+k) = n$, where n is an integer. Therefore $k = (1-n)/n$, or $k = (1-2n)/(1+2n)$.

If k is rational, then the Tchebycheff integral (23) *is expressible in terms of elementary functions only in the cases $k = (1-n)/n$, or $k = (1-2n)/(1+2n)$, where n is an integer.*

The following result may be readily established:

The only systems S_k where $k(\neq -1)$ is an integer and which are expressible in terms of elementary functions are (a) S_0, *dynamical trajectories*; (b) S_1, *catenaries*; (c) S_{-2}, *brachistochrones* (*ordinary cycloids in this case*); (d) S_∞, *velocity curves*; (e) S_{-3}, *circles.*

The proof of the above result is as follows. In the first case, $1/(1+k) = n$ is an integer only when $k = 0$ or $k = -2$. These give the systems S_0 of dynamical trajectories and S_{-2} of brachistochrones.

In the second case, it is desired to solve the equation

$$(24) \qquad \frac{1-k}{2(1+k)} = n,$$

for integral values of k and n. Evidently k is odd, say $k=2m-1$. Then $n=(1-m)/2m$. Thus m is an odd integer such that $1-m$ is divisible by m. This can happen only when $m=1$ or $m=-1$. Hence $n=0$ or $n=-1$. Therefore $k=1$ or $k=-3$. The first solution gives S_1, the system of catenaries, and the second solution gives S_{-3}, a system which we now examine.

Substitute $k=-3$ into (22). The parametric equations of S_{-3} are

$$\begin{aligned} x &= x_0 - \frac{v_0^2}{2g}\tan\theta_0 + \frac{v_0^2}{2g\cos\theta_0}\sin\theta, \\ y &= y_0 + \frac{v_0^2}{2g} - \frac{v_0^2}{2g\cos\theta_0}\cos\theta. \end{aligned} \tag{25}$$

These are circles. The center (X, Y) and the radius R of any such circle are

$$X = x_0 - \frac{v_0^2}{2g}\tan\theta_0, \qquad Y = y_0 + \frac{v_0^2}{2g}, \qquad R = \frac{v_0^2}{2g\cos\theta_0}. \tag{26}$$

Substitute $k=0$ into (22). The system S_0 of dynamical trajectories is

$$\begin{aligned} x &= x_0 + \frac{v_0^2}{g}\sin\theta_0\cos\theta_0 - \frac{v_0^2}{g}\cos^2\theta_0\tan\theta, \\ y &= y_0 + \frac{v_0^2}{2g} - \frac{v_0^2}{2g}\cos^2\theta_0\sec^2\theta. \end{aligned} \tag{27}$$

These are the vertical parabolas

$$y = y_0 + \frac{v_0^2}{2g}\sin^2\theta_0 - \frac{g}{2v_0^2\cos^2\theta_0}\left(x - x_0 - \frac{v_0^2}{g}\sin\theta_0\cos\theta_0\right)^2. \tag{28}$$

Substitute $k=1$ into (22). The system S_1 of catenaries is

$$\begin{aligned} x &= x_0 - \frac{v_0^2}{2g}\cos\theta_0\log\frac{\sec\theta + \tan\theta}{\sec\theta_0 + \tan\theta_0}, \\ y &= y_0 + \frac{v_0^2}{2g}(1 - \cos\theta_0\sec\theta). \end{aligned} \tag{29}$$

In explicit form, S_1 consists of the catenaries

$$y = y_0 + \frac{v_0^2}{2g}\left[1 - \cosh\frac{2g(x - x_0)}{v_0^2\cos\theta_0} + \sin\theta_0\sinh\frac{2g(x - x_0)}{v_0^2\cos\theta_0}\right]. \tag{30}$$

These are catenaries of the reversed gravitational field of force. If v_0^2 is taken to be negative, then we obtain the catenaries of the given gravitational field of force.

Finally substitute $k=-2$ into (22). The system S_{-2} of brachistochrones is

$$x = x_0 - \frac{v_0^2}{4g\cos^2\theta_0}(2\theta_0 + \sin 2\theta_0) + \frac{v_0^2}{4g\cos^2\theta_0}(2\theta + \sin 2\theta),$$
$$(31)\qquad y = y_0 + \frac{v_0^2}{2g}\left[1 - \left(\frac{\cos\theta}{\cos\theta_0}\right)^2\right].$$

These are ordinary cycloids.

References

1. E. Kasner, Differential geometric aspects of dynamics, Amer. Math. Soc. Colloquium Publ. 1913, 1934, 1938. See especially pp. 91–97.
2. See the following series of papers by Kasner and De Cicco: Physical curves, Proc. Nat. Acad. Sci., 34, 68 (1948); A generalized theory of dynamical trajectories, Trans. Amer. Math. Soc. 54, 23–38 (1943). Physical systems of curves in space, Proc. Nat. Acad. Sci. 35, 106–108 (1949).
3. Sir Edmund Whittaker, Analytical dynamics, Dover Publications, (1944). See p. 29.
4. J. F. Ritt, Integration in finite terms, Columbia University Press, (1948). See pp. 37–39.

PLANE AREAS BY COMPLEX INTEGRATION

J. D. MANCILL, University of Alabama

1. Introduction. The purpose of this paper is to show how the theory of functions of a complex variable can be employed to determine the area enclosed by simple closed curves in the complex plane. The real line integral

$$(1)\qquad A(C) = 1/2\int_C [-ydx + xdy],$$

which represents the area in the real plane enclosed by the simple closed curve C, is very difficult, if not impossible, to evaluate in practice except for the simpler curves. We shall show how this line integral can be expressed as a complex integral of a very simple type. Under certain conditions the evaluation of this complex integral can be greatly facilitated by the theory of residues and conformal transformations. In this way the problem of determining the area enclosed by a simple closed curve is made to depend upon the theory of residues and conformal mapping, like so many other problems in applied mathematics. This application of complex integration seems to have been overlooked in the literature.

We shall illustrate the facility of this procedure by finding the area of any Joukowski aerofoil section. The result is a remarkable property of these aerofoils.

2. Statement of the problem. Suppose that the region whose area we seek is enclosed by the simple closed curve C, whose equation is

$$\bar{z} = f(z), \tag{2}$$

in the isotropic coordinates $(z, \bar{z})$ where $z=x+iy$ and $\bar{z}=x-iy$. Since

$$x = (z + \bar{z})/2, \qquad y = (z - \bar{z})/2i,$$

any function of the variables (x, y) can be expressed in terms of z and $\bar{z}$. For example, any circle with center at (h, k) and radius r can be represented by the equation

$$z\bar{z} - \overline{P}z - P\bar{z} + K = 0, \quad \text{or} \quad \bar{z} = (\overline{P}z - K)/(z - P), \tag{3}$$

where $P=h+ik$, $\overline{P}=h-ik$, and $K=h^2+k^2-r^2$.

Consider the integral

$$\int_C \bar{z}dz = \int_C (xdx + ydy) + i\int_C (-ydx + xdy),$$

where we assume that the integral is taken along the closed curve C in the positive sense. Since C is closed the first integral in the second member of the last equation is zero, and the second integral is $2i$ times the area integral (1). Therefore, we have

$$A(C) = 1/2i \int_C \bar{z}dz, \tag{4}$$

as a formula for the area enclosed by the curve C. This formula can be expressed in terms of z alone for a given curve C by substituting for $\bar{z}$ from the equation (2) of C, which gives

$$A(C) = 1/2i \int_C f(z)dz. \tag{5}$$

Now, if we assume that the function $f(z)$ is single-valued and meromorphic interior to C and analytic on C, then we may apply the theory of residues to the evaluation of the integral (5).

As a simple example, let us find the area of the circle (3) by this method. In this case, $f(z)=(\overline{P}z-K)/(z-P)$ which is single-valued and meromorphic interior to C, the only singularity being a simple pole at $z=P$, and is analytic on C. Therefore, from the theory of residues, we have

$$\begin{aligned} A(C) &= 1/2i \int_C \frac{(\overline{P}z - K)dz}{z - P} \\ &= \left[2\pi i \times \operatorname*{Res.}_{z=P} \frac{\overline{P}z - K}{z - P}\right]/2i \\ &= \pi(\overline{P}P - K) = \pi r^2. \end{aligned}$$

The hypothesis that $f(z)$ in equation (2) is single-valued involves a considerable restriction on the curve C. For example, the equation (2) for the simple ellipse $x^2+4y^2=4$ is

$$\bar{z} = [5z \pm 4(z^2 - 3)^{1/2}]/3$$

and $f(z)$ is not single-valued inside the ellipse. Consequently, the theory of residues is not applicable to the evaluation of the integral (5) for this curve. However, this ellipse is transformed into the circle C' with equation $z'\bar{z}'=9/4$ by means of the transformation

$$z = z' + \frac{3}{4z'}\cdot$$

Since this transformation satisfies the reflection principle

$$\bar{z} = \bar{z}' + \frac{3}{4\bar{z}'},$$

the integral (4) is transformed into the integral

$$A(C) = 1/2i \int_{C'} \left(\bar{z}' + \frac{3}{4\bar{z}'}\right)\left(1 - \frac{3}{4z'^2}\right) dz' = 2\pi,$$

which is easily evaluated by the use of residue theory. The details of this example are omitted here since they follow from a more general problem which is treated in the next section. For example, we shall see that the ellipse under discussion is also transformed into the circle $z'\bar{z}'=1/4$ by the above transformation, and that either circle may be used to evaluate the integral.

In general, then, if the function $f(z)$ in the equation (2) of the curve C is either extremely complicated or multiple-valued, we seek a transformation defined by a single-valued and meromorphic function,

$$z = T(z'),$$

which transforms the curve C into a closed curve C' for which the function $F(z')$ in its equation

$$\bar{z}' = F(z')$$

is single-valued. If this transformation satisfies the reflection principle

$$\bar{z} = T(\bar{z}'),$$

the area integral (4) becomes

$$A(C) = 1/2i \int_C \bar{z}dz = 1/2i \int_{C'} T(\bar{z}')T'(z')dz',$$

where $T'(z')$ denotes the derivative of $T(z')$. If the function $F(z')$ is mero-

morphic interior to C' and analytic on C', the theory of residues is applicable to the evaluation of this integral.

Obviously, one might consider any integral

$$\int_C F(z, \bar{z})dz$$

evaluated along a curve C, closed or not, represented in the form (2). Use is made of this concept in those problems where a transformation is made in order to evaluate the area integral (4).

3. Application to area of Joukowski's aerofoil sections. The transformation

$$w = z + a^2/z, \tag{6}$$

where $z=x+iy$, $w=u+iv$, and a is any positive real constant, maps suitably chosen circles in the z-plane into curves in the w-plane which resemble the cross section of a modern aerofoil. Cylinders having these images as sections are known as Joukowski's aerofoils.

Suppose that the circle in the z-plane is given by equation (3) and let L denote the transformed curve in the w-plane. According to Green,* when the circle C passes through the point $A(z=-a)$ and just encloses the point $B(z=a)$ the form of the curve L is that of an aerofoil section. The shape and dimensions of the aerofoil depend upon the position and radius of the circle C. The equation of the curve L is isotropic coordinates is of the form†

$$\begin{aligned} a_0(w^2\bar{w}^2) &+ a_1(w^2\bar{w} + w\bar{w}^2) + a_2(w^2\bar{w} - w\bar{w}^2) + a_3(w^2 + \bar{w}^2) \\ &+ a_4(w^2 - \bar{w}^2) + a_5(w\bar{w}) + a_6(w + \bar{w}) + a_7(w - \bar{w}) + a_8 = 0, \end{aligned}$$

where a_i, $i=0, 1, \cdots, 8$ are known functions of a, h, k, and K. The curve L becomes an aerofoil section if h and k are sufficiently small and K is determined so that C passes through the point A.

Obviously, the formula

$$A(L) = 1/2i \int_L \bar{w}dw \tag{7}$$

for the area enclosed by L in the w-plane is extremely difficult to evaluate, because the equation of L is complicated and does not define $\bar{w}$ as a single-valued function of w. Therefore, we shall transform the profile L into a circle C in the z-plane by means of the transformation (6).

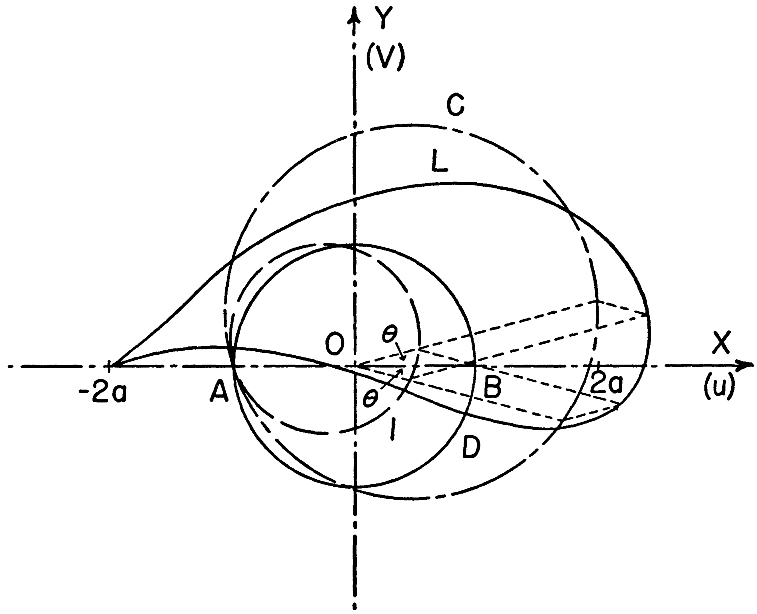
In order to study this mapping more closely, let D denote the circle with center at the origin and passing through A and B. Let I denote the circle which

---

* Hydro- and Aerodynamics, Pitman, 1937, pp. 62–67.

† Mancill and Thomas, On the equation of Joukowski's aerofoils, this MONTHLY, vol. 53, 1946, p. 148.

is the inversion of C with respect to D, as shown in the figure. Now, it is easily seen from the geometric construction of the point by point mapping defined by (6) that the region enclosed by L maps into the two regions, not containing the origin, in the z-plane bounded by the circles C and D and by the circle I and D. The profile itself maps into the two circles C and I.



The formula (7) for the area enclosed by the profile L becomes

$$(8)\quad \begin{aligned} A(L) &= 1/2i\int_L \bar{w}dw = 1/2i\int_C\left(\bar{z}+\frac{a^2}{\bar{z}}\right)\left(1-\frac{a^2}{z^2}\right)dz \\ &= 1/2i\int_C \bar{z}dz - a^4/2i\int_C dz/(z^2\bar{z}) + a^2/2i\int_C dz/\bar{z} - a^2/2i\int_C \bar{z}dz/z^2, \end{aligned}$$

since the transformation (6) satisfies the reflection principle $\bar{w}=\bar{z}+a^2/\bar{z}$. The first integral on the right in the last expression for $A(L)$ in (8) is the area of the circle C. The second integral is the area of the circle I of inversion of C with respect to D, because

$$a^4/2i\int_{C\circlearrowleft} dz/(z^2\bar{z}) = -1/2i\int_{I\circlearrowright} \bar{z}'dz' = A(I)$$

by the transformation $z=a^2/z'$, since the circle C and I are traced in opposite senses by means of this transformation.

Now we wish to show that the other two integrals in (8) are zero. It follows easily that

$$\int_C dz/\bar{z} = \int_C \frac{z - P}{\bar{P}z - K}\, dz = 0,$$

because the point $z = K/\bar{P}$ is outside the circle C and the integrand is a regular function of z in the circle C. The point $z = K/\bar{P}$ is outside the circle C because

$$\begin{aligned} z\bar{z} - \bar{P}z - P\bar{z} + K &= (K/\bar{P})(K/P) - \bar{P}(K/\bar{P}) - P(K/P) + K \\ &= K^2/P\bar{P} - K > 0, \end{aligned}$$

since $K = h^2 + k^2 - r^2 < 0$ under our definition of the aerofoil.

The last integral in (8) is

$$\begin{aligned} a^2/2i \int_C \bar{z}dz/z^2 &= a^2/2i \int_C \frac{\bar{P}z - K}{z^2(z - P)}\, dz \\ &= a^2[2\pi i \times \text{Res. at } z = 0 + 2\pi i \times \text{Res. at } z = P]/2i \\ &= a^2\pi\left(\frac{K - \bar{P}P}{P^2} + \frac{P\bar{P} - K}{P^2}\right) \\ &= 0. \end{aligned}$$

Thus we have proved the following

THEOREM. *The area of any Joukowski aerofoil section is equal to the area of the circle C minus the area of its inversion I, or*

$$\begin{aligned} A(L) &= \pi r^2 - \pi a^4 r^2/(h^2 + k^2 - r^2)^2 \\ &= 4\pi h(h + a)[(h + a)^2 + k^2]/(2h + a)^2. \end{aligned} \tag{9}$$

The last form of the formula for $A(L)$ is obtained by determining r so that the circle C passes through the point $A(z = -a)$.

Two special cases of the aerofoil section are of interest. If $k = 0$, the resulting profile L is symmetric to the u-axis and its area is

$$4\pi h(h + a)^3/(2h + a)^2.$$

If $h = k = 0$, the resulting profile L is a degenerate ellipse, composed of the segment of the u-axis between $z = -2a$ and $z = 2a$, and its area is zero. This checks with formula (9) for this case.

As a final example to verify the formula (9), consider the non-aerofoil section determined by the circle with radius $r \neq a$ and $h = k = 0$. The curve L in the w-plane is the ellipse.

$$\frac{u^2}{(r + a^2/r)^2} + \frac{v^2}{(r - a^2/r)^2} = 1, \tag{10}$$

whose area is

$$\pi(r + a^2/r)(r - a^2/r) = \pi(r^4 - a^4)/r^2.$$

This agrees with the first form of (9) for $h=k=0$.

Also, the example of the ellipse discussed in the preceding section is a special case of these results. For, the ellipse $u^2+4v^2=4$ results from (10) for $a^2=3/4$ and $r=3/2$ or $r=1/2$. In this connection, it should be pointed out that since the profile L maps into both the circles C and I in the z-plane, either circle could have been employed to evaluate the formula (8). The results would have been the same since the roles of C and I would have been merely interchanged.

AN APPLICATION OF DETERMINANTS TO THE PROBABILITY OF MATED PAIRS

JULIA W. BOWER, Connecticut College

1. Introduction. In this paper we are concerned first with the terms of a determinant, that is, with the combinations of elements which compose them. We wish to know how many terms in the expansion contain a given number of elements from the principal diagonal.

We know that the only term of a first order determinant consists of its one principal diagonal element, while the two terms of a second order determinant consist respectively of a product of elements of and a product of elements exclusive of the principal diagonal. Likewise in a third order determinant it is easy to find that one term contains only principal diagonal elements, three terms contain one, and two terms contain none. For higher order determinants the actual forming and counting of terms is impractical. We consider then that for an nth order determinant all terms containing a specified group of r diagonal elements are the products of those elements by the terms of the $(n-r)$th order minor formed by deleting from the determinant the rows and columns of those elements. If now the minor is made zero-axial, that is, has its diagonal elements replaced by zeros, then the non-vanishing products will contain no diagonal elements except those specified. The problem thus requires us to find the number of non-vanishing terms of a zero-axial determinant.

2. The non-vanishing terms of a zero-axial determinant. Let a determinant of order n in which all but r of the principal diagonal elements are set equal to zero be called a determinant of Type R and let the number of its non-vanishing terms be $T_{n,r}$. We seek a formula for $T_{n,0}$.

Consider first that an nth order determinant of Type I has in its expansion the $T_{n,0}$ non-vanishing terms which contain no diagonal element together with the $T_{n-1,0}$ terms formed when the one non-zero diagonal element is multiplied into the non-vanishing terms of its $(n-1)$st order minor. It follows that

$$T_{n,1} = T_{n,0} + T_{n-1,0}. \tag{1}$$

We digress briefly to remark that in like manner the non-vanishing terms of an nth order determinant of Type R will consist of the $T_{n,0}$ terms containing no elements from the principal diagonal plus the ${}_nC_1T_{n-1,0}$ terms containing precisely one element from the principal diagonal, and so on. Hence,

$$T_{n,r} = T_{n,0} + {}_nC_1T_{n-1,0} + \cdots + {}_nC_rT_{n-r,0}.$$

Let a zero-axial determinant of order $n+1$ be expanded in terms of elements of its first row. In the expansion the upper left-hand element, 0, times its minor contributes no non-vanishing terms. Each of the other n elements of the first row times its nth order minor of Type I contributes $T_{n,1}$ non-vanishing terms. Hence we have the relation

$$T_{n+1,0} = n(T_{n,0} + T_{n-1,0}). \tag{2}$$

From the obvious values $T_{1,0}=0$ and $T_{2,0}=1$ it follows that $T_{3,0}=2$, $T_{4,0}=9$, and $T_{5,0}=44$. These numbers strongly suggest the more easily applicable recursion formula below, which can be established readily by induction.

$$T_{n,0} = nT_{n-1,0} + (-1)^n. \tag{3'}$$

Repeated application of formula (3′) gives the relation

$$T_{n,0} = n!\left\{1 - \frac{1}{1!} + \frac{1}{2!} - \cdots + \frac{(-1)^{n-1}}{(n-1)!} + \frac{(-1)^n}{n!}\right\}. \tag{4'}$$

The sum in the braces consists of the first n terms in the series expansion for e^{-1}; hence it is alternately larger and smaller than e^{-1}. For $n\neq 0$, $T_{n,0}$ is the integer nearest in value to the product $n!\,e^{-1}$.

The relation (4′) may be derived directly from known properties of expansion of determinants by considering the number of terms omitting the first diagonal element, then of those omitting both the first and the second, and so on. A development and discussion of this is given in Metzler's revision of Muir's treatise on determinants [1]. The recursion formula (3′) then follows as a corollary from (4′).

We have now found an efficient formula for $T_{n-r,0}$ which, as we have seen above, gives the number of terms of an nth order determinant which contain a specified set of r diagonal elements. Since there are ${}_nC_r$ ways of choosing the r elements, it follows that the number of terms containing precisely r diagonal elements is ${}_nC_rT_{n-r,0}$.

If now we use the notation $N_{n,r}$ to indicate the number of terms of an nth order determinant which contain exactly r elements from the principal diagonal, then it is clear that $N_{n,0}=T_{n,0}$ and the formulas (2), (3′), and (4′) above can be written in terms of $N_{n,0}$. For the sake of simplification in an expression below let $N_{0,0}=1$. This definition produces the proper value for $N_{1,0}$ when used in the analogue of formula (3′). We summarize results.

THEOREM I. *The number, $N_{n,r}$, of terms of an nth order determinant containing precisely r elements from the principal diagonal is given by the formula*

$$N_{n,r} = {}_nC_r N_{n-r,0}, \tag{5}$$

where $N_{0,0}=1$ and for $n\neq 0$, $N_{n,0}$ is defined by formula

$$N_{n,0} = nN_{n-1,0} + (-1)^n, \tag{3}$$

or by formula

$$N_{n,0} = n!\left\{1 - \frac{1}{1!} + \frac{1}{2!} - \cdots + \frac{(-1)^n}{n!}\right\}, \tag{4}$$

or by taking the integer nearest in value to the product $n!e^{-1}$.

3. The application to mated pairs. We turn now to the second problem in the paper, the application of Theorem I to a problem in probability. Consider an experiment in which n samples of handwriting are to be matched with photographs of the individuals making them. When correctly matched a pair is said to be mated, otherwise mismated. What is the probability, $p_{n,r}$, due to chance alone, that if n pairs are formed, r will be mates and $n-r$ will be mismates?

The statistical problem is analogous to the following one. A bag contains n balls each marked with a different element of the sequence $A_1, A_2, \cdots, A_n$. Another bag contains n balls each marked with a different element of the sequence $B_1, B_2, \cdots, B_n$. A ball is chosen at random from each bag. The balls in the resulting pair are mates in case they have like subscripts, otherwise mismates. The drawing continues without replacements until all n pairs have been withdrawn from the bags. What is the probability $p_{n,r}$ that in the resulting set of n pairs r pairs are mates and $n-r$ pairs are mismates?

The probability desired is the quotient of the number of sets of n pairs containing exactly r mates by the total number of possible sets of n pairs. The denominator is $(n!)^2$, for there are n^2 ways of drawing balls in the first pair, $(n-1)^2$ ways of drawing balls in the second pair, and so on. Two sets of n pairs are distinct provided the first contains a pair whose subscripts differ from those of any pair in the second set.

To find the numerator we form the square array below in which the principal diagonal elements give mates and the remaining elements give mismates.

$$\begin{vmatrix} A_1B_1 & A_1B_2 & \cdots & A_1B_n \\ A_2B_1 & A_2B_2 & \cdots & A_2B_n \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ A_nB_1 & A_nB_2 & \cdots & A_nB_n \end{vmatrix}$$

The $n!$ terms of this determinant represent all possible sets of n pairs since each letter with each subscript is used once and only once in each term. How many terms contain exactly r diagonal elements, that is, omit precisely $n-r$ diagonal

elements? The answer is given by $N_{n,r}$ in Theorem I. Each of these sets of n terms can be drawn in $n!$ ways; hence the numerator of the probability fraction is $N_{n,r}n!$ It follows that

$$p_{n,r} = \frac{N_{n,r}n!}{(n!)^2} = \frac{N_{n,r}}{n!} = \frac{N_{n-r,0}}{r!(n-r)!}.$$

A brief computation shows the correctness of the recursion formula (7) stated in the theorem below.

THEOREM II. *The probability $p_{n,r}$ that in drawing n pairs there will be precisely r mates and $n-r$ mismates is given by formula*

$$p_{n,r} = \frac{N_{n-r,0}}{r!(n-r)!}, \tag{6}$$

where $N_{n-r,0}$ is given in Theorem I. Computation may be shortened by noting that $p_{1,1}=1$, $p_{n,0}=N_{n,0}/n!$, and for $n>1$ and $r>0$,

$$p_{n,r} = \frac{p_{n-1,r-1}}{r}. \tag{7}$$

Using the expression in (4) for $N_{n,0}$ we have the formula

$$p_{n,r} = \frac{1}{r!}\left\{1 - \frac{1}{1!} + \frac{1}{2!} - \cdots + \frac{(-1)^{n-r}}{(n-r)!}\right\}. \tag{8}$$

The expansion in the braces differs from the value of e^{-1} by an amount less than $1/(n-r+1)!$. Hence the approximation

$$p_{n,r} = \frac{e^{-1}}{r!} \tag{9}$$

differs from the correct value by an error less than

$$E = \frac{1}{r!(n-r+1)!}.$$

It is evident that once the desired number of correct decimal places in expressing the probability has been determined, then the "long method" of calculation by Theorem II or formula (8) may be replaced, after certain values of n, by the "short method" of formula (9). The following table indicates some degrees of accuracy.

Value of n	Number of correct decimal places given by formula (9)
10	4
15	8
20	13
25	18

Inasmuch as such experiments are seldom done with less than ten and usually more than twenty-five balls in each bag, the "short method" will be satisfactory generally for computation. Results are summarized in the following theorem.

THEOREM III. *For values of n greater than* 10 *the probability* $p_{n,r}$ *that in drawing n pairs there will be precisely r mates and* $n-r$ *mismates is given correct to at least* $n-7$ *decimal places by the approximation*

$$p_{n,r} = \frac{e^{-1}}{r!}. \tag{9}$$

If now all members of a class of s students are given n samples of handwriting and asked to match the samples with n photographs shown seriatim on a screen, each photograph appearing once and only once, what is the probability $p_{s,t,r}$ that exactly t of the students will mate r pairs and mismate $n-r$ pairs? The answer to this problem is given by the well-known formula

$$p_{s,t,r} = {}_sC_t(p_{n,r})^t(1 - p_{n,r})^{s-t}, \tag{10}$$

where $p_{n,r}$ is found by use of Theorem II or III.

4. Further extensions to statistical theory. The third part of the paper will tell of other studies of this problem and indicate further statistical extensions. Two discussions of interest to the elementary student are those by Whitworth and Cheney. Both considered the problem of derangements of n ordered objects. W. A. Whitworth [2] called the number $N_{n,0}$ (or $T_{n,0}$) by the name "subfactorial n," defined it by use of formula (3), stated recursion relation (2), and worked out a table of values of $N_{n,0}$ for $n=1, \cdots, 12$. He developed formula (4), noted the approximation of the number value of $N_{n,0}$ by $n!e^{-1}$, and in a problem used relation (5). Whitworth also considered a number of related problems which are of interest but not germane to our discussion. By actual listing and counting of possible derangements, Professor W. F. Cheney, Jr.,* of the University of Connecticut made a table of values of $N_{n,r}$ for n and r between 1 and 10. Using the relations evident in the table he developed the recursion formula (3) and formulas (4), (5), (6), (8), and (9).

Psychologists would find a discussion by D. W. Chapman [3] couched in familiar terms. He considered an x and a y series of t elements each in which the order of the x series remains fixed, but that of the y series may be varied. If precisely s of the elements are to be mated, then s of the y series are also fixed, and the remaining $t-s$ are capable of permutation. From the $(t-s)!$ permutations of the remaining elements must be removed all those in which a y element is mated with its corresponding x element, since the inclusion of these would give arrangements in which $s+1$ of the x and y elements are mated. The permutations to be removed are first reckoned to be $(t-s)!$ in number. That is, any of the $t-s$ elements can be the one to mate, and for each mated

* Mathematics Club Lecture, Connecticut College, 1946.

one there are $(t-s-1)!$ permutations of the remaining elements. But this number of permutations includes also the number of those arrangements with two mated pairs, each counted twice, once for each pair; those containing three mated pairs; *etc.* A series of corrections is thus necessary. When these are made, and when t and s are replaced by n and r, respectively, formula (8) is derived. Chapman then worked out a table correct to four decimal places of the numerical values of the probabilities for $r=0, 1, \cdots, 10$ and $n=1, 2, \cdots, 8$, and $\geqq 9$.

In a later article [**4**] he generalized the problem to the case in which the x and y series are of different lengths so that one contains extraneous items with no mates in the shorter series. The expression for the probability that a single random arrangement will result in s correct matchings again reduces for our simpler case to the analogue of formula (8).

The remaining discussions are of particular interest to the mathematical statistician. A generalization of our simple problem given by T. N. E. Greville [5] considered the case in which $i_1, i_2, \cdots, i_t$ of the balls in the first bag and $j_1, j_2, \cdots, j_t$ of the balls in the second bag all have subscripts $1, 2, \cdots, t$, respectively, where $i_1+i_2+\cdots+i_t=j_1+j_2+\cdots+j_t=n$. Greville determined the number W_s of sets of n pairs having s mates and $n-s$ mismates. In order to apply his results to our problem take the i's and j's each equal to 1 and the c's each equal to 1 or 0. His formula (3) reduces to

$$V_s = (n - s)!\,{}_nC_s = \frac{n!}{s}\cdot$$

Replacing W_s and s by our equivalent notation $N_{n,r}$ and r, Greville's formula (4) becomes

$$N_{n,r} = \frac{n!}{r!}\left\{1 - \frac{1}{1!} + \frac{1}{2!} - \cdots + \frac{(-1)^{n-r}}{(n-r)!}\right\}.$$

This is our formula (5) when $N_{n-r,0}$ is replaced by the value given in (4).

S. S. Wilks [7] used a different analysis to find the same result. For our simple problem his $n_{dj}=1$ where $d=1, 2$, and $j=1, 2, \cdots, k=n$, so that each $s_i=1$ or 0. His formula (a) reduces to $N=(n!)^2$, the denominator of the probability fraction. For $h=r$, his formulas (h) and (i) combine to give the numerator of the probability fraction $N(r)$, when $N_{n,0}$ is given by formula (4).

$$N(r) = n!\,{}_nC_r\sum_{g=0}^{n-r}(-1)^{n-r-g}\,{}_{n-r}C_g\,g! = n!\,{}_nC_r N_{n-r,0}.$$

Many other aspects of the problem immediately present themselves. What is the significance of the mean number of correct matchings as the result of i independent trials? What is the nature of the probability function? What are the formulas for moments? What results are obtained when there are three or more decks of arbitrary composition or when all decks but one have fixed

orders in relation to each other? Readers interested in these and other pertinent extensions of the simple problem will profit by further reading of the works of Greville [6] and Wilks as well as an article by I. L. Battin [8] and the papers listed in his careful bibliography. Finally, I. Kaplansky and J. Riordan [9] applied symbolic methods to the problem of multiple matchings which resulted in a simplification of analysis and consequent expedition in obtaining results.

References

1. Thomas Muir and William H. Metzler, *A treatise on the theory of determinants*, New York, 1933, # 124–130.
2. William Allen Whitworth, *Choice and chance*, New York, 1925, pp. 107–110.
3. D. W. Chapman, *American Journal of Psychology*, vol. 46, 1934, pp. 287–298.
4. D. W. Chapman, *Annals of Mathematical Statistics*, vol. 6, 1935, pp. 85–95.
5. T. N. E. Greville, *Annals of Mathematical Statistics*, vol. 12, 1941, pp. 350–354.
6. T. N. E. Greville, *Annals of Mathematical Statistics*, vol. 15, 1944, pp. 432–434.
7. S. S. Wilks, *Mathematical Statistics*, Princeton, 1944, pp. 208–213.
8. I. L. Battin, *Annals of Mathematical Statistics*, vol. 13, 1942, pp. 294–305.
9. I. Kaplansky and J. Riordan, *Annals of Mathematical Statistics*, vol. 16, 1945, pp. 272–277.

NORMAL TRIGRADE AND CYCLIC QUADRILATERAL WITH INTEGRAL SIDES AND DIAGONALS[1]

A. GLODEN, University of Brussels

1. Complete solution, by normal trigrade, of cyclic quadrilateral with integral sides and diagonals. We will here indicate how one obtains *all* the cyclic quadrilaterals with integral sides and diagonals. We will show, in fact, that this problem is equivalent to that of solving the normal trigrade.

Let us recall a definition.[2] The diophantine system

$$\sum_{i=1}^{4} A_i^k = \sum_{i=1}^{4} B_i^k,$$

where the exponent k takes successively the values 1, 2, 3, is called the normal trigrade, and we represent it by the abridged notation

$$A_1, A_2, A_3, A_4 \overset{3}{=} B_1, B_2, B_3, B_4. \tag{1}$$

Before showing how to obtain all the solutions of (1) we effect several transformations on the unknowns.

[1] Translated from the French by Howard Eves.

[2] For notation and definition of multigrades see the author's Mehrgradige Gleichungen, Noordhoff, Groningen, 1944.

First of all, by a general theorem on multigrades of order n, we are always able to put the trigrade system (1) in a form in which the sum of the terms in each member is zero; one merely subtracts $S_1/4$ from each term, where

$$S_1 = \sum_{i=1}^{4} A_i = \sum_{i=1}^{4} B_i.$$

We obtain

$$\begin{aligned} &3A_1 - A_2 - A_3 - A_4, \quad - A_1 + 3A_2 - A_3 - A_4, \\ &\quad - A_1 - A_2 + 3A_3 - A_4, \quad - A_1 - A_2 - A_3 + 3A_4 \\ &\overset{3}{=} 3B_1 - B_2 - B_3 - B_4, - B_1 + 3B_2 - B_3 - B_4, \\ &\quad - B_1 - B_2 + 3B_3 - B_4, \quad - B_1 - B_2 - B_3 + 3B_4. \end{aligned}$$

An obvious linear change gives the simpler form

$$A_1', A_2', A_3', \quad -(A_1' + A_2' + A_3') \overset{3}{=} B_1', B_2', B_3', \quad -(B_1' + B_2' + B_3').$$

Next we introduce new unknowns by means of the linear system

$$\begin{aligned} a_1 &= A_2' + A_3', & b_1 &= B_2' + B_3', \\ a_2 &= A_3' + A_1', & b_2 &= B_3' + B_1', \\ a_3 &= A_1' + A_2', & b_3 &= B_1' + B_2', \end{aligned}$$

from which one easily eliminates the A_i' and the B_i'. This linear transformation finally reduces the solution of (1) to that of the system

$$(2) \qquad \begin{aligned} a_1^2 + a_2^2 + a_3^2 &= b_1^2 + b_2^2 + b_3^2, \\ a_1a_2a_3 &= b_1b_2b_3, \end{aligned}$$

of which the complete solution[3] is given by the following formulas of sixth degree in six arbitrary parameters:

$$\begin{aligned} a_1/r = b_1/s &= p(p^2 - q^2)m^2 - 2r(p^2 - q^2)mn + p(r^2 - s^2)n^2, \\ a_2/ps = b_2/qr &= (p^2 - q^2)m^2 - (r^2 - s^2)n^2, \\ a_3/q = b_3/p &= r(p^2 - q^2)m^2 - 2p(r^2 - s^2)mn + r(r^2 - s^2)n^2. \end{aligned}$$

Another linear transformation

$$\begin{aligned} a_1 + b_1 &= \mu, & a_1 - b_1 &= \nu, \\ b_2 + a_2 &= \pi, & b_2 - a_2 &= \kappa, \\ b_3 = a_3 &= \rho, & b_3 - a_3 &= \sigma, \end{aligned}$$

leads to the diophantine system, equivalent to (2) and consequently to (1),

[3] *Loc. cit.*, pp. 36–37.

$$\mu\nu = \pi\kappa + \rho\sigma,$$
$$\mu(\pi\sigma + \kappa\rho) = \nu(\pi\rho + \kappa\sigma).$$

But these two relations characterize[4] the cyclic quadrilateral having π and κ, ρ and σ as pairs of opposite sides, and μ and ν as diagonals.

One thus obtains all cyclic quadrilaterals whose sides and diagonals can be expressed as integers.

2. Cyclic heronian quadrilaterals. A polygon is said to be *heronian* if its sides and area can be expressed rationally. One is able to obtain, by means of the normal trigrade, a cyclic quadrilateral with integral sides, diagonals, and area.

Here is a solution with four independent parameters m, n, p, q, of system (2). Let a, b, c and α, β, γ represent two pythagorean triples, so that we have

$$a^2 + b^2 = c^2 \quad \text{and} \quad \alpha^2 + \beta^2 = \gamma^2,$$

where

$$a = m^2 - n^2, \qquad b = 2mn, \qquad c = m^2 + n^2,$$
$$\alpha = p^2 - q^2, \qquad \beta = 2pq, \qquad \gamma = p^2 + q^2$$

the second member of the expressions for a, b, c being able to be multiplied by a factor of proportionality l, those for α, β, γ by a factor λ. We then have

$$a_1 = (a + b)(\alpha + \beta), \qquad b_1 = (a - b)(\beta - \alpha),$$
$$a_2 = (a - b)\gamma, \qquad b_2 = (a + b)\gamma,$$
$$a_3 = (\beta - \alpha)c, \qquad b_3 = (\alpha + \beta)c.$$

We may then take

$$\mu = a\beta + b\alpha, \qquad \nu = a\alpha + b\beta,$$
$$\pi = a\gamma, \qquad \kappa = b\gamma,$$
$$\rho = c\beta, \qquad \sigma = c\alpha.$$

These are the diagonals (μ, ν) and opposite sides (π, κ), (ρ, σ) of a cyclic quadrilateral. The diagonals are perpendicular to each other,[5] whence, if S is the area of the quadrilateral[6]

$$S = \mu\nu/2 = (a\beta + b\alpha)(a\alpha + b\beta)/2 = (ab\gamma^2 + \alpha\beta c^2)/2.$$

Also,[7] $R = c\gamma/2$, and the circumdiameter of the quadrilateral is integral.

3. Numerical example. Consider the normal trigrade with sum of terms equal to zero:

---

[4] *Each* of the relations characterizes the quadrilateral as cyclic. (H. Eves)

[5] Since $\pi^2+\kappa^2=\rho^2+\sigma^2$. (H. Eves)

[6] S is an integer because b and β are each even. (H. Eves)

[7] $4RS=\mu(\pi\sigma+\rho\kappa)$, whence $2R=(\pi\sigma+\rho\kappa)/\nu=c\gamma$. (H. Eves)

$$-167,\ -71,\ 97,\ 141 \overset{3}{=} -183,\ 1,\ 13,\ 169,$$

$$S_1 = 0, \qquad S_2 = 62{,}220, \qquad S_3 = -\ 1{,}299{,}480.$$

The corresponding solution of system (2) may be taken as

$$(a_1, a_2, a_3) = (119, 13, 35), \qquad (b_1, b_2, b_3) = (7, 91, 85).$$

The pairs of opposite sides of the corresponding cyclic quadrilateral may then be taken as (52, 39), (60, 25), the diagonals as 63 and 56, the area $S = 1764 = 42^2$, and $R = 65/2$.[8]

MATHEMATICAL NOTES

EDITED BY E. F. BECKENBACH, *University of California*

Material for this department should be sent to F. A. Ficken, Institute of Mathematics and Mechanics, New York University, 45 Fourth Ave., New York 3, New York.

SYSTEMS OF HYPERBOLOIDS AND OF QUARTIC CURVES CIRCUMSCRIBED ABOUT A TETRAHEDRON*

VICTOR THÉBAULT, Tennie, Sarthe, France

1. **Notation.** Following custom, we shall designate by a, a', b, b', c, c' the lengths of the edges BC, DA, CA, DB, AB, DC, and also the dihedral angles along these edges; by A, B, C, D the areas of the faces BCD, CDA, DAB, ABC; and by d_1, d_2, d_3 the distances between the opposite edges a and a', b and b', c and c' of a tetrahedron $T \equiv ABCD$.

2. **Circumscribed hyperboloids and quartic curves.** We first establish the following theorem.

THEOREM. *If the planes BCP, CDP, BDP, ADP, ACP, ABP, which pass through the edges of a tetrahedron T and through an arbitrary point P, cut the opposite edges DA, AB, AC, BC, BD, CD in N, U, S, M, Q, W, then the radical axes of the triples of spheres ($ABCN$, $ACDU$, $ABDS$), ($BCDU$, $BACQ$, $BADM$), ($CBDS$, $CABW$, $CADM$), ($DBCM$, $DACQ$, $DABW$) form a hyperbolic group of lines.*

For, if $S = 0$ and $\pi = 0$ are the equations of the sphere $ABCD$ and of the plane at infinity, those of the spheres $DBCN$, $DACQ$, $DABW$ are

$$(A\alpha + D\delta)S - (D\delta Axa'^2)\pi = 0,$$
$$(B\beta + D\delta)S - (D\delta Byb'^2)\pi = 0,$$
$$(C\gamma + D\delta)S - (D\delta Czc'^2)\pi = 0,$$

---

[8] This same quadrilateral can be arrived at by starting with the two pythagorean triples (3, 4, 5) and (5, 12, 13). (H. Eves)

* Translated from the French by Howard Eves, Oregon State College.

α, β, γ, δ denoting the normal coordinates of the point P with respect to the tetrahedron T. Hence, equations of the radical axes of the triples of spheres under consideration are

$$\Delta_a \equiv Byc^2/(B\beta + A\alpha) = Czb^2/(C\gamma + A\alpha) = Dta'^2/(D\delta + A\alpha),$$
$$\Delta_b \equiv Cza^2/(C\gamma + B\beta) = Dtb'^2/(D\delta + B\beta) = Axc^2/(A\alpha + B\beta),$$
$$\Delta_c \equiv Dtc'^2/(D\delta + C\gamma) = Axb^2/(A\alpha + C\gamma) = Bya^2/(B\beta + C\gamma),$$
$$\Delta_d \equiv Axa'^2/(A\alpha + D\delta) = Byb'^2/(B\beta + D\delta) = Cxc'^2/(C\gamma + D\delta).$$

It follows [1] that these four lines form a hyperbolic group.

In order that the lines Δ_a, Δ_b, Δ_c, Δ_d be concurrent, it is necessary and sufficient that the point P be on the quartic curve

$$(1)\qquad \begin{aligned}(A\alpha + D\delta)(B\beta + C\gamma)/(aa')^2 &= (A\alpha + C\gamma)(B\beta + D\delta)/(bb')^2\\ &= (A\alpha + B\beta)(C\gamma + D\delta)/(cc')^2,\end{aligned}$$

which passes through the vertices A, B, C, D, and through the harmonic conjugates of the centroid G of T with respect to the medians AG_a, BG_b, CG_c, DG_d of T as segments.

Curve (1) belongs to the pencil of quartic curves

$$(2)\qquad \begin{aligned}(A\alpha + D\delta)(B\beta + C\gamma)/m &= (A\alpha + C\gamma)(B\beta + D\delta)/n\\ &= (A\alpha + B\beta)(C\gamma + D\delta)/p,\end{aligned}$$

where m, n, p are arbitrary parameters. This pencil of curves passes through the eight vertices of the parallelepiped (P) circumscribed about the tetrahedron T; for the planes $A\alpha+D\delta=0$ and $B\beta+C\gamma=0$ are parallel, *etc.* Any curve $\mathcal{B}$ of this pencil is, then, the common intersection of three second degree cylinders having for axes the bimedians of T (obtained by setting equal in pairs the three fractions (2)) and of a second degree cone with vertex at G and having for equation

$$(n - p)Q_1 + (p - m)Q_2 + (m - n)Q_3 = 0,$$

where Q_1, Q_2, Q_3 are the numerators of the fractions (2).

3. Another definition of $\mathcal{B}$. Let us denote by x and x' the distances of a point from those faces of (P) which pass through the edges a and a', and similarly define y and y', z and z'. Then we have

$$A\alpha + D\delta = x(A^2 + D^2 - 2AD\cos a)^{1/2} = (x/2)aa'\sin(a, a') = 3Vx/d_1,$$
$$B\beta + C\gamma = x'(B^2 + C^2 - 2BC\cos a')^{1/2} = (x'/2)aa'\sin(a, a') = 3Vx/d_1.$$

The numerators of the fractions (2) are thus proportional to

$$xx'/d_1^2, \qquad yy'/d_2^2, \qquad zz'/d_3^2.$$

Consequently, *the quartic curve $\mathcal{B}$ is the locus of points for which the products of the distances to the pairs of parallel faces of the parallelepiped* (P) *divided by the squares of the distances between these pairs of faces are proportional to* m, n, p.

These distances can, moreover, be taken either perpendicular to the faces or in an arbitrary direction, for x, x', d_1 vary proportionally when one changes the direction. For example, the distances may follow the directions of those edges of (P) not parallel to the pairs of respective faces.

4. Special cases. When m, n, p are proportional to $(aa')^2$, $(bb')^2$, $(cc')^2$, one obtains the quartic curve (1). In order to characterize it in the pencil of curves $\mathcal{B}$, it suffices to fix one point.

If the tetrahedron T is *orthocentric*, a quartic curve $\mathcal{B}$ passes through the orthocenter H, and consequently through the circumcenter, because, for H, we have

$$xx' = yy' = zz'.$$

By performing two dilatations of the parallelepiped (P) relative to any tetrahedron T in the directions of the axes of (P), *i.e.*, the bimedians of T, in such a fashion as to render these bimedians equal to each other, one obtains an orthocentric tetrahedron T'. Under the transformation, a quartic $\mathcal{B}$ of T goes into a quartic $\mathcal{B}'$ of T'. The correspondent of the orthocenter of T' gives, then, a point through which $\mathcal{B}$ passes in the general tetrahedron.

If the tetrahedron T is *isodynamic*, the denominators of the fractions (1) are equal, and *the quartic curve* (1) *degenerates into the four diagonals of* (P), (*medians of* T), *and passes through the centroid* G.

5. Note. The fact that in the quartic curve (1), aa', bb', cc' are proportional to the sides of antiparallel sections of T [2] ought perhaps to confer some special properties with respect to these sections, and consequently to the *second Lemoine point* of T.

References

1. Koehler, Exercises de Géométrie, t. II (coordonées tétraèdriques).
2. Rouché et de Comberousse, Traité de Géométrie, deuxième partie (1891), p. 599.

CHANGE IN POTENTIAL DUE TO A DIELECTRIC SPHERE

G. POWER, The University of Nottingham

1. Introduction. When a sphere of radius a, filled with homogeneous dielectric of specific inductive capacity k, is placed with its center at the origin of coordinates in any electrostatic field whose potential function is $\phi(x, y, z)$, having no singularities inside or on the sphere, then, by considering the boundary values at the surface of the sphere, we can determine the potential functions inside and outside the sphere in terms of $\phi(x, y, z)$. These potential functions are respectively

$$(1) \qquad \phi_1 \equiv \frac{2}{(k+1)}\phi(x, y, z) + \frac{(k-1)}{(k+1)^2}\int_0^1 t^{-k/(k+1)}\phi(xt, yt, zt)dt,$$

and

$$\begin{aligned}\phi_2 \equiv \phi(x, y, z) &- \frac{(k-1)}{(k+1)}\,\frac{a}{r}\,\phi(x_1, y_1, z_1) \\ &+ \frac{(k-1)}{(k+1)^2}\,\frac{a}{r}\int_0^1 t^{-k/(k+1)}\phi(x_1 t, y_1 t, z_1 t)\,dt,\end{aligned} \tag{2}$$

where

$$x_1 = \frac{a^2 x}{r^2}, \qquad y_1 = \frac{a^2 y}{r^2}, \qquad z_1 = \frac{a^2 z}{r^2}, \qquad r^2 = x^2 + y^2 + z^2.$$

It is assumed that there are no other boundaries present.

2. Proof. To prove (1) and (2), we expand $\phi(x, y, z)$ in the form

$$\phi \equiv \sum_{n=0}^{\infty} A_n r^n S_n,$$

where S_n is, as usual, a surface harmonic.

When the sphere is placed with its center at the origin, the potential function ϕ_1 inside the dielectric is given by

$$\phi_1 \equiv \sum_{n=0}^{\infty} B_n r^n S_n, \qquad r \leqq a.$$

Outside the dielectric the potential function ϕ_2 will be given by

$$\phi_2 \equiv \phi + \sum_{n=0}^{\infty} C_n r^{-n-1} S_n, \qquad r \geqq a,$$

the latter term representing the interference potential of the dielectric sphere (it must be of this form because it satisfies Laplace's equation and also tends to zero as $r \to \infty$).

The conditions to be satisfied are

$$\phi_1 = \phi_2 \qquad \text{on } r = a,$$

and

$$k\frac{\partial \phi_1}{\partial r} = \frac{\partial \phi_2}{\partial r} \qquad \text{on } r = a.$$

Thus

$$\sum_{n=0}^{\infty} B_n a^n S_n = \sum_{n=0}^{\infty} A_n a^n S_n + \sum_{n=0}^{\infty} \frac{C_n S_n}{a^{n+1}},$$

and

$$k\sum_{n=1}^{\infty} nB_n a^{n-1}S_n = \sum_{n=1}^{\infty} nA_n a^{n-1}S_n - \sum_{n=1}^{\infty} \frac{(n+1)C_n S_n}{a^{n+2}}.$$

Therefore

$$B_n a^{2n+1} - C_n = A_n a^{2n+1},$$

and

$$knB_n a^{2n+1} + (n+1)C_n = nA_n a^{2n+1}.$$

From these two equations we obtain

$$B_n = \frac{(2n+1)A_n}{n(k+1)+1} \quad \text{and} \quad C_n = \frac{-n(k-1)A_n a^{2n+1}}{n(k+1)+1}.$$

Writing

$$B_n = \frac{2A_n}{(k+1)} + \frac{(k-1)}{(k+1)}\frac{A_n}{(nk+n+1)},$$

we see that

$$\phi_1 = \frac{2}{(k+1)}\sum_{n=0}^{\infty} A_n r^n S_n + \frac{(k-1)}{(k+1)^2}\sum_{n=0}^{\infty} \frac{A_n r^n S_n}{n+1-\dfrac{k}{k+1}}.$$

Now $\phi(x, y, z) \equiv \sum_{n=0}^{\infty} A_n r^n S_n$ is a sum of terms homogeneous in x, y, z of degree n, $n=0, 1, 2, \cdots$, so that (1) holds.

Again writing C_n in the form

$$\begin{aligned} C_n &= -\frac{(k-1)}{(k+1)}A_n a^{2n+1} + \frac{(k-1)}{(k+1)}\frac{A_n a^{2n+1}}{(nk+n+1)}, \\ &= -\frac{(k-1)}{(k+1)}A_n a^{2n+1} + \frac{(k-1)}{(k+1)^2}\frac{A_n a^{2n+1}}{\left(n+1-\dfrac{k}{k+1}\right)}, \end{aligned}$$

we see that

$$\sum_{n=0}^{\infty} C_n \frac{S_n}{r^{n+1}} = -\frac{(k-1)}{(k+1)}\sum_{n=0}^{\infty}\frac{A_n a^{2n+1}}{r^{n+1}} + \frac{(k-1)}{(k+1)^2}\sum_{n=0}^{\infty}\frac{A_n a^{2n+1}}{r^{n+1}\left(n+1-\dfrac{k}{k+1}\right)}.$$

If (x_1, y_1, z_1) is the inverse point of (x, y, z) with regard to the sphere, then $rr_1 = a^2$, and so

$$\sum_{n=0}^{\infty} \frac{C_n S_n}{r^{n+1}} = \sum_{n=0}^{\infty} \frac{C_n S_n r_1^{n+1}}{a^{2n+2}}$$

$$= -\frac{(k-1)}{(k+1)} \sum_{n=0}^{\infty} \frac{A_n r_1^{n+1} S_n}{a} + \frac{(k-1)}{(k+1)^2} \sum_{n=0}^{\infty} \frac{A_n r_1^{n+1} S_n}{a\left(n+1-\dfrac{k}{k+1}\right)}$$

$$= -\frac{(k-1)}{(k+1)} \frac{a}{r} \sum_{n=0}^{\infty} A_n r_1^n S_n + \frac{(k-1)}{(k+1)^2} \frac{a}{r} \sum_{n=0}^{\infty} \frac{A_n r_1^n S_n}{\left(n+1-\dfrac{k}{k+1}\right)}$$

$$= -\frac{(k-1)}{(k+1)} \frac{a}{r} \phi(x_1, y_1, z_1) + \frac{(k-1)}{(k+1)^2} \frac{a}{r} \int_0^1 t^{-k/(k+1)} \phi(x_1 t, y_1 t, z_1 t) dt.$$

Thus (2) holds.

Equations (1) and (2) give the new potentials inside and outside the sphere respectively.

For a solid spherical conductor, we let $k \to \infty$ so that

$$\phi_1 = 0, \qquad \phi_2 = \phi(x, y, z) - \frac{a}{r} \phi(x_1, y_1, z_1).$$

3. Examples. We present two examples.

(a) Let a charge e be placed at $A(c, 0, 0)$, $c > a$. Here

$$\phi(x, y, z) \equiv \frac{e}{[(x-c)^2 + y_1^2 + z_1^2]^{1/2}}.$$

The second term in ϕ_2 is

$$-\frac{(k-1)}{(k+1)} \cdot \frac{ea}{r} \cdot \frac{1}{[(x_1 - c)^2 + y^2 + z^2]^{1/2}}$$

$$= -\frac{(k-1)}{(k+1)} \cdot \frac{ea}{c} \cdot \frac{1}{\left[\left(x - \dfrac{a^2}{c}\right)^2 + y^2 + z^2\right]^{1/2}}.$$

The last term in ϕ_2 is

$$\frac{(k-1)}{(k+1)^2} \frac{a}{r} \int_0^1 \frac{t^{-k/(k+1)} dt}{[(x_1 t - c)^2 + y_1^2 t^2 + z_1^2 t^2]^{1/2}}$$

$$= \frac{(k-1)}{(k+1)^2} \frac{ea}{c} \int_0^1 \frac{t^{-k/(k+1)} dt}{\left[\left(x - \dfrac{a^2 t}{c}\right)^2 + y^2 + z^2\right]^{1/2}}.$$

Thus the image system of a charge e with regard to a dielectric sphere is a charge of strength $-(k-1)ea/(k+1)c$ at A_1, the inverse point of A, and a line distribution of sources extending from the center 0 to A_1, the strength per unit length of which, at a distance b from 0, is

$$\frac{(k-1)}{(k+1)^2}\frac{e}{a}\left(\frac{cb}{a^2}\right)^{-k/(k+1)}.$$

(b) As a hydrodynamical problem, consider a sphere of radius a placed with its center at the origin in a fluid whose velocity potential is $\phi(x, y, z)$. These conditions correspond to $k=0$ in the previous work for the velocity potential outside the sphere. From (2) we see that the new velocity potential is

$$\phi(x, y, z) + \frac{a}{r}\phi(x_1, y_1, z_1) - \frac{a}{r}\int_0^1 \phi(x_1t, y_1t, z_1t)dt.*$$

CLASSROOM NOTES

EDITED BY C. B. ALLENDOERFER, Haverford College

All material for this department should be sent to C. B. Allendoerfer, Haverford College, Haverford, Penna.

THE DIOPHANTINE EQUATION OF A CARELESS ERROR

J. D. SWIFT, University of California, Los Angeles

1. Introduction. A rather common error committed by students on all levels from the elementary school to sophomore mathematics and beyond takes the general form: $(a+b)/(c+d)=a/c+b/d$. During the later stages, this mistake is somewhat unusual when the quantities involved are numerical, but is still frequently applied to algebraic or analytical expressions which later in the problem may be numerically evaluated. Many instructors have had the somewhat disheartening experience of having students achieve the correct result after commission of this gross error. This will be the case if $cd\neq 0$, and $ad^2+bc^2=0$ either identically or for the applicable numerical values.

If a, b, c, d are integral, the Diophantine equation, $ad^2=-bc^2$, results. This equation is of the general form, $\prod x_i^{\alpha_i}=\prod y_i^{\beta_i}$, to which the general method of E. T. Bell† is immediately applicable. Although this method has been termed

---

* P. Weiss, *On hydrodynamical images. Arbitrary irrotational flow disturbed by a sphere*, Cambridge Philos. Soc. vol. 40, 1945, pp. 259–261.

† Bell, E. T., Diophantine equations from algebraic invariants and covariants. Annals of Math. vol. 34 (2nd ser.) 1933, pp. 450–60.

the most general one for the solution of nonlinear Diophantine equations, the instances of interesting or motivated equations to which it is directly applicable are quite rare. Normally, an equation must be first drastically rearranged and substitutions made to bring it into the desired form. It is therefore believed that instructors who wish to include some introduction to modern methods in Diophantine Equations in courses in Elementary Number Theory will be interested in this example.

2. Solution of $ad^2=-bc^2$ in integers. The general theorem states: All solutions of $xy=zw$ are given by $x=ab$, $y=fg$, $z=af$, $w=bg$, with $(b, f)=1$. In the successive applications of this theorem, parameters appearing in the final result will be in Latin letters, temporary parameters in Greek. Following each major step, all g.c.d. conditions applicable will be listed in the parameters valid at that point. A symmetric solution may be obtained as follows:

a) $d^2=\alpha\beta$, $a=\gamma u$, $c^2=\alpha\gamma$, $-b=\beta u$; $(\beta, \gamma)=1$.

b) If $d^2=\alpha\beta$; $d=\delta s=\zeta\theta$, $\alpha=\delta\zeta$, $\beta=s\theta$; $(s, \zeta)=1$ .Then in $\delta s=\zeta\theta$, $s|\theta$ and $\zeta|\delta$; therefore $\theta=s\kappa$ and $\delta=\zeta\kappa$. We now have: $d=s\zeta\kappa$, $\alpha=\zeta^2\kappa$, $\beta=s^2\kappa$; $(s, \gamma)=(s, \zeta)=(\kappa, \gamma)=1$.

c) Similarly, from $c^2=\alpha\gamma$, $c=t\lambda\mu$, $\alpha=\lambda^2\mu$, $\gamma=t^2\mu$; $(s, t)=(s, \mu)=(\kappa, t)=(\kappa, \mu)=(s, \zeta)=(t, \lambda)=1$.

d) The values of α give: $\lambda^2\mu=\zeta^2\kappa$. Since $(\kappa, \mu)=1$, $\mu=\pm n^2$, $\kappa=\pm r^2$. Thus $\lambda n=\zeta r$ (the case $\lambda n=-\zeta r$ will yield nothing new in the final result), and $(n, r)=1$. Therefore $\zeta=mn$ and $\lambda=mr$, so that:

$$a = n^2t^2u, \qquad b = - r^2s^2u$$
$$c = mn^2rt \qquad d = mnr^2s;$$

$(s, t)=(s, n)=(r, t)=(r, n)=(s, m)=(t, m)=1$. The g.c.d. conditions in the final result serve only to avoid repetition.

If $d^2=\alpha\beta$ and $c^2=\alpha\gamma$ are solved successively, rather than simultaneously as above, a non-symmetric set of solutions is obtained involving one less parameter: $a=u^2v$, $b=-r^2t^2v$, $c=stu$, $d=rst^2$; $(r, s)=(r, u)=(t, u)=1$.

3. Remarks. By virtue of the fact that the basic theorem is valid in any integral domain with unique factorization, the same set of solutions, with appropriate provision for units, is complete in any such domain, e. g., polynomials with coefficients in a field.

The value of the expressions $(a+b)/(c+d)$ and $a/c+b/d$ is, using the symmetric set, $u(nt-rs)/mnr$. It is interesting to note that any rational number, x/y, can be obtained, and, if the trivial restriction, $(x, y)=1$ is made, with the 'common multipliers' u and m each equal to one. It is merely necessary to factor y into co-prime factors n and r, and then solve the linear Diophantine equation $nt-rs=x$ for s and t. This can be done within the g.c.d. conditions since $(x, nr)=1$.

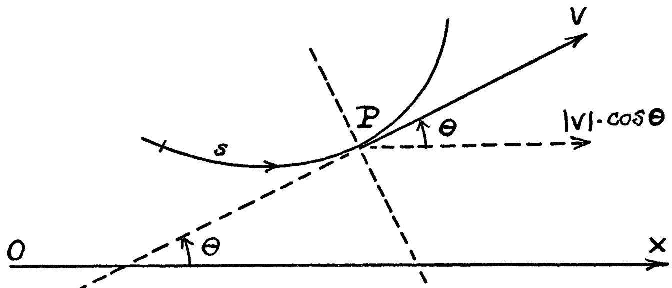
Finally, it may be that the values derived in this note may be of use in con-

vincing those students given to 'proving' theorems by citing special cases, that in mathematics frequently even an infinity of swallows does not make a summer.

NORMAL AND TANGENTIAL ACCELERATION

R. C. YATES and C. P. NICHOLAS, USMA

As a first approach to acceleration in the study of plane motion both student and teacher would be happy to avoid the customary routine of converting from components in an arbitrary rectangular reference system to the more useful normal and tangential ones. The following discussion, essentially intrinsic in character, has this appeal.



At a point P of the path let θ be the angle made by the velocity vector V with any direction OX. This angle, measured *from* the direction line OX *to* the velocity vector, is counterclockwise-positive. The component of V in this direction is

$$V_\theta = |V| \cdot \cos\theta.$$

The acceleration component in this same direction is the time derivative:

$$a_\theta = \frac{dV_\theta}{dt} = \frac{d|V|}{dt}\cdot\cos\theta - |V|\cdot(\sin\theta)\cdot\frac{d\theta}{dt}$$
$$= \frac{d|V|}{dt}\cdot\cos\theta - |V|\cdot(\sin\theta)\cdot\frac{d\theta}{ds}\cdot\frac{ds}{dt},$$

or

$$a_\theta = \frac{d|V|}{dt}\cos\theta - |V|^2\sin\theta\frac{d\theta}{ds}\cdot$$

If we select the direction OX parallel to the tangent by taking $\theta = 0$, we obtain the *tangential acceleration*:

$$a_T = \frac{d|V|}{dt}\cdot$$

Supposing that $d\theta/ds \neq 0$ at P, we now select OX along the normal at P pointing toward the concave side of the curve. If $d\theta/ds > 0$ at P, we have: $\theta = -\pi/2$ and $d\theta/ds = 1/R$ where R is the radius of curvature. If $d\theta/ds < 0$ at P, we have $\theta = \pi/2$ and $d\theta/ds = -1/R$. Then in either case:

$$a_N = \frac{|V|^2}{R},$$

the *normal* component of the acceleration measured in the direction of the center of curvature.

NOTE ON VECTOR PRODUCTS

M. E. SHANKS, Purdue University

Vector analysis books define the vector product geometrically. The definition is justified solely by its later usefulness. Frequently students are repelled by the arbitrariness of the definition and feel that there should be some inherent necessity for such a peculiar product. It is the purpose of this note to supply an algebraic argument, based on a simple physical assumption, which shows that the ordinary product is, to within a multiplicative constant, the only one possible.*

For the following discussion familiarity with the scalar or dot "product" is necessary as well as the fact that the scalar product is invariant under rotation of axes. Guided by a feeling for algebra we seek a definition for a true product, $\mathbf{u} \times \mathbf{v}$, of two vectors $\mathbf{u}$ and $\mathbf{v}$. We use the common notation and denote a right-hand set of orthonormal base vectors by $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$. We require of our product only the distributive law: $\mathbf{u} \times (\mathbf{v} + \mathbf{w}) = \mathbf{u} \times \mathbf{v} + \mathbf{u} \times \mathbf{w}$, and $(\alpha\mathbf{u}) \times (\beta\mathbf{v}) = \alpha\beta\mathbf{u} \times \mathbf{v}$, where α, β are real numbers. Since distributivity is assumed it suffices to define products of base vectors only. Stated otherwise, we seek to make an algebra out of a real 3-dimensional vector space with an inner product, or a real 3-dimensional unitary space.

A vector space becomes an algebra by giving an arbitrary multiplication table for the base vectors. Some restriction must therefore be placed on the product and since vectors are intended for use in physics we examine the physical requirements. Now in physics there are no preferred orientations; therefore, since any right-hand set of base vectors $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ can be transformed by rotation into any other set $\mathbf{i}'$, $\mathbf{j}'$, $\mathbf{k}'$, it follows that the formula for the product must be the same for all sets of base vectors. It is an easy consequence that, if R is a rotation, then $R(\mathbf{u} \times \mathbf{v}) = (R\mathbf{u} \times R\mathbf{v})$, or *products are invariant under rotation.*

* While this result does not seem to be explicitly stated in the literature on vector analysis, it is implicit in the fact that the basic invariants of the orthogonal group are the scalar product and triple product. See H. Weyl, "The Classical Groups," p. 53.

THEOREM 1. *The only vector product which is invariant under rotation is given by*

$$\mathbf{i} \times \mathbf{j} = \lambda\mathbf{k} = -\mathbf{j} \times \mathbf{i}, \qquad \mathbf{j} \times \mathbf{k} = \lambda\mathbf{i} = -\mathbf{k} \times \mathbf{j},$$
$$\mathbf{k} \times \mathbf{i} = \lambda\mathbf{j} = -\mathbf{i} \times \mathbf{k}, \qquad \mathbf{i} \times \mathbf{i} = \mathbf{j} \times \mathbf{j} = \mathbf{k} \times \mathbf{k} = 0,$$

where λ *is an arbitrary real number.*

Consider $\mathbf{i}\cdot\mathbf{i}\times\mathbf{i}$ and the rotation R: $R\mathbf{i}=-\mathbf{i}$, $R\mathbf{j}=-\mathbf{j}$, $R\mathbf{k}=\mathbf{k}$. Since both scalar and vector products are invariant under rotation, $\mathbf{i}\cdot\mathbf{i}\times\mathbf{i}=R\mathbf{i}\cdot R(\mathbf{i}\times\mathbf{i}) =R\mathbf{i}\cdot R\mathbf{i}\times R\mathbf{i}=-\mathbf{i}\cdot\mathbf{i}\times\mathbf{i}$. Hence $\mathbf{i}\cdot\mathbf{i}\times\mathbf{i}=0$. Similarly it follows that $\mathbf{j}\cdot\mathbf{i}\times\mathbf{i}=0$. With a different rotation one obtains $\mathbf{k}\cdot\mathbf{i}\times\mathbf{i}=0$. Therefore $\mathbf{i}\times\mathbf{i}=\mathbf{0}$, and because of the invariance of products under rotation $\mathbf{j}\times\mathbf{j}=\mathbf{k}\times\mathbf{k}=\mathbf{0}$.

Consider now $\mathbf{i}\times\mathbf{j}$. We compute $\mathbf{i}\cdot\mathbf{i}\times\mathbf{j}$ using the rotation R: $R\mathbf{i}=-\mathbf{i}$, $R\mathbf{j}=-\mathbf{j}$, $R\mathbf{k}=\mathbf{k}$. Then $\mathbf{i}\cdot\mathbf{i}\times\mathbf{j}=R\mathbf{i}\cdot R(\mathbf{i}\times\mathbf{j})=R\mathbf{i}\cdot R\mathbf{i}\times R\mathbf{j}=-\mathbf{i}\cdot\mathbf{i}\times\mathbf{j}$ and $\mathbf{i}\cdot\mathbf{i}\times\mathbf{j}=0$. Similarly $\mathbf{j}\cdot\mathbf{i}\times\mathbf{j}=0$. Therefore $\mathbf{i}\times\mathbf{j}=\lambda\mathbf{k}$ where λ is a real number.

The rotation R: $R\mathbf{i}=\mathbf{j}$, $R\mathbf{j}=\mathbf{k}$, $R\mathbf{k}=\mathbf{i}$, gives $\mathbf{j}\times\mathbf{k}=\lambda\mathbf{i}$ and $\mathbf{k}\times\mathbf{i}=\lambda\mathbf{j}$.

By the same argument it follows that $\mathbf{j}\times\mathbf{i}=\mu\mathbf{k}$, $\mathbf{k}\times\mathbf{j}=\mu\mathbf{i}$, $\mathbf{i}\times\mathbf{k}=\mu\mathbf{j}$, where μ is real.

Finally, consider $\mathbf{k}\cdot(\mathbf{i}\times\mathbf{j}+\mathbf{j}\times\mathbf{i})$ and the rotation R: $R\mathbf{i}=\mathbf{j}$, $R\mathbf{j}=-\mathbf{i}$, $R\mathbf{k}=\mathbf{k}$. We have $\mathbf{k}\cdot(\mathbf{i}\times\mathbf{j}+\mathbf{j}\times\mathbf{i})=\mathbf{k}\cdot(-\mathbf{j}\times\mathbf{i}-\mathbf{i}\times\mathbf{j})$, whence $\mathbf{k}\cdot(\mathbf{i}\times\mathbf{j}+\mathbf{j}\times\mathbf{i})=0=\lambda+\mu$ or $\mu=-\lambda$. This completes the proof. It may be noted that the only rotations used in the proof are those generated by rotations through 90° about the coördinate axes.

Thus in a real 3-dimensional unitary space products invariant under rotation are essentially unique. It is natural then to inquire into the possibilities when the dimension is not 3. A 1-dimensional space is isomorphic to the real line, and the only "rotation" is the identity. If $\mathbf{i}$ is a unit base vector then $\mathbf{i}\times\mathbf{i}=\lambda\mathbf{i}$ gives a product. But for other dimensions it is easy to prove, using the same arguments as above, the following theorem.†

THEOREM 2. *In a real n-dimensional unitary space, the only product invariant under orthogonal transformations of determinant* 1 *is the trivial one* $u\times\mathbf{v}=0$, *if* $n\neq 1, 3$.

ANGLE WITH RESPECT TO A FAMILY OF PLANE CURVES

L. V. TORALBALLA, Marquette University

On writing the equation of the circle $x^2+y^2=a^2$ in the parametric form $x=a\cos\theta$, $y=a\sin\theta$, one observes that the area swept by the radius vector from the parameter value $\theta=0$ to the parameter value $\theta=\theta_0$ is $\frac{1}{2}a^2\theta_0$. One could single out this particular property of θ and consider it the defining property of a generalized angle.

To make the analogy with the above circle more complete we shall consider only one-parameter families of curves. That is to say, we shall deal only with

† In n dimensions a "product" of $n-1$ vectors may be formed which is invariant under the proper orthogonal group. See Weyl, *loc. cit.*, p. 49.

$$x = a \cos(\sqrt{2}\,\theta); \qquad y = \frac{a}{\sqrt{2}} \sin(\sqrt{2}\,\theta).$$

For the hyperbola $x^2 - 2y^2 = a^2$, the representation is

$$x = a\left[\frac{e^{\sqrt{2}\,\theta} - e^{-\sqrt{2}\,\theta}}{2}\right]; \qquad y = a\left[\frac{e^{\sqrt{2}\,\theta} + e^{-\sqrt{2}\,\theta}}{2}\right].$$

For the parabola $y^2 = 2ax$, the representation is

$$x = \tfrac{1}{2}a\sqrt[3]{36\theta^2}; \qquad y = a\sqrt[3]{-6\theta}$$

For the circle $x = a\cos\theta$, $y = a\sin\theta$, the parameter θ has also the property that the arc length from $\theta = 0$ to $\theta = \theta_0$ is $a\theta_0$. One could adopt this property as the defining property of the generalized angle, instead of the property discussed above.

Then for the curve family $h(x, y, a) = 0$, of the type described above, one should, obviously, use the differential relation $f'^2(\theta) + g'^2(\theta) = 1$, instead of the relation $f(\theta)g'(\theta) - g(\theta)f'(\theta) = 1$. However, in general, the functions $f(\theta)$ and $g(\theta)$ thus obtained are more complex than those obtained before.

For an arbitrary curve, it is not possible to choose $f(\theta)$ and $g(\theta)$ so that the area be $\frac{1}{2}a^2\theta_0$ and at the same time the arc length be $a\theta_0$. In fact this can be done only for the family of circles with center at the origin and for the families of parallel straight lines.

ELEMENTARY PROBLEMS AND SOLUTIONS

EDITED BY HOWARD EVES, Oregon State College

Send all communications concerning Elementary Problems and Solutions to Howard Eves, Mathematics Department, Oregon State College, Corvallis, Oregon. This department welcomes problems believed to be new, and demanding no tools beyond those ordinarily furnished in the first two years of college mathematics. To facilitate their consideration, solutions should be submitted on separate, signed sheets, within three months after publication of problems.

PROBLEMS FOR SOLUTION

E 952 [1951, 108]. *Correction*

Change the direction of the inequality signs.

E 961. *Proposed by Leo Moser, Texas Technological College*

Find the probability that if the digits 0, 1, 2, $\cdots$, 9 be placed in random order in the blank spaces of

5_383_8_2_936_5_8_203_9_3_76

the resulting number will be divisible by 396. (By permission of Prof. E. P. B. Umbugio, April 1, 1951.)

E 962. *Proposed by B. H. Brown, Dartmouth College*

A cask of unit volume is full of wine. A man withdraws from the cask an amount a $(0<a<1)$, and then adds amount a of *water*, which is assumed to mix perfectly with the wine. He continues this process b times. He then withdraws amount a and adds amount a of *wine*, continuing this process b times.

(1) Derive a general formula for the amount of wine in the cask after the $2b$ operations.

(2) If $1/a=A$ is an integer, and if B is that corresponding (integral) value of b for which the eventual wine-content is a minimum, so determine A and B that the wine-content is (a) the minimum minimorum (b) the maximum minimorum.

(3) If $A=51$, find B and the wine-content.

(4) Find $\lim_{A\to\infty}(B/A)$.

E 963. *Proposed by M. S. Klamkin, Polytechnic Institute of Brooklyn*

It had started snowing before noon and three snow plows set out at noon, 1 o'clock, and 2 o'clock, respectively, along the same path. If at some later time they all came together simultaneously, find the time of meeting and also the time it started snowing.

E 964. *Proposed by C. W. Trigg, Los Angeles City College*

(1) Determine the relationship between the sides of a triangle ABC if the line joining the Gergonne point P to the Nagel point Q is parallel to side c.

(2) Show that, given a and b, such a triangle can always be constructed with straight edge and compasses.

(3) Find the smallest scalene triangle of this type with integer sides.

E 965. *Proposed by R. K. Meany, Purdue University*

A function $f(x)$ is continuous in the open interval (a, b). Its derivative, $f'(x)$ exists at every point of (a, b) except perhaps at c, but $\lim_{x\to c} f'(x)$ exists and equals A. Show that then $f'(c)$ also exists and equals A.

SOLUTIONS

Enumeration of Triangles in a Polygon

E 927 [1950, 483]. *Proposed by J. H. Braun, Illinois Institute of Technology*

In a regular polygon of $2n+1$ sides all the diagonals are drawn. Find, as a function of n, the total number of triangles of all shapes and sizes thus formed.

Solution by G. W. Walker, Buffalo, N. Y. If the sides of any triangle are extended, they will intersect the circle circumscribing the polygon in 3, 4, 5, or 6 distinct points, each point a vertex of the polygon, according as the triangle has 3, 2, 1, or 0 corners on the perimeter of the polygon. In the first case, the points may be chosen in C_3^{2n+1} ways. In the second case, the 4 points may be chosen in C_4^{2n+1} ways, and the 2 (of the 4) may be chosen (cyclically adjacent)

for the vertices of the triangle in 4 ways. In the third case, the 5 points may be chosen in C_5^{2n+1} ways, and the point for the vertex of the triangle may be chosen (from the 5) in 5 ways. In the last case, the 6 points may be chosen in C_6^{2n+1} ways.

Therefore, there are, all together,

$$C_3^{2n+1} + 4C_4^{2n+1} + 5C_5^{2n+1} + C_6^{2n+1} = n(2n-1)(2n+1)(2n^3 + 21n^2 - 2n + 9)/90$$

different triangles.

Also solved by Aaron Buchman, Roger Lessard, and the proposer.

Tetrahedron and Concurrent Cevians

E 928 [1950, 483]. *Proposed by Victor Thébault, Tennie, Sarthe, France*

Given a tetrahedron $ABCD$ and a point O. Denote by A', B', C', D' the intersections of AO, BO, CO, DO with the corresponding faces of the tetrahedron, and set $x=AO/A'O$, $y=BO/B'O$, $z=CO/C'O$, $t=DO/D'O$. Show that

$$xyzt = 3 - 2(x+y+z+t) + (xy + xz + xt + yz + yt + zt).$$

Solution by M. S. Klamkin, Polytechnic Institute of Brooklyn. The altitude of $OBCD$ from O is easily shown to be $h_A/(1-x)$, where h_A is the altitude of $ABCD$ from A. Therefore, if V is the volume of $ABCD$ and b_A is the area of the face opposite A,

$$V = \Sigma(h_A b_A)/3(1-x) = \Sigma V/(1-x).$$

That is

$$1 = 1/(1-x) + 1/(1-y) + 1/(1-z) + 1/(1-t),$$

or

$$xyzt = 3 - 2(x+y+z+t) + (xy + xz + xt + yz + yt + zt).$$

Also solved by Joseph Langr, Roger Lessard and G. S. Mahajani.

A Product of Reflections

E 929 [1950, 483]. *Proposed by H. S. Shapiro, Massachusetts Institute of Technology*

Given three non-concurrent straight lines l_1, l_2, l_3 in the plane. Let T_i denote reflection in l_i and set $T=T_1T_2T_3$. Show that T^2 is a translation.

Solution by B. D. Roberts, New Mexico Highlands University. The double reflection T_1T_2 in the two lines l_1, l_2 (whether intersecting or not) is a rigid sense-preserving transformation involving a rotation through twice the angle from l_1 to l_2. But T^2 is the product of three such double reflections, the sum of the angles involved being a multiple of π. The total rotation due to T^2 is then through an angle which is a multiple of 2π. This means that T^2 produces a translation only. It is easy to show that this translation reduces to the identity transformation if and only if l_1, l_2, l_3 are concurrent or parallel.

Also solved by J. M. Feld, H. C. Kranzer, W. J. Nemerever, C. S. Ogilvy, O. Dale Smith, and the proposer.

Feld used the known fact that the product of three reflections is equivalent to the commutative product of a reflection L in a line l and a translation A parallel to l. Then $T^2 = LALA = ALLA = A^2$, a translation.

Editorial Note. It can be shown that the magnitude of the translation T^2 is equal to twice the perimeter of the orthic triangle of the triangle formed by l_1, l_2, l_3, where, if the latter triangle is obtuse, the side of the orthic triangle corresponding to the obtuse angle is taken as negative. Thus the magnitude of the translation is independent of the order of reflections in the set T. See, *e.g.*, Morley and Morley, *Inversive Geometry*, p. 35.

An interesting consequence of the problem, pointed out by the proposer, is that a plane point set with three non-concurrent and non-parallel axes of symmetry contains an infinite lattice-work of points.

Inverse Tangents

E 930 [1950, 483]. *Proposed by Gordon Raisbeck, Bell Telephone Laboratories.*

If

$$\prod_{i=1}^{n}(x + r_i) \equiv \sum_{j=0}^{n} a_j x^{n-j},$$

show that

$$\sum_{i=1}^{n} \tan^{-1} r_i = \tan^{-1} \frac{a_1 - a_3 + a_5 - \cdots}{a_0 - a_2 + a_4 - \cdots}$$

and

$$\sum_{i=1}^{n} \tanh^{-1} r_i = \tanh^{-1} \frac{a_1 + a_3 + a_5 + \cdots}{a_0 + a_2 + a_4 + \cdots}.$$

Solution by T. M. Apostol, California Institute of Technology. Let p_k be the kth elementary symmetric function of the n variables $r_1, r_2, \cdots, r_n$. Then the formula

$$\tan(x_1 + x_2) = (\tan x_1 + \tan x_2)/(1 - \tan x_1 \tan x_2)$$

is merely the case $n = 2$ of the more general formula

$$\text{(1)} \qquad \tan\left(\sum_{i=1}^{n} x_i\right) = (p_1 - p_3 + p_5 - \cdots)/(1 - p_2 + p_4 - p_6 + \cdots),$$

where $x_i = \tan^{-1} r_i$. Equation (1) can be proved by induction.

Expressing the coefficients of the polynomial

$$\prod_{i=1}^{n}(x + r_i) = \sum_{j=0}^{n} a_j x^{n-j}$$

in terms of the symmetric functions p_k and substituting in (1) gives the desired result at once.

In the case of the hyperbolic tangent, all the signs on the right of (1) are $+$.

Also solved by Vern Hoggatt, M. S. Klamkin, Roger Lessard, H. F. Mattson, Azriel Rosenfeld, M. R. Spiegel, O. E. Stanaitis, and the proposer. The proposer used the identities

$$\tan^{-1} a = (-i/2) \log [(1 + ia)/(1 - ia)],$$
$$\tanh^{-1} a = (1/2) \log [(1 + a)/(1 - a)].$$

Drawing Balls from an Urn

E 931 [1950, 556]. *Proposed by H. D. Larsen, Albion College*

Ten balls numbered from 0 to 9 inclusive are placed in an urn. Five of the balls are then drawn at random (without replacement) and arranged in a row. What is the probability that the number thus formed is divisible by 396?

Solution by C. W. Trigg, Los Angeles City College. Since N $(=abcde)$ is divisible be 396 $(=4\cdot 9\cdot 11)$ it follows that $S=a+b+c+d+e\equiv 0 \pmod 9$, so $S=18$ or 27. Also, $(a+c+e)-(b+d)\equiv 0 \pmod{11}$. Now if $S=18$, then $b+d=9$, so (b, d) or $(d, b)=(0, 9)$, $(1, 8)$, $(2, 7)$, $(3, 6)$, or $(4, 5)$. There are ten sets of five distinct digits with $S=18$ which contain one of the eligible pairs (b, d) and at least one additional even digit, so that the possibility exists that the terminal pair of a permutation of the set may be divisible by 4. From these we find 64 permutations that are divisible by 396.

If $S=27$, $b+d=8$, so (b, d) or $(d, b)=(0, 8)$, $(1, 7)$, $(2, 6)$, or $(3, 5)$. From the eight sets of five distinct digits which meet these restrictions we find 32 values of N. Therefore the probability that N is divisible by 396 is $96/(10!/5!)$ or $1/315$.

The same result may be obtained by listing all the multiples of 396 between 01188 and 99792 (easily done since $396=400-4$) and deleting those containing duplicate digits.

If k balls are drawn at random from the urn, the probability p that the number found is divisible by 396 may be found by using the divisibility criteria as above. These probabilities are:

k	P
3	$2/(10!/7!) = 1/490$
4	$18/(10!/6!) = 1/280$
6	$360/(10!/4!) = 1/420$
7	$1488/(10!/3!) = 31/12600$
8	$5328/(10!/2!) = 37/12600$
9	$15984/(10!) = 37/8400$
10	$78336/(10!) = 34/1575$

Also solved by Ferrel Atkins, Monte Dernham, J. B. Friedman, Vern Hoggatt, Roger Lessard, Leo Moser, L. A. Ringenberg, and the proposer.

Triangle with Orthogonal Circum- and Nine Point Circles

E 933 [1950, 557]. *Proposed by Joseph Langr, Prague, Czechoslovakia*

Let A', B', C' be the feet of the altitudes of a triangle ABC whose circumcircle and nine point circle are orthogonal. Show that (1) the areas of triangles ABC and $A'B'C'$ are equal, (2) $(AC')(BA')(CB')=(AB)(BC)(CA)/2$. Construct the triangle ABC given side BC and the position of A' on BC.

Solution by A. Sisk, Maryville, Tenn. Let points A, B, C have coordinates $(0, a)$, $(b, 0)$, $(c, 0)$ respectively. Then A' is at the origin and the coordinates of B' and C' are

$$(c(a^2 + bc)/(a^2 + c^2),\ ac(c - b)/(a^2 + c^2)),$$
$$(b(a^2 + bc)/(a^2 + b^2),\ ab(b - c)/(a^2 + b^2)).$$

The equations of the circumcircle (O) and the nine point circle (N) are found to be

$$x^2 + y^2 - (b + c)x - (a^2 + bc)y/a + bc = 0$$

and

$$x^2 + y^2 - (b + c)x/2 + (bc - a^2)y/2a = 0.$$

The condition that (O) and (N) be orthogonal is

$$\text{(A)} \qquad a^4 + a^2(b - c)^2 - b^2c^2 = 0.$$

The area ABC is $a(c-b)/2$, and the area of $A'B'C'$ is

$$abc(b - c)(a^2 + bc)/(a^2 + b^2)(a^2 + c^2),$$

which, by use of (A), reduces to $a(b-c)/2$. This proves the first part.

$$(AB)^2(BC)^2(CA)^2/4 = (AC')^2(BA')^2(CB')^2 = bc(b - c)^2(a^2 + bc)/2,$$

and the second part is established.

As for the third part, when we are given BC and the position of A' on BC, we are given b and c. Therefore, by (A), a can be found and the triangle constructed.

Also solved by the proposer, who pointed out that the triangle with orthogonal circum- and nine point circles was studied by V. Thébault in *Mathesis*, tome LVIII, 1949, Question 3409, p. 102 and p. 363. If H is the orthocenter, R the circumradius, and a, b, c the sides of such a triangle, Thébault found that

$$(HA)(HB)(HC) = 4R^3 \quad \text{and} \quad (HA)^2(HB)^2(HC)^2 = a^6 + b^6 + c^6.$$

To these relations the proposer added

$$\cos A \cos B \cos C = -1/2 \quad \text{and} \quad \sin^2 A + \sin^2 B + \sin^2 C = 1.$$

Curves with the "Focal" Property

E 934 [1950, 557]. *Proposed by C. O. Oakley, Haverford, College*

Let C_1 and C_2 be two arbitrarily given curves whose parametric equations are $x_1=x_1(t)$, $y_1=y_1(t)$ and $x_2=x_2(t)$, $y_2=y_2(t)$. Show that there exists a third curve C, $x=x(t)$, $y=y(t)$, with the "focal" property that, if P_1, P_2, and P are corresponding points on C_1, C_2, and C respectively, then P_1P and P_2P make equal angles with the tangent to C at P.

Solution by the Proposer. To prove this we apply the "tangent of the angle between two lines theorem" from analytic geometry. In our notation

$$\left(\frac{y-y_1}{x-x_1}-\frac{y'}{x'}\right)\Bigg/\left(1+\frac{y-y_1}{x-x_1}\,\frac{y'}{x'}\right)=\left(\frac{y'}{x'}-\frac{y-y_2}{x-x_2}\right)\Bigg/\left(1+\frac{y-y_2}{x-x_2}\,\frac{y'}{x'}\right),$$

where the primes indicate differentiation with respect to the parameter t. Reducing this we obtain

$$\begin{aligned}(1)\quad &(y'^2/x'^2-1)[(x-x_1)(y-y_2)+(x-x_2)(y-y_1)]\\ &\qquad +2(y'/x')[(x-x_1)(x-x_2)-(y-y_1)(y-y_2)]=0.\end{aligned}$$

The (arbitrary) function $x(t)$ and a corresponding solution $y(t)$ of the non-linear differential equation (1) will thus yield the parametric equations of a curve C satisfying the conditions of the theorem. In actual practice it would be very difficult to carry out the integration except in the very simplest cases.

If each of the two curves be reduced to a single point, say $x_1=-k$, $y_1=0$ and $x_2=k$, $y_2=0$, respectively, equation (1) reduces to

$$(2)\qquad xy(dy/dx)^2+(x^2-y^2-k^2)dy/dx-xy=0,$$

where we have written dy/dx for y'/x'. Equation (2) is the familiar differential equation of confocal conics, in standard position, and the theory shows that for two fixed points the confocal conics are the only curves enjoying this focal property.

In many instances the theorem above can be strengthened. Consider, as before, the two given curves C_1 and C_2 and let a third curve C, $y=f(x)$, be given in cartesian form. A parametric representation of C, $x=x(t)$, $y=y(t)$, often exists such that the focal property holds. For, in this case, equation (1) becomes

$$\begin{aligned}(3)\quad &(f'^2-1)[(x-x_1)(f-y_2)+(x-x_2)(f-y_1)]\\ &\qquad +2f'[(x-x_1)(x-x_2)-(f-y_1)(f-y_2)]=0,\end{aligned}$$

a non-differential equation in x and t. From a real solution $x=x(t)$ of (3), when such exists, we obtain the parametrization sought since $y=y(t)=f(x(t))$.

Also solved by A. Sisk.

ADVANCED PROBLEMS AND SOLUTIONS

EDITED BY E. P. STARKE, Rutgers University

Send all communications concerning Advanced Problems and Solutions to E. P. Starke, Rutgers University, New Brunswick, New Jersey. All manuscripts should be typewritten, with double spacing and margins at least one inch wide. Problems containing results believed to be new or extensions of old results are especially sought. Proposers of problems should also enclose any solutions or information that will assist the editor. In general, problems in well known textbooks or results in readily accessible sources should not be proposed.

PROBLEMS FOR SOLUTION

4433. *Proposed by J. W. Gaddum, University of Missouri*

Let $f(z)$ be a continuous function of a complex variable, finite at z_0. Then in every neighborhood of z_0 there are distinct z_1, z_2 such that $|f(z_1)| = |f(z_2)|$.

4434. *Proposed by M. R. Spiegel, Rensselaer Polytechnic Institute, Troy, N. Y.*

Prove

$$\sum_{k=1}^{n-1} \frac{\sin^2 (kr\pi/n)}{\sin^2 (k\pi/2n)} = 2r(n-r), \qquad 0 \leqq r \leqq n,$$

and thus evaluate

$$\int_0^1 \frac{\sin^2 rx}{\sin^2 \frac{1}{2}x}\, dx.$$

4435. *Proposed by J. P. Ballantine, University of Washington, Seattle*

Find the locus of points in xyz-space from which the conic $b^2x^2+a^2y^2=a^2b^2$ in the plane $z=0$ looks like a circle.

4436. *Proposed by D. J. Newman, New York University*

What is the probability that an arbitrary integer have a prime divisor which is larger than its square root?

4437. *Proposed by Paul Erdös, University of Aberdeen, Scotland*

Here is an old problem of Turán: Let $z_1, z_2, \cdots, z_n$ be n complex numbers, $|z_1|=1$. Put $s_k^{(n)}=z_1^k+z_2^k+\cdots+z_n^k$. Does an absolute constant c exist, independent of n, so that

$$\max_{1\leqq k\leqq n} |s_k^{(n)}| = d_n > c?$$

SOLUTIONS

Isosceles Tetrahedron

4354 [1949, 414]. *Proposed by Victor Thébault, Tennie, Sarthe, France*

A necessary and sufficient condition for a tetrahedron to be isosceles is that each of two bialtitudes of the tetrahedron divide the opposite edges proportion-

ally.

Solution by L. M. Kelly, Michigan State College. Consider the parallelepiped circumscribed about the tetrahedron $ABCD$. If $ABCD$ is isosceles, the diagonals of the various faces are pairwise equal so that the parallelepiped is rectangular and the bialtitudes clearly bisect the opposite edges.

Conversely, let P and Q be the feet of a bialtitude, P being on AB and Q on CD, and suppose $AP/PB=CQ/QD$. Let the second diagonal in the face with AB be labelled $C'D'$, and locate point Q' on $C'D'$ such that $Q'D'=QD$. Then QQ' is parallel to DD' and $Q'P$ is parallel to $D'B$. Thus the face $DD'B$ is parallel to the plane $QQ'P$ containing the bialtitude and is therefore perpendicular to the face $ABD'C'$.

If another bialtitude divides its opposite edges proportionally then the parallelopiped will be rectangular and the tetrahedron isosceles.

Also solved by the Proposer.

Another Sequence of Remarkable Squares

4355 [1949, 479]. *Proposed by Victor Thébault, Tennie, Sarthe, France*

Solve the equation

$$5m^2+2m+1=n^2$$

in integers and show that, provided $m\equiv -1 \bmod 3$, there exists in every system of numeration of base $B=3m+1$ at least one pair of perfect squares having the form $aabb=(cc)^2$, $bbaa=(dd)^2$. There are infinitely many such systems of numeration. (See also the Proposer's paper, "Two Classes of Remarkable Perfect Square Pairs," this MONTHLY, 1949, pp. 443–448).

Solution by Daniel Block, Yeshiva College, New York City. Solving $5m^2+2m+1=n^2$ for m, we obtain

$$m=(-1+k)/5, \qquad k^2=5n^2-4,$$

or $(k/2)^2-5(n/2)^2=-1$. Since the convergents for the continued fraction of $\sqrt{5}$ are 2/1, 9/4, 38/17. 161/72, 682/305, $\cdots$, we have solutions $k/2=p$, $n/2=q$, where p/q is any convergent of odd order. Thus

$$n=2,\quad k=-4,\quad m=-1;\qquad n=34,\quad k=76,\quad m=15;$$
$$n=610,\quad k=-1364,\quad m=-273;\cdots.$$

In order that perfect squares exist as prescribed, there must be integers a, b, c, d which satisfy conditions (1), (2), (3), (4) of the Proposer's paper [1949, 444]. These are obtained with little effort by setting

$$c=n,\qquad d=2m+1,\qquad a=(5m+2)/3,\qquad b=(4m+4)/3.$$

a and b will be integers by virtue of the condition $m\equiv -1 \bmod 3$.

Also solved by Roger Lessard and by the Proposer.

Editorial Note. It is easily verified that if k_i, n_i are any positive integers satisfying $k^2=5n^2-4$, then k_{i+1}, n_{i+1} will also satisfy if

$$(1)\qquad k_{i+1} = 9k_i + 20n_i, \qquad n_{i+1} = 4k_i + 9n_i;$$

furthermore, that k_i, n_i may be obtained from $(k_1, n_1)=(1, 1)$ or $(4, 2)$ or $(11, 5)$ by repeated use of (1). There are therefore three infinite sequences of values of k_i and n_i. A recursion formula for k_i alone, obtained by eliminating the n's, is

$$(2)\qquad k_{i+1} = 18k_i - k_{i-1}.$$

To meet the requirement that m and a, as given above, be positive integers we must have $k_i \equiv -4 \bmod 15$. From (2) and the values of (k_1, n_1) it is evident that the values of k_i in each sequence are congruent mod 15 to $1, -1, -4, 4, 1, -1, \cdots$ in cyclic succession. Hence every fourth value in each sequence (starting at the proper member) provides a satisfactory set of squares. The simplest is, to base 7, $4444=(55)^2$. The next is given by $B=313$, $a=174$, $b=140$, $c=233$, $d=209$. The smallest given by the sequence in the above solution has $B=14686$.

Two Summations

4356 [1949, 479]. *Proposed by P. A. Piza, San Juan, Puerto Rico*

Prove the relations

$$(a)\qquad x^{2n+1} - (x-1)^{2n+1} = \sum_{a=0}^{n}\left[\binom{n+a}{2a+1} + \binom{n+1+a}{2a+1}\right](x^2-x)^{n-a}$$

$$(b)\qquad x^{2n+2} - (x-1)^{2n+2} = (2n-1)\sum_{a=0}^{n}\binom{n+1}{2a+1}(x^2-x)^{n-a}.$$

I. *Solution by Max LeLeiko, Brooklyn, New York.* These formulas are special cases of more general formulas given in Chrystal, *Algebra.* From part II, page 203 we take the identity

$$\alpha^{2n+1} + \beta^{2n+1} = (-1)^n(2n+1)\sum_{a=0}^{n}(-1)^a\frac{\binom{n+a}{2a}}{2a+1}p^{2a+1}q^{n-a},$$

where $p=\alpha+\beta$, $q=\alpha\beta$. Now let $\alpha=x$, $\beta=1-x$. Then $p=1$, $q=x-x^2$, and substituting we get

$$x^{2n+1} + (1-x)^{2n+1} = (2n+1)\sum_{a=0}^{n}(-1)^{n+a}\frac{\binom{n+a}{2a}}{2a+1}(x-x^2)^{n-a}.$$

Using

$$\frac{2n+1}{2a+1}\binom{n+a}{2a} = \binom{n+a}{2a+1} + \binom{n+1+a}{2a+1}$$

we find proposed relation (a).

To prove (b) we make use of another identity (page 204 of the same reference):

$$\alpha^{2m} - \beta^{2m} = (\alpha - \beta)(-1)^{m-1}\sum_{a=0}^{m-1}(-1)^a\binom{m+a}{2a+1}p^{2a+1}q^{m-a-1}.$$

With $\alpha = x$, $\beta = 1-x$, $m = n+1$, $p = 1$, $q = x - x^2$, this is easily reduced to the desired relation.

II. *Solution by M. S. Klamkin, Polytechnic Institute of Brooklyn, New York.* Assume (b) is true for $n = k$. Then by integrating between 1 and x we obtain

$$x^{2k+3} - (x-1)^{2k+3} = 1 + \sum_{a=0}^{k}\frac{2k+3}{k+1-a}\binom{k+1+a}{2a+1}(x^2 - x)^{k+1-a}$$

which is easily shown to be equivalent to (a) for $n = k+1$. Further, upon multiplying (a) for $n = k+1$ by $2x-1$, multiplying (b) for $n = k$ by $x^2 - x$, and subtracting, we get exactly (b) for $n = k+1$. Since (a) and (b) are obviously true for $n = 0$, it follows by induction that they are true for all integral n.

Also solved by Roger Lessard.

A Summation Equivalent to an Integration

4357 [1949, 479]. *Proposed by R. D. Stalley, University of Arizona*

Reduce the problem of summing the series

$$\sum_{k=1}^{\infty} k^{-n}x^k$$

where n is a positive integer $\geqq 2$ to numerical integration of a function over a finite range.

Solution by the Proposer. Assume $|x| < 1$. From the relation

$$x\frac{d}{dx}\sum_{k=1}^{\infty} k^{-n}x^k = \sum_{k=1}^{\infty} k^{-n+1}x^k,$$

we have

$$\sum_{k=1}^{\infty} k^{-n}x^k = \int_0^x \frac{1}{x}\sum_{k=1}^{\infty} k^{-n+1}x^k dx,$$

the constant of integration being 0. If $n = 1$, this equation yields

$$\sum_{k=1}^{\infty} k^{-1}x^k = \int_0^x \frac{1}{x}\sum_{k=1}^{\infty} x^k dx = \int_0^x \frac{1}{x}\cdot\frac{x}{1-x}dx = \log\frac{1}{1-x}\cdot$$

If $n = 2$, we have

$$\sum_{k=1}^{\infty} k^{-2}x^k = \int_0^x \frac{1}{x} \log \frac{1}{1-x}\, dx.$$

Repeating this process for successive integers we see inductively that

$$\sum_{k=1}^{\infty} k^{-n}x^k = \int_0^x \frac{1}{x} \int_0^x \frac{1}{x} \cdots \int_0^x \frac{1}{x} \log \frac{1}{1-x} (dx)^{n-1}.$$

Round-the-World Flight

4360 [1949, 556]. *Proposed by Free Jamison, San Jose State College, California*

Any one of a group of airplanes may be refueled from any other. Each has a fuel capacity sufficient for a flight one-fifth the distance around the earth. Assuming that all have the same constant ground speed and the same rate of fuel consumption, that the only landing place and the only available fuel supply are at the home base, and that refueling time is negligible, find the minimum number of planes necessary so that one plane may fly around the earth and all return home safely. (Compare *The Jeep Problem* [1947, 24], and [1947, 458].)

Comments by the Proposer. The minimum number of planes required is not more than 75.

It seems reasonable that a sufficient number of planes would depart from the home base so that, by refueling, one plane would be fueled to capacity when two-fifths of the distance around the earth and be met there by a plane flying in the opposite direction. Let two-fifths of the distance be divided into twelve equal legs. (Each leg is one-thirtieth of the earth's circumference, a plane fueled to capacity has enough fuel for six legs.) A schedule is given to show how 77 planes can accomplish the desired feat. Thirty-two planes depart at the same time, at the end of the first leg 25 are fueled to capacity and 7 return, at the end of the second leg 5 turn back after refueling 20 to capacity but at the same instant 9 depart from the home base, and so on. Let the amount of fuel necessary for one plane for one leg be called a portion. Subscripts on numbers of returning planes indicate portions of fuel remaining at the end of the leg, negative subscripts on numbers of outbound planes indicate portions of fuel below capacity at the beginning of the leg.

After six legs have been completed by the first flight of planes it would be necessary to send planes from the home base in the opposite direction to meet the plane flying around the earth. For this operation we have only to use the lines of the schedule in reverse order. The numbers of planes in the air at various times are now easily computed, and it is found that no more than 77 are required at any one time. However this number can be reduced by one if on the eighth outbound flight of ten planes, one plane turns back at the midpoint of the first leg after refueling the other nine planes (with a similar change in the analogous flight which is to meet the home-coming plane). A further reduction of one is possible if on the sixth, seventh, ninth and tenth flights each, one plane is turned back at the midpoint of the first leg; and if on the eight flight

one plane is turned back at the quarter point and one at the three-quarter point on the first leg.

It is reasonable to suppose that the minimum number demanded in the proposal is less than 75, since the schedule here presented makes the simplifying assumptions that changes of direction and refueling occur only when an integral number of legs have been traversed and that all planes flying at any instant are flying in the same direction (indicated by the arrow.)

Somewhat similar comments were received from W. E. Buker and J. V. Pennington.

32 →												
7 ←	25 →											
9 →	5_0 ←	20 →										
8 ←	6 →	4_0 ←	16 →									
18 →	6_0 ←	4 →	4_4 ←	12 →								
11 ←	13 →	5_1 ←	3 →	3_3 ←	9 →							
13 →	9_3 ←	9 →	4_2 ←	2 →	2_1 ←	7 →						
13 ←	9 →	7_4 ←	6 →	3_2 ←	1 →	2_3 ←	5 →					
15 →	10_3 ←	6 →	5_3 ←	4 →	2_1 ←	1_{-1} →	1_0 ←	4 →				
15 ←	10 →	7_2 ←	4 →	3_0 ←	3 →	2_2 ←	→	1_1 ←	3 →			
14 →	10_0 ←	7 →	5_3 ←	2 →	3_2 ←	2 →	1_0 ←	→	1_2 ←	2 →		
15 ←	9 →	7_1 ←	5 →	4_2 ←	1 →	1_0 ←	2_{-3} →	1_1 ←	→	←	2_{-2} →	
12 →	10_0 ←	6 →	6_3 ←	3 →	1_0 ←	1_{-2} →	2_0 ←	1 →	←	→	1_1 ←	1 →
15 ←	7 →	8_1 ←	4 →	2_1 ←	2 →	3_0 ←	→	←	1_{-1} →	1_0 ←		
10 →	11_1 ←	4 →	3_0 ←	3 →	4_0 ←	1 →	←	→	2_2 ←			
15 ←	6 →	5_3 ←	2 →	6_3 ←	1 →	←	1_{-1} →	2_0 ←				
9 →	7_2 ←	4 →	7_0 ←	1 →	←	1_{-1} →	3_1 ←					
10 ←	6 →	9_0 ←	2_{-1} →	←	1_{-1} →	4_1 ←						

$$\begin{array}{cccccc}
7 & 12_0 & 3 & & 2_{-3} & 5_0 \\
\rightarrow & \leftarrow & \rightarrow & \leftarrow & \rightarrow & \leftarrow \\
16 & 3 & 1_2 & 2 & 7_0 & \\
\leftarrow & \rightarrow & \leftarrow & \rightarrow & \leftarrow & \\
4 & 2_3 & 2 & 9_1 & & \\
\rightarrow & \leftarrow & \rightarrow & \leftarrow & & \\
3 & 3 & 11_0 & & & \\
\leftarrow & \rightarrow & \leftarrow & & & \\
4 & 14_1 & & & & \\
\rightarrow & \leftarrow & & & & \\
18 & & & & & \\
\leftarrow & & & & &
\end{array}$$

Units of time are indicated vertically, distances (in legs) horizontally.

An Application of Helly's Theorem

4361 [1949, 557]. *Proposed by Melvin Dresher, The Rand Corporation.*

If $S_1, S_2, \cdots, S_m$ are m line segments parallel to the y-axis such that through every set of $n+2$ of them the locus of an nth degree polynomial can be passed, then there exists some nth degree polynomial whose locus intersects all m segments.

I. *Solution by Robert Steinberg, University of California, Los Angeles.* The theorem is true for $m=n+2$. Hence, we take $m>n+2$ and assume that the theorem is true for $m-1$ as a basis for an inductive proof.

Let f_1 be the polynomial through the segments $S_2, S_3, \cdots, S_m$, and let $f_2, f_3, \cdots, f_m$ be similarly defined. Consider the equations:

$$\sum_{i=1}^{m} a_i f_i = 0, \qquad \sum_{i=1}^{m} a_i = 0,$$

where the first equation is considered to be an identity in x. These give $n+2$ linear homogeneous equations to determine m a_i's, with $m>n+2$, and hence there is a solution in which the a_i's are not all zero. Let all the positive a_i's be included in $a_1, a_2, \cdots, a_r$, and the negative a_i's in $a_{r+1}, \cdots, a_m$.

Then, the polynomial

$$f = \frac{\sum_{i=1}^{r} a_i f_i}{\sum_{i=1}^{r} a_i} = \frac{\sum_{i=r+1}^{r} (-a_i) f_i}{\sum_{i=r+1}^{r} (-a_i)}$$

is a polynomial which intersects all m segments. For, from its two representations, it is obviously intermediate to $f_1, f_2, \cdots, f_m$ for any value of x; and, from the first representation, it is seen to intersect $S_{r+1}, \cdots, S_m$ and, from the second, $S_1, S_2, \cdots, S_r$.

This completes the proof.

II. *Solution by S. H. Gould, Purdue University.* Let S_k have the end-points (x_k, p_k) and (x_k, q_k) with $p_k < q_k$. Interpret the coefficients a_i of the polynomial

$$f(x) \equiv a_0x^n + a_1x^{n-1} + \cdots + a_n$$

as coördinates of a point A in $(n+1)$-Euclidean space. Then the set of points A for which

$$p_k \leqq f(x_k) \leqq q_k \tag{1}$$

is convex; for, if $a_i'' = (1-\lambda)a_i + \lambda a_i'$, with $0 \leqq \lambda \leqq 1$, and if A and A' satisfy (1), then clearly A'' also satisfies (1).

Thus, the problem becomes a particular case of Helly's Theorem (see e.g. Alexandroff-Hopf, *Topologie*, p. 297): *in order that any finite number m of convex point-sets in $(n+1)$-Euclidean space shall have a point in common, it is necessary and sufficient that every set of $(n+2)$ of them shall have a point in common.* Simple proofs were given by Radon (*Math. Ann.* v. 83, 1921, p. 113), Konig (*Math. Zeitschr.* v. 14, 1922, p. 203), and Helly (*Jber. Deutsch. Math.-Vereinig.* v. 32, 1923, p. 175).

Evaluation of Determinant

4362 [1949, 557]. *Proposed by D. J. Newman, New York University*

Evaluate

$$\begin{vmatrix} 1 & 1/2 & 1/3 & \cdots & 1/n \\ 1/2 & 1/3 & 1/4 & \cdots & 1/(n+1) \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ 1/n & 1/(n+1) & 1/(n+2) & \cdots & 1/(2n-1) \end{vmatrix}.$$

I. *Solution by N. D. Lane, Acadia University, Nova Scotia.*

Subtract the nth column from the 1st, 2nd, $\cdots$, $(n-1)$th columns in turn, and factor out the common factors of the rows and of the columns. Now subtract the nth row from the 1st, 2nd, $\cdots$. $(n-1)$th rows in turn and again factor out the common factors of the rows and of the columns. This gives the recurrence formula

$$F(n) = \frac{[(n-1)!]^4}{(2n-1)!(2n-2)!} \cdot F(n-1),$$

where $F(n)$ is the given determinant. Thus we obtain

$$F(n) = \frac{[1!2!3! \cdots (n-1)!]^4}{1!2!3! \cdots (2n-1)!}.$$

The same procedure will give the evaluation of a determinant whose ele-

ments a_{ij} are the reciprocals of the $(i+j-1)$th terms of any arithmetic progression. The determinant has the value

$$F(n) = [1!2!3! \cdots (n-1)!]^2 d^{n(n-1)} \prod,$$

where $\prod$ is the product of all the elements of the determinant.

II. *Solution by P. W. M. John, University of Oklahoma.* The following proof employs an idea suggested by A. C. Aitken. Consider the determinant $|A|$, (the Cauchy double alternant)

$$|A| = |(a_i - b_j)^{-1}|.$$

Clearing the fractions by multiplying each row by the continued product of the denominators of that row, we have

$$|A| = \frac{1}{\prod_{i,j}(a_i - b_j)} |C|$$

where the (ij)th element of C is $[\prod_j(a_i-b_j)]/(a_i-b_j)$. Then C is a polynomial of order $n(n-1)$ in the a_i, b_j. It vanishes when $a_i=a_j$ and when $b_i=b_j$ $(i\neq j)$. Hence it is divisible by the two difference products, $\Delta(a_1, a_2, \cdots, a_n)$ and $\Delta(b_1, b_2, \cdots, b_n)$. Since their combined degree is $n(n-1)$, the remaining factor is numerical. To find it, put $a_i=b_i$, when all terms except those in the leading diagonal vanish. The continued product of the diagonal elements gives the difference product twice. However, with each factor (a_i-a_j) occurs also (a_j-a_i), so that the numerical term is seen to be $(-1)^{n(n-1)/2}$. Then

$$|A| = \frac{|C|}{\prod_{i,j}(a_i - b_j)} = \frac{(-1)^{n(n-1)/2}\Delta(a_1, \cdots, a_n)\Delta(b_1, \cdots, b_n)}{\prod_{i,j}(a_i - b_j)}.$$

The value of the proposed determinant may be obtained easily by setting $a_i=n+1, n+2, \cdots, 2n$ and $b_j=n, n-1, \cdots, 2, 1$.

Also solved by A. C. Aitken, F. Bagemihl, Sol Ciolkowski, A. R. Erskine, Daniel Finkel, H. L. Krall, Emma Lehmer, Julius Lieblein, C. D. Olds, E. G. Olds, Ingram Olkin, R. H. Pennington, D. W. and J. R. Pounder, H. D. Ruderman, W. Seidel, J. H. Simester, O. E. Stanaitis, Robert Steinberg, H. E. Stelson, and the Proposer.

The following references were noted:

Bourbaki, N., Algebra, p. 90, ex. 5

Cauchy, Œuvres complètes, 2 serie, v. xii, p. 177.

Hilbert, D, Ein Beitrag zur Theorie des Legendreschen Polynoms, Gesammelte Abhandlungen, v. 2, p. 369.

Ligowski, Archiv. d. Math. u. Phys. v. xxxvi, 1861, pp. 181–185.

Muir, T., The Theory of Determinants, III, 1920, p. 311.

Muir and Metzler, The Theory of Determinants, 1930, pp. 429–431.

Schrader. W., Beiträge zur Theorie der Determinanten, Halle, 1887.

RECENT PUBLICATIONS

EDITED BY E. P. VANCE, Oberlin College

All books for review should be sent directly to the editor of this department, American Mathematical Monthly, 531 West 116th Street, New York 27, N. Y. and not to any of the other editors or officers of the Association.

Elements of Analytical Geometry. By W. L. Hart. D. C. Heath and Company, Boston, 1950. 264 pages. $2.75.

This book is a new addition to W. L. Hart's list of carefully written texts for freshman mathematics. Those teachers of freshman mathematics who have enjoyed his *College Algebra* and his *Trigonometry*, will welcome this book into the field of available texts in analytical geometry.

This book cannot be cited for any radical departures from the customary content of the course in analytical geometry as taught in most colleges and uni versities in the United States. It gives an adequate treatment of a minimum course in plane analytical geometry, followed by a treatment of solid analytical geometry, which should meet the needs of most students of first year calculus. The last chapter of the book, considers, in an elementary way, the problem of fitting a curve to empirical data.

The author has given considerable attention to the need for carefully defined mathematical terms. He has also seen the need for providing a summarizing chapter on the function concept and the definition of terms usually encountered in freshman mathematics when studying this topic. A short chapter, (Chapter 2) entitled, "Related Variables," presents the minimum content pertaining to this topic.

It is becoming increasingly popular to present the subject-matter of freshman mathematics in as unified a manner as possible. This objective has not been neglected by the author. In his own words, "As a unifying aim, the book endeavors to present a foundation for the joint use of analytic methods and graphical representation, oriented with respect to their later applications. In particular, this viewpoint induces emphasis on fundamental terminology relating to variables, functions and their graphs, and informal but accurate use of the language of limits on a few occasions." However, in the opinion of the reviewer, this book, and almost all other books on analytic geometry, could achieve greater unity if the authors had made more use of such ideas as direction cosines in plane analytical geometry thereby more readily leading to the study of lines and planes in space and achieving an even greater unity than can be achieved by introducing such ideas very late in the course.

HENRY VAN ENGEN

Giant Brains, or Machines That Think. By C. Berkeley. John Wiley and Sons, Inc., New York, 1949. 7+270 pages. $4.00.

Giant Brains, or Machines That Think is a semi-popular book about large scale computing machines. The book begins by discussing the provocative subtitle, "machines that think." While the author makes a good case (and an even better one can now be made), it seems to the reviewer that the definition of "think" will usually be framed, humans being ego-centric, to exclude machines, much as "soul" is usually defined to exclude animals.

For the beginner in the field, the author devotes a chapter and two appendices to the language of computing machines. Chapter 3 describes a simple machine, called Simple Simon, designed by the author to illustrate the nature of digital computing machines, and as such, the machine is an excellent example.

The bulk of the book is devoted to descriptions of the major computing machines which were completed and in use during the years 1942–6. These machines include IBM accounting equipment, the MIT Differential Analyser (No. 2), the Harvard Automatic Sequence-Controlled Calculator (Mark I), the Moore School ENIAC (now at Aberdeen Proving Grounds), the Bell Laboratories' General-Purpose Relay Calculators (Models 5), and the Kalin-Burkhart Logical-Truth Calculator. One chapter is devoted to each of these machines, and in each case, the author shows in simple colorful language the main features of the machine. Each of these chapters is a mine of information, and collectively (along with a very excellent set of references in the back of the book) they form a valuable contribution to the literature of large scale computing machines.

An additional chapter is devoted to recent developments in the art of designing and building these machines. The chapter includes a brief description of the Harvard Mark II, the IBM S-SEC, and the Eckert-Mauchly BINAC.

In the next to the last chapter, the author extrapolates from recent developments in all directions—many of which extrapolations are provocative and some of which may well be true! On the other hand, the last chapter on Social Control might well have been left out.

In conclusion, it is only fair to say that the book represents an enormous labor of love on the part of the author, and that in spite of its elementary style of writing and occasional technical inaccuracies, it will be useful to the beginner in the field as well as to the professional.

R. W. Hamming

Fourier Methods. By Philip Franklin. McGraw Hill, New York, 1949. 10+289 pages. $4.00.

This is a book for engineers and other technicians. It is designed to show them how to operate with complex quantities and how to solve problems which depend for their solution on the use of Fourier series and integrals, and Laplace transforms. It is vigorous mathematics at its extreme; the emphasis is not on the concepts involved, but on how to use these concepts to work problems.

The author makes no attempt to lean on the mathematical rigor, but nevertheless the analysis is done in orderly fashion; the arguments are clear and plausible, and important facts, such as those concerning the convergence of Fourier series are plainly stated without claiming to be proved.

The book consists of five large chapters. The first is devoted to manipulation of complex numbers and of the elementary transcendental functions of a complex variable. Complex power series and the notion of the derivative of a complex function are used.

The second chapter is devoted mainly to Fourier series, although about five pages near its end are used to define and describe Fourier and Laplace transforms. The author states in his introduction that the approach to Fourier series is through averages and root mean square values; by this is meant that he regards the process of finding the coefficients as getting the averages of certain trigonometric products. The notions of orthogonality and least square fit are not at all exploited; as far as the reviewer can tell, the words do not appear in the book. It is probably the book's weakest feature that these ideas, which seem so essential for the understanding of Fourier series, and which unite so many superficially different ideas and processes in mathematical analysis, are completely ignored. When the harmonic analysis of a function given numerically is taken up, it is considered as a problem in finite curve fitting, although the notion of curve fitting is used nowhere else, and in spite of the fact that the formulas obtained for the coefficients in this way are essentially the usual Fourier formulas, with the integrals replaced by Riemann sums. The treatment of the process of expanding a function is very down to earth; various kinds of symmetry are treated in detail, and all discussions are in terms of functions of period p instead of 2π. For this latter to be done throughout a large work, unless it is a reference manual, seems to the reviewer to be a waste of typography and a confusion to the eye. An engineer who really has difficulty in managing a linear transformation ought have no business operating with anything as gaudy as a Laplace transform.

Chapter III is devoted to the derivation and/or description of a large number of partial differential equations, mainly those of mathematical physics, and to some elementary methods of solution-partial integration, operator factorization, *etc.* Certain parts of the chapter use vector analysis considerably, in particular and of necessity the part dealing with Maxwell's equations. Chapter IV contains the solutions of many of these equations, together with the appropriate boundary conditions, in terms of Fourier series. Problems considered include heat flow, vibrating string, power transmission, and wave guides.

In Chapter V, which is the largest chapter in the book, is found the application of the Laplace transform to the solution of ordinary and partial differential equations. There is a small table of transforms and a considerable discussion of partial fractions, with what amounts to residue computation without being called such. The main applications are to electrical networks and power transmission problems.

There is an extremely large collection of problems. The problem set on the computation of Laplace transforms, for example, contains sixty-four problems, ranging from straightforward definite integrations to significant theoretical results requiring considerable epsilon and delta analysis.

In his introduction, the author states that the student should have a working knowledge of elementary calculus, and that most of the topics can be covered in one semester. The reviewer is of the opinion that it would be a very busy semester.

J. W. GREEN

Gelöste und Ungelöste Mathematische Probleme aus Alter und Neuer Zeit. By Heinrich Tietze (Professor of Mathematics, University of Munich, Bavaria, Germany). Biederstein Publishers, Munich, 1949. Volume I: xx, 256 pp., 8°, 115 figs., 10 plates; Volume II: iv, 305 pp., 8°, 41 figs., 8 plates. Price: Paper Bound 18 DM, Linen Bound 25 DM (German Marks).

These two volumes are based on a series of lectures given repeatedly to students of all curricula, at the University of Munich, by the author. They are intended for readers interested in mathematical problems, as suggested by the subtitle. Although the author states specifically that the work is not a treatise on mathematics, the student and advanced reader will find an abundance of information in the "Remarks" which are gathered at the end of each volume.

The following chapter titles serve to show the coverage: Prime Numbers and Prime Twins; Journeys on Surfaces (Geodesy); Trisection of Angles; Adjoining Regions (Topology); Squaring the Circle; Three and More Dimensions; More About Prime Numbers; Counts and Calculations; the Regular 17-Gon; Solution of Algebraic Equations by Radicals; The Four-Color Problem; The Infinite in Mathematics; Fermat's Last Theorem; Curvature of Space.

A considerable amount of effort is devoted to the thorough development of each problem. This is done with the same eloquence to which Professor Tietze's students are accustomed. Numerous biographies of great mathematicians and their portraits provide a rich historical setting for the understanding of the growth of the problems. Additional illustrations are contained in each chapter; the color plates for the Möbius strips and for the Four-Color Problem are extremely well designed and harmonious in the choice of their hues. The tabulations, special problems, detailed proofs, historical notes, literature, bibliography, *etc.*, which comprise the "Remarks" are a veritable fountain of information. This work will prove to be a prized possession in any mathematical library.

CARL HAMMER

CLUBS AND ALLIED ACTIVITIES

EDITED BY L. F. OLLMANN, Hofstra College

Send reports of all activities, such as club reports, special features, topics with references, student papers, and other material of interest to L. F. Ollmann, Hofstra College, Hempstead, New York.

CLUB REPORTS, 1949–50

Pi Mu Epsilon, Carnegie Institute of Technology

The *Pennsylvania Epsilon* chapter of *Pi Mu Epsilon* reports that, for the academic year 1949–50, six meetings were held, including a business meeting and the annual initiation and banquet. The following papers were presented:

The so-called crisis in the foundations of mathematics and logic, by Prof. Abraham Fraenkel of the Hebrew University at Jerusalem

The mathematical theory of games, by Prof. H. A. Simon

Four-space representations of functions of a complex variable, by Prof. James Taylor of the University of Pittsburgh

Applications of electronic computing equipment to technical and scientific problems, by C. C. Hurd of the International Business Machines Corporation

Mathematical and inductive proofs, by Visiting Professor Albert Schaeffer, formerly of Purdue University and now of the University of Wisconsin.

The initiation of new members and the annual banquet of the chapter were held on April 26, 1950. Fifty-seven persons were initiated, of which twenty-three were undergraduates, twenty were graduates, and fourteen were faculty members.

The officers for 1949–50 were: Director, Richard DiPrima; Vice-Director, Douglas Shaffer; Secretary, William Warner; Treasurer, F. B. Smith, Jr.; Faculty Adviser, Prof. J. B. Rosenbach; Executive Committee, Richard Cutkosky, Robert McKelvey, and William Simon.

Mathematics Club, Hunter College

The *Mathematics Club* of Hunter College held bi-weekly meetings during the Spring semester of 1950. The following talks were presented:

The lighter side of mathematics, by Marie Hyman

Mathematical treasures at Hunter College, by Carolyn Eisele

Hyperbolic functions, by Marjorie Hull and Jerry Greyson

Some topological ideas, by Marion Walter

Finite geometry, by Prof. Jewell H. Bushey.

In addition to the above, Mr. Rutherford Boyd gave an illustrated lecture on *Mathematical forms in art*. His film, *Parabola*, was also shown. A few weeks later, members of the Club and Faculty spent a delightful afternoon at his home learning more about his mathematical art—or artistic mathematics.

During the term a dinner, a theatre party, and a picnic were also enjoyed by the members.

The officers for the semester were: President, Marion Walter; Vice-President,

Dorothy Freudenberger; Secretary, Anita Friedman; Treasurer, Evelyn Hovrath; Faculty Advisor, Prof. Anita Tuller.

Kappa Mu Epsilon, Pomona College

Papers presented to the *California Alpha* chapter of *Kappa Mu Epsilon* during the academic year 1949–50 were:

Method of casting out nines, by Donald Benson
Sir Isaac Newton, by Carolyn Grove
Linkages, by John Dienes
Basic problems of number theory, by Ralph Vernon
Conic sections, with illustrations in plastic, by Dr. C. G. Jaeger and Mr. Charles Halberg
Navigation, by Gilbert Madden
Continued fractions, by Donald Benson
Polygonal numbers, by Dr. Jean Walton
Diophantine equations, by William Paxton
Existence proof for transcendental numbers, by Richard Edelstein
Boolean algebra, by Rodney Weldon
The numbers racket, by Dr. Aubrey Kempner
Rigidity of the earth, by Dr. Walter Whitney.

The annual banquet of the chapter followed the initiation of twenty-three new members. Dr. Aubrey Kempner spoke on *What is truth in mathematics?*

Officers elected for 1950–51 are: President, Walter Rosenow; Vice-President, Rodney Weldon; Secretary-Treasurer, Zoe Lindberg; Social Chairman, Richard Edelstein.

Kappa Mu Epsilon, Central Missouri State College

The *Missouri Beta* chapter of *Kappa Mu Epsilon* held regular monthly meetings during the academic year 1949–50 and the summer session of 1950. The following talks were presented by members of the Society:

Denumerable and non-denumerable infinities, by Sammy Vaughn
The Euclidean algorithm, by Wayne Vanderlinden
Types of discontinuities, by Philip Burford
Nomography, by Rex Wyrick
Complex numbers and their geometric interpretation, by Kathryn Lou Baker
Puzzles based on binary and ternary number systems, by Martin Rowland
Short-cuts in multiplication, by Peggy June Taylor
The fourth dimension, by Barbara Wurth
An oblique coordinate system, by Mrs. Marcia Jackson
Cryptography, by Wilfred Poese
The maneuvering board, by Charles Sigrist
The expression of $\cos n\theta$ *as* $f(n, \cos\theta)$, by Charles Warden
Two theorems on prime numbers, by Michael Stratton
A minimum body of mathematical concepts and information for high school graduates, by Mrs. Margaret Shrake

Volumes and surface areas of regular polyhedra, by Charles Warden.

At the annual banquet in the Spring, Mr. Ray Watson, a sales and market analyst for Hallmark Brothers of Kansas City, Missouri, spoke to the society on *Market potentials—their use and instructions*. The annual Summer picnic was held in May.

Mr. Keith Stumpff was elected President for 1950–51. Interim officers serving during the Summer of 1950 were: Vice-President, Isabella Clarke; Secretary, Mrs. Marcia Jackson; Treasurer, Charles Warden; Faculty Sponsor and Corresponding Secretary, Mr. Loren W. Akers.

Pi Mu Epsilon, Oklahoma Agricultural and Mechanical College

The following discussions were made at the regular meetings of the *Oklahoma Beta* chapter of *Pi Mu Epsilon* during the year 1949–50:

The foundations of mathematics, by Prof. Arnold Dresden of Swarthmore College

The elements of Hilbert space, by Prof. Ainsley Diamond

Non-Euclidean geometry, by Prof. J. V. Robinson

The relation of mathematics to the work of an engineer, by Prof. R. R. Rogosinski, Kings College, Newcastle-upon-Tyne, England

Some aspects of the history of science, by Prof. Orville Schultz of the department of botany.

The chapter had 75 members in residence at the beginning of the year and initiated 18 members. The total membership of the chapter since its installation in 1938 is approximately 350.

Kappa Mu Epsilon, Mount Mary College

The *Wisconsin Alpha* chapter of *Kappa Mu Epsilon* held monthly meetings during the year 1949–50. Besides movies on the solar system and on weather conditions, the following papers were read:

Non-Euclidean geometry, by Joan Daley

The fourth dimension, by Mary Kilkelly

History of the calculus, by Mary Hund

Relativity, by Kathleen Hanley

Number systems, by Wanda Kropp

The slide rule, by Janet Haig

Planetariums, by Betty Prossen.

Dorothy Karner read a paper on *Inversion* at the *Kappa Mu Epsilon* meeting which was held in Chicago in connection with the meeting of the National Council of Teachers of Mathematics.

Twelve new members were initiated at a dinner meeting in May.

Officers for 1949–50 were: President, Dorothy Karner; Vice-President, Rosemary White; Secretary, Mary Hund; Treasurer, Mary Kilkelly; Corresponding Secretary and Sponsor, Sister Mary Felice.

Joan Bakle was elected President for the year 1950–51 and Sister Mary Petronia has taken over the office of Corresponding Secretary.

Mathematics Club, Harvard University

The following is a summary of the activities of the Harvard *Mathematics Club* for 1949–50. The talks presented were:

The Banach-Tarski paradox, by Prof. Lynn Loomis

Infinite-dimensional representations of the Lorentz group, by Dr. Harish-Chandra

A probability model for simple learning, by Prof. F. Mosteller

A miniature theory, by Prof. D. V. Widder

A functional equation, by Prof. G. deRham

Integral geometry, by Prof. L. Ahlfors

The four-color problem, by Mr. Reese Prosser

Communication and code, by Mr. Ariel Zemach

The odd number six, by Prof. W. V. D. Hodge.

The *Club* participated in the meetings of the Intercollegiate Mathematics Conference of the Boston Area, and sponsored a picnic for faculty and students.

The officers elected for 1950–51 are: President, W. Turanski; Vice-President, A. Zemach; Secretary, H. Gonshor; Treasurer, H. Royden.

The Rogers prizes for student speakers at club meetings were awarded jointly to Mr. Prosser and Mr. Zemach.

Kappa Mu Epsilon, Montclair State Teachers College

Included in the regular monthly meetings of the *New Jersey Beta* chapter of *Kappa Mu Epsilon* were the following programs:

The algebra of logic, by Werner Schanzenbach

Appreciation of elementary mathematics, by Prof. Howard Fehr of Columbia University Teacher's College

Polyhedrons, by Ramon Steinen

Trisection of an angle by means of various curves, by Audrey Jensen

Applications of mathematics to electricity, by Prof. C. Sensale.

The first meeting of the year was an initiation meeting held at the home of Dr. V. S. Mallory at which five members were taken into the organization.

At most of the meetings, members of the organization spoke more fully on topics which they had presented for their initiation. Members of the faculty also presented informal talks on such subjects as hints for the teaching of mathematics. The more formal talks outlined above completed the regular meetings of the chapter.

Members of the *New Jersey Alpha* chapter of *Kappa Mu Epsilon* at Upsala College as well as the mathematics majors and minors from Montclair heard the inspiring talk by Dr. Fehr.

The social program for the year consisted of a Christmas Party held with *Sigma Phi Mu*, the mathematics club; a picnic in May; and finally the annual Spring Banquet held as a farewell to the seniors and to welcome returning alumni. At this banquet copies of Archibald's *History of Mathematics* were awarded to Jack King, Charles Paglieri, and Ramon Steinen, three seniors

having the highest average in mathematics this year.

Officers for 1950–51 are: President, Robert Lynch; Vice-President, Werner Schanzenbach; Secretary, Catherine Buce; Treasurer, Leonard Nichols.

Pi Mu Epsilon, University of Alabama

The *Alabama Alpha* chapter of *Pi Mu Epsilon* held five program meetings during the academic year, 1949–50, in addition to the annual business meeting. Two social functions were held, a Christmas Party and a Spring picnic, on the occasions of the semi-annual initiation ceremonies.

The following papers were presented:

Lattice theory, by Haskell Cohen

New derivations of Taylor's expansion, by Prof. C. L. Seebeck

A method of graphical integration, by Jack Cothren

Mathematics and the sciences, by Dr. A. S. Householder of the Oak Ridge Laboratories.

At one of the meetings the program consisted in discussions of certain problems in this MONTHLY and the *Pi Mu Epsilon Journal*, led by Dr. J. C. Eaves and Mr. Ben Green.

For the second consecutive year, the chapter sponsored a campus-wide competitive examination, designed this year to test the reasoning powers of the competitors rather than their mastery of techniques. Prize winners were William McKee, first prize, Robert Asquith and Donald Swenson, tied for second prize, Wallace Driggers and Walter B. Mitchell.

The following officers were elected for the academic year, 1950–51: Director, Susie Lee Ward; Vice-Director, Dr. Ferdinand Mitchell; Secretary, Mrs. A. M. Jones; Treasurer, Hasell Palmer; Librarian, Dr. J. D. Mancill; Publicity Chairman, Kathleen Cannon; Social Chairman, Ella Jones.

Mathematics Club, University of Buffalo

The activities of the *Mathematics Club* of the University of Buffalo included a business meeting, a picnic, and the following program meetings:

The mathematician in optics, by Dr. Wilkins

The application of mathematics and statistics to the analysis of the economic cycle, by Dr. Zennon Szatrowski, of the Statistics Department

Phyllotaxis, by Dr. Robert Gordon

Intrinsic equations, by Norman Severo

The nomograph, by Henry Hollwedel

Elementary modular arithmetic, by Phyllis Schwartz

Mathematic puzzles and games, by William Schulze

The mathematician in the industrial world, by Dr. Welmers, of the Bell Aircraft Corporation.

The officers elected in April are: President, Paul Schillo; Vice-President, Harlan Stevens; Secretary, Phyllis Schwartz; Treasurer, Bernadine Lippert; Refreshments Chairmen, Norma Wilson and Lois Hunt.

NEWS AND NOTICES

EDITED BY EDITH R. SCHNECKENBURGER, University of Buffalo

Readers are invited to contribute to the general interest of this department by sending news items to Edith R. Schneckenburger, University of Buffalo, Buffalo 14, New York. Items must be submitted at least two months before publication can take place.

FOURTH SYMPOSIUM ON APPLIED MATHEMATICS

The Fourth Symposium on Applied Mathematics of the American Mathematical Society, cosponsored by the University of Maryland and the United States Naval Ordnance Laboratory, will be held at the University of Maryland, College Park, Maryland, and the United States Naval Ordnance Laboratory, White Oak, Maryland, on June 22–23, 1951. Sessions will begin at 10:00 A.M. Friday.

The general topic of the Symposium is Fluid Dynamics. Sessions are planned on Turbulence, Foundations, Compressible and Incompressible Flow. Participation in the program is by invitation of the Committee on Arrangements.

Detailed information concerning the program, registration, rooms, meals and travel will be included in a formal announcement of the Symposium to be sent out in May. Inquiries concerning the Symposium should be addressed to Professor M. H. Martin, Chairman of the Committee on Arrangements, University of Maryland, College Park, Maryland.

SUMMER COURSES

The following institutions announce advanced courses in mathematics for the summer of 1951:

Boston University. May 28 to July 7: Professor Chand, seminar in statistics; Dr. Giever, theory of equations; Dr. Scheid, vector analysis. July 9 to August 18: Professor Sobczyk, seminar in analysis, introduction to modern mathematics, advanced analytic geometry; Professor Mode, introduction to mathematical statistics.

Catholic University of America. July 2 to August 11: Dr. Moller, theory of equations; Professor Finan, theory of numbers; Professor Rice, solid analytic geometry and advanced calculus; Mr. Clark, partial differential equations of mathematical physics; Professor Ramler, college geometry, differential equations and analytic projective geometry.

Cornell University. July 2 to August 11: Professor Hunt, projective geometry; Professor Fuchs, teachers' course.

Duke University. June 12 to July 21: Professor Thomas, solid geometry and spherical trigonometry from an advanced standpoint, abstract algebra I; Professor Gergen, algebra from an advanced standpoint, integral equations; Associate Professor Dressel, thesis seminar. July 21 to August 31: Professor Carlitz, abstract algebra II, probability. August 7 to 17: Director W. W. Rankin, Institute for Teachers of Mathematics.

Northwestern University. June 22 to August 25: Differential equations; fundamental concepts of analysis; definite integrals; theory of equations; analytic geometry; theory of statistics; introduction to the theory of numbers; geometry for teachers; functions of a complex variable; topics in linear operators; introduction to the theory of groups; vector analysis; seminar; independent study; thesis; teaching of general mathematics (six-week session only); workshop course in multisensory aids in the teaching of mathematics (last three weeks).

Ohio State University. June 19 to August 31: Professor Rechard, advanced calculus; Professor Mickle, introduction to the theory of complex variables, differential geometry; Professor Miller, projective geometry, fundamental ideas in algebra and geometry; Professor Lazar, history of mathematics.

State University of Iowa. June 12 to August 8: Professor Chittenden, differential equations, theory of manifolds; Professor Wylie, astronomy; Professor Woods, theory of equations, constructive geometry; Professor Price, supervision of mathematics; Professor Muhly, analytic trigonometry, advanced calculus; Professor Hogg, introduction to statistics, matrices and determinants; Professor Oberg, elementary theoretical mechanics, numerical integration of ordinary and partial differential equations.

Syracuse University. July 2 to August 10: Introduction to higher algebra I; college plane geometry; introduction to modern mathematics; history of mathematics; analysis of elementary mathematics; teaching high school mathematics; workshop in mathematics education.

University of Buffalo. July 2 to August 11: Professor Gehman, theory of probability; Professor Montague, curves and their properties, higher algebra; Professor Schneckenburger, differential equations, non-Euclidean geometry.

University of California, Los Angeles. June 18 to August 11: functions of a complex variable; partial differential equations. The following courses are offered under sponsorship of the Institute for Numerical Analysis: sampling methods and stochastic processes; difference and differential equations; asymptotic expansions.

University of Chicago. June 26 to September 1: Professor Albert, linear associative algebras, reading and research in algebra and the theory of numbers; Professor Halmos, Boolean algebras, reading and research in measure theory and functional analysis; Professor Kaplansky, ideals in rings, reading and research in algebra; Professor Graves, existence theorems in the calculus of variations, reading and research in analysis; Professor Segal, unbounded operators on Hilbert space, reading and research in abstract analysis; Professor Chern, differentiable manifolds, reading and research in differential geometry; Professor MacLane, reading and research in algebra and topology.

University of Colorado. June 18 to July 20: Professor Roy, mathematical statistics; Dr. Johnson, workshop in mathematics, advanced course in teaching of mathematics; Professor Kendall, analytic projective geometry, history of mathematics; Professor Britton, introduction to modern algebra, functions of a

complex variable (first and second terms); Mr. Hunt, advanced calculus (first and second terms); Mr. Wagner, vector analysis (first and second terms). July 23 to August 24: Professor Jones, theory of numbers, analytic projective geometry.

University of Detroit. June 25 to August 3: Professor Smith, theory of groups and theory of higher plane curves; Professor Markle, seminar in mathematics; Professor McCarthy, differential equations; Professor Mehlenbacher, theory of functions of a complex variable.

University of Kansas. June 7 to August 4: Professor Bell, projective geometry, matrix and tensor calculus; Mr. Bradt, elementary statistics; Professor Scott, set theory; Professor Wolontis, Fourier series and boundary value problems.

University of Kentucky. June 18 to August 11: Professor Davis, intermediate calculus; Professor Goodman, theory of equations; Professor Downing, advanced calculus (second part); Professor Pence, college geometry; Professor South, calculus of finite differences; Professor Ward, introduction to theory of numbers; Professor Cowling, integral equations.

University of Maryland. June 23 to August 3: Professor Good, higher algebra; Professor Jackson, introduction to projective geometry.

University of Michigan. June 25 to August 17: Professor Artin, arithmetic of rings; Professor Bartels, modern operational mathematics, Fourier series; Professor Carver, mathematical statistics II, finite differences; Professor Craig, theory of estimation; Professor Copeland, theory of probability, mechanics; Professor Darling, significance tests, theory of statistics I; Professor Dwyer, multivariate analysis, computational methods; Professor Hay, vector analysis, theory of plates and shells; Professor Jones, history of algebra, teaching of collegiate mathematics; Professor Kaplan, functions of complex variable with applications, non-linear differential equations; Dr. Leisenring, higher geometry for teachers; Professor Nesbitt, intermediate mathematics of life insurance; Professor Rothe, methods of partial differential equations, linear integral equations; Professor Samelson, functions of a real variable, integral geometry; Professor Thrall, Galois theory, analytic projective geometry; Professor Wilder, foundations of mathematics, general spaces.

University of Minnesota, Department of Mathematics. June 18 to July 28: Professor Hatfield, mathematical recreations, vector analysis; Professor Carlson, theory of geometric constructions, solid analytic geometry; Professor Cameron, probability, calculus of variations. July 30 to September 1: Professor Olmsted, theory of equations; Professor Nering, foundations of geometry, mathematical theory of games of strategy; Professor Kalisch, the real number system; Professor Loud, the mathematics of small vibrations, Laplace transforms.

University of Minnesota, Institute of Technology. June 18 to July 27: Professor Koehler, intermediate and advanced calculus; Professor Munro, vector analysis and theory of complex variable. July 30 to August 31: Professor Warschawski, advanced calculus; Professor Turrittin, vector analysis, dyadics

with applications.

University of Nebraska. June 5 to July 27: Professor Camp, differential equations; Professor Leavitt, vector analysis; Professor Ribeiro, theory of equations; Professor Basoco, differential geometry.

University of North Carolina. June 11 to July 19: Professor Whyburn, foundations of geometry; Professor Cameron, introduction to modern algebra; Professor Hill, elementary mathematical statistics; Professor Garner, calculus of finite differences; Professor Brauer, some recent results in algebra. July 20 to August 28: Professor Lasley, synthetic projective geometry; Professor Linker, differential equations; Professor Jones, general topology.

University of Oklahoma. June 8 to August 3: Mr. LaFon, elementary differential equations, theory of equations; Professor Hassler, fundamental concepts and methods (teachers' course); Dr. Huff, ordinary and partial differential equations; Professor Brixey, higher algebra; Dr. Bernhart, analytic mechanics; Professor Springer, metric differential geometry; Professor Goffman, integral equations.

University of Virginia. June 25 to August 18: Professor Floyd, advanced calculus and applied mathematics; Professor McShane, differential equations and applied mathematics, advanced analysis; Professor Whyburn, transformation theory.

University of Wisconsin. June 25 to August 17: Professor Bing, elementary plane topology; Professor Bruck, survey of the foundations of algebra, rings and fields; Professor Buck, determinants and matrices; Professor Eberlein, advanced calculus; Professor Kleene, higher mathematics for engineers, introduction to the theory of probability; Professor Mayor, college geometry; Professor Schaeffer, higher analysis, potential theory; Professor Young, higher mathematics for engineers, calculus of variations.

University of Wyoming. June 18 to July 20: Professor Steen, theory of equations; Professor S. R. Smith, differential equations, Fourier series; Professor Calvert, projective geometry; Professor Barr, seminar in geometry. July 23 to August 24: Professor Varineau, theory of numbers, fundamental concepts of mathematics; Professor W. N. Smith, partial differential equations, mathematical theory of probability; Professor Schwid, seminar in analysis, college geometry; Professor Neubauer, history of mathematics.

West Virginia University. June 6 to July 17: Professor Cunningham, modern geometry, higher plane curves; Professor Peters, linear algebra; Professor Vehse, advanced calculus. July 18 to August 24: Professor Vest, theory of equations, advanced differential equations; Professor Stewart, advanced calculus, higher plane curves.

PERSONAL ITEMS

Professor C. H. Gingrich of Carleton College has been elected to honorary membership in the American Association of Variable Star Observers.

Mr. J. H. Johnston of Orkney, Scotland has received the Annual Award of

the Duodecimal Society of America.

Assistant Professor Edith R. Schneckenburger, University of Buffalo, was the representative of the Association at the inauguration of Chancellor T. R. McConnell of the University of Buffalo on January 5–6, 1951.

Dr. W. J. Youden of the National Bureau of Standards has been elected to Fellowship in the New York Academy of Sciences.

Carleton College announces: Professor C. H. Gingrich is the Chairman of the Department of Astronomy and Mathematics; Associate Professor K. W. Wegner has been appointed Registrar but is continuing to teach part-time in the Department.

At Eastern Illinois State College: Dr. D. J. Davis, formerly of the Michigan Children's Institute, has been appointed to an assistant professorship; Instructor L. R. Van Deventer has been promoted to an assistant professorship.

Los Angeles City College reports that the Mathematics Department has sponsored the following series of lectures for mathematics teachers of the Los Angeles School System: Theory of Games by Professor H. F. Bohnenblust, California Institute of Technology; Difference Equations by Professor J. W. Green, University of California at Los Angeles; Series and Sums by Professor Hugh Hamilton, Pomona College; Topics in Applied Mathematics by Professor Tobias Dantzig; Recent Developments in Geometric Paper Folding by C. W. Trigg of Los Angeles City College. Also the College sponsored a lecture, Man and Mathematics, by Professor H. W. Turnbull of the University of St. Andrews, Scotland.

Oklahoma Agricultural and Mechanical College announces the following: Dr. A. D. Ziebur, previously graduate assistant at the University of Wisconsin, has been appointed to an assistant professorship; Mr. W. J. Nemerever, formerly an AEC Fellow at the University of Michigan, has been appointed to an instructorship; Lecturer F. F. Bonsall of King's College, Durham University, England, has been appointed Visiting Associate Professor; Mrs. Gillian Ronsall of Dame Allen's School, England, has been appointed Visiting Assistant Professor; Associate Professor H. W. Smith has been promoted to a professorship; Professor O. H. Hamilton is on leave and is at King's College, Durham University, England; Associate Professor Bee Chrystal has retired with the title of Associate Professor Emeritus.

Suffolk University announces the following appointments: Professor N. J. Anderson of Evansville College to the position of Professor of Chemistry and Mathematics; Mr. Harvey Blend, graduate student at the University of Texas, as an Assistant Professor of Mathematics and Physics.

At the University of British Columbia: Assistant Professor F. M. C. Goodspeed of Queens University has been appointed to an associate professorship; Dr. S. W. Nash has been appointed to an assistant professorship; Professor H. A. Heilbronn of the University of Bristol, England, was Visiting Professor during the first semester of the academic year 1950–51.

University of Colorado announces the following appointments: Professor

Emeritus A. J. Kempner as Visiting Lecturer for the year 1950–51; Professor S. N. Roy of the University of North Carolina and Dr. D. A. Johnston of the University of Minnesota as members of the staff of the Summer Session.

University of Delaware reports the following: Dr. E. V. Lewis, associate chemist with the DuPont Company, has been appointed to an assistant professorship; Instructor A. C. Nelson, Jr., has left the University and is now a graduate student at the University of North Carolina.

University of Detroit makes the following announcements: Assistant Professor Emily C. Pixley has been promoted to an associate professorship; Mr. E. R. Lancaster, graduate student engineer with the Chrysler Corporation, has been appointed to an instructorship.

University of Kentucky announces: Dr. M. S. Davis, previously fellow at Yale University, and Dr. J. C. Flack have been appointed to instructorships; Mr. L. F. Boron of the University of Maine and Mr. R. C. Brown, Jr., West Virginia University, have been appointed to part-time instructorships; Mrs. Carolyn S. Leo, Mr. R. H. Sprague, formerly graduate assistant at Ohio State University, Mr. L. O. Thompson of the University of Detroit and Mr. W. M. Zaring have been appointed to graduate assistantships.

At the University of Massachusetts: Mr. Edward Halpern has been appointed to an instructorship; Instructor P. D. Ritger is on leave of absence and is engaged in graduate study at New York University.

University of Oregon reports: Mr. I. J. Christopher, formerly graduate assistant, Mr. K. G. Clemens of Willamette University, Mr. J. E. Maxfield, previously graduate assistant, Mr. O. S. Rothaus, graduate student at Princeton University, and Mr. J. H. Skelton of Southwest Missouri State College have been appointed to instructorships; Instructor F. H. Young is now an AEC Fellow at the University.

University of Pennsylvania announces the following: Professor L. J. Mordell of St. John's College, Cambridge University, was Visiting Professor during the first semester of the current academic year; Professor F. W. Beal has retired with the title of Professor Emeritus.

University of Wichita makes the following announcements: Assistant Professor E. B. Wedel has been promoted to an associate professorship; Miss Sabrina Morlan of Kansas State College has been appointed to an instructorship.

Wellesley College announces the following: Assistant Professor J. W. Warwick of the Department of Astronomy is a part-time member of the Mathematics Department; Mrs. Mary L. Boas has resigned her position as Lecturer.

Mr. A. H. Albert, formerly teaching fellow at the University of Michigan, has been appointed to an instructorship at Kansas State Teachers College.

Assistant Professor K. J. Arrow of Stanford University has been promoted to an associate professorship.

Dr. George Copp of the University of Texas has been appointed to an assistant professorship at North Texas State College.

Assistant Professor W. H. Fagerstrom of City College of New York has been promoted to an associate professorship.

Associate Professor B. E. Gatewood, USAF Institute of Technology, has been promoted to the position of Professor and Head of the Department of Mechanics.

Mr. Martin Goland has been appointed Editor of *Applied Mechanics Reviews*. The editorial office of this journal is located now at Midwest Research Institute, Kansas City, Missouri.

Mr. L. A. Jehn, formerly teaching fellow at the University of Michigan, has been appointed to an assistant professorship at the University of Dayton.

Mr. J. E. Lesch, graduate assistant at Purdue University, has been appointed to an instructorship at Cornell College.

Miss Betty McKnight of Centenary College, Louisiana, has been promoted to an assistant professorship.

Assistant Professor N. S. Mendelsohn of the University of Manitoba has been promoted to an associate professorship.

Assistant Professor Ruth E. O'Donnell of Duquesne University has been promoted to an associate professorship.

Associate Professor C. D. Olds, San Jose State College, has been promoted to a professorship.

Assistant Professor L. L. Rauch of the Department of Aeronautical Engineering, University of Michigan, has been promoted to an associate professorship.

Associate Professor O. E. Stanaitis of the University of Vilna has been appointed to an instructorship at St. Olaf College.

Professor L. W. Swanson of Coe College has accepted a position as Mathematician in the Applied Science Department of International Business Machines Corporation, Chicago, Illinois.

Associate Professor J. F. Wardwell of Colgate University has been promoted to a professorship.

Mr. G. P. Weeg has been appointed to an instructorship at St. Ambrose College.

Instructor William Wells of State Teachers College, Mankato, Minnesota, has been promoted to an assistant professorship.

Dr. D. M. Young, Jr., has been appointed to an instructorship at Harvard University.

Professor Emeritus Lennie P. Copeland of Wellesley College died on January 11, 1951. She was a charter member of the Association.

Assistant Professor L. C. Dawson of Colorado Agricultural and Mechanical College died on May 29, 1950.

Emeritus Assistant Professor L. C. Knight of the College of Wooster died on December 24, 1950. He had been a member of the Association for thirty years.

THE MATHEMATICAL ASSOCIATION OF AMERICA

Official Reports and Communications

THE NEW EDITOR-IN-CHIEF

At a recent meeting of the Board of Governors of the Association, Professor C. B. Allendoerfer of Haverford College was elected Editor-in-Chief of the MONTHLY for a five-year period beginning January 1, 1952. Professor Allendoerfer has been an Associate Editor of the MONTHLY since 1947 and has been in charge of the Department of Classroom Notes since that time. He is now a member of the Board of Governors and has been a Vice-President of the Association. He brings to his new position a wide experience in the affairs of the Association.

After August 1, 1951, articles intended for publication in the MONTHLY should be sent to Professor Allendoerfer at the address: Haverford College, Haverford, Pennsylvania.

H. M. GEHMAN, *Secretary-Treasurer*

NEW MEMBERS

Professor H. M. Gehman, Secretary-Treasurer, announces that the following seventy-nine persons have been elected to membership by the Board of Governors on applications duly certified.

C. B. ABLETT, M.S. (Southern Illinois) Asst. Professor, Lebanon Valley College, Annville, Pa.

MRS. EDITH W. AINSWORTH, M.A. (California) Tuscaloosa, Ala.

DOROTHY J. ANDERSON, B.A. (Macalester) Instr., Macalester College, St. Paul, Minn.

FLORENCE R. ANDERSON, M.A. (U.C.L.A.) Research Laboratory Analyst, Northrop Aircraft, Hawthorne, Calif.

J. M. ANDERSON, Student, South Dakota School of Mines and Technology, Rapid City, S. D.

C. B. BAXTER, (Kansas State T. C.) Partner, C. B. Baxter and Company, Kansas City, Mo.

WILLIAM BECK, Student, Kent State University, Ohio.

E. L. BROOKS, B.S. (Alabama Poly.) Biloxi, Miss.

ARTHA J. BURINGTON, M.E. (Maryland) Teacher, Mount Vernon High School, Alexandria, Va.

M. O. BURRELL, B.A. (Emory) Grad. Student, Emory University, Ga.

C. M. CALLAHAN, B.S. (Lincoln Memorial) Teaching Fellow, Alabama Polytechnic Institute, Auburn, Ala.

S. H. CHASEN, B.C.E. (Georgia Tech.) Grad. Student, Emory University, Gla.

R. H. CHU, Student, Wayne University, Detroit, Mich.

J. W. CREELY, M.S. (Pennsylvania) Research Chemist, American Cyanamid Company, Bound Brook, N. J.

A. C. DAUGHTRY, Student, Alabama Polytechnic Institute, Auburn, Ala.

B. C. DELOACH, JR., Student, Alabama Polytechnic Institute, Auburn, Ala.

G. M. EDDINGTON, M.A. (George Peabody) Asso. Professor, Tusculum College, Greeneville, Tenn.

KARL EIDE, Student, Emmanuel Missionary College, Berrien Springs, Mich.

MARY T. FINIGAN, Student, Regis College, Weston, Mass.

A. D. FLESHLER, M.S. (N.Y.U.) Asst. Professor, Champlain College, Plattsburg, N. Y.

A. J. FLYNN, B.S. (Illinois State Normal) Grad. Student, Illinois State Normal University, Normal, Ill.

L. C. GRAUE, Ph.D. (Indiana) Instr., Sacramento State College, Calif.

EDWARD HALPERN, M.A. (Columbia) Instr., University of Massachusetts, Amherst, Mass.

F. C. HATFIELD, M.S. (Minnesota) Instr., Bemidji State Teachers College, Minn.

G. P. HENDERSON, Ph.D. (Toronto) Lecturer, University of Western Ontario, London, Ont.

JESSIE M. HOAG, M.A. (Alabama) Asst. Professor, Southwestern Louisiana Institute, Lafayette, La.

F. X. HOLZHAUER, B.S. (Detroit) Teaching Fellow, University of Detroit, Mich.

J. G. HORNE, JR., M.S. (Tulane) Grad. Fellow, Tulane University, New Orleans, La.

GRACE M. HYDER, B.S. (Massachusetts) Bookkeeper, Bay State Merchants National Bank, Lawrence, Mass.

JANE C. INGERSOLL, M.A. (Michigan) Research Assistant, Los Alamos Scientific Laboratory, N. M.

J. M. IVANOFF, Student, Gonzaga University, Spokane, Wash.

MRS. RUTH KISSEL, M.A. (Houston) Teacher, University of Houston, Tex.

A. R. KNEER, Student, Gonzaga University, Spokane, Wash.

R. C. KROEGER, B.A. (Buffalo) Teaching Fellow, University of Buffalo, N. Y.

MRS. MERRY M. KRUSE, B.S. (Iowa S. C.) Mathematics Analyst, Sandia Corporation, Albuquerque, N. M.

PAOLO LANZANO, Ph.D. (Rome) Instr., St. Louis University, Mo.

D. B. LARSON, A.B. (Houghton) Grad. Student, University of Buffalo, N. Y.

ROGER LESSARD, B.A.Sc. (Montreal) Asst. Professor, Ecole Polytechnique, Montreal, Que.

W. W. LEUTERT, Dr.Sc.Math. (Zurich) Asst. Professor, University of Maryland, College Park, Md.

F. H. LLOYD, B.S. (Westminster) Asst. Instructor, University of Missouri, Columbia, Mo.

LEE LORCH, Ph.D. (Cincinnati) Asso. Professor, Fisk University, Nashville, Tenn.

J. L. MADDOX, JR., B.E.E. (Alabama Poly.) Teaching Fellow, Albama Polytechnic Institute, Auburn, Ala.

F. J. MALAK, B.S. (Ohio State) Instr., Youngstown College, Ohio.

P. J. MCCARTHY, B.S. in E.E. (Notre Dame) Teaching Fellow, University of Notre Dame, Ind.

W. C. MCCLUNG, Student, College of Wooster, Ohio.

EUGENE MCGILLICUDDY, B.S. (Worcester Poly.) Instr., College of the Holy Cross, Worcester, Mass.

MRS. ANITA B. MILAM, A.B. (Washington Missionary) Instr., Washington Missionary College, Takoma Park, Md.

H. S. MOREDOCK, JR., Ph.D. (California) Asst. Professor, Sacramento State College, Calif.

PRASERT NA NAGARA, B.Sc.Agr. (Philippines) Registrar, College of Agriculture, Bangkhen, Bangkok, Thailand.

FRANCES A. NORTON, A.B. (Birmingham Southern) Teaching Fellow, Alabama Polytechnic Institute, Auburn, Ala.

DANIEL ORLOFF, S.M. (Chicago) Insti., Southern Illinois University, Carbondale, Ill.

MRS. ERNA H. PEARSON, M.A. (Texas) Atlanta, Ga.

G. M. PETERSEN, M.A. (Stanford) Grad. Student, University of Toronto, Ont.

O. A. PLAIT, B.S. (Illinois Tech.) U. S. Navy, Chicago, Ill.

J. E. PRYOR, Ph.D. (Louisiana State) Professor, Harding College, Searcy, Ark.

J. W. RICHARDSON, Student, South Dakota School of Mines and Technology, Rapid City, S. D.

J. R. RICUPITO, Student, University of Florida, Gainesville, Fla.

N. W. RIEBE, B.S. (New Mexico) Research Assistant, University of California, Los Alamos, N. M.

J. F. RIES, M.A. (Delaware) Instr., Kings College, Del.

B. D. RUDIN, B.S. (C.I.T.) Grad. Assistant, University of Southern California, Los Angeles, Calif.

W. A. RUTLEDGE, Ph.D. (Tennessee) Asst. Professor, Alabama Polytechnic Institute, Auburn, Ala.

J. M. SANDY, B.S. in Ed. (Missouri) Asst. Instructor, University of Missouri, Columbia, Mo.

P. T. SCHAEFER, Student, University of Rochester, N. Y.

E. J. SCHWEPPE, B.S. (Missouri Valley) Grad. Student, University of Illinois, Urbana, Ill.

ESTHER SEIDEN, Ph.D. (California) Asst. Professor, University of Buffalo, N. Y.

MAE DELL SENTELL, M.S. (Alabama Poly.)

Instr., Alabama Polytechnic Institute, Auburn, Ala.

S. W. SHELTON, JR., M.A. (George Peabody) Asso. Professor, Northwestern State College, Natchitoches, La.

J. A. SILVA, M.A. (Duke) Instr., Duke University, Durham, N. C.

SISTER MARY SYLVESTER, M.S. (St. Louis) Instr., Mundelein College, Chicago, Ill.

W. I. STEINKAMP, Student, Gonzaga University, Spokane, Wash.

T. R. STENGLE, Student, Franklin and Marshall College, Lancaster, Pa.

W. J. STRAUSS, M.S. (Chicago) Chicago, Ill.

W. G. THORNTON, M.S. (Florida State) Teacher, Madison High School, Fla.

C. H. TROSS, B.S. (Illinois) Mathematician, U. S. Naval Proving Ground, Dahlgren, Va.

T. E. VICHICH, B.S. (Michigan Tech.) Asst. Professor, Michigan College of Mining and Technology, Houghton, Mich.

G. E. WALDMAN, M.A. (Columbia) Teacher, Tottenville High School, Staten Island, N. Y.

C. R. WALKER, M. in E.E. (Brooklyn Poly.) Engineer, American Telephone and Telegraph Company, New York, N. Y.

ANDREW WEISS, Student, University of Detroit, Mich.

EDMUND WESOLOWSKI, Student, Elmhurst College, Ill.

REPORT OF THE TREASURER FOR THE YEAR 1950

Following is a summary of the report of Professor H. M. Gehman as Treasurer of the Association for the year 1950. The complete report has been approved by the Finance Committee and accepted by vote of the Board of Governors. Any member of the Association who wishes the complete report of the Treasurer may obtain it by writing to the office of the Association.

I. TOTAL FUNDS OF THE ASSOCIATION ON JANUARY 1, 1950

M & T Trust Co., Buffalo	$ 7,926.08	Current Fund	$ 7,926.08
Securities	63,600.63	Carus Fund	10,337.74
		Chace Fund	9,003.16
		Houck Fund	9,861.11
		Chauvenet Fund	653.95
		General Fund	33,744.67
	$71,526.71		$71,526.71

II. CURRENT FUND

Balance, January 1, 1950	$ 7,926.08	MONTHLY	
Dues	16,830.03	Publication	$12,750.66
Initiation fees	890.00	Reprints (net)	227.73
Subscriptions	3,618.10	Editor's Office	448.65
Sales of back numbers	1,127.87	Secretary-Treasurer's Office	
Advertisements	3,392.00	Clerical help	5,963.33
Contributions for publication of the MONTHLY	404.00	Postage	768.10
Sale of exchange periodicals	262.75	Printing	538.00
Interest on General Fund	1,088.75	Office expenses	248.51
Income from Hardy Fund	120.00	Bank fees	109.54
		Board of Governors	2,146.50
		Meetings	572.82
		Representatives	122.85
		Subventions	983.39
		List of Officers and Members	1,610.47
		Transfer to General Fund	464.97
		Balance, December 31, 1950	8,705.06

III. Carus Fund

Balance January 1, 1950	$10,337.74	Honoraria, Monographs 9, 10	$ 600.00
Sales of Monographs	3,926.96	Printing Monograph 9	3,165.93
Interest	328.68	Printing Monograph 10	3,150.55
Increase in value of securities	24.98	Balance, December 31, 1950	7,701.88

IV. Chace Fund

Balance, January 1, 1950	$ 9,003.16	Balance December 31, 1950	$10,117.07
Sales of Papyrus	290.00		
Sale of Slaught Papers	514.46		
Interest	287.60		
Increase in values of securities	21.85		

V. Houck Fund

Balance, January 1, 1950	$ 9,861.11	Clerical and editorial expense, Index	175.00
Interest	328.68	Printing Index	1,827.60
Sale of Index	2.00	Balance, December 31, 1950	8,214.71
Increase in value of securities	24.98		

VI. Chauvenet Fund

Balance, January 1, 1950	635.95	Award of Chauvenet Prize	50.00
Interest	20.54	Balance, December 31, 1950	626.05
Increase in value of securities	1.56		

VII. General Fund

Balance, January 1, 1950	$33,744.07	Contribution to International Congress	$ 1,000.00
Increase in value of securities	82.73	Balance, December 31, 1950	33,292.37
Transfer from Current Fund	464.97		

VIII. Total Funds of the Association on December 31, 1950

Current Fund	$ 8,705.06	M & T Trust Co., Buffalo	$ 8,705.06
Carus Fund	7,701.88	Securities	59,952.08
Chace Fund	10,117.07		
Houck Fund	8,214.71		
Chauvenet Fund	626.05		
General Fund	33,292.37		
	$68,657.14		$68,657.14

MAY MEETING OF THE WISCONSIN SECTION

The eighteenth annual meeting of the Wisconsin Section of the Mathematical Association of America was held at Marquette University, Milwaukee, Wisconsin, on Saturday, May 13, 1950. Professor J. R. Mayor, Chairman of the Section, presided at the morning and afternoon sessions.

About one hundred persons were present, including the following forty-seven members of the association: K. J. Arnold, R. H. Bardell, R. D. Bartz, Leon Battig, L. J. Berner, A. C. Berry, Leonard Bristow, R. C. Buck, B. H. Colvin, Rev. L. A. V. DeCleene, W. S. Ericksen, H. P. Evans, E. R. Finkbeiner, Harold

Glander, W. A. Golomski, E. G. Harrell, Martha Hildebrandt, R. T. Hood, C. C. Hsiung, R. C. Huffer, Rev. M. L. Jautz, J. B. Kelly, J. F. Kenney, R. E. Langer, Vivian E. Larson, C. C. MacDuffee, A. E. May, J. R. Mayor, Edward McGaughy, P. E. Meadows, Genevieve S. Meyer, Elli Otteson, L. E. Overn, G. A. Parkinson, H. P. Pettit, Sister M. Mirabella, Sister Mary Felice, Sister Mary Petronia, Elizabeth S. Sokolnikoff, Abraham Spitzbart, J. V. Talacko, J. R. Thompson, R. D. Wagner, N. M. Watermolen, Frances Weisbecker, Louise A. Wolf, Lillian B. Zarling.

At the business meeting held after the morning program, the following officers were elected for the coming year: Chairman, E. G. Harrell, Platteville State Teachers College; Secretary, Louise A. Wolf, University of Wisconsin in Milwaukee; Program Committee Chairman, Rev. L. A. V. DeCleene, St. Norbert's College. The 1951 meeting of the Section will be held at Carroll College.

The program consisted of the following papers:

1. *Arithmetic density*, by Professor R. C. Buck, University of Wisconsin.

On the set of positive integers I, both a finitely additive measure D^*, and a topology may be defined by recourse to the arithmetic progressions $(\lambda n - b)$ as intervals. The problem of the measurability of a set of integers, and its topological properties, lead to new multiplicative and semi-multiplicative functions, and become problems in the theory of numbers. As examples, we have: (i) the set P of primes, of squares, and of cubes, has measure 0; (ii) the set of squares is closed, the set $P \cup \{1\}$ is closed, but there exist polynomials Q such that $Q(\mathrm{I})$ is not closed; (iii) the quasi-progression $([\beta n])$, for irrational β, is everywhere dense in I, and has inner density 0.

2. *On letting a coin decide*, by Professor A. C. Berry, Lawrence College.

3. *The Mathematical Association of America*, by Professor R. E. Langer, University of Wisconsin.

4. *The Wisconsin Mathematics Council*, by Miss Margaret Striegl, Wauwatosa High School, introduced by the Secretary.

This speaker, who represented the Wisconsin Mathematics Council at the Delegate Assembly sponsored by the National Council of Teachers of Mathematics in Chicago, April 13–15, reported on the work of that group. Thirty-nine official delegates were in attendance to discuss problems which large area groups face when organizing to carry on a better program in the field of mathematics. Decisions as to eligibility requirements, group responsibilities, group privileges, and National Council obligations were submitted as recommendations to the board of directors. The members of the board, pleased with the work of the Delegate Assembly, arranged for another meeting at the next annual convention. The Wisconsin Mathematics Council affiliated with the National Council in December, 1949, after having reorganized its own group during the previous year. The membership has increased considerably. The publication, *Wisconsin Teacher of Mathematics*, to be issued four times a year, will help to keep the members informed of current activities, and will give suggestions for teachers of mathematics at all levels.

5. *Mathematics teacher training programs in America*, by Mr. George Bullis, State Teachers College, Platteville, introduced by the Chairman.

This paper gave a summary of results of a questionnaire on the pre-service training of mathematics teachers. The survey was designed to show how inadequately the present programs in our colleges for teacher education conform to the recommendations of the Joint Commission Report of 1940.

6. *An analysis of some issues and problems faced by the secondary teacher of mathematics*, by Miss Martha Hildebrandt. Proviso Township High School, Maywood, Illinois.

After a brief resume of some suggested "new college admissions requirements recommended" and of an administration suggested program for secondary schools, such other issues as college going and non-college going mathematics, teaching for meaning and understanding, isolation of concepts and courses, using teaching aids for developmental purposes, teaching reading in mathematics, guidance and psychological development through mathematics, were discussed.

LOUISE A. WOLF, *Secretary*

CALENDAR OF FUTURE MEETINGS

Joint meeting with American Society for Engineering Education, Michigan State College, East Lansing, June 25–26, 1951.

Thirty-second Summer Meeting, University of Minnesota, Minneapolis, September 3–4, 1951.

Thirty-fifth Annual Meeting, Brown University, Providence, Rhode Island, December 29, 1951.

The following is a list of the Sections of the Association with dates of future meetings so far as they have been reported to the Associate Secretary.

ALLEGHENY MOUNTAIN, Duquesne University, Pittsburgh, Pennsylvania, May 5, 1951.

ILLINOIS, University of Illinois, Urbana, May 11–12, 1951.

INDIANA, May 5, 1951.

IOWA, Wartburg College, Waverly, April 20–21, 1951.

KANSAS, University of Kansas, Lawrence, April 7, 1951.

KENTUCKY, Eastern Kentucky State College, Richmond, April 28, 1951.

LOUISIANA-MISSISSIPPI

MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA, United States Naval Academy, Annapolis, Maryland, April 28, 1951.

METROPOLITAN NEW YORK, Manhattan College, April 7, 1951.

MICHIGAN

MINNESOTA, College of St. Benedict, St. Joseph, April 28, 1951.

MISSOURI, Central College, Fayette, April 6, 1951.

NEBRASKA, University of Nebraska, Lincoln, April 21, 1951.

NORTHERN CALIFORNIA

OHIO, Ohio State University, Columbus, April 21, 1951.

OKLAHOMA

PACIFIC NORTHWEST, State College of Washington, Pullman, June 15, 1951.

PHILADELPHIA, University of Pennsylvania, Philadelphia, November 24, 1951.

ROCKY MOUNTAIN, Colorado State College of Education, Greeley, April 20–21, 1951.

SOUTHEASTERN

SOUTHERN CALIFORNIA

SOUTHWESTERN

TEXAS, Southern Methodist University, Dallas, April 27–28, 1951.

UPPER NEW YORK STATE, Hamilton College, Clinton, May 5, 1951.

WISCONSIN, Carroll College, Waukesha, May 12, 1951.

GEORGE BANTA PUBLISHING COMPANY, MENASHA, WISCONSIN

THE AMERICAN MATHEMATICAL MONTHLY

THE OFFICIAL JOURNAL OF

THE MATHEMATICAL ASSOCIATION OF AMERICA, INC.

VOLUME 58 NUMBER 5

CONTENTS

MAY 1951

The AMERICAN MATHEMATICAL MONTHLY

(FOUNDED IN 1894 BY BENJAMIN F. FINKEL)

EDITORIAL CORRESPONDENCE should be addressed to the Editor, C. V. NEWSOM, State Education Building, Albany 1, N. Y.

ADVERTISING CORRESPONDENCE should be addressed to J. F. RANDOLPH, Morey Hall, University of Rochester, Rochester 3, N. Y.

NOTICE OF CHANGE OF ADDRESS by members of the Association as well as correspondence regarding subscriptions to the MONTHLY should be sent to the Secretary-Treasurer, H. M. GEHMAN, University of Buffalo, Buffalo 14, N. Y. Change of address must reach the Secretary-Treasurer about six weeks before the change can become effective.

Entered as second class matter at the post office at Menasha, Wis. Acceptance for mailing at special rate of postage provided for in the Act of February 28, 1925, embodied in Paragraph 4, Section 538, P. L. and R., authorized April 1, 1926.

SUBSCRIPTION PRICE: To Members, \$4 a Year, To Others, \$5 a Year.

PUBLISHED BY THE ASSOCIATION at Menasha, Wisconsin, and Albany, N. Y. during the months of January, February, March, April, May, June-July, August-September, October, November, December.

WHAT ARE TENSORS?*

PETER SCHERK and MICHAEL KWIZAK, University of Saskatchewan

Generally speaking, the content of present day college mathematics was established some two hundred years ago. We therefore find an ever widening gap between the mathematics that we teach and current creative work. The mere existence of this gap may not necessarily be deplorable. A theory has to undergo a process of ageing during which it is polished and simplified until it acquires that apparently immediate appeal which, for example, calculus and analytic geometry now have for our more gifted beginners. But the growing size of this gap should induce us to examine continually the possibility of revitalizing our curriculum.

Let us have a glance at analytic geometry. Its core consists of the theories of points, straight lines, and planes, of coordinate systems and their transformation, of conics in the plane and of quadrics in three-space. After the beginning of this century, these theories merged with certain algebraic theories dealing, for example, with systems of linear equations and with quadratic forms, into one new and powerful discipline, linear algebra. In the last twenty years linear algebra has developed rapidly. Becoming more and more abstract, it is covering an ever wider range of phenomena, and its applications reach into all branches of mathematics. We find an exposition of the present state of linear algebra in two French treatises, Bourbaki's *Algèbre Linéaire* of 1947 and his *Algèbre Multilinéaire* of 1948.

Geometry has fared poorly in this recent pursuit of abstraction and generality, and so the geometrical aspects of linear algebra are somewhat neglected. But I think analytic geometry could profit from the newer theory in two ways: (1) By making use of some of its ideas in the treatment of the standard subject matter, and (2) by incorporating some of its achievements in an advanced course. The latter would not necessarily have to lead the student through the latest abstractions of the theory of modules over rings. But it could give him a sound grounding in the now classical theory of finite dimensional vector spaces and its algebraic and geometric applications.

The chapter in this theory that we now discuss is the so-called tensor algebra. It is neglected in the textbooks on analytic geometry, and we have to turn to books on Riemannian geometry for an account of it. But it seems to me that it is bona fide analytic geometry and good geometry at that. Its most modern and abstract form is Bourbaki's multilinear algebra. And it is from the two Bourbaki treatises that we are borrowing what seems to us one of the fundamental ideas of modern analytic geometry, namely, the elimination of

* This paper is a revised version of a talk delivered by Mr. Scherk before a meeting of the Minnesota section of the M.A.A. at Grand Forks, N. D., on October 15, 1949. It is based on Mr. Kwizak's master's thesis at the University of Saskatchewan.

coordinates.†

We start with ordinary 3-space with its points $A, B, C, \cdots$. A segment AB is determined by its end-points A and B. By calling one of them, say A, the initial point of our segment and the other point B its end-point, we arrive at the directed segment or "vector" $\overrightarrow{AB}$. Thus, every ordered pair of points A, B determines a vector. If $A \neq B$, we may imagine the vector $\overrightarrow{AB}$ as an arrow leading from A to B. Now we are not really interested in the initial point and the end-point of a vector, but only in its direction and its length. Thus, we shall call two vectors $\overrightarrow{AB}$ and $\overrightarrow{CD}$ equal if they have the same direction and the same length. Obviously this definition of equality satisfies the usual requirements, namely,

(1) Reflexivity: Every vector is equal to itself.

(2) Symmetry: If $\overrightarrow{AB}$ is equal to $\overrightarrow{CD}$, then $\overrightarrow{CD}$ is also equal to $\overrightarrow{AB}$.

(3) Transitivity: If two vectors are both equal to a third vector, then they are equal to one another.

Our definition of the equality of vectors enables us to move a given vector about, keeping it parallel to itself. In particular, any point may become its initial point. Its end-point is then determined.

We have neglected those vectors $\overrightarrow{AA}$ whose end-points coincide with their initial points. These vectors have the length zero and no specific direction. We identify all these vectors and consider them as a single vector, the "null-vector" $\mathfrak{o}$.

If two vectors $\mathfrak{a}$ and $\mathfrak{b}$ are given, we may choose the initial point A of the first vector arbitrarily, say $\mathfrak{a} = \overrightarrow{AB}$. We now choose B as the initial point of the second vector $\mathfrak{b}$. This determines its end-point C; thus $\mathfrak{b} = \overrightarrow{BC}$. The point C is determined by the two vectors $\mathfrak{a}$ and $\mathfrak{b}$ and by the choice of the point A. The same will hold true of the vector $\mathfrak{c} = \overrightarrow{AC}$. In fact, $\mathfrak{c}$ depends only on the vectors $\mathfrak{a}$ and $\mathfrak{b}$. We write

$$\mathfrak{c} = \mathfrak{a} + \mathfrak{b} \quad \text{or} \quad \overrightarrow{AC} = \overrightarrow{AB} + \overrightarrow{BC}.$$

This vector addition has the following properties:

(I) Each pair of vectors $\mathfrak{a}$, $\mathfrak{b}$ determines one and only one third vector $\mathfrak{a}+\mathfrak{b}$.

(II) $\mathfrak{o}+\mathfrak{v}=\mathfrak{v}+\mathfrak{o}=\mathfrak{v}$ for every vector $\mathfrak{v}$.

Indeed, $\overrightarrow{AB}+\overrightarrow{BB}=\overrightarrow{AA}+\overrightarrow{AB}=\overrightarrow{AB}$. This states that the null-vector $\mathfrak{o}$ plays a role in vector addition analogous to that of the number 0 in ordinary addition.

† As late as twenty years ago, coordinates were considered the very essence of analytic geometry. An excellent textbook on this subject, published in 1931, contains the following observation: "It is the basic principle of analytic geometry to refrain from studying straight lines, planes, and space directly, and to examine the corresponding numerical manifolds instead. It is only afterwards that the algebraic results thus obtained are translated into results on manifolds of points." Cf. Schreier-Sperner, Analytische Geometrie und Algebra, vol. I, p. 8, Leipzig, 1931.

(III) To every vector $\mathfrak{a}$ there exists a vector denoted by $-\mathfrak{a}$ such that

$$\mathfrak{a} + (-\mathfrak{a}) = (-\mathfrak{a}) + \mathfrak{a} = \mathfrak{o}.$$

Obviously $\overrightarrow{AB}+\overrightarrow{BA}=\overrightarrow{BA}+\overrightarrow{AB}=\mathfrak{o}$. This states that every vector has an inverse.

(IV) $$\mathfrak{a} + (\mathfrak{b} + \mathfrak{c}) = (\mathfrak{a} + \mathfrak{b}) + \mathfrak{c},$$

the associative law.

Finally, the commutative law,

(V) $$\mathfrak{a} + \mathfrak{b} = \mathfrak{b} + \mathfrak{a}.$$

Any set of elements that satisfy these five conditions is called a commutative or abelian group. Thus, we may state: The vectors in 3-space form an abelian group.

There exists a second important operation on vectors, *i.e.*, their multiplication with real numbers. Let $\mathfrak{a}\neq\mathfrak{o}$, and let λ be a real number. If $\lambda>0$, we define $\lambda\mathfrak{a}$ to be the vector whose direction is the same as that of $\mathfrak{a}$ and whose length is λ times that of $\mathfrak{a}$. If $\lambda<0$, then $\lambda\mathfrak{a}$ is defined to be the vector whose direction is opposite to that of $\mathfrak{a}$ and whose length is $|\lambda|$ times the length of $\mathfrak{a}$. Finally, we define $0\cdot\mathfrak{a}=\mathfrak{o}$ and $\lambda\cdot\mathfrak{o}=\mathfrak{o}$. This multiplication satisfies the following additional rules:

(VI) $$1\cdot\mathfrak{a} = \mathfrak{a},$$

(VII) $$\lambda(\mu\mathfrak{a}) = (\lambda\mu)\cdot\mathfrak{a},$$

(VIII) $$(\lambda + \mu)\mathfrak{a} = \lambda\mathfrak{a} + \mu\mathfrak{a}, \qquad \lambda(\mathfrak{a} + \mathfrak{b}) = \lambda\mathfrak{a} + \lambda\mathfrak{b}.$$

Any set of elements that satisfy the conditions (I)–(VIII) is called a vector-space, or more accurately, a vector space over the real field. Thus, we have seen that the vectors in 3-space form such a vector space. A second example of a vector space is the set of all polynomials $f(x)$ in one variable x. Still another example is the set of all ordered n-tuples of real numbers $(a^1, \cdots, a^n)$ where n is a fixed positive integer, and our vector operations are defined through

$$(a^1, a^2, \cdots, a^n) + (b^1, b^2, \cdots, b^n) = (a^1 + b^1, a^2 + b^2, \cdots, a^n + b^n)$$

and

$$\lambda(a^1, a^2, \cdots, a^n) = (\lambda a^1, \lambda a^2, \cdots, \lambda a^n).$$

It may not be very important that the vector space of all vectors in ordinary 3-space could be constructed without reference to coordinates. What does appear essential is this: once the concept of a vector space has been arrived at, it is studied using its definition rather than any coordinates that could be introduced. It is this axiomatic or conceptual approach that has led to the unification of theories that originally appeared far removed from each other.

The vectors in euclidean 3-space permit still another operation: Each pair

of vectors $\mathfrak{a}$, $\mathfrak{b}$ determines a real number $\mathfrak{ab}$, their scalar product. We can define it in the following fashion: $\mathfrak{a}^2$ is the square of the length of the vector $\mathfrak{a}$. We define $\mathfrak{ab}$ in such a way that if,

$$(\mathfrak{a} + \mathfrak{b})^2 = \mathfrak{a}^2 + \mathfrak{b}^2 + 2\mathfrak{ab},$$

that is

$$\mathfrak{ab} = \tfrac{1}{2}[(\mathfrak{a} + \mathfrak{b})^2 - \mathfrak{a}^2 - \mathfrak{b}^2]. \tag{1}$$

From the Pythagorean Theorem $\mathfrak{ab} = 0$ if and only if either $\mathfrak{a} = \mathfrak{o}$ or $\mathfrak{b} = \mathfrak{o}$ or the vectors $\mathfrak{a}$ and $\mathfrak{b}$ are perpendicular. It is easy to see that the perpendicular projection of the vector $\mathfrak{a}$ on the vector $\mathfrak{b}$ is the vector

$$\frac{\mathfrak{ab}}{\mathfrak{b}^2}\cdot\mathfrak{b} \qquad [\mathfrak{b} \neq \mathfrak{o}]. \tag{2}$$

Our scalar product has the following properties:

$$\mathfrak{ab} = \mathfrak{ba}, \tag{IX}$$

$$(\lambda\mathfrak{a})\cdot\mathfrak{b} = \lambda\cdot\mathfrak{ab}, \tag{X}$$

$$(\mathfrak{a} + \mathfrak{b})\mathfrak{c} = \mathfrak{ac} + \mathfrak{bc}. \tag{XI}$$

The commutative law (IX) follows directly from (1). The other two rules can be interpreted geometrically. Thus (XI) states that the orthogonal projection of the vector $\mathfrak{a}+\mathfrak{b}$ on $\mathfrak{c}$ is equal to the sum of the projections of the vectors $\mathfrak{a}$ and $\mathfrak{b}$. Finally, our scalar product satisfies the following condition:

(XII) If $\mathfrak{ab} = \mathfrak{o}$ for a given vector $\mathfrak{a}$ and for every vector $\mathfrak{b}$, then $\mathfrak{a} = \mathfrak{o}$.

Suppose each pair of vectors $\mathfrak{a}$, $\mathfrak{b}$ of some vector space determines a real number $\mathfrak{ab}$ and suppose this mapping satisfies the rules (IX)–(XI). Then we call $\mathfrak{ab}$ the scalar product of the vectors $\mathfrak{a}$ and $\mathfrak{b}$. Any finite dimensional vector space can be provided with a scalar product in a great many different ways. We can introduce a scalar product into the vector space R_n of all ordered n-tuples by defining, for example,

$$(a^1, a^2, \cdots, a^n)\cdot(b^1, b^2, \cdots, b^n) = a^1b^1 + a^2b^2 + \cdots + a^nb^n, \tag{i}$$

or

$$(a^1, a^2, \cdots, a^n)\cdot(b^1, b^2, \cdots, b^n) = -a^1b^1 + a^2b^2 + \cdots + a^nb^n, \tag{ii}$$

or

$$(a^1, a^2, \cdots, a^n)(b^1, b^2, \cdots, b^n) = a^2b^2 + \cdots + a^nb^n. \tag{iii}$$

A vector space with a scalar product that also satisfies condition (XII) is sometimes called complete. Thus the vectors in euclidean 3-space form a complete vector-space. We can make R_n complete by using the definitions (i) or (ii).

From now on we consider an arbitrary complete vector space V. It does not matter how we have obtained V and its scalar product. All we need is the fact

that V satisfies the axioms (I)–(XII). In particular, the dimension of V need not be finite. We want to define tensors and tensor operations in V.

A simple tensor

$$\mathfrak{A} = \{\mathfrak{a}_1, \mathfrak{a}_2, \cdots, \mathfrak{a}_p\} \tag{3}$$

of order p is an ordered p-tuple of vectors in V. Let

$$\mathfrak{B} = \{\mathfrak{b}_1, \mathfrak{b}_2, \cdots, \mathfrak{b}_p\} \tag{4}$$

be another simple tensor of order p. If none of the vectors $\mathfrak{a}_1, \mathfrak{a}_2, \cdots, \mathfrak{a}_p$ is the null-vector, then we call $\mathfrak{A}$ and $\mathfrak{B}$ equal if corresponding vectors are parallel and if the products of their lengths are equal:

$$\mathfrak{A} = \mathfrak{B} \leftrightarrow \mathfrak{b}_1 = \lambda^1\mathfrak{a}_1,\ \mathfrak{b}_2 = \lambda^2\mathfrak{a}_2, \cdots, \mathfrak{b}_p = \lambda^p\mathfrak{a}_p$$

where

$$\lambda^1\cdot\lambda^2\cdot \cdots \cdot\lambda^p = 1.$$

We complete this definition by identifying all those simple tensors (3) of order p for which at least one of the vectors $\mathfrak{a}_i$ is the null-vector. This simple tensor will be called the null-tensor $\mathfrak{O}_p$ of order p.

These definitions may be satisfactory from a geometric point of view, but they will lead to awkward proofs. We can overcome this difficulty by a more algebraic approach. Before defining the equality of the two simple tensors (3) and (4) of order p, we define their scalar product by means of

$$\mathfrak{A}\mathfrak{B} = \mathfrak{a}_1\mathfrak{b}_1\cdot\mathfrak{a}_2\mathfrak{b}_2\cdot \cdots \cdot\mathfrak{a}_p\mathfrak{b}_p. \tag{5}$$

And now we define:

(6) $\mathfrak{A} = \mathfrak{B} \leftrightarrow \mathfrak{A}\mathfrak{C} = \mathfrak{B}\mathfrak{C}$ for every simple tensor $\mathfrak{C}$ of order p.

It is not difficult to verify that these two definitions of equality are equivalent, that they satisfy the usual requirements, and that equal simple tensors have equal scalar products, that is,

(7) $\mathfrak{A} = \mathfrak{A}'$, $\mathfrak{B} = \mathfrak{B}'$ implies $\mathfrak{A}\mathfrak{B} = \mathfrak{A}'\mathfrak{B}'$.

We want to extend our vector operations to tensors. The product of a simple tensor with a scalar, *i.e.*, number can be readily defined:

$$\lambda\{\mathfrak{a}_1, \mathfrak{a}_2, \cdots, \mathfrak{a}_p\} = \{\lambda\mathfrak{a}_1, \mathfrak{a}_2, \cdots, \mathfrak{a}_p\}.$$

Since the latter tensor is equal to

$$\{\mathfrak{a}_1, \lambda\mathfrak{a}_2, \cdots, \mathfrak{a}_p\} = \cdots = \{\mathfrak{a}_1, \mathfrak{a}_2, \cdots, \mathfrak{a}_{p-1}, \lambda\mathfrak{a}_p\},$$

the lack of symmetry in this definition is only apparent.

In order to define addition, we have to introduce more general tensors. A tensor of order p is a formal finite sum of simple tensors of order p:

$$\mathfrak{A} = \{\mathfrak{a}_1^{(1)}, \mathfrak{a}_2^{(1)}, \cdots, \mathfrak{a}_p^{(1)}\} + \cdots + \{\mathfrak{a}_1^{(r)}, \mathfrak{a}_2^{(r)}, \cdots, \mathfrak{a}_p^{(r)}\}. \tag{8}$$

Let

$$\mathfrak{B} = \{\mathfrak{b}_1^{(1)}, \mathfrak{b}_2^{(1)}, \cdots, \mathfrak{b}_p^{(1)}\} + \cdots + \{\mathfrak{b}_1^{(s)}, \mathfrak{b}_2^{(s)}, \cdots, \mathfrak{b}_p^{(s)}\} \tag{9}$$

be another tensor of the same order. Then we define their scalar product through

$$\begin{aligned}\mathfrak{A}\mathfrak{B} &= \{\mathfrak{a}_1^{(1)}, \mathfrak{a}_2^{(1)}, \cdots, \mathfrak{a}_p^{(1)}\}\{\mathfrak{b}_1^{(1)}, \mathfrak{b}_2^{(1)}, \cdots, \mathfrak{b}_p^{(1)}\} + \cdots \\ &\quad + \{\mathfrak{a}_1^{(r)}, \mathfrak{a}_2^{(r)}, \cdots, \mathfrak{a}_p^{(r)}\}\{\mathfrak{b}_1^{(s)}, \mathfrak{b}_2^{(s)}, \cdots, \mathfrak{b}_p^{(s)}\} \\ &= \sum_{\rho=1}^{r}\sum_{\sigma=1}^{s}\{\mathfrak{a}_1^{(\rho)}, \mathfrak{a}_2^{(\rho)}, \cdots, \mathfrak{a}_p^{(\rho)}\}\{\mathfrak{b}_1^{(\sigma)}, \mathfrak{b}_2^{(\sigma)}, \cdots, \mathfrak{b}_p^{(\sigma)}\} \\ &= \sum_{\rho=1}^{r}\sum_{\sigma=1}^{s}\sum_{\pi=1}^{p} \mathfrak{a}_\pi^{(\rho)}\mathfrak{b}_\pi^{(\sigma)}.\end{aligned}$$

Thus each summand of $\mathfrak{A}$ is multiplied with each summand of $\mathfrak{B}$. The equality of tensors of order p can now be defined in a manner similar to (6): $\mathfrak{A}=\mathfrak{B}\leftrightarrow\mathfrak{A}\mathfrak{C}=\mathfrak{B}\mathfrak{C}$ for every tensor $\mathfrak{C}$ of order p. We can justify this definition by showing that it satisfies the usual requirements, that it is equivalent to the previous definitions in the case of simple tensors and that finally equal tensors have equal scalar products, *i.e.*, (7) holds.

It is now obvious how we have to define the addition of tensors of the same order p and the multiplication of tensors by scalars; namely, if $\mathfrak{A}$ and $\mathfrak{B}$ are given by (8) and (9), then $\lambda\mathfrak{A}$ and $\mathfrak{A}+\mathfrak{B}$ are defined to be

$$\begin{aligned}\lambda\mathfrak{A} &= \{\lambda\mathfrak{a}_1^{(1)}, \mathfrak{a}_2^{(1)}, \cdots, \mathfrak{a}_p^{(1)}\} + \cdots + \{\lambda\mathfrak{a}_1^{(r)}, \mathfrak{a}_2^{(r)}, \cdots, \mathfrak{a}_p^{(r)}\}, \\ \mathfrak{A}+\mathfrak{B} &= \{\mathfrak{a}_1^{(1)}, \mathfrak{a}_2^{(1)}, \cdots, \mathfrak{a}_p^{(1)}\} + \cdots + \{\mathfrak{a}_1^{(r)}, \mathfrak{a}_2^{(r)}, \cdots, \mathfrak{a}_p^{(r)}\} \\ &\quad + \{\mathfrak{b}_1^{(1)}, \mathfrak{b}_2^{(1)}, \cdots, \mathfrak{b}_p^{(1)}\} + \cdots + \{\mathfrak{b}_1^{(s)}, \mathfrak{b}_2^{(s)}, \cdots, \mathfrak{b}_p^{(s)}\}.\end{aligned}$$

Again, we can verify that equal tensors have equal multiples, and that sums of equal tensors are equal, that is,

$$\mathfrak{A} = \mathfrak{A}', \mathfrak{B} = \mathfrak{B}' \quad \text{implies} \quad \lambda\mathfrak{A} = \lambda\mathfrak{A}' \quad \text{and} \quad \mathfrak{A}+\mathfrak{B} = \mathfrak{A}'+\mathfrak{B}'.$$

We have constructed a set of elements, the tensors of order p, and we have introduced three operations into this set: multiplication with scalars, addition, and scalar products. These were the same operations we used in the definition of the complete vector space from which we started. In fact, it can be proved that the tensors of order p form a complete vector space V_p. This space depends on our original vector space V and on the value of p. Thus V_1 differs from V in nothing but its notation. It therefore may be identified with V. The real numbers are called tensors of order zero.

So far, the various V_p's are connected with each other only by their common dependence on V. However, there are two operations on tensors that tie these V_p's together, as indicated in the following. The general product of the

tensor (8) of order p by a tensor

$$\mathfrak{C} = \sum_{\tau=1}^{t} \{\overset{(\tau)}{\mathfrak{c}_1}, \overset{(\tau)}{\mathfrak{c}_2}, \cdots, \overset{(\tau)}{\mathfrak{c}_q}\}$$

of order q is the following tensor of order $p+q$:

$$\{\mathfrak{A}, \mathfrak{C}\} = \sum_{\rho=1}^{r} \sum_{\tau=1}^{t} \{\overset{(\rho)}{\mathfrak{a}_1}, \overset{(\rho)}{\mathfrak{a}_2}, \cdots, \overset{(\rho)}{\mathfrak{a}_p}, \overset{(\tau)}{\mathfrak{c}_1}, \overset{(\tau)}{\mathfrak{c}_2}, \cdots, \overset{(\tau)}{\mathfrak{c}_q}\}.$$

Again, general products of equal tensors are equal, i.e.,

$$\mathfrak{A} = \mathfrak{A}', \ \mathfrak{C} = \mathfrak{C}' \quad \text{implies} \quad \{\mathfrak{A}, \mathfrak{C}\} = \{\mathfrak{A}', \mathfrak{C}'\}.$$

Let $p \geqq 2$. We contract the tensor $\mathfrak{A}$ of order p by first selecting two indices α, β, say $1 \leqq \alpha < \beta \leqq p$, and then forming the new tensor

$$\mathfrak{A}^* = \sum_{\rho=1}^{r} \overset{(\rho)}{\mathfrak{a}_\alpha} \overset{(\rho)}{\mathfrak{a}_\beta} \{\overset{(\rho)}{\mathfrak{a}_1}, \cdots, \overset{(\rho)}{\mathfrak{a}_{\alpha-1}}, \overset{(\rho)}{\mathfrak{a}_{\alpha+1}}, \cdots, \overset{(\rho)}{\mathfrak{a}_{\beta-1}}, \overset{(\rho)}{\mathfrak{a}_{\beta+1}}, \cdots, \overset{(\rho)}{\mathfrak{a}_p}\}.$$

Thus the αth and the βth vector are taken out of each brace. The contracted tensor $\mathfrak{A}^*$ has the order $p-2$. It will, as a rule, depend on the choice of the indices α and β. But it is important that equal tensors contracted in the same fashion yield equal tensors.

We have developed tensor algebra for arbitrary complete vector spaces V. If the dimension of V is finite, then we can choose a basis of V and introduce covariant and contravariant components of a vector. After this we can define the various types of components of a tensor and follow our tensor operations in terms of these components. We then arrive at the standard formulas we find in any textbook on tensor analysis or Riemannian geometry.

Appendix

The reader may be interested in a few remarks on some topics not mentioned in the above discussion.

1. Tensors over finite dimensional vector spaces. Let V be an n-dimensional complete vector space, and let

$$\mathfrak{B} \equiv \mathfrak{b}_1, \mathfrak{b}_2, \cdots, \mathfrak{b}_n$$

be a basis of V. Put $g_{ik} = \mathfrak{b}_i\mathfrak{b}_k$.† Then the matrix (g_{ik}) is regular and has an inverse (g^{ik}). Define

† Thus each of the numbers g_{ik} is the scalar product of the corresponding vectors $\mathfrak{b}_i$ and $\mathfrak{b}_k$. The last formula of this section shows that the g_{ik} are the covariant components of the fundamental tensor $\mathfrak{G}$. It should be noted however that usually the *set* of numbers g_{ik} itself is called a covariant tensor.

The basis $\mathfrak{B}^1$ and hence the various types of tensor components can be constructed without referring to the g_{ik} and g^{ik}: The vector $\mathfrak{b}^k$ is the one and only vector whose scalar product with $\mathfrak{b}_k$ is equal to one, while its scalar products with the other $\mathfrak{b}_i$'s vanish.

$$\mathfrak{b}^k = \sum_1^n \lambda\, g^{k\lambda}\mathfrak{b}_\lambda.$$

The vectors $\mathfrak{b}^1, \mathfrak{b}^2, \cdots, \mathfrak{b}^n$ form another basis $\mathfrak{B}^1$ of V.

By means of $\mathfrak{B}$ and $\mathfrak{B}^1$ we can readily construct bases of the vector space V_p of all tensors of order p over V. Thus either of the following sets of simple tensors form such a basis:

$$\begin{array}{ll} \mathfrak{B}_p\colon & \{\mathfrak{b}_{i_1}, \mathfrak{b}_{i_2}, \cdots, \mathfrak{b}_{i_p}\} \\ \mathfrak{B}^p\colon & \{\mathfrak{b}^{i_1}, \mathfrak{b}^{i_2}, \cdots, \mathfrak{b}^{i_p}\} \end{array} \qquad [i_1, i_2, \cdots, i_p = 1, 2, \cdots, n].$$

In particular V_p has the dimension n^p. The p-times contravariant components $p^{i_1,i_2,\cdots,i_p}$ of a tensor $\mathfrak{T}$ with respect to $\mathfrak{B}$ are its contravariant components if we consider $\mathfrak{T}$ as an element of the vector space V_p with the basis $\mathfrak{B}_p$. Thus

$$\mathfrak{T} = \sum_{i_1,i_2,\cdots,i_p}^{n} P^{i_1,i_2,\cdots,i_p}\{\mathfrak{b}_{i_1}, \mathfrak{b}_{i_2}, \cdots, \mathfrak{b}_{i_p}\}.$$

Similarly, the p-times covariant components $P_{i_1i_2,\cdots,i_p}$ of $\mathfrak{T}$ with respect to $\mathfrak{B}$ are its covariant components with respect to $\mathfrak{B}_p$. Thus

$$P_{i_1,i_2,\cdots,i_p} = \mathfrak{T}\cdot\{\mathfrak{b}_{i_1}, \mathfrak{b}_{i_2}, \cdots, \mathfrak{b}_{i_p}\}.$$

The $P^{i_1,i_2,\cdots,i_p}$ and $P_{i_1i_2,\cdots,i_p}$ are respectively the covariant and contravariant components of $\mathfrak{T}$ with respect to the basis $\mathfrak{B}^p$. The mixed components of $\mathfrak{T}$ can be introduced in a similar fashion, for example,

$$P_{i_1}^{\cdot i_2,\cdots,i_p} = \mathfrak{T}\cdot\{\mathfrak{b}_{i_1}, \mathfrak{b}^{i_2}, \cdots, \mathfrak{b}^{i_p}\}$$

and

$$\mathfrak{T} = \sum_{i_1,i_2,\cdots,i_p}^{n} P_{i_1}^{\cdot i_2,\cdots,i_p}\{\mathfrak{b}^{i_1}, \mathfrak{b}_{i_2}, \cdots, \mathfrak{b}_{i_p}\}.$$

Instead of introducing the basis $\mathfrak{B}^1$ most textbooks prefer working with the fundamental tensor

$$\mathfrak{G} = \sum_{i,k=1}^{n} g^{ik}\{\mathfrak{b}_i, \mathfrak{b}_k\} = \sum_{i,k=1}^{n} g_{ik}\{\mathfrak{b}^i, \mathfrak{b}^k\} = \sum_{i=1}^{n}\{\mathfrak{b}_i, \mathfrak{b}^i\}.$$

$\mathfrak{G}$ is independent of the choice of the basis $\mathfrak{B}$.

2. Multivectors. Let V be any complete vector space. Interesting subspaces of V_p are formed by those tensors of order p that satisfy suitable symmetry conditions. The following example may suffice:

We map each tensor

$$\mathfrak{T} = \sum_{\lambda}\{\mathfrak{t}_1^{(\lambda)}, \mathfrak{t}_2^{(\lambda)}, \cdots, \mathfrak{t}_p^{(\lambda)}\}$$

on V_p on the tensor

$$A\mathfrak{T} = \frac{1}{p!}\sum_{\lambda}\sum_{(i_1,i_2,\cdots,i_p)} \pm \{\mathfrak{t}_{i_1}^{(\lambda)}, \mathfrak{t}_{i_2}^{(\lambda)}, \cdots, \mathfrak{t}_{i_p}^{(\lambda)}\}.$$

The inner sum is extended over all the permutations $(i_1, i_2, \cdots, i_p)$ of the numbers 1, 2, $\cdots$, p, and the plus sign holds for even permutations, the minus sign for the odd ones. This mapping

$$A: \ \mathfrak{T} \to A\mathfrak{T}$$

of V_p into itself is linear; more accurately, it is a projection of V_p on a subspace $A\,V_p$; that means, a tensor $\mathfrak{V}$ lies in $A\,V_p$ if and only if $\mathfrak{V} = A\mathfrak{V}$. A tensor in $A\,V_p$ which is the image of a simple tensor is called a p-vector. Thus, every tensor in $A\,V_p$ is a sum of p-vectors. The subspace $A\,V_p$ is again complete.

Since both p-vectors and determinants of order p are invariant under the alternating group of order p, their theories are closely related. Only a few results can be quoted:

Let $\mathfrak{V}$ and $\mathfrak{W}$ be two p-vectors, say

$$\mathfrak{V} = A\{\mathfrak{v}_1, \cdots, \mathfrak{v}_p\} \quad \text{and} \quad \mathfrak{W} = A\{\mathfrak{w}_1, \cdots, \mathfrak{w}_p\}.$$

Then

$$\mathfrak{V}\mathfrak{W} = \frac{1}{p!}\begin{vmatrix} \mathfrak{v}_1\mathfrak{w}_1 & \cdots & \mathfrak{v}_1\mathfrak{w}_p \\ \vdots & & \vdots \\ \mathfrak{v}_p\mathfrak{w}_1 & \cdots & \mathfrak{v}_p\mathfrak{w}_p \end{vmatrix}.$$

$\mathfrak{V}$ is equal to the null-tensor [cf. p. 301] if and only if the vectors $\mathfrak{v}_1, \mathfrak{v}_2, \cdots, \mathfrak{v}_p$ are linearly dependent. If $\mathfrak{V} \neq \mathfrak{O}_p$, then $\mathfrak{V} = \mathfrak{W}$ if and only if the $\mathfrak{w}_i$'s are linear combinations of the $\mathfrak{v}_k$'s with determinant 1, *i.e.*, if the two parallelotopes determined by the vectors $\mathfrak{v}_1, \mathfrak{v}_2, \cdots, \mathfrak{v}_p$, respectively, $\mathfrak{w}_1, \mathfrak{w}_2, \cdots, \mathfrak{w}_p$ span the same p-space and are equal in volume and orientation.

Let $\mathfrak{S}$ and $\mathfrak{T}$ be tensors of order p, respectively q and let $\{\mathfrak{S}, \mathfrak{T}\}$ denote their general product. Then

$$A\{\mathfrak{S}, \mathfrak{T}\} = A\{A\mathfrak{S}, A\mathfrak{T}\}.$$

This sum of $(p+q)$-vectors is called the exterior product of $A\mathfrak{S}$ by $A\mathfrak{T}$. Obviously, the exterior product of two multivectors is again a multivector.

Finally, given a p-vector $\mathfrak{V}$ and a q-vector $\mathfrak{W}$; $q < p$. Then there exists one and only one sum $\mathfrak{V}\mathfrak{W}$ of $(p-q)$-vectors such that

$$\mathfrak{V}\cdot A\{\mathfrak{W}, \mathfrak{U}\} = \mathfrak{V}\mathfrak{W}\cdot\mathfrak{U}$$

for every $(p-q)$-vector $\mathfrak{U}$. Up to a numerical factor, this "inner product" $\mathfrak{V}\mathfrak{W}$ is equal to the q-times contracted general product $\{\mathfrak{V},\mathfrak{W}\}$.

GEOMETRICAL EXTREMA SUGGESTED BY A LEMMA OF BESICOVITCH

PAUL BATEMAN, The Institute for Advanced Study, and PAUL ERDÖS, Syracuse University

1. Introduction. In [1] Besicovitch needed as a lemma a result of the following type.

THEOREM 1. *Given a set Γ of coplanar circles, the center of no one of them being in the interior of another, and U the circle (or a circle) of Γ whose radius does not exceed the radius of any other circle of Γ, then the number of circles meeting U does not exceed* 18.

Besicovitch proved the weaker theorem obtained from this one by replacing 18 by 21. In this paper we shall prove Theorem 1 as it stands. The number 18 cannot be replaced by a smaller number, as is shown by the example in which all of the circles have radius 1 and the centers are at the following points in a polar coordinate system: the origin, the points $(1, h\cdot 60°)$ where $h=0, 1, \cdots, 5$, and the points $(2\cos 15°, (2k+1)\cdot 15°)$ where $k=0, 1, \cdots, 11$.

We prove Theorem 1 by establishing its equivalence with Theorem 2 and then proving the latter.

THEOREM 2. *It is impossible to have* 20 *points in* a circle of radius* 2 *such that one of the points is at the center and all of the mutual distances are at least* 1.

Naturally one can ask the general question: For any positive integer n, what is the radius $r(n)$ of the smallest† circle containing n points one of which is at the center and all the mutual distances between which are at least 1? Thus Theorem 2 says that $r(20)>2$. A related question is the following: Of all sets of n points in the plane such that the mutual distances are all at least 1 (with no restriction on the arrangement of the points) what set has the minimum diameter‡ $D(n)$? The following theorem answers this question for $n=7$.

THEOREM 3. *A set of seven points in the plane whose mutual distances are all at least* 1 *has diameter at least* 2, *with this value being attained only by the set of points consisting of the vertices and circumcenter of a regular hexagon of side-length* 1.

The asymptotic behavior of $r(n)$ and $D(n)$ is well-known§; in fact, for large n the regular hexagonal lattice gives about the best results, so that $D(n)\sim 2r(n)\sim(12/\pi^2)^{1/4}n^{1/2}$ as $n\to\infty$. However we are interested here in small values of n.

* The term "in" is supposed to include the boundary.

† It is not difficult to see that the greatest lower bound is attained here.

‡ The diameter of a set of points is the least upper bound of all the mutual distances. For a finite set this is simply the greatest mutual distance. Also note that the diameter of a polygon is equal to the diameter of the point-set consisting of the vertices.

§ See [3], [4], [5], [6].

For each n the values of $r(n)$ and $D(n)$ can be calculated to any desired degree of accuracy by constructing a sufficiently fine mesh, but the proofs of our theorems will show that to accomplish this practically is not easy.

The first few values of $r(n)$ are as follows: $r(2)=r(3)=\cdots=r(7)=1$, $r(8)=\frac{1}{2}$ cosec $(180°/7)=1.15\cdots$, $r(9)=\frac{1}{2}$ cosec $22°.5=1.30\cdots$, $r(10)=\frac{1}{2}$ cosec $20°=1.46\cdots$, $r(11)=\frac{1}{2}$ cosec $18°=1.61\cdots$, $r(12)=1.68\cdots$, and $r(13)=\sqrt{3}=1.73\cdots$. The evaluation of $r(8)$ through $r(11)$ will come out in the proof of Theorem 2; the evaluation of $r(12)$ and $r(13)$ can be effected by similar methods. Further, the example before the statement of Theorem 2 shows that $r(19)\leqq 2\cos 15°=1.93\cdots$. Also $r(20)>2$.

The first few values of $D(n)$ are also easy to find: $D(2)=D(3)=1$, $D(4)=\sqrt{2}=1.41\cdots$, $D(5)=\frac{1}{2}(1+\sqrt{5})=1.61\cdots$, $D(6)=2\sin 72°=1.90\cdots$. Further, Theorem 3 gives $D(7)=2$. For $n=3, 4, 5$ the vertices of the regular n-gon of side-length 1 give the minimum diameter; for $n=6$ the best result is given by the set of points consisting of the vertices and circumcenter of a regular pentagon of circumradius 1.

2. Proof that Theorem 1 is equivalent to Theorem 2. To show that Theorem 1 implies Theorem 2 we proceed as follows. Suppose we have k points in a circle of radius 2 such that one of the points is at the center and all the mutual distances are at least 1. If we construct a circle of unit radius about each point and then apply Theorem 1 to this set of k circles, with U as the circle around the central point, we get $k\leqq 19$.

In showing that Theorem 2 implies Theorem 1 (and, in fact, in proving Theorem 2) we use a polar coordinate system with radial distance ρ and amplitude θ. The set of points such that $\rho\leqq t$ we denote by $C(t)$. Now we may assume that the U of Theorem 1 is the unit circle of our polar coordinate system. Thus we have a set Δ of $k-1$ circles of radius at least 1 each one of which meets the the unit circle U, the center of no one of these k circles (including U) lying in the interior of another. Then it suffices to show that we can construct a set Δ^* of $k-1$ points in $C(2)$ whose mutual distances and distances from the origin are all at least 1. We do this by choosing a point for Δ^* corresponding to each circle D of Δ in the following way: if the center of D lies in $C(2)$, we pick the center; if the center X of D lies outside of $C(2)$, we pick the point R lying on the circle $\rho=2$ and having the same amplitude as X. As a result of this correspondence the circle of radius 1 about a point Q of Δ^* is contained in the corresponding circle of Δ; hence Q is at distance at least 1 from the points of Δ^* which were originally centers of circles of Δ. Thus it remains only to prove that if two circles of Δ have centers X and Y both of which lie outside of $C(2)$, then the corresponding points R and S of Δ^* have mutual distance at least 1. Let $OX=x$, $OY=y$, angle $XOY=\psi$. Then by the properties of Δ we have $\overline{XY}^2\geqq\max\{(x-1)^2, (y-1)^2\}$; that is,

$$x^2 + y^2 - 2xy\cos\psi \geqq \max\{(x-1)^2, (y-1)^2\}.$$

Now if we suppose that $y\geqq x$ we have

$$\cos\psi \leqq \frac{x^2+y^2-(y-1)^2}{2xy} = \frac{1}{x} + \frac{x^2-1}{2xy}$$
$$\leqq \frac{1}{x} + \frac{x^2-1}{2x^2} = 1 - \frac{1}{2}\left(1-\frac{1}{x}\right)^2 \leqq \frac{7}{8},$$

since $x>2$. Therefore $\overline{RS}^2 = 8-8\cos\psi \geqq 1$. Thus the points chosen for Δ^* have the desired properties and our proof of equivalence is finished.

3. Proof of Theorem 2. The proof is based upon the following lemmas, of which the first is basic and the others are simple corollaries thereof.

LEMMA 1. *Let r and R be such that $0<R-1\leqq r\leqq R$ and suppose that we have two points P and Q which lie in the annulus $r\leqq\rho\leqq R$ and which have mutual distance at least* 1. *Then the minimum $\phi(r, R)$ of the angle POQ has the following values*

$$\phi(r, R) = \arccos\frac{R^2+r^2-1}{2Rr}, \qquad \textit{if } R-1\leqq r\leqq R-\frac{1}{R};$$

$$\phi(r, R) = \arccos\left(1-\frac{1}{2R^2}\right) = 2\arcsin\frac{1}{2R}, \qquad \textit{if } R-\frac{1}{R}\leqq r\leqq R.$$

In particular

$\phi(1, 1.10) > 54\overset{\circ}{.}0$	$\phi(1.10, 2) > 16\overset{\circ}{.}8$
$\phi(1, 1.15) > 51\overset{\circ}{.}5$	$\phi(1.15, 2) > 20\overset{\circ}{.}0$
$\phi(1, 1.20) > 49\overset{\circ}{.}2$	$\phi(1.20, 2) > 22\overset{\circ}{.}3$
$\phi(1, 1.25) > 47\overset{\circ}{.}1$	$\phi(1.25, 2) > 24\overset{\circ}{.}1$
$\phi(1, 1.30) > 45\overset{\circ}{.}2$	$\phi(1.30, 2) > 25\overset{\circ}{.}5$
$\phi(1, 1.45) > 40\overset{\circ}{.}3$	$\phi(1.45, 2) > 28\overset{\circ}{.}3$
$\phi(1, 1.60) > 36\overset{\circ}{.}4$	$\phi(1.60, 2) > 28\overset{\circ}{.}9$

Proof. It suffices to consider the case in which $OQ=R$ and $PQ=1$. If we put $OP=\rho$, then our problem is to find the minimum of $f(\rho)=$ angle POQ $=\arccos\{(R^2+\rho^2-1)/(2R\rho)\}$ for ρ in the interval $r\leqq\rho\leqq R$. By differential calculus we see that although $f(\rho)$ may have an interior maximum for $\rho=(R^2-1)^{1/2}$, it cannot have an interior minimum. If we compare $f(r)=\arccos\{(R^2+r^2-1)/(2Rr)\}$ and $f(R)=\arccos\{1-1/(2R^2)\}$, we see that when $R-1\leqq r\leqq R-1/R$ the minimum is achieved for $OQ=R$, $PQ=1$, and $OP=r$, while when $R-1/R\leqq r\leqq R$ the minimum is achieved for $OQ=R$, $PQ=1$, and $OP=R$.

We shall find it convenient in what follows to use the term admissible points to refer to a set of points in $1\leqq\rho\leqq 2$ whose mutual distances are all at least 1.

LEMMA 2. *There are at most* 12 *admissible points in the annulus* $1.45 \leqq \rho \leqq 2$.

Proof. This follows from the fact that $13\phi(1.45, 2) > 360°$. (Note that the constant 1.45 could be replaced by any number r such that $\phi(r, 2) > 360°/13$, for example by 1.402.)

LEMMA 3. *It is impossible to have* 7 *admissible points in* $C(1.15)$, 8 *admissible points in* $C(1.30)$, 9 *admissible points in* $C(1.45)$, *or* 10 *admissible points in* $C(1.60)$.

Proof. In fact, $7\phi(1, 1.15) > 360°$, $8\phi(1, 1.30) > 360°$, $9\phi(1, 1.45) > 360°$, $10\phi(1, 1.60) > 360°$. (Note that the constants 1.15, 1.30, 1.45, 1.60 could be replaced by any values less than $\frac{1}{2}$ cosec $(180°/7)$, $\frac{1}{2}$ cosec $22°.5$, $\frac{1}{2}$ cosec $20°$, $\frac{1}{2}$ cosec $18°$, respectively. This remark, along with the fact that the vertices of the regular k-gon of side length 1 with center at the origin do constitute a set of k admissible points in $C\{\frac{1}{2} \text{ cosec } (180°/k)\}$, provides the evaluation of $r(8)$, $r(9)$, $r(10)$, $r(11)$).

LEMMA 4. *It is impossible to have* 7 *admissible points in* $C(1.30)$ *such that* 6 *of them lie in* $C(1.10)$.

Proof. If we had 7 admissible points in $C(1.30)$, 6 of which lay in $C(1.10)$, then of the 7 angles subtended at O by pairs of consecutive points (considered in order of amplitude) 5 would be each at least $\phi(1, 1.10)$ and the other two would be each at least $\phi(1, 1.30)$. But $5\phi(1, 1.10) + 2\phi(1, 1.30) > 360°.4$.

LEMMA 5. *It is impossible to have* 8 *admissible points in* $C(1.45)$ *such that* 7 *of them lie in* $C(1.25)$. *It is impossible to have* 8 *admissible points in* $C(1.45)$ *such that* 6 *of them lie in* $C(1.15)$.

Proof. The proof is similar to that of Lemma 4. The first part follows from the fact that $6\phi(1, 1.25) + 2\phi(1, 1.45) > 363°.2$; the second from the fact that $4\phi(1, 1.15) + 4\phi(1, 1.45) > 367°.2$.

Now we come to the proof of Theorem 2. We suppose that we have 19 admissible points and seek to get a contradiction. By Lemma 2 there are at most 12 admissible points outside $C(1.45)$ and by Lemma 3 at most 8 admissible points in $C(1.45)$. Thus there are two cases to consider, according to whether we have 7 admissible points in $C(1.45)$ and 12 admissible points outside, or 8 admissible points in $C(1.45)$ and 11 admissible points outside. The first case we subdivide further, according to whether one of the 7 points lies outside of $C(1.30)$ or not.

Case Ia: 12 admissible points $B_1, \cdots, B_{12}$ outside of $C(1.45)$, 7 admissible points $A_1, \cdots, A_7$ in $C(1.45)$, one of the A_i, say A_k, outside of $C(1.30)$. All the angles B_iOB_{i+1} exceed $\phi(1.45, 2) > 28°.3$. (We mean to include the angle $B_{12}OB_1$, of course; similarly in the sequel.) But for any i the angle A_kOB_i exceeds $\phi(1.30, 2) > 25°.5$ and so one of the angles B_iOB_{i+1} exceeds $2\phi(1.30, 2) > 51°.0$. However $11\ (28°.3) + 51°.0 = 362°.3$, a contradiction.

Case Ib: 12 admissible points $B_1, \cdots, B_{12}$ outside of $C(1.45)$, 7 admissible points $A_1, \cdots, A_7$ in $C(1.30)$. By Lemma 3 one of the A_i, say A_k, lies outside of $C(1.15)$ and by Lemma 4 another one of the A_i, say A_m, lies outside of $C(1.10)$. For each i the angle A_kOB_i exceeds $\phi(1.15, 2)>20°.0$ and the angle A_mOB_i exceeds $\phi(1.10, 2)>16°.8$. Hence an angular sector of more than $40°.0$ and another sector of more than $33°.6$ are ruled out as possible locations for points B_i. These sectors cannot overlap, since the angle A_kOA_m is at least $\phi(1, 1.30)>45°.2$. If one of the angles B_iOB_{i+1}, say B_nOB_{n+1}, includes both these sectors, then B_nOB_{n+1} exceeds $40°.0+33°.6=73°.6$ and we have a contradiction, since $11\phi(1.45, 2)+73°.6>384°.9$. Otherwise we see that out of the 12 angles B_iOB_{i+1} one exceeds $40°.0$ and another exceeds $33°.6$. Since at most 9 admissible points can lie in $C(1.60)$, at least 10 of the 12 points B_i lie outside $C(1.60)$. Hence at least 8 of the 12 angles B_iOB_{i+1} exceed $\phi(1.60, 2)>28°.9$. But $40°.0+33°.6+6(28°.9)+4\phi(1.45, 2)>360°.2$, a contradiction.

Case II: 11 admissible points $B_1, \cdots, B_{11}$ outside $C(1.45)$, 8 admissible point $A_1, \cdots, A_8$ in $C(1.45)$. By Lemma 3 one of the A_i, say A_k lies outside $C(1.30)$; by Lemma 5 a second one of the A_i, say A_m, lies outside of $C(1.25)$ and a third one of the A_i, say A_n, lies outside of $C(1.15)$.

Now for each i the angle B_iOA_k exceeds $\phi(1.30, 2)>25°.5$, the angle B_iOA_m exceeds $\phi(1.25, 2)>24°.1$, and the angle B_iOA_n exceeds $\phi(1.15, 2)>20°.0$. Hence three angular sectors of more than $51°.0$. $48°.2$, and $40°.0$, respectively, are ruled out as possible locations for the points B_i. If one of the angles B_iOB_{i+1}, say B_pOB_{p+1}, includes two of these sectors, then B_pOB_{p+1} exceeds $20°.0+24°.1+40°.3=84°.4$, since A_kOA_m, A_mOA_n, and A_nOA_k are each at least $\phi(1, 1.45)>40°.3$; this gives a contradiction, since $10\phi(1.45, 2)+84°.4>367°.4$.

On the other hand if no angle B_iOB_{i+1} includes two of the proscribed sectors, then we see that out of the 12 angles B_iOB_{i+1} one exceeds $51°.0$, another exceeds $48°.2$, another exceeds $40°.0$, and each of the 8 remaining exceeds $\phi(1.45, 2)>28°.3$. But $51°.0+48°.2+40°.0+8(28°.3)=365°.6$, a contradiction.

4. Proof of Theorem 3. We require the following lemmas, of which the first three are well-known.

LEMMA 6. *The area of a triangle does not exceed* $\frac{1}{4}\sqrt{3}$ *times the square of the longest side.*

LEMMA 7. *The product of the diagonals of a quadrilateral is at least twice the area.*

LEMMA 8. *If a convex polygon has perimeter not less than* 2π, *its diameter exceeds* 2.

LEMMA 9. *If there is a point in a triangle whose distance from each vertex is at least* 1, *then some side of the triangle has length at least* $\sqrt{3}$.

Proof. One of the sides of the triangle must subtend an angle of 120° or more at the point in question.

LEMMA 10. *If* $0<\xi\leqq\frac{1}{3}\pi$ *and* $ABCD$ *is a convex quadrilateral with* $\sphericalangle ABC=\frac{2}{3}\pi+\xi$, $\sphericalangle BCD\geqq\frac{2}{3}\pi-\frac{2}{3}\xi$, *and* $1\leqq AB, BC, CD\leqq 2$, *then* $AD>2$.

Proof. Put $AB=x$, $BC=y$, $CD=z$, $\sphericalangle BCD=\theta$. Then

$$\begin{aligned}\overline{AD}^2 &= \left\{y - x\cos\left(\tfrac{2}{3}\pi+\xi\right) - z\cos\theta\right\}^2 + \left\{x\sin\left(\tfrac{2}{3}\pi+\xi\right) - z\sin\theta\right\}^2 \\ &= x^2+y^2+z^2-2yz\cos\theta+2zx\cos\left(\theta+\tfrac{2}{3}\pi+\xi\right)-2xy\cos\left(\tfrac{2}{3}\pi+\xi\right).\end{aligned}$$

Considering this expression as a function of x, y, z, θ over the domain $1\leqq x, y, z\leqq 2$, $\frac{2}{3}\pi-\frac{2}{3}\xi\leqq\theta\leqq\pi$, we see from the positivity of the partial derivatives that the smallest value of $\overline{AD}^2$ occurs for $x=y=z=1$, $\theta=\frac{2}{3}\pi-\frac{2}{3}\xi$. It suffices therefore to prove that

$$f(\xi) = 3 - 2\cos\left(\tfrac{2}{3}\pi-\tfrac{2}{3}\xi\right) + 2\cos\left(\tfrac{4}{3}\pi+\tfrac{1}{3}\xi\right) - 2\cos\left(\tfrac{2}{3}\pi+\xi\right)$$

exceeds 4 for $0<\xi\leqq\frac{1}{3}\pi$; but this follows from the fact that $f(0)=4$, $f(\frac{1}{3}\pi)=5-4\cos(4\pi/9)$, and $f''(\xi)<0$ for $0\leqq\xi\leqq\frac{1}{3}\pi$.

LEMMA 11. *If* $ABCDE$ *is a convex pentagon such that* $AB, BC, CD, DE\geqq 1$, $AE\geqq\sqrt{3}$, *and* $\sphericalangle C>120°$, *then either the diagonal* $AD>2$ *or the diagonal* $BE>2$.

Proof. If $\sphericalangle A>90°$, then $BE>2$; if $\sphericalangle E>90°$, then $AD>2$. Suppose then that neither angle adjacent to AE exceeds 90°. Then if $\sphericalangle C=120°+\xi$, we see that either $\sphericalangle B\geqq 120°-\frac{1}{2}\xi$ or $\sphericalangle D\geqq 120°-\frac{1}{2}\xi$, for otherwise the angle sum of the pentagon would be less than $90°+90°+(120°+\xi)+2(120°-\frac{1}{2}\xi)=540°$. Hence by Lemma 10 either $AD>2$ or $BE>2$.

Now we turn to the proof of Theorem 3. Suppose then that we have seven points in the plane such that the mutual distances are all at least 1. In proving that the diameter of the set of seven points is at least 2, we consider five cases, according as the convex hull of the seven points is a triangle, quadrilateral, pentagon, hexagon, or heptagon. As stated in the theorem we shall find that the diameter actually exceeds 2 throughout, except for the subcase of the hexagonal case in which the angles of the hexagon are all 120°, the sides all have length 1, and the seventh point is at the center.

The triangular case. Suppose that a circle of radius $\frac{1}{2}$ is drawn about each of the seven points as center. Then no two of these circles can properly intersect. If one side of the triangle is intersected by the circles around two of the four inner points, then that side has length at least $3\sqrt{3}/2$ by the Pythagorean theorem. If no side of the triangle is intersected by two of the circles about inner points, then the area of the triangle is greater than the area of three circles of radius $\frac{1}{2}$, namely $3\pi/4>\sqrt{3}$; thus by Lemma 6 at least one side of the triangle has length greater than 2.

The quadrilateral case. Again we construct a circle of radius $\frac{1}{2}$ about each of the seven points. If one side of the quadrilateral is intersected by the circles about two of the three inner points, then that side has length at least $3\sqrt{3}/2>2$. Suppose then that no side is intersected by more than one of the circles about

the inner points. If just one or two sides are intersected by inner circles, then the area of the quadrilateral is greater than the area of three circles of radius $\frac{1}{2}$, namely $3\pi/4>2$, and thus by Lemma 7 at least one of the diagonals of the quadrilateral has length greater than 2.

Now notice that if a side of the quadrilateral is intersected by the circle around one of the inner points, then that side has length at least $\sqrt{3}$ by the Pythagorean theorem. If three sides are intersected by inner circles and the quadrilateral is a rectangle, then all sides of the rectangle have length at least $\sqrt{3}$ and hence the diagonals have length at least $\sqrt{6}$. If three sides are intersected by the circles about inner points and one of the angles of the quadrilateral exceeds 90°, then at least one of the sides adjoining that angle has length $\sqrt{3}$ or greater, while the other side adjoining it has length at least 1; hence (by the law of cosines) the diagonal spanning this angle has length greater than 2.

The pentagonal case. Again if the circles of radius $\frac{1}{2}$ about the two inner points intersect the same side of the pentagon, then that side has length at least $3\sqrt{3}/2$. On the other hand if these two circles intersect two different sides of the pentagon, then both these sides have lengths at least $\sqrt{3}$, the pentagon has perimeter at least $3+2\sqrt{3}>2\pi$, and thus by Lemma 8 the diameter of the pentagon exceeds 2.

Thus we assume that at least one of the inner points, say K, has a distance from the perimeter of the pentagon greater than $\frac{1}{2}$. Then the other inner point, say L, must lie in one of the triangles formed by K and consecutive vertices of the pentagon, say the triangle KAB. By Lemma 9 one of the sides of the triangle KAB has length at least $\sqrt{3}$. If either AK or BK has length $\sqrt{3}$ or more, the diameter of the pentagon exceeds $\sqrt{3}+\frac{1}{2}>2$. We suppose then that $AB \geqq \sqrt{3}$. Since one of the angles ALK and BLK is at least 90°, either $AK\geqq\sqrt{2}$ or $BK\geqq\sqrt{2}$. Suppose $AK\geqq\sqrt{2}$. Then prolong AK until it meets the pentagon again at a point X. If $AX>2$, we are finished. If $AX\leqq 2$, then $KX\leqq 2-\sqrt{2}$, the side on which X lies has length at least $2\{1-(2-\sqrt{2})^2\}^{1/2}>1.62$, the pentagon has perimeter greater than $3+1.73+1.62=6.35>2\pi$, and thus the pentagon has diameter more than 2 by Lemma 8.

The heptagonal case. This case is settled immediately by Lemma 8. Actually by pushing our methods further it is possible to prove that of all convex heptagons with sides of length at least 1 the minimum diameter occurs for the regular heptagon of side-length 1, in which case it is $1+2\cos(2\pi/7)=2.24\cdots$.

The hexagonal case. Dismissing the case in which a side of the hexagon has length greater than 2, we separate the proof into cases according to the number and arrangement of those angles of the hexagon which exceed 120°. First of all we note that if two adjacent angles of the hexagon exceed 120°, the diagonal spanning these two angles has length greater than 2. This case occurs, for example, if there are four or five angles of the hexagon exceeding 120°.

If exactly one angle of the hexagon exceeds 120°, say is $120°+\xi$, then one of the angles adjacent to it is at least $120°-\frac{1}{2}\xi$; the proof is completed in this case by Lemma 10. If exactly two angles of the hexagon exceed 120°, say have the

values $120°+\alpha$ and $120°+\beta$, and these two angles are non-adjacent, then somewhere in the hexagon we have a pair of adjacent angles of which one is $120°+\xi$, $\xi>0$, and the other is at least $120°-\frac{2}{3}\xi$; for otherwise the pentagon would have an angle sum less than $(120°+\alpha)+(120°+\beta)+(120°-\frac{2}{3}\alpha)+(120°-\frac{2}{3}\beta)+(120°-\frac{2}{3}\max\{\alpha, \beta\})+120°\leqq 720°$. Again the proof is completed by Lemma 10.

In case that exactly three of the angles of the hexagon exceed 120° and no two of these are consecutive, let A, B, C, D, E, F be the vertices of the hexagon and let A, C, E be those at which the angles exceed 120°. First we remark that if one of the sides of the hexagon has length greater than $\frac{1}{2}(\sqrt{13}-1)$, we have a diagonal of length greater than 2 by the law of cosines; thus we may assume all sides of the hexagon to have length not exceeding $\frac{1}{2}(\sqrt{13}-1)<\sqrt{3}$. The three diagonals AC, CE, EA divide the hexagon up into four triangles. One of these four triangles contains the seventh point and hence one of the three diagonals AC, CE, EA has length at least $\sqrt{3}$ by Lemma 9. Suppose $EA\geqq\sqrt{3}$. Then by applying Lemma 11 to the pentagon $ABCDE$ we find a diagonal of length greater than 2.

The only case left is that in which no angle of the hexagon exceeds 120°, *i.e.*, all angles are 120°. We easily see that every diagonal spanning two angles has length at least 2 and that the diameter is exactly 2 if and only if all the sides of the hexagon have length 1 (with the seventh point at the center naturally).

5. Further questions. Another function which we could consider is the diameter $d(n)$ of the smallest circle containing n points whose mutual distances are all at least 1, without the restriction that one point be at the center. Obviously $D(n)\leqq d(n)\leqq 2r(n)$ and the same general remarks that were made about $D(n)$ and $r(n)$ could be made about $d(n)$. The first few values of $d(n)$ are $d(2)=1$, $d(3)=2/\sqrt{3}=1.15\cdots$, $d(4)=\sqrt{2}=1.41\cdots$, $d(5)=\operatorname{cosec} 36°=1.70\cdots$, $d(6)=2$, $d(7)=2$.

Naturally we can consider the analogue of $d(n)$ for other figures than the circle, for example, the side-length $t(n)$ of the smallest equilateral triangle containing n points whose mutual distances are all at least 1. An interesting unsolved question about $t(n)$ is whether or not $t(\frac{1}{2}k\{k+1\}+1)>k-1$ for k a positive integer. Obviously $t(\frac{1}{2}k\{k+1\})\leqq k-1$, since the regular hexagonal lattice gives $\frac{1}{2}k\{k+1\}$ admissible points in a triangle of side-length $k-1$.

In the proof that $D(7)=2$ we remarked that of all convex heptagons whose sides have length 1 or more the minimum diameter is assumed by the regular heptagon of side-length 1. An analogous statement can be made for triangles, quadrilaterals, and pentagons. However for hexagons the minimum diameter is assumed by the equilateral hexagon of side-length 1 whose angles are alternately 90° and 150°. The situation for n-gons with $n>7$ is an open question.

Another interesting problem is whether or not $r(n+1)>r(n)$ for $n\geqq 7$ and whether $D(n+1)>D(n)$ for $n\geqq 3$. It follows from Theorem 3 that $D(8)>D(7)$

$=2$, since the diameter of a set of seven points (with mutual distances 1 or more) is greater than 2 except for one special configuration (for which we can make a separate argument).

Of course we can consider the analogues of $r(n)$ and $D(n)$ in k dimensions, say $r_k(n)$ and $D_k(n)$. Clearly $D_k(n)=1$ for $n \leqq k+1$. It would be interesting to have a good estimate for $D_k(k+2)$ from below. It is probably true that $D_k(k+2) \geqq \sqrt{4/3}$, but this seems difficult to prove. On the other side it is easy to give an example to show that $D_k(k+2) \leqq \sqrt{4k/(3k-2)}$. As for $r_k(n)$, we do not even know the smallest value of n such that $r_k(n)>1$, except for $k=2$. For $n \to \infty$ there are asymptotic relations of the form (cf. [2])

$$\tfrac{1}{2}D_k(n) \sim r_k(n) \sim c_k n^{1/k},$$

but c_k is not known for $k>2$.

6. Added in Proof: After this paper was submitted for publication, a proof of our Theorems 1 and 2 was published by E. F. Reifenberg, Math. Gaz., vol. 32, 1948, pp. 290–292. His proof is in general similar to ours, although considerably different in detail.

The problem mentioned briefly in the third paragraph of §5 has been discussed recently in a paper by S. Vincze, Acta Univ. Szeged. (Sect. Sci. Math.) vol. 12, part A, 1950, pp. 136–142. For n not a power of 2 he shows that the minimum diameter possible for convex n-gons whose sides have length at least 1 is $\frac{1}{2}\operatorname{cosec}(\pi/2n)$. In particular for even n having at least one odd prime factor the regular n-gon of side length 1 does not give the minimal diameter. Vincze points out that a closely related problem was considered earlier by K. Reinhardt, Jber. Deutsch. Math. Verein., vol. 31, 1922, pp. 251–270.

References

1. A. S. Besicovitch, A general form of the covering principle and relative differentiation of additive functions, Proc. Cambridge Philos. Soc., vol. 41, 1945, pp. 103–110.

2. R. A. Rankin, On the closest packing of spheres in n dimensions, Ann. of Math. (2), vol. 48, 1947, pp. 1062–1081.

3. B. Segre and K. Mahler, On the densest packing of circles, This Monthly, vol. 51, 1944, pp. 261–270.

4. A. Thue, Über die dichteste Zusammenstellung von kongruenten Kreisen in einer Ebene, Christiana Videnskabs-Selskabets Skrifter 1910, no. 1.

5. L. Fejes (Tóth), Extremal distributions of points in the plane, on the surface of the sphere and in space, Acta Sci. Math. Nat. (Univ. Francisco-Josephina, Kolozsvar) no. 23, 1944, iv+55 pp.

6. L. Fejes (Tóth), Über einem geometrischen Satz, Math. Zeit., vol. 46, 1940, pp. 83–85.

AREA IN NON-EUCLIDEAN GEOMETRY

KENNETH LEISENRING, University of Michigan

1. Introduction. Because of its origin in connection with the problem of parallelism, hyperbolic geometry, the non-Euclidean geometry of Lobatchewsky and Bolyai, has from the beginning been treated in as close analogy as possible with Euclidean; particularly in synthetic discussions. However, it has long been known that hyperbolic geometry is perfectly realized on a sphere of imaginary radius. The writer believes that the key to a quick intuitive grasp of this geometry is to minimize the analogy with the Euclidean plane, and to stress instead the analogy with the real sphere.

On the sphere (or in the "elliptic plane," which may be regarded as the sphere with opposite points abstractly identified) any two "lines" both intersect and have a unique common perpendicular. In the hyperbolic plane line pairs either intersect or have a unique common perpendicular, "parallels" being the singular, intermediate case. On the sphere the common normal of two lines is greater than any other distance to one from a point of the other, while in the hyperbolic plane the common normal of two "ultra-parallels" is the shortest distance between them, and away from it in either direction the lines diverge indefinitely. Parallels are asymptotic.

The angular excess of a spherical triangle is replaced, in the hyperbolic plane, by angular deficiency. The theorem that the defect of the sum of two polygons is the sum of their defects is proved precisely as in the case of the excess. As regards area, however, there is an important difference, for the hyperbolic plane is infinite in extent. This means that the fundamental theorem, that area is proportional to deficiency, must be handled differently.

The customary treatment is in part remarkably like that of Euclid, an essential procedure being to dissect the triangle in such a way as to show it equivalent to a quadrilateral with two right angles. This achieves the object neatly enough, however, and particularly as handled by Liebmann [1] seems entirely satisfactory.* But the same argument as treated by Carslaw [2] and others [3], and outlined by Wolfe in his recent textbook [4], seems to the writer less acceptable. These writers introduce the equivalence relation used by Hilbert [5] in his *Foundations of Geometry* to relate area to congruence. This relation makes two polygons equivalent if they can be partitioned into the same finite number of triangles, congruent in pairs. For Hilbert this relation serves the useful purpose of economizing on axioms, but by itself it is not an altogether suitable basis for a theory of area, for the area concept does not inherently require such a strong equivalence. The fact that polygons of equal area can be thus finitely partitioned has no more *a priori* force than its analogue in three dimensions for equal polyhedral volumes, which, as is well known, does not hold.

The present paper employs a more direct axiomatic approach. It also puts to use the analogy with the elliptic case, this being our chief motivation. There

* However this argument requires the use of equidistant curves.

is a further point of interest in the use that is made of a class of polygons most characteristic of the non-Euclidean planes, but frequently neglected, the orthogons—polygons all of whose angles are right. In the elliptic plane ortho-n-gons exist for $n=2$ and $n=3$; in the hyperbolic plane for $n \geqq 5$. The hyperbolic plane can be tessellated with duplicates of an arbitrary ortho-n-gon, $n \geqq 5$, without the requirement of regularity.

What makes the area problem so simple on the sphere is the finiteness of the total area and the existence of a triangle one angle of which is proportional to its area, the triangle with two right angles. The figure corresponding to this in the hyperbolic plane is not a triangle but a quadrilateral with three right angles. The area is not proportional to the remaining acute angle, but is proportional to its complement. This is the key theorem in the argument which follows; the fundamental theorem for the triangle is a simple consequence.

2. The area function. By the term "area" we shall mean here any function μ satisfying the following postulates:

1. The function μ is defined for every simple closed polygon.
2. The function μ is a real number, positive and finite.
3. If the simple closed polygons s and t are congruent, then $\mu(s)=\mu(t)$.
4. If the simple closed polygon w is the sum of two others, s and t, then $\mu(w)=\mu(s)+\mu(t)$.

For such a function we shall use the conventional terminology. We shall say that a polygon has "an area," one polygon has "greater area" than another, *etc.*

In the Euclidean plane there exists no intrinsic function μ. Euclid, in effect, assumed the existence of such functions and then showed that any two can differ only by a factor of proportionality. In the non-Euclidean geometries the situation is different: angular excess in the elliptic plane, and defect in the hyperbolic, satisfy these postulates—the argument being familiar. What remains to be shown, in each case, is that *any* function satisfying them is proportional to the excess or defect. We are concerned with the hyperbolic plane only. To be completely explicit: writing $d(s)$ for the angular defect of s, we have to show that for any function μ, we can write $\mu(s)=k^2d(s)$. Besides the existence of the angular defect, we shall use the axiom of Archimedes, and the basic theorems on lines, such as the uniqueness of the common normal of two ultra-parallels, and the theorem that away from their common normal ultra-parallels diverge.

3. The area theorems.

THEOREM 1. *If two trirectangles have equal acute angles, then they have equal area.*

Proof: Let the two trirectangles be $t=ABCD$ and $t'=KLMN$, the equal acute angles being A and K, and let us suppose $\mu(t) \geqq \mu(t')$.

Let the figures be superimposed so that KL falls along AB and KN along AD. (See Figure 1.) If L now coincides with B, then M must coincide with C,

for CD is the unique common normal of the ultra-parallels AD and BC. The trirectangles are thus congruent and have the same area by Postulate 3. If L falls between A and B, then D will lie between A and N. For, the only alternative is that N lie between A and D; but then M would lie within $ABCD$, and the hexagon $LBCDNM$ would have no angular defect.

Let us suppose that L does fall between A and B, and let LB and CD intersect in E. We see that in this case t and t' are simultaneously decomposed into new quadrilaterals: a birectangular part in which they overlap, and new trirectangles, BE and EN, which have equal acute angles at E.

These new trirectangles can be superimposed in the same way and the procedure can be carried on indefinitely, unless brought to a stop after a finite number of steps by the coincidence of the last pair of trirectangles. If it is thus brought to a stop we have $\mu(t)=\mu(t')$ by Postulates 3 and 4.

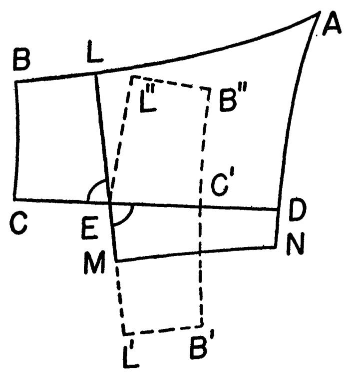


FIG. 1

To obtain a proof when the process is infinite, we must be more precise about the manner in which the successive superpositions are carried out. We shall arrange it in such a way that the process falls into distinct stages, each of which contains a finite number of superpositions. At the end of each stage it will be possible to show that the trirectangles which remain are each less than half the trirectangles with which that stage began.

Suppose in the initial superposition $EC<ED$ (Figure 1), and let the trirectangle EB be rotated through a straight angle about E to the position EB'. Let EB' be reflected in ED, giving rise to the trirectangle $EL''B''C'$. This reflection lies wholly within $ALED$. Otherwise EL'' or $C'B''$ must have a point in common with AL, which is impossible. $EL''=EL$, and the perpendicular is shorter than any oblique segment from a point to a line. Also, by the divergence of ultraparallels, mentioned above, $C'B''=CB$ is shorter than any other perpendicular to CD from a point of AL.

By Postulate 4, $EL''B''C'$ has a smaller area than $ALED$. And since, by assumption, $\mu(EB)\geqq\mu(EN)$, we have $\mu(BE)<1/2\cdot\mu(t)$ and $\mu(EN)<1/2\cdot\mu(t')$.

This is the goal of the first stage. We now let $BE=t_1$ and $EN=t_1'$, and proceed in the same way with the second stage.

Suppose, on the other hand, $EC>ED$ in the initial superposition. In this case, the first stage must contain further superpositions, as indicated in Figure 2. Upon rotation of BE about E, the vertex E_1 of the new acute angles will fall on AN.

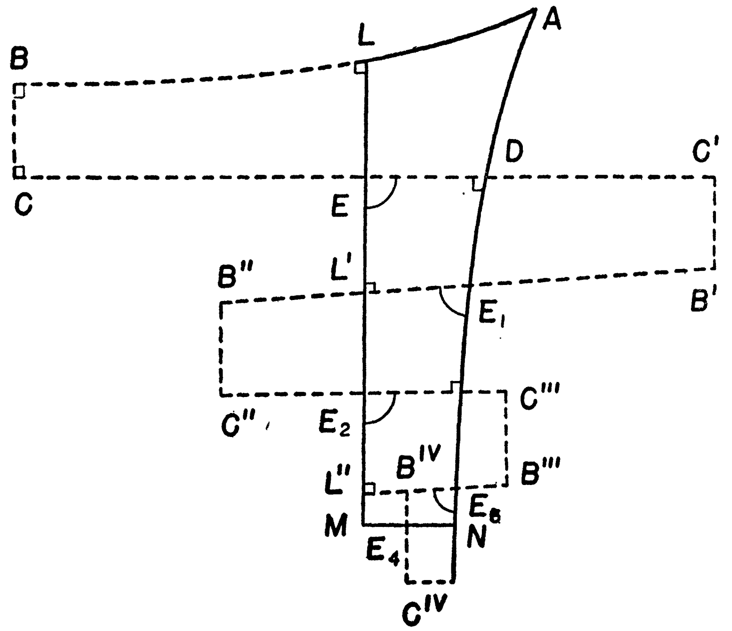


FIG. 2

We proceed to rotate E_1C' about E_1, and compare E_1B' with E_1L'. If $E_1B'>E_1L'$, as in the figure shown, we rotate again about E_2, *etc.* At each step in this process the sides of the successive trirectangles which are adjacent to BC, $B'C'$, *etc.*, are diminished by an amount greater than MN. It follows from the axiom of Archimedes that the sides will become in a finite number of steps less than MN. If we consider the various possible resulting configurations and argue from the angular defect as in the first paragraph of this proof, we can rule out all alternatives but that, for some k we shall find E_{k+1} on MN. By the same reasoning as before it will follow that $\mu(E_kC^k)<\mu(E_kE_{k-1})<\mu(E_kL)$. And since $\mu(E_kM)<\mu(E_kC^k)$ we have again reached our goal. We let $E_kC^k=t_1$, and $E_kM=t_1'$. After n stages, we can write for the difference of the original areas:

$$\mu(t) - \mu(t') = \mu(t_n) - \mu(t_n') < \mu(t_n) < 1/2\cdot\mu(t_{n-1}) < 1/2^n\cdot\mu(t).$$

The fixed numbers $\mu(t)$ and $\mu(t')$ therefore differ by as little as we please and must be equal.

For later reference, we state the following obvious corollary:

COROLLARY 1a. *In a variable trirectangle of fixed acute angle, either side adjacent to the acute angle can be arbitrarily increased without altering the area.*

(Note that the side in question cannot be arbitrarily diminished, since the adjacent sides must remain ultra-parallel.)

THEOREM 2. *If the acute angle of one trirectangle is greater than that of another, then its area is less.*

Proof: Let the trirectangle with the greater acute angle be $ABCD$, the acute angle being at A. Consider the side AB and one of the sides adjacent to the acute angle in the other trirectangle. Let them be made equal by extending the shorter one by Corollary 1a, and let these equal sides be superimposed as in Figure 3, where the second trirectangle is represented by $ABRS$. Segment BR

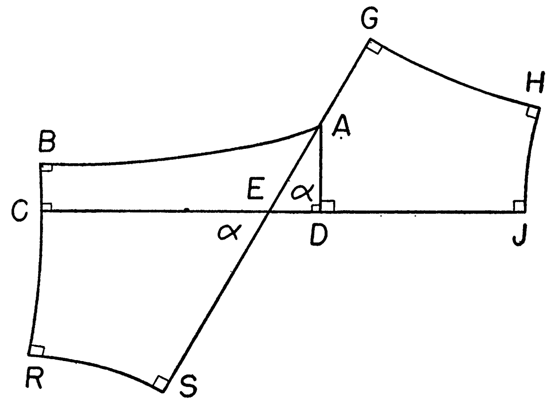


FIG. 3

will then be longer than BC. Let AS meet CD at E and call the vertical acute angles α. On EA let segment EG be marked off such that $EG > EA$ and also greater than the segment whose angle of parallelism is α. Let GH be perpendicular to EG. The line GH is then ultra-parallel to CD; let the common normal be HJ. Segment EJ is then definitely longer than ED. By Theorem 1 the trirectangles $ECRS$ and $EGHJ$ have the same area. Since triangle EAD is a proper part of $EGHJ$, the theorem follows.

The following theorem is now evidently true:

THEOREM 3. *If two trirectangles have the same area they have the same acute angle.*

Thus the area of a trirectangle is a monotonic decreasing function of its acute angle, and a monotonic increasing function of its angular defect, which is just the complement of the acute angle. It remains to show that the latter function is a proportionality.

THEOREM 4. *All orthopentagons have the same area.*

Proof: If two orthopentagons have a side of one equal to a side of the other they can be superimposed, and the conclusion then follows by Theorem 1. But any side of an orthopentagon can be arbitrarily increased without changing the area; hence, the superposition can always be carried out.

Remark: For the area of an orthopentagon we shall use the letter U.

COROLLARY 4a. *The area of an ortho-n-gon is* $(n-4)U$.

Proof: If s_1, s_2, s_3, s_4 are four consecutive sides of any ortho-n-gon, then s_1 and s_4 are ultra-parallel. Their common normal lies inside the figure, and is the side opposite s_2 and s_3 of an orthopentagon. Removing from the figure this orthopentagon leaves an ortho-$(n-1)$-gon and diminishes the area by U.

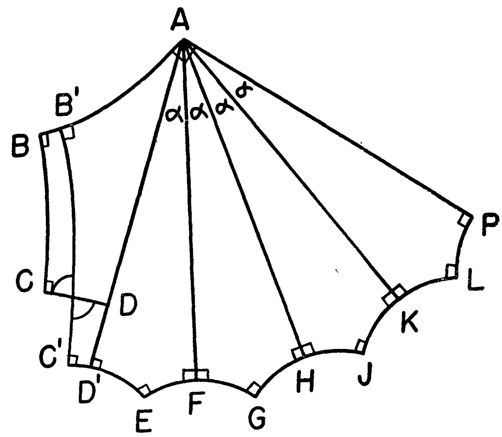


FIG. 4

THEOREM 5. *The area of a trirectangle is proportional to its angular defect.*

Proof: It is sufficient to show that if the ratio of the defects of two given trirectangles is m/n then this is also the ratio of the areas.

Let the trirectangles be t and t', and let $d(t)/d(t')=m/n$. Let $t=ABCD$, with the acute angle at A. Let AP be perpendicular to AB, as in Figure 4. The defect of t is then angle DAP. Let that angle be divided into m equal parts by the lines AF, AH, AK, *etc.*, the equal angles having the value α. Now if side AD is not greater than the segment whose angle of parallelism is α, let it be extended by Corollary 1a, so that $C'D'$ is ultra-parallel to AF. Let the common normal be EF; EF is then ultra-parallel to AH, by the equality of the angles α, and similarly GH to AK, *etc.* The resulting figure, represented in our figure by $AB'C'EGJLP$, is an ortho-$(m+4)$-gon whose area, by Corollary 4a, is mU.

If we write μ_α for the area of one of the trirectangles with acute angle α,

we have $\mu(t) = mU - m\mu_\alpha = m(U - \mu_\alpha)$. Now consider the trirectangle t'; let the complement of the acute angle be divided into n equal parts—by hypothesis these equal angles are again α. Proceeding as above, we obtain $\mu(t') = n(U - \mu_\alpha)$, which proves the theorem. For the trirectangle t we may write $\mu(t) = k^2 d(t)$.

COROLLARY 5a. *The area unit U can be written $k^2 \cdot \pi/2$.*

Proof: It is easily seen that an orthopentagon is the sum of two trirectangles whose total defect is $\pi/2$.

THEOREM 6. *The area of a triangle is proportional to its angular defect.*

Proof. Let ABC be any (proper) triangle and let the sides be extended, AB through B, BC through C, and CA through A. (See Figure 5). On these exten-

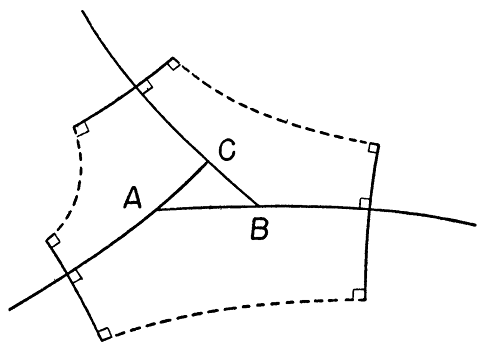


FIG. 5

sions let perpendiculars be erected, sufficiently far out to be pair-wise ultra-parallel. With the common normals, these then make an orthohexagon. Each vertex of the triangle is then the vertex of a pentagon with four right angles, a figure which is the sum of two trirectangles whose total defect is just the corresponding angle of the triangle. The area of the hexagon is $k^2\pi$. Subtracting the three pentagons, we have $\mu(\triangle ABC) = k^2(\pi - A - B - C)$.

4. The factor of proportionality. In the hyperbolic plane the factor of proportionality is strictly arbitrary; but it is natural, and customary, to make it agree with the otherwise locally Euclidean character of the plane. In a hyperbolic plane of parameter ρ the factor is then ρ^2. This can be shown in various ways; the following argument is simpler than some.

Consider an isosceles trirectangle t, the length of the sides opposite the acute angle being a. Let the complement of the acute angle be θ. By trigonometry we have then

$$\sin \theta = \sinh^2 \frac{a}{\rho} \cdot$$

The area of t is then

$$A = k^2 \arcsin\left(\sinh^2 \frac{a}{\rho}\right).$$

We want it to be true that

$$\lim_{a\to 0} A/a^2 = \lim_{a\to 0} 1/a^2 \cdot k^2 \arcsin\left(\sinh^2 \frac{a}{\rho}\right) = 1.$$

Since the sine and the hyperbolic sine both differ from their arguments by terms of the third degree and higher, we have at once $k^2 = \rho^2$.

References

1. Liebmann, Nichteuklidische Geometrie, Leipzig, First Edition, 1905. pp. 69–75. Also: Summerville, The Elements of Non-Euclidean Geometry, London, 1914, pp. 77–84.

2. Carslaw, The Elements of Non-Euclidean Plane Geometry and Trigonometry, London, 1916, pp. 84–90.

3. See A. Finzel, Mathematische Annalen, Leipzig, 1912, pp. 262–284. Also, but with a difference: H. S. M. Coxeter, Non-Euclidean Geometry, Toronto, 1942, pp. 241–251. Professor Coxeter gives an interesting alternative to the usual treatment—it is something of a tour de force, beginning with the area of a horocyclic sector, a figure both curvilinear and unbounded.

4. Wolfe, Introduction to Non-Euclidean Geometry, New York, 1945, pp. 122–127.

5. Hilbert, Foundations of Geometry, Reprint Edition, The Open Court Publishing Co., LaSalle, Illinois, 1947, pp. 66–77.

FURTHER EXPERIENCE WITH UNDERGRADUATE MATHEMATICAL RESEARCH*

F. L. GRIFFIN, Reed College

At the 1941 summer meeting of the Association, I reported briefly on the experience of sixteen colleges and universities with undergraduate mathematical research, and described more fully the experience at Reed College with the senior thesis which is required for graduation. At that time there had been 94 theses in mathematics, of which approximately 80 were of a research type in the sense that they explored some new problem or a new aspect of an old problem and got some apparently new results. Details of my report can be found in this MONTHLY for June–July, 1942.†

Since 1941 we have had 41 mathematical theses, of which 34 have been of a research type. The average level of the research seems possibly a little higher than that reported on before. In the few minutes available I shall try to suggest the range of the problems considered, to characterize a few of the papers, and to

* Read before Sec. VII, International Congress of Mathematicians, Sept. 2, 1950.

† Vol. 49, No. 6, pages 379–385.

indicate how our department views the role of the thesis in undergraduate education.

Of thirteen theses in analysis, five were on problems in the calculus of variations. One of these gave a definitive treatment of the isoperimetric brachistochrone problem. Another dealt with constrained motion on a paraboloid, under gravity, exhibited possible paths by several systems of mapping, and determined the brachistochrone on the paraboloid. Three other theses investigated the geodesics on particular surfaces, one of which is an elliptical torus and another is the surface generated by revolving a lemniscate about its principal axis. Three theses dealt with the numerical or graphical solution of the differential equation $dy/dx=f(x, y)$. One of these set up very accurate formulas, of which Runge's and Kutta's formulas are special cases. Another, which generalized the type of formula used by Milne and others, got seven-place accuracy in the test case proposed by the National Research Council's committee on numerical integration, and without much labor. Three other theses dealt with special functions. One of these defined, analyzed, and applied hypocycloidal functions analogous to the circular and hyperbolic functions. Another, on functions with recurring derivatives, studied the solutions of $d^ny/dx^n=\pm y$, obtaining many identities and investigating the solutions of a related functional equation for the case $n=3$. (It was subsequently learned that about half of the results had been found previously by Glaisher and others.) Another thesis in analysis studied an infinite sequence generated by an exponential recurrence relation. Another, on the student's own initiative, constructed an extensive function theory in a hypercomplex number field.

Of twelve theses in geometry, four dealt with special surfaces and space curves. One of these obtained various properties of three-dimensional Lissajou curves, and from some of these properties derived much information about related plane curves. It also gave examples of the classification of non-intersecting, non-degenerate Lissajou space curves as topological knots by the calculation of torsion numbers. Another thesis in this group examined analytically some metrical properties of a two-parameter family of surfaces related to the Möbius strip, and suggested a method of classifying topologically this family and its intersections with some common surfaces. Another determined the complicated centro-surface for a special hypocycloidal surface. Still another geometrical thesis analyzed some alleged methods of trisecting an angle. Three others dealt with plane transformations and correspondences. One of these investigated the geometric properties of the analytic duals of several subgroups of the group of projective transformations, examining seven groups of affine transformations, and also showing a method of dualizing metric theorems. Another thesis defined pencils of curves of degree n and order m, also the cross-ratio of four curves of such a pencil, and derived some properties of such cross-ratios. Still another constructed and studied a family of porisms relating to polygons circumscribed about a given polygon. (This thesis is part of a project now being carried on by one of the staff.)

Of two theses in algebra and number theory, one used the concept of the cycle of an integer a (mod m) to develop the problem of power residues by what appears to be a new approach. It deduced some basic properties of the cycle and applied these to certain classes of integers.

In the field of foundations and abstract logic there have been four theses. One of these derived Hilbert's axioms, except that on parallels, from Pieri's postulates. Another was a particularly mature study in the algebra of logic. Among other contributions it constructed three systems of propositional calculi with multiple truth values, one a Boolean algebra and the others non-distributive lattices.

There have been ten theses in applied mathematics: two of these in actuarial science, two in mathematics of finance, four in theoretical or applied statistics and two in physical science. One thesis fitted Makeham's formula to a mortality table by least squares, and in solving the normal equations had to determine the location of the real roots of an equation of degree 266. Two others, supervised mainly by an economist, made valuable contributions to the mathematical study of statistical inference, examining the extent to which the Z and F tests of significance are applicable to non-normal populations and to binomial distributions. Still another thesis studied the effect of a cylindrical source of light within a parabolic reflector; and another made a highly technical analysis of sound-ranging in warfare, based on data from World War I.

You may be wondering what the seniors who wrote these and the earlier theses have done since graduation. A large majority of the men and many of the women have gone into mathematical, scientific, and actuarial pursuits. It recently came to our attention that the percentage of Reed male graduates since 1924 who have gone on to the doctorate in mathematics is the highest in the United States. Indeed, the absolute number of 1936–1945 mathematics doctors stemming from Reed is as large as from some of our noted universities. This may be due in part to the stimulus given by our thesis program, in part to the general intellectual atmosphere of the college, and perhaps in part to the interest aroused by our introductory work in mathematics which has led considerable numbers to select mathematics as their college major. Though a small college of about 600 students, we have 18 students registered this fall for an upper-division year-course in Theory of Equations and Modern Algebra.

Notwithstanding the types of career chosen by our graduates, we have never regarded the thesis project as training for graduate work or for a profession. Rather, it is the culmination of a program of general education with a special emphasis. The primary objective of all departments at Reed is to give students a broad perspective and a familiarity with fundamental problems, rather than to train them for a particular vocation. For some routine jobs specific training is, of course, indispensable; but for many vocations *education* is vastly more important than *training*; and a man with a sound grasp of basic theory and its significance will soon master the technical requirements of his particular job. And he will see beyond the job. For our seniors, the thesis topic, whether pro-

posed by the student himself or chosen by him from a wide range of suggestions, represents an expression of a scholarly interest. Advanced theoretical research usually arises out of intellectual curiosity rather than from the exigencies of a job; and the senior thesis project, on its more modest level, shares this quality.

While most of the chosen topics naturally relate to classical mathematics, there is opportunity, if the student so desires, to explore some phase of recent development. We realize that the content of undergraduate courses inevitably involves some lag behind recent discoveries, and we are glad when the thesis gives a student an opportunity for even a slight correction of that lag.

MATHEMATICAL NOTES

EDITED BY E. F. BECKENBACH, *University of California*

Material for this department should be sent to F. A. Ficken, Institute of Mathematics and Mechonics, New York University, 45 Fourth Ave., New York 3, New York.

ARITHMETICO-GEOMETRICAL PROGRESSIONS HAVING THREE COMMON TERMS*

ABDUR RAHMAN NASIR, Punjab College of Engineering and Technology, Pakistan

1. Introduction. A series is called an arithmetico-geometrical progression (AGP) if each term of it is the product of corresponding terms of an arithmetic and a geometric progression [1]. It has been observed that different AGP's may have three corresponding terms alike. For example, AGP's having first, third, and fourth terms alike are

$$3+\frac{47}{5}+19-45-\cdots, \qquad R=5, \qquad D=-\frac{28}{25};$$

$$3-\frac{31}{4}+19-45+\cdots, \qquad R=-2,\ D=\frac{7}{8};$$

$$3-\frac{23}{3}+19-45+\cdots, \qquad R=-3,\ D=-\frac{4}{9};$$

here R denotes the common ratio of the GP component of the AGP, and D the common difference of its AP component.

This note continues previous work [2, 3] of the author and establishes a theorem on the number AGP's having three given terms in common. We shall use the following lemma which is easily proved:

LEMMA. *If a, b, c are respectively the fth, gth, and hth terms, $f<g<h$, of an AGP whose $R=t$, then at^{h-f}, bt^{h-g}, c are respectively the fth, gth, and hth terms of a certain AP.*

* Revised by P. A. Clement.

2. Number of such AGP. We shall establish the following result.

THEOREM. *Let l, m, n where $ln \neq 0$ be respectively the pth, qth, and rth terms, $p<q<r$, of an AGP. Then the number of distinct such AGP's is $r-p$ provided $m^{r-p} \neq n^{q-p}l^{r-p}$; if $m^{r-p}=n^{q-p}l^{r-p}$, the number of AGP's is $r-p-k$ where $k=(q-p, r-q)$, the greatest common divisor of $q-p$ and $r-q$.*

Proof. With R denoting the common ratio of the GP component of the AGP, it follows from the lemma that lR^{r-p}, mR^{r-q}, n are respectively the pth, qth, and rth terms of a certain AP. Letting A denote the first term of this AP and B its common difference, we have

$$\begin{aligned} lR^{r-p} &= A + (p-1)B, \\ mR^{r-q} &= A + (q-1)B, \\ n &= A + (r-1)B. \end{aligned}$$

When these equations are multiplied respectively by $q-r$, $r-p$, and $p-q$ and then added, we get

(1) $$l(q-r)R^{r-p} + m(r-p)R^{r-q} + n(p-q) = 0.$$

It is clear that l, m, and n are common to AGP's whose R's satisfy (1); hence these quantities are respectively the pth, qth, and rth terms of AGP's whose R's are the distinct roots of the equation

(2) $$F(x) \equiv l(q-r)X^{r-p} + m(r-p)X^{r-q} + n(p-q) = 0.$$

If (2) has a double root, it must satisfy $F'(x)=0$ and hence, since $x\neq 0$, the equation

(3) $$lx^{q-p} - m = 0.$$

Moreover, this same root must satisfy the equation $(r-p)F(x) - xF'(x) = 0$, which yields

(4) $$mx^{r-q} - n = 0.$$

Elimination of x from (3) and (4) presents the condition

(5) $$m^{r-p} - n^{q-p}l^{r-q} = 0.$$

Hence the number of distinct AGP's is less than $r-p$ if (5) holds; otherwise it is exactly $r-p$.

Now assume that (5) holds and that $(q-p, r-q)=k$. Then an application of the Euclidean algorithm to the polynomial functions of (3) and (4) readily determines their highest common factor to be $x^k - b$, where b is a constant. This factor gives all the double roots of (2), and thus (2) admits exactly k double roots. Thus the number of distinct AGP's for this case is $r-p-k$.

3. Remark. It may be verified that repeated roots occur only when the given

terms l, m, n belong to a GP, and that then the k roots of the equation $X^k - (m/l)^{k/q-p} = 0$ are all the double roots of (2).

References

1. Hall & Knight, *Higher Algebra*, Fourth Edition, pp. 44–45.
2. Abdur Rahman Nasir, Current Science (India), vol. 16, No. 1, p. 19.
3. Abdur Rahman Nasir, Bull. Calcutta Math. Society, vol. 39, No. 2, pp. 79–81.

POLYNOMIAL PARAMETRIZATIONS

J. L. BRENNER, State College of Washington

1. Introduction. If $P(t)$ is any polynomial in t with coefficients 0 or 1, then the 2×2 matrix

$$U = \begin{pmatrix} P, & 1+P \\ 1+P, & 1 \end{pmatrix}$$

is orthogonal, modulo 2; that is, $UU^T = 1 \pmod 2$. However, for prime modulus p, $p>2$, there can be no non-constant 2×2 orthogonal matrices, the elements of which are modular polynomials. This fact follows from the theorem established in this note; the theorem and its corollaries seem to have interest in themselves. The truth of the theorem was conjectured by D. H. Lehmer in conversation.

2. Theorem. In the following, R denotes a unique factorization domain (that is, a commutative ring with unit element in which the cancellation law holds and factorization is unique up to associates); $R[x, y, t]$ is the ring of polynomials in x, y, t with coefficients in R; and C_p is the finite field with p elements, where p is a prime.

THEOREM. *Let $f(x, y)$ be a homogeneous polynomial of degree m, $f(x, y) \in R[x, y]$; suppose that non-constant polynomials $P = P(t)$, $Q = Q(t)$ exist, P, $Q \in R[t]$, such that $G(t) = f(P, Q)$ is a non-zero constant in $R[t]$. Then $f(x, y) = c(bx - ya)^m$; that is, $f(x, y)$ is a constant times the mth power of a linear form.*

Proof. Assume that m is positive. Let $f(x, y)$, P, Q, satisfy the hypotheses of the theorem, where

$$P(t) = \sum a_i t^i, \qquad Q(t) = \sum b_i t^i. \tag{1}$$

It is easy to see that degree P = degree $Q = n$, say. It will turn out that $b_n = b$, $a_n = a$. The coefficient of t^{mn} in $G(t)$ must be zero; therefore $f(a_n, b_n) = 0$. Thus $f(x, y)$ is divisible by $b_n x - y a_n$.

If the quotient $g(x, y)$ in this division has positive degree, then the quotient must be divisible by $b_n x - y a_n$, since $g(P, Q)$ must be a non-zero constant. The theorem follows by induction.

3. **Corollaries.** We have the following consequent results.

COROLLARY 1. *If $f(x, y)$ is any homogeneous polynomial of degree m, and h is any constant, $f(x, y)$, $h \in R[x, y, t]$, and if the equation $f(x, y) = h$ has polynomial parametrizations $x = P(t)$, $y = Q(t)$, then the same equation has parametrizations of degrees* 1, 1.

COROLLARY 2. *In Corollary* 1, *if $h \neq 0$, then all polynomial parametrizations, if there are any, are given by $x = aS(t) + cw$, $y = bS(t)$, where a, b, c are constants whose values are fixed once $f(x, y)$ and h are given; $S(t)$ is an arbitrary polynomial, and w is an arbitrary mth root of unity.*

Corollary 2 gives a parametrization of the parametrizations.

COROLLARY 3. *The conclusions of the above theorem and corollaries are true in the more general case in which t is a set of indeterminates rather than a single one.*

Proof. Let $f(x, y)$ be a homogeneous polynomial of positive degree; let t stand for a set $t_1, t_2, \cdots, t_s$ of indeterminates; let $P(t)$, $Q(t)$ be polynomials in these indeterminates. Let $G(t) = f[P(t), Q(t)]$ be a constant in $R[x, y, t]$. If $P(t)$, $Q(t)$ did not have the same degree, the coefficient of the first* term in $G(t)$ could not be zero. Indeed the first term of $P(t)$ must be the same (except for the coefficient) as the first term of $Q(t)$. Let the coefficients of these first terms be a, b respectively; let the terms themselves be called w, v. Then $f(w, v) = 0$, so that $f(a, b) = 0$, and $f(x, y)$ must be divisible by $bx - ya$, as before.

COROLLARY 4. *Let $f(x, y)$ be a homogeneous polynomial of discriminant D with integral coefficients, not identically an integer times the mth power of a linear form. Then the congruence $f(x, y) \equiv K \pmod{p}$, $K \neq 0$, can have polynomial parameterizations for only a finite number of primes.*

This is the form in which Lehmer made his conjecture. Clearly $p \mid DK$ for only a finite number of primes.

COROLLARY 5. *The only 2×2 orthogonal matrices with elements in $C_p[t]$, $p > 2$, are the constant matrices. When $p = 2$, the only 2×2 orthogonal matrices are*

$$\begin{pmatrix} 1 + P & P \\ P & 1 + P \end{pmatrix},$$

where $P(t)$ is an arbitrary polynomial.

Proof. If $U = (u_{ij})$ is orthogonal, then $u_{11}^2 + u_{12}^2 = 1$. The polynomial $x^2 + y^2$ has discriminant 4, so $x^2 + y^2$ does not have multiple factors in $C_p[x, y]$, $p > 2$. Moreover, $x^2 + y^2 = (x - y)^2 \pmod{2}$, from which all statements in the corollary follow. Here again, t may be a set of indeterminates, rather than a single one.

* The term $\alpha t_1^{e_1} t_2^{e_2} \cdots$ precedes $\beta t_1^{f_1} t_2^{f_2} \cdots$ if the degree of the first term exceeds the degree of the second term, or if the degrees are the same and the first non-zero number in the sequence $e_1 - f_1$, $e_2 - f_2$, $\cdots$ is positive.

Since there exist non-constant 2×2 orthogonal matrices over $C_2[t]$, there exist also non-constant $r\times r$ orthogonal matrices over $C_2[t]$. Some of these can be obtained by bordering 2×2 orthogonal matrices; others are products of such bordered matrices. I do not know whether there are still other $r\times r$ orthogonal matrices over $C_2[t]$.

4. Further questions. Questions concerning homogeneous polynomials in more than two variables, analogous to the questions considered in this note when the number of variables is 2, are not amenable to the same method of attack. The congruences below seem to show that the situation there is more complicated:

$$(t^3 + t^2 + t)^2 + (t^2 - 1)^2 = (t^3 + t^2 + 1)^2 \pmod 3,$$
$$(2t^2 + 2t + 1)^2 + (t^2 + t + 1)^2 + (2t + 1)^2 = 3 \pmod 5.$$

SOME PROPERTIES OF D NUMBERS

D. C. MORROW, Wayne University

1. Introduction. The well-known Fermat Theorem states that for any positive integer a which is not divisible by the odd prime p, we have $a^{p-1}\equiv 1 \pmod p$. In 1912, R. D. Carmichael* proved the existence of composite positive integers N which have the property that for any positive integer a which is relatively prime to N, we have $a^{N-1}\equiv 1 \pmod N$. This is known as the *Fermat property* and such integers have been called *F numbers.*

An integer N, such as 195, 399 or 1023, which is greater than 3 and which has the property that for any positive integer a which is relatively prime to N, we have $a^{N-3}\equiv 1 \pmod N$, we shall call a *D number.*

2. Theorem. We shall first establish the following result.

THEOREM 1. *If N is a D number, it must be of one of the following forms:*

$$9p_1p_2\cdots p_s, \qquad 3p_1p_2\cdots p_s, \qquad p_1p_2\cdots p_t,$$

where the p_i are distinct primes >3, $s\geqq 1$, and $t>2$.

Proof. First we note that since every prime possesses a primitive root, N cannot be a prime. If $N=2^t$, where $t\geqq 2$, then

$$3^{N-3} - 1 = 3^{2^t-3} - 1 = (3 - 1)(3^{2^t-4} + 3^{2^t-5} + \cdots + 1).$$

The second factor of this product is the sum of an odd number of odd terms. Hence $3^{N-3}-1$ is not divisible by 4. It follows that N must contain an odd prime factor. Suppose $N=rp^k$, where p is an odd prime and r is not divisible by p. We may choose s so that $x=w+sp^k$, where w is a primitive root of p^k, is a prime $>N$. Then x is relatively prime to N and $x^{N-3}\equiv w^{N-3}\equiv 1 \pmod{p^k}$. Since w is a primitive root of p^k, $N-3$ is divisible by $\phi(p^k)=p^k-p^{k-1}$. But $N-3$ is relatively prime to p unless $p=3$. It follows that if $p\neq 3$, then $k=1$ and $N-3$

* This MONTHLY, vol. 19 (1912), pp. 22–27.

CLASSROOM NOTES

EDITED BY C. B. ALLENDOERFER, Haverford College

All material for this department should be sent to C. B. Allendoerfer, Haverford College, Haverford, Pa.

ANOTHER VARIATION OF NEWTON'S METHOD

J. K. STEWART, West Virginia University

1. Introduction. Hamilton [1] has recently presented a generalization of Newton's Method of which the formula of Frame [2] and Wall [3] is a special case. The recent index of the MONTHLY [4] reveals a considerable literature on this topic, as well as some duplication. For example, Frame and Wall rediscovered in 1944–48 the method known to Halley [5] in 1694 and published in substantially the same form by Kobald in 1891. The general theory of successive approximation formulae has been discussed by Ford [7] and by Hamilton [8].

It is thus with some misgivings that one ventures to add to a literature well within the grasp of students who have completed an elementary course in the calculus. However, a variation of Newton's method which involves a very simple method of estimating the error in any approximation might justify itself.

2. A Review. The equation $f(x)=0$ has a root $x=r$ if $f(r)=0$. For $f(x)$ regular at $x=x_1$, we have, by Taylor's expansion:

$$f(x) = f(x_1) + f'(x_1)(x - x_1) + \frac{f''(x_1)}{2!}(x - x_1)^2 + \cdots . \tag{1}$$

If we write a_n for $f^{(n)}(x_1)/n!$, then

$$f(x) = a_0 + a_1(x - x_1) + a_2(x - x_1)^2 + \cdots + a_n(x - x_1)^n + \cdots . \tag{1'}$$

The curve whose equation is

$$y = a_0 + a_1(x - x_1) + a_2(x - x_1)^2 + \cdots + a_{k-1}(x - x_1)^{k-1} \tag{2}$$

has contact with the graph of $y=f(x)$ of order k, and the problem of determining a k-order iteration formula is that of finding the intercept x_2 of (2) to within powers of $(x_2-x_1)^k$. But this is identical with the process called inversion (or reversion) of series [9]. By putting $x=r$ and $y=f(r)=0$ in (2) then, when a finite number of terms in the solution of (2) are retained, r becomes the second approximation x_2. Thus

$$a_0 + a_1(x_2 - x_1) + a_2(x_2 - x_1)^2 + \cdots + a_{k-1}(x_2 - x_1)^{k-1} = 0. \tag{2'}$$

Newton's method follows at once from (2′) with $k=2$:

$$a_0 + a_1(x_2 - x_1) = 0,$$
$$x_2 = x_1 - a_0/a_1.$$

Frame's method follows from $k=3$, and (2) is geometrically an osculating parabola. The approximate intercept was found by writing

$$(x_2 - x_1) = \frac{-a_0}{a_1 + a_2(x_2 - x_1)} \tag{3}$$

and using Newton's value $-a_0/a_1$ for (x_2-x_1) on the right:

$$x_2 = x_1 - \frac{a_0 a_1}{a_1^2 - a_0 a_2}.$$

Hamilton solves (3) by constructing from (2′) the system:

$$\begin{aligned} a_1(x_2 - x_1) + a_2(x_2 - x_1)^2 &= -a_0 - a_3(x_2 - x_1)^3 - \cdots \\ a_0(x_2 - x_1) + a_1(x_2 - x_1)^2 &= -a_2(x_2 - x_1)^3 - \cdots . \end{aligned} \tag{4}$$

Since $k=3$, we neglect powers of $(x_2-x_1)^k$, $k \geqq 3$, and solve by Cramer's rule:

$$x_2 - x_1 = \frac{\begin{vmatrix} -a_0 & a_2 \\ 0 & a_1 \end{vmatrix}}{\begin{vmatrix} a_1 & a_2 \\ a_0 & a_1 \end{vmatrix}} = \frac{-a_0 a_1}{a_1^2 - a_0 a_2}.$$

Hamilton generalizes the system (4) to obtain iterative formulas of any desired order.

3. Another Variation. The approximate error in a k-order formula is given by the term in $(x_2-x_1)^k$ in the solution of (2). In Hamilton's system corresponding to (4), this is found by retaining one more term on the right of each of the k equations. This means that each of the equations contributes to the error. We can avoid this by constructing the system from (2′) in such a way that the error enters only once:

$$\begin{aligned} & a_1(x_2 - x_1) + a_2(x_2 - x_1)^2 \\ & \quad + \cdots + a_{k-1}(x_2 - x_1)^{k-1} && = -a_0 - a_k(x_2 - x_1)^k - \cdots \\ & a_0(x_2 - x_1) + a_1(x_2 - x_1)^2 \\ & \quad + \cdots + a_{k-2}(x_2 - x_1)^{k-1} + a_{k-1}(x_2 - x_1)^k && = -a_k(x_2 - x_1)^{k+1} - \cdots \\ & a_0(x_2 - x_1)^2 \\ & \quad + \cdots + a_{k-3}(x_2 - x_1)^{k-1} + a_{k-2}(x_2 - x_1)^k && = -a_{k-1}(x_2 - x_1)^{k+1} - \cdots \\ & \quad \cdots && = \quad \cdots \end{aligned} \tag{5}$$

Let the principal determinant be

$$D_k = \begin{vmatrix} a_1 & a_2 \cdots a_{k-1} & 0 \\ a_0 & a_1 \cdots a_{k-2} & a_{k-1} \\ 0 & a_0 \cdots a_{k-3} & a_{k-2} \\ 0 & 0 \cdots a_0 & a_1 \end{vmatrix} \tag{6}$$

and let M_k be the minor of the upper left-hand element of D_k, then

$$x_2 - x_1 = \frac{-[a_0 + a_k(x_2 - x_1)^k]M_k}{D_k}.$$

If x_2 is replaced by $x+\epsilon_2$ and x_1 by $x+\epsilon_1$, then the lowest powers of ϵ_1 in the expression a_kM_k/D_k come from the principal diagonals, and we have

$$x_2 = x_1 - \frac{a_0M_k}{D_k} - \frac{a_k}{a_1}(x_2 - x_1)^k. \tag{7}$$

When $a_k=0$, $k\geqq 3$, the error term reduces to $(-a_2/a_1)^k(x_2-x_1)^{k+1}$.

A numerical example for illustration. A suitable first approximation may be found by any of many methods of estimating. To calculate $\sqrt{2}$, we may start with $x_1=7/5$, then $a_0=-1/25$, $a_1=14/5$, $a_2=1$, $a_k=0$, $k\geqq 3$. Then with $k=5$,

$$x_2 = \frac{7}{5} + \frac{7801}{548842},$$

and before reducing to decimal form we may determine the number of places to retain in the division. The error is

$$E \sim (-5/14)^5(.014)^6 = -4.37(10^{-14}).$$

Then

$$\begin{aligned} x_2 &= 1.4142135623731419 \\ &\underline{- \qquad\qquad\qquad\quad 437} \\ &= 1.4142135623730982 \end{aligned}$$

which we know to be true for fourteen places.

4. Conclusion. The method works equally well for transcendental equations, and for imaginary as well as real roots. As in the derivation of other iterative formulas, there are several approaches. For example, the case $k=3$ follows from two applications of Newton's method with terms involving derivatives of higher order than the second being omitted; or again, by using Frame's formula instead of Newton's in (3).

References

1. Hamilton, H. J. "A Type of Variation on Newton's Method," this MONTHLY, Vol. 57 (1950) pp. 517–522.

2. Frame, J. S. "A Variation of Newton's Method," this MONTHLY, Vol. 51 (1944) pp. 36–38.

3. Wall, H. S. "A Modification of Newton's Method," this MONTHLY, Vol. 55 (1948) pp 90–94.

4. This MONTHLY, Vol. 57, No. 7 (1950) pp. 2–3.

5. Bateman, Harry, "Halley's Methods of Solving Equations," this MONTHLY, Vol. 45 (1938) p. 11.

6. Kobald, E. "Notiz, betreffend die Berechnung der Wurzeln numerischer Gleichung," Monatshefte für Math. u. Physik, Vol. 2 (1891) pp. 331–332.

7. Ford, L. R. "The Solution of Equations by the Method of Successive Approximation," this MONTHLY, Vol. 32 (1925) pp. 272–287.

8. Hamilton, H. J. "Roots of Equations by Functional Iteration," Duke Math. Jour., Vol. 13 (1946) pp. 113–121.

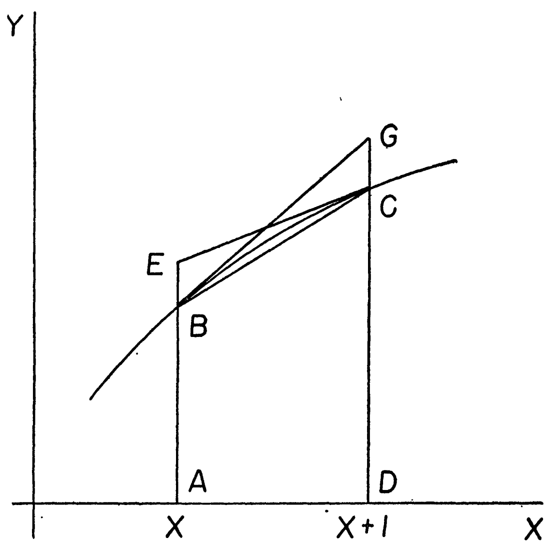
9. Franklin, Philip, "Methods of Advanced Calculus," McGraw-Hill (1944) p. 27.

A SIMPLE PROOF OF STIRLING'S FORMULA

A. J. COLEMAN, University of Toronto

Stirling's formula for the asymptotic value of factorial N is widely used in physics and many different proofs of it are known. The following, which is a slight variation on an old approach, is so simple as to be useful as an exercise in integration and the convergence of montone, bounded sequences. It could therefore be used in a first course in Calculus.

Consider the graph of $y=\ln x$, where by $\ln x$ we mean the natural logarithm of x. Since $y'=1/x$, the slope between x equal 1 and N is positive and decreas-



ing. Thus chords, such as BC in the accompanying figure, are below the curve and therefore the area of the trapezoid $ABCD$ is less than the area under the curve. The area of the trapezoid is

$$\tfrac{1}{2}[\ln x + \ln (x+1)],$$

and therefore the area of the $N-1$ trapezoids from x equal 1 to $N-1$ is

$\frac{1}{2}\{[\ln 1 + \ln 2] + [\ln 2 + \ln 3] + \cdots + [\ln (N-1) + \ln N]\} = \ln N! - \frac{1}{2}\ln N.$

Thus

$$(1) \qquad c_N = \int_1^N \ln x dx - \ln N! + \tfrac{1}{2}\ln N = (N + \tfrac{1}{2}) \ln N - N + 1 - \ln N!$$

is a positive number equal to the area under the curve $y = \ln x$ from $x = 1$ to $x = N$ minus the trapezoidal approximation to this area. The sequence c_N therefore increases monotonely with N.

On the other hand, the area under the curve is clearly less than the average of the areas of the trapezoids $AECD$ and $ABGD$, where BG is the tangent to the curve at B and CE is the tangent at C. From the known value of the slope, we have DG equal to $\ln x + 1/x$ and AE equal to $\ln (x+1) - 1/(x+1)$. Thus the average area of the two trapezoids is

$$\frac{1}{2}\left[\ln x + \ln (x+1) + \frac{1}{2}\left(\frac{1}{x} - \frac{1}{x+1}\right)\right].$$

Summing as before, we have

$$\int_1^N \ln x dx < \ln N! - \frac{1}{2}\ln N + \frac{1}{4} - \frac{1}{4N},$$

and, therefore,

$$c_N < \frac{1}{4} - \frac{1}{4N} < \frac{1}{4}.$$

Thus c_N, which is a monotone increasing sequence, is bounded above and therefore has a limit c such that $0 \leqq c \leqq \frac{1}{4}$. If we set $e^{1-c} = a$, so that $2.11 < a < 2.72$, it follows from (1) that

$$(2) \qquad \frac{N!}{\sqrt{N}\, N^N e^{-N}} \to a$$

as N tends to infinity. This is the essential part of Stirling's formula. However, we must still find the value of a. By geometrical intuition our method of approximation from below seems about twice as accurate as the approximation from above, so we shall expect a to be closer to 2.72 than to 2.11, perhaps around 2.5.

There is a well-known trick for finding the exact value of a, by making use of Wallis' formula. To prove the latter, we set $I_r = \int_0^{\pi/2} \sin^r \phi d\phi$, and since in the range $0 < \phi < \pi/2$,

$$\sin^{2k-1}\phi > \sin^{2k}\phi > \sin^{2k+1}\phi$$

we have

$$(3) \qquad I_{2k-1} > I_{2k} > I_{2k+1}.$$

Integrating I_r by parts, one easily shows that $rI_r=(r-1)\ I_{r-2}$, and therefore I_r may be evaluated by recursion. The inequalities (3) then immediately give Wallis' Formula:

$$\frac{[2^k(k!)]^4}{[(2k)!]^2(2k+1)} \to \frac{\pi}{2}$$

as k approaches infinity.* But by (2) the left-hand side of this approaches $a^2/4$. Thus $a=\sqrt{2\pi}\doteq 2.50$, and we finally have:

$N!$ is asymptotically equal to $\sqrt{2\pi N}\, e^{-N}N^N$.

BRINGING IN DIFFERENTIALS EARLIER

W. R. Ransom, Tufts College

Suppose we skip the usual chapter on functions and continuity, and start with the concept of variables connected by equations:

$$x^2 + y^2 = r^2, \qquad x = r\cos\theta, \cdots . \tag{1}$$

Not variables as "symbols to which numerical values can be assigned," but variables as Newton (and beginners in calculus) might conceive them—the measures of quantities in nature and geometry. And not continuity as unnaturally defined for a point only, but the sort of continuity implied in the ancient doctrine "natura non agit per saltum."

A set of variables connected by equations will have sets of simultaneous numerical values, and such a set constitutes a "state." In an interval from one state to another, the variables receive increments (denoted by $\Delta x, \Delta y, \cdots$ *etc.*) and the new values, $x+\Delta x, \cdots$ are connected by the same relations as before. The intervals between states are taken small enough so that no question of multiple values arises.

By subtractions a new set of equations is obtained which describe relations between the variables and their increments. If these equations are all divided by the same increment, we get relations between the variables for some state and some mean rates of increase for an interval from that state to another. We define exact rate (now simply called rate) as the limit which the mean rate approaches when the interval shrinks so that the second state approaches the first. These rates are then adjoined to the set of variables, thus enlarging the state.

Now we come to differentials. In the equations which connect the variables with the rates, express the rates as fractions with a common denominator. This denominator is called the differential of that variable whose increment was used as a divisor, and the numerators are called the differentials of the variables in whose rates they appear. That is, we have symbolized the rates so that $D_xy=dy/dx$, $D_xr=dr/dx, \cdots$. This defines the differentials, one as a denominator

* See Courant and Robbins, *What is Mathematics?*, pp. 509–510, for details.

of rates, and the others as numerators of rates. Now adjoin these differentials to the state containing variables and rates.

By the argument which appears in current texts under the title "function of a function," it is then to be shown that not only are

$$D_x y = dy/dx, \qquad D_x r = dr/dx, \cdots$$

but also:

$$D_y x = dx/dy, \qquad D_r y = dy/dr, \cdots .$$

Accordingly the distinction relating to one differential as denominator of rates may, and should forthwith be laid aside: any two differentials in a state may be taken as numerator and denominator for the corresponding rate.

It is desirable to frame rules for writing down the equations that connect variables with their differentials, as, from (1):

$$2xdx + 2ydy = 2rdr, \; dx = dr \cos\theta + r \sin\theta d\theta, \cdots .$$

From these we get the information about dy/dx, dr/dy, $d\theta/dy$, $\cdots$.

The definition commonly given in current texts for dy is $dy = (D_x y)\Delta x$, whence $dx = \Delta x$. This does not make sense, since Δx is not defined for the state to which the *rates* belong, but only for the interval to which mean rates belong, Graphically dy and dx are the rise and run of a piece of the tangent, and have nothing to do with two points on the curve.

Since differentials are determined numerically only to within a common factor, they cannot themselves be differentiated. There is no d^2y. The symbol d^2y/dx^2 is shorthand for $d(dy/dx)/dx$ just as $D_x^2 y$ is shorthand for $D_x(D_x y)$.

By using differentials from the start, we get rid of the clumsy phrase "with respect to" in both differentiation and integration. The dx in the integral tables becomes a differential, a factor of the differential to be integrated, and not a part of a symbol "$\int(\ )dx$." It seems scandalous to the writer that the dx in dy/dx should be equal to Δx, while the dx in $\int dx$ should not be equal to dx.

Editorial Note: This paper by Professor Ransom is the outgrowth of extended discussion between him and the editor arising from a discussion group at Tufts College in August, 1950 on the "Teaching of Calculus." Although the editor does not agree with Professor Ransom at many points, he is publishing this paper as a stimulus to discussion of an area in which there seems to be great confusion among textbook writers and teachers. Classroom Notes will welcome a limited number of replies to Professor Ransom's paper, particularly those which discuss the questions:

(1) What is a differential?

(2) Does the "dx" of differential calculus have the same meaning as the "dx" in the integral: $\int_a^b f(x)\, dx$?

Prospective commentators will be interested in the various articles on this subject in earlier volumes of this MONTHLY. These are listed under "Differentiation" in the Index which was recently issued to readers of the MONTHLY.

ELEMENTARY PROBLEMS AND SOLUTIONS

EDITED BY HOWARD EVES, Oregon State College

Send all communications concerning Elementary Problems and Solutions to Howard Eves, Mathematics Department, Oregon State College, Corvallis, Oregon. This department welcomes problems believed to be new, and demanding no tools beyond those ordinarily furnished in the first two years of college mathematics. To facilitate their consideration, solutions should be submitted on separate, signed sheets, within three months after publication of problems.

PROBLEMS FOR SOLUTION

E 966. *Proposed by C. W. Bruce, Tennessee Polytechnic Institute.*

Any integer N may be written in the form $2^k(a^2-b^2)$, where k, a, b are non-negative integers. When are k, a, b unique? When not unique, how many sets of k, a, b are there?

E 967. *Proposed by L. A. Ringenberg, Eastern Illinois State College.*

Let $P_1, P_2, \cdots, P_n$ $(n>4)$ be n points equally spaced and arranged in order of subscripts on a circle of radius 1, and let S_n be the area of the star formed by the line segments $P_1P_3, P_2P_3, \cdots, P_{n-1}P_1, P_nP_2$. (a) Find a formula for S_n. (b) Find the smallest positive integer n such that $nS_n>(n-1)\pi$. (c) Show that if k is any positive integer greater than 1, then $n^kS_n<(n^k-1)\pi$ for $n=5, 6, 7, \cdots$.

E 968. *Proposed by R. M. Redheffer, University of California.*

If $f(s)=\int_{-\pi}^{\pi} e^{\cos x} \cos (\sin x) \cos sx\, dx$, find $\sum_1^\infty f(n)$.

E 969. *Proposed by Solomon Golomb, The Johns Hopkins University.*

Show that a necessary and sufficient condition that there be infinitely many twin primes is that there be infinitely many numbers n not of the form $6ab\pm a\pm b$.

E 970. *Proposed by Victor Thébault, Tennie, Sarthe, France.*

Show that in triangle ABC side BC bisects the segments intercepted by the other two sides on the tangents to the inscribed parabola having for focus the point A_1 where the symmedian AA_1 intersects the circumcircle.

SOLUTIONS

Integers Less Than N as Sum of Divisors of N

E 932 [1950, 556]. *Proposed by Paul Erdös, University of Illinois.*

(1) Prove that every positive integer not exceeding $n!$ is the sum of n or fewer distinct divisors of $n!$.

(2) Denote by $E(n)=2^{a_2}3^{a_3}\cdots p^{a_p}$ the least common multiple of the positive integers not exceeding n. Prove that every positive integer not exceeding $E(n)$ is the sum of $a_2+a_3+\cdots+a_p$ or fewer distinct divisors of $E(n)$.

Solution by Joseph Oppenheim, Des Plaines, Illinois. (1) Let us assume the theorem is true for $n=k$, and let x be an integer not exceeding $(k+1)!$. We can

write $x=p(k+1)+q$, where $q<k+1$. Then

$$(k+1)k! \geqq x \geqq p(k+1),$$

or $p\leqq k!$. By the induction hypothesis $p=a_1+a_2+\cdots+a_m$, where $a_i\neq 0$, $a_i \mid k!$, $m\leqq k$, and $a_i\neq a_j$ if $i\neq j$. Hence

$$x = (k+1)\sum_1^m a_i + q = \sum_1^m (k+1)a_i + q.$$

The number of summands is $m+1\leqq k+1$. Since $q<k+1$, $q\neq(k+1)a_i$ for all i. Further, since the a_i's are distinct, so are all the $a_i(k+1)$'s.

Finally, for $k=1, 2, 3$ the theorem is obvious.

(2) Let us make the notation a little more explicit by setting

$$E(n) = LCM\{1, 2, \cdots, n\} = 2^{a\,(n)}3^{a_3(n)}\cdots p^{a_p(n)}$$

and

$$E^*(n) = a_2(n) + a_3(n) + \cdots + a_p(n).$$

We first establish the

LEMMA: *$J(n)\equiv E^*(n)-E^*(n-1)=0$ or 1. If $J(n)=1$ then $n=p^{a_p(n)}$* and *$E(n)=pE(n-1)$. If $J(n)=0$ then $E(n)=E(n-1)$.*

Clearly $E(n)$, $E^*(n)$, and $a_p(n)$ for $p=2, 3, 5, \cdots$, are all monotonically increasing functions. If $J(n)\neq 0$ there exists a p such that $a_p(n-1)<a_p(n)$. Now

$$p^{a_p(n-1)} \leqq n-1 < p^{a_p(n-1)+1} \leqq p^{a_p(n)} \leqq n.$$

It follows that

$$p^{a_p(n-1)+1} = p^{a_p(n)} = n.$$

Therefore $E(n)=pE(n-1)$ and p is the only prime whose exponent is altered. Hence $J(n)=1$. If $J(n)=0$, then $a_p(n)-a_p(n-1)=0$ for all p, and $E(n)=E(n-1)$.

We now prove the theorem by induction. We assume, for any $k\leqq n$, that $x\leqq E(k)$ implies that x is the sum of at most $E^*(k)$ distinct divisors of $E(k)$.

Take $x\leqq E(n+1)$ and let m be the least integer such that $E(m)=E(m+1)$. By the lemma we have $m=p^{ap(m)}$ and $E(m)=pE(m-1)=E(n+1)$. Now $x=qp+r$, where $r<p$ and $q\leqq E(m-1)$ and $m-1\leqq n$. Hence $q\sum_1^h\alpha_i$, where $\alpha_i\neq\alpha_j$ for $i\neq j$, each $\alpha_i \mid E(m-1)$, and $h\leqq E^*(m-1)$. Therefore $x=\sum_1^h p\alpha_i+r$. Now $r<p\leqq m$, whence $r \mid E(m)=E(n+1)$. Also, $r\neq p\alpha_i$ and $p\alpha_i \mid pE(m-1)=E(m)=E(n+1)$. Finally, $h+1\leqq E^*(m-1)+1=E^*(m)=E^*(n)$. Since the theorem is obviously true for $n=1, 2, 3$, it now follows for all n.

Also solved by Leopold Flatto, Roger Lessard, and Fred Marer.

An Interesting Function

E 935 [1950, 557]. *Proposed by C. D. Olds, San Jose State College*

Given an example of a function which is discontinuous in an everywhere dense set, and which is also differentiable in an everywhere dense set.

Solution by J. W. Gaddum, University of Missouri. We define $f(x)$ for $x\epsilon[0, 1]$. If $x=k/2^n$, where k is an odd integer less than 2^n, set $f(x)=1/2^{2n-1}$. Otherwise set $f(x)=0$.

For 2^{n-1} values of x, $f(x)=1/2^{2n-1}$. Hence the variation of $f(x)$ on $[0, 1]$ is

$$\sum_{n=1}^{\infty} 2^{n-1}/2^{2n-1} = \sum_{n=1}^{\infty} 1/2^n = 1/2.$$

Hence $f(x)$ is of bounded variation and is differentiable almost everywhere. But $f(x)$ is discontinuous for $x=k/2^n$.

Also solved by Leopold Flatto, Norman Miller, Leo Moser, C. S. Ogilvy, L. A. Ringenberg, J. B. Rosser, O. E. Stanaitis, Albert Wilansky, and the proposer.

Several solutions offered $f(x)=0$ if x is irrational and $f(x)=q^{-q}$ if $x=p/q$, $(p, q)=1$. Then $f(x)$ is discontinuous for every rational x and differentiable for every algebraic irrationality. A very elementary, but perhaps somewhat unsportsmanlike, example is $f(x)$ defined on $[-1, 1]$ by $f(x)=1$ for negative rational x and $f(x)=0$ otherwise.

F. Bagemihl located this problem, with a solution, in K. Knopp, *Aufgabensammlung zur Funktionentheorie*, vol. 2, Berlin (1949), p. 9, prob. 2; p. 55.

The Scratched Lawn Edger

E 936 [1950, 632]. *Proposed by R. E. Horton, Los Angeles City College.*

A wheel of radius a with a flange of radius b is rolling without slipping along a straight rail. On the corner of the rail a piece projects which scratches a mark on the face of the flange. Identify the curve of the mark so formed. What conditions on a and b are necessary in order that the mark on the flange will be a simple closed curve?

Solution by C. S. Ogilvy, Columbia University. Considering the wheel fixed and the rail rotating around it, the mark is at once identified as two arcs of involutes of the wheel.

For a simple closed curve, the point $(-b, 0)$ satisfies the parametric equations of the involute, giving

$$-b = a(\cos\phi + \phi\sin\phi), \qquad 0 = a(\sin\phi - \phi\cos\phi).$$

Squaring and adding we find that $b/a=(1+\phi^2)^{1/2}$, where, from the second equation, ϕ satisfies the relation $\phi=\tan\phi$. An approximate numerical value for b/a is 4.60.

Also solved by Alan Berndt, E. A. Franz (partially), and the proposer.

A Generalization of Wilson's Theorem

E 937 [1950, 632]. *Proposed by I. N. Herstein, University of Kansas.*

If p is a prime and $n \geqq p$, then

$$n\,!\sum_{pi+j=n} 1/p^i i!j! \equiv 0, \text{ mod } p.$$

Solution by Kirk Stewart, College of Puget Sound. We observe that

$$n!\sum_{pi+j=n} 1/p^i i!j! = \sum_{i=0}^{[n/p]} n!/p^i i!(n-ip)!.$$

Now consider

$$n!/p^i i!(n-ip)! = (n-ip+1)(n-ip+2)\cdots(n-1)n/p^i i!.$$

In the numerator on the right side there are: (a) ip factors, and hence i complete systems of residues modulo p, (b) i multiples of p, namely $p[n/p]$, $p([n/p]-1)$, $\cdots$, $p([n/p]-i+1)$. Denote a complete system of residues by ρ_j, $j=0, 1, \cdots$, $p-1$, with $\rho_0 \equiv 0$, mod p. Then

$$\prod_{j=1}^{p-1} \rho_j \equiv (p-1)! \equiv -1, \text{ mod } p,$$

by Wilson's theorem, and consequently

$$\begin{aligned} n!/p^i i!j! &\equiv [n/p]([n/p]-1)\cdots([n/p]-i+1)(-1)^i/i! \\ &\equiv \binom{[n/p]}{i}(-1)^i, \text{ mod } p. \end{aligned}$$

It follows that

$$n!\sum_{pi+j=n} 1/p^i i!j! \equiv \sum_{i=0}^{[n/p]} (-1)^i\binom{[n/p]}{i} = (1-1)^{[n/p]} \equiv 0, \text{ mod } p.$$

Also solved by B. W. Brewer, Roger Lessard, Azriel Rosenfeld, and the proposer.

Euler's Constant

E 938 [1950, 632]. *Proposed by H. F. Sandham, Trinity College, Ireland.*

If $x=\sum_{i=1}^{n} 1/i$, prove that $n=[e^{x-\gamma}]$, where γ is Euler's constant, and $[a]$ denotes the integral part of a.

Solution by S. H. Gould, Purdue University. If

$$f(y) = 1/(y+1) - \log (y+1)/y,$$

then $f(1)<0, f'(y)>0$, $\lim_{y\to\infty} f(y)=0$, so that $f(y)<0$ for all $y \geqq 1$. Thus $x-\log n$

is monotone decreasing. Similarly, $x-\log(n+1)$ is monotone increasing. Thus $\log n<x-\gamma<\log(n+1)$, or $n<e^{x-\gamma}<n+1$.

Also solved by Philip Anselone and Vern Hoggatt (jointly), P. T. Bateman, D. H. Browne, Leopold Flatto, P. G. Kirmser, M. S. Klamkin, Norman Miller, Leo Moser, C. F. Pinzka, Alex Rosenberg, O. E. Stanaitis, and the proposer.

A Chain of Circles

E 939 [1950, 632]. *Proposed by P. J. Schillo, University of Buffalo.*

Circles $C_0, C_1, C_2, \cdots$, having radii $r_0, r_1, r_2, \cdots$, are tangent to a circle C, of radius $2r$, and to a diameter of C. Circle C_0 passes through the center of C, and C_n is tangent externally to C_{n-1}. Show that

$$r_n = 4w^n r/(w^n + 1)^2,$$

where $w=3+2\sqrt{2}$, and that r/r_n is an integer.

Solution by W. O. Pennell, Exeter, N. H. Take the diameter of C as x-axis and the center of C as origin. Then it is easily found that the coordinates of the centers of C_k and C_{k+1} are

$$(r_k,\ 2(r^2 - rr_k)^{1/2}) \quad \text{and} \quad (r_{k+1},\ 2(r^2 - rr_{k+1})^{1/2}).$$

Since the distance between these centers must be r_k+r_{k+1} we have

$$f(r_k, r_{k+1}) \equiv (r_k + r_{k+1})^2 - (r_k - r_{k+1})^2 - 4[(r^2 - rr_k)^{1/2} - (r^2 - rr_{k+1})^{1/2}]^2 = 0.$$

Now, for a given value of r_k, there is a unique value of r_{k+1} such that $f(r_k, r_{k+1})=0$ and $r_{k+1}<r_k$. By substitution we find that the values

$$r_k = 4w^k r/(w^k + 1)^2, \qquad r_{k+1} = 4w^{k+1}r/(w^{k+1} + 1)^2,$$

where $w=3+2\sqrt{2}$, are such that $f(r_k, r_{k+1})=0$ and $r_{k+1}<r_k$. Therefore if r_k has the value given above, then r_{k+1} must also have the value given above. But we do have $r_0=2r=4w^0r/(w^0+1)^2$. Hence the first part of the problem follows by mathematical induction.

We next observe that

$$\begin{aligned} r/r_n = (w^n + 1)^2/4w^n &= [(3 + 2\sqrt{2})^n + 2 + (3 - 2\sqrt{2})^n]/4 \\ &= [2(3^n + 1) + \cdots\ ]/4, \end{aligned}$$

where the unwritten terms contain even powers of $2\sqrt{2}$ as factors. It follows that r/r_n is an integer.

Also solved by A. L. Epstein, Vern Hoggatt, M. S. Klamkin, Roger Lessard, O. Dale Smith, and the proposer. Smith solved the first part of the problem by the method of inversion, using one end of the diameter of C as center of inversion.

ADVANCED PROBLEMS AND SOLUTIONS

EDITED BY E. P. STARKE, Rutgers University

Send all communications concerning Advanced Problems and Solutions to E. P. Starke, Rutgers University, New Brunswick, New Jersey. All manuscripts should be typewritten, with double spacing and margins at least one inch wide. Problems containing results believed to be new or extensions of old results are especially sought. Proposers of problems should also enclose any solutions or information that will assist the editor. In general, problems in well known textbooks or results in readily accessible sources should not be proposed for this department.

PROBLEMS FOR SOLUTION

4438. *Proposed by H. F. Sandham, Trinity College, Ireland.*

Show that the polynomial of degree $2n$ defined by the hypergeometric function $F(-2n, \frac{1}{2}; n+1; x)$ takes on the value 1 when $x=4$.

4439. *Proposed by Josef Langr, Prague, Czecho-Slovakia.*

A line m cuts the sides BC, CA, AB of a triangle ABC at L, M, N. Show that the midpoints of the segments AL, BM, CN are on a line n, and find the locus of the point of intersection of m and n as m moves parallel to itself.

4440. *Proposed by R. G. Stoneham, University of California, Berkeley.*

Evaluate

$$\sum_{k=1}^{\infty} \frac{(-1)^{k+1}B_{2k}\Pi^{2k}}{(2k+1)!}$$

where B_{2k} are the even Bernoulli numbers.

4441. *Proposed by R. M. Cohn, Rutgers University.*

If a polynomial equation $f(x)=0$, with integral coefficients, has no positive roots, there exists a polynomial $g(x)$ with integral coefficients such that all coefficients of $f(x)\cdot g(x)$ are positive.

4442. *Proposed by Michael Golomb, Purdue University*

If a polynomial (not a constant) with complex coefficients is of the same modulus at two points z_1, z_2 of the z-plane, show that it has at least one zero in the open half-plane

$$R\{\bar{z}(z_1 - z_2)(|z_1| - |z_2|) > 0$$

if $|z_1| \neq |z_2|$, and at least one zero in each of the closed half-planes

$$R\{\bar{z}(z_1 - z_2)\} \geqq 0, \qquad R\{\bar{z}(z_1 - z_2)\} \leqq 0$$

if $|z_1| = |z_2|$.

SOLUTIONS

A Factor Problem

4339 [1949, 187]. *Proposed by Paul Erdös, University of Aberdeen, Scotland.*

Prove that if $n=2^\mu(2k+1)$, $k>1$ and μ being positive integers, then 2^n-1 has a composite factor congruent to 1 modulo n.

Solution by N. G. W. H. Beeger, Amsterdam, Holland. We make use of the well known theorems:*

(1) 2^m+1, $m\neq 3$, is divisible by a prime of the form $2mx+1$.

(2) 2^m-1 is divisible by a prime of the form $mx+1$.

(3) $2^{4m}+1$ is divisible by a prime of the form $16mx+1$.

For $n=2^\mu(2k+1)$, $k>1$, $\mu>0$, we have

$$\text{(A)} \qquad 2^n - 1 = (2^d - 1)(2^d + 1)(2^{2d} + 1) \cdots (2^{2^{\mu-1}d} + 1),$$

in which we have put d for the odd number $2k+1$. From (1) the last factor of (A) is divisible by a prime $X=nx+1$. From (3) with $m=2^{\mu-4}d$, $\mu\geqq 4$, the factor $2^{2^{\mu-2}d}+1$ is divisible by a prime $W=nw+1$. Hence XW is the required composite factor and the theorem is proved for $\mu\geqq 4$.

If $\mu=1$, the right member of (A) has only two factors, the second of which is divisible by a prime $X=nx+1$ from (1). The first factor is divisible by a prime $dy+1$ from (2); hence this (odd) prime has the form $Y=2dy+1=by+1$. So XY is the required composite factor.

If $\mu=2$, the right-hand member of (A) has three factors, the last of which is divisible by a prime $X=nx+1$ from (1). The first and second factors are divisible by primes $Y=2dy+1$, $Z=2dz+1$, respectively. If y or z is even the theorem is proved with XY or XZ; if y and z are both odd, then YZ has the form $nu+1$, and the theorem is also proved.

If $\mu=3$, the right-hand side of (A) has four factors, the last of which is divisible by a prime $X=nx+1$ from (1). The other three factors are divisible respectively by primes $Y=2dy+1$, $Z=2dz+1$, $U=4du+1$. It is easily verified that the desired composite factors may be selected as follows:

$$\begin{array}{lll}
u \equiv 0 \pmod 2, & XU; & \\
y \equiv 0 \pmod 4, & XY; & \\
u \equiv 1 \pmod 2, & y \equiv z \pmod 4, & XYZ; \\
u \equiv 1 \pmod 2, & y \equiv 2 \pmod 4, & XZU; \\
u \equiv 1 \pmod 2, & y \equiv 1, \quad z \equiv 3 \pmod 4, & XYZU.
\end{array}$$

Since y and z are interchangeable this gives all the cases and the proof is complete.

---

* These are corollaries of Bang's theorem. See Kraitchik, *Theorie des Nombres*, v. 1, p. 129; also Dickson, *History of the Theory of Numbers*, v. 1, chapt. xvi. (3) was noted by Lucas.

A Sum of Reciprocals

4365 [1949, 637]. *Proposed by Paul Erdös, University of Aberdeen, Scotland.*

Let $a_1<a_2<a_3< \cdots <a_k \leqq n$ be such that the least common multiple of any two a's exceeds n. Prove that

$$\sum_{i=1}^{k} \frac{1}{a_i} < 2.$$

Solution by R. S. Lehman, Stanford University. The problem is trivial if $a_1=1$. Let us take $a_1 \geqq 2$ and let b_m be the number of a's satisfying the condition

$$\frac{n}{m} < a_i \leqq \frac{n}{m-1} \cdot \tag{1}$$

Then exactly b_m of the a's have $(m-1)$ multiples each among the first n integers, $1, 2, \cdots, n$; and no two a's have a common multiple among these n integers. Therefore

$$b_2 + 2b_3 + 3b_4 + \cdots + nb_{n+1} = n - r_1, \qquad r_1 \geqq 0. \tag{2}$$

For any given set of a's, $a_1<a_2< \cdots <a_k$, we may, without loss of generality, choose n to be equal to a_k. Then from (1) it follows that

$$\begin{aligned}\sum_{i=1}^{k} \frac{1}{a_i} &< \frac{1}{n} + (b_2 - 1)\frac{2}{n} + b_3\frac{3}{n} + b_4\frac{4}{n} + \cdots + b_{n+1}\frac{n+1}{n} \\ &= \frac{1}{n}(-1 + 2b_2 + 3b_3 + 4b_4 + \cdots + (n+1)b_{n+1}).\end{aligned}$$

Substituting the value for b_2 from (2), we obtain

$$\sum_{i=1}^{k} \frac{1}{a_i} < 2 - \frac{1}{n} - \frac{2r_1}{n} - \frac{1}{n}(b_3 + 2b_4 + 3b_5 + \cdots + (n-1)b_{n+1}), \tag{3}$$

which proves the proposed result.

To improve this estimate, consider the integers $1, 2, 3, \cdots, [n/2]$. Of the a's, b_3+b_4 have one multiple each among these integers; b_5+b_6 have two multiples each among them, and so forth. Then

$$b_3 + b_4 + 2b_5 + 2b_6 + 3b_7 + 3b_8 + \cdots = \left[\frac{n}{2}\right] - r_2, \qquad r_2 \geqq 0. \tag{4}$$

Further, if $p > [n/2]$ is not divisible by any of the a's, we can include p among the a's, since any multiple of p other than p itself is greater than n. (Such an addition to the set of a's merely increases $\sum 1/a_i$.) But $r_1 - r_2$ is the number of integers which are greater than $[n/2]$ and less than n, and which are not multiples of a's. If all such p's have been included we have consequently $r_1 - r_2 = 0$. Substituting the value of b_3 given by (4) into (3) and using the fact

that $r_1 - r_2 = 0$ we have

$$\sum_{i=1}^{k} \frac{1}{a_i} < 2 - \frac{1}{n} - \frac{r_1}{n} - \frac{1}{n}\left[\frac{n}{2}\right] - \frac{1}{n}(b_4 + b_5 + 2b_6 + 2b_7 + 3b_8 + \cdots) < 1\tfrac{1}{2}. \tag{5}$$

Further consider the integers 1, 2, 3, $\cdots$, $[n/3]$. Of the a's, $b_4+b_5+b_6$ have one multiple each among these integers; $b_7+b_8+b_9$ have two, and so forth. Therefore

$$b_4 + b_5 + b_6 + 2b_7 + 2b_8 + 2b_9 + 3b_{10} + 3b_{11} + \cdots = \left[\frac{n}{3}\right] - r_3, \quad r_3 \geqq 0. \tag{6}$$

Substituting from (6) into (5) we obtain

$$\sum_{i=1}^{k} \frac{1}{a_i} < 2 - \frac{1}{n} - \frac{r_1}{n} - \frac{1}{n}\left[\frac{n}{2}\right] - \frac{1}{n}\left[\frac{n}{3}\right] + \frac{r_3}{n} - \frac{1}{n}(b_6 + b_8 + b_9 + b_{10} + b_{11} + 2b_{12} + \cdots),$$

where all the terms inside the parenthesis are positive. Since

$$\frac{1}{n}\left[\frac{n}{2}\right] \geqq \frac{1}{n}\left(\frac{n}{2} - \frac{1}{2}\right), \qquad \frac{1}{n}\left[\frac{n}{3}\right] \geqq \frac{1}{n}\left(\frac{n}{3} - \frac{2}{3}\right),$$

we have

$$\sum_{i=1}^{k} \frac{1}{a_1} < 2 - \frac{1}{n} - \frac{r_1}{n} + \frac{r_3}{n} - \frac{1}{2} - \frac{1}{3} + \frac{1}{2n} + \frac{2}{3n}.$$

Since r_1 is the number of integers $\leqq n$ that are not multiples of any a, and r_3 is the number of integers $\leqq [n/3]$ having the same property, it is clear that $r_1 \geqq r_3$. It follows that

$$\sum_{i=1}^{k} \frac{1}{a_i} < \frac{7}{6} + \frac{1}{6n}.$$

As an easy corollary, $\sum 1/a_i < 6/5$.

Also solved by G. F. D. Duff, A. W. Fuller, D. J. Newman, and the Proposer.

Tetrahedron with Concentric Inscribed and Circumscribed Spheres

4366 [1949, 637]. *Proposed by Joseph Rosenbaum, Hartford, Connecticut.*

Determine the condition which two concentric spheres must satisfy in order that a tetrahedron can be simultaneously inscribed in one and circumscribed about the other. Give a construction for the tetrahedron.

Solution by the Proposer. It will be seen that the only requirement is

$$R \geqq 3r, \tag{1}$$

where R, r are the radii of the spheres S, s, having O as their common center. The argument is based on the known theorem that if any pair of the three centers —circumcenter, incenter, centroid—of a tetrahedron coincide, then all three coincide and, further, the four faces are congruent.

Now if $ABCD$ is a tetrahedron satisfying the given requirements, then the circumcircle C_1 of the face ABC is the intersection of S with a plane M tangent to s. The center O_1 of C_1 is the point of contact of M and s, and its radius is

$$R_1 = \sqrt{R^2 - r^2}. \tag{2}$$

Since O is the centroid of $ABCD$, the line DO pierces the face ABC in its centroid G, and

$$DO = 3 \cdot OG. \tag{3}$$

Let the plane N through D parallel to M cut S in the circle C_2. The center O_2 of C_2 is collinear with O_1 and O, and it follows from (3) that $OO_2 = 3 \cdot O_1O = 3r$, so that, since OO_2D is a right triangle, the radius O_2D of C_2 is

$$R_2 = \sqrt{R^2 - (3r)^2}. \tag{4}$$

The similarity of OO_1G and OO_2D gives

$$O_1G = R_2/3, \tag{5}$$

where R_2 is given by (4).

A construction for the tetrahedron is now easy. On a circle C_1 with radius R_1 as given by (2), take an arbitrary point for vertex C, and locate any point G satisfying (5). These can always be done if condition (1) is satisfied. Prolong CG half its length to M. From the preceding analysis it is easy to show that $CM < 2R_1$, so that M always falls within circle C_1. Through M draw the chord AB perpendicular to O_1M. Finally locate D as the intersection of GO with S. It follows that when condition (1) is satisfied, there exist in general infinitely many noncongruent tetrahedrons inscribed in S and circumscribed about s.

Considerable interest attaches to the problem of the set of tetrahedrons belonging to a given pair of circumscribed and inscribed spheres. The special case here treated points to the likelihood that, unlike its analogue in plane geometry the conditions on two spheres that they be the circumscribed and inscribed spheres of a tetrahedron (and perhaps also any polyhedron) is not an equality, but rather an inequality—something like $F(R, r, d) > 0$, where d is the distance between the two centers.

A Step Function Connected with the Law of the Mean

4371 [1949, 695]. *Proposed by Albert Wilansky, Lehigh University.*

Define $\theta(a, h)$ as the largest number θ satisfying

(i) $$0 < \theta < 1,$$

(ii) $$f(a + h) = f(a) + hf'(a + \theta h),$$

where $f(x) = x^2 \sin(1/x)$ for $x \neq 0, f(0) = 0$.

Now set $\lambda(h) = [h \cdot \theta(0, h)]^{-1}$. Prove that as h tends to zero, $\lambda(h)$ tends to infinity in a step function manner; specifically, given $\epsilon > 0$, there is a number $H(\epsilon)$ such that for every h with $|h| < H$ there is an integer $n(h)$ such that

$$|\lambda(h) - (n + \tfrac{1}{2})\pi| < \epsilon.$$

Solution by L. A. Ringenberg, Eastern Illinois State College. With $a = 0, h \neq 0$, (ii) becomes

$$h \sin(1/h) = 2\theta h \sin(1/\theta h) - \cos(1/\theta h).$$

Since $h \sin(1/h)$ and $2\theta h \sin(1/\theta h)$ each approach zero with h, it follows that $\cos(1/\theta h) = \cos\lambda$ also approaches zero with h. Given $\delta > 0$, there is an $H(\delta) > 0$ such that $|h| < H$ implies $|\cos\lambda| < \delta$. Given $\epsilon > 0$, there is a $\delta = \delta(\epsilon) > 0$ such that $|\cos\lambda| < \delta$ implies $|\lambda - (n + \frac{1}{2})\pi| < \epsilon$ for some integer n. Thus $|h| < H[\delta(\epsilon)] = H(\epsilon)$ implies

$$|\lambda - (n + \tfrac{1}{2})\pi| < \epsilon$$

for some integer n. Since $\lambda = \lambda(h)$ approaches infinity as h approaches zero, it follows that n is a function of λ, and hence of h.

We observe that if $\theta(a, h)$ is any number (not necessarily the largest) satisfying conditions (i) and (ii) then the conclusion of the problem is still valid except that n depends upon h and θ.

Also solved by N. J. Fine and E. G. Kimme.

Editorial Note. Fine, by similar analysis, shows that if $f(x)$ is $x^\alpha \sin(1/x^\beta)$, where $x > 0$, $\beta > 0$, and α is an arbitrary real number, then

$$|\lambda(h) - \{(n + \tfrac{1}{2})\pi\}^{1/\beta}| < \epsilon.$$

Sequence of Sines

4372 [1949, 695]. *Proposed by Ky Fan, University of Notre Dame.*

For what real values of x does the sequence

$$f_n(x) = \sin 7^n \pi x$$

converge and what is the limit?

Solution by Robert Steinberg, University of California, Los Angeles. Consider the more general problem with 7 replaced by any odd positive integer $q \neq 1$, and let angles congruent mod 2π be considered equivalent. If $f_n(x)$ converges, then

(I) The sequence of angles $s_n = q^n \pi x$ must either have a single limit t (in

the sense of the preceding sentence), or it must separate into two subsequences approaching t and $\pi-t$, respectively. In the former case, t satisfies the relation $qt\equiv t$ (mod 2π) and hence $t=2m\pi/(q-1)$, where m is any integer. In the latter case the subsequences must be $s_{2n-1}\to t$, $s_{2n}\to\pi-t$. (Otherwise it would be possible to select a subsequence of pairs of consecutive terms

$$s_{n_k},\ s_{n_k+1}\to t, \qquad k=1,2,\cdots,$$

whence t would satisfy $qt\equiv t$, and we have only the first case.) Therefore $qt+t\equiv\pi$ (mod 2π) and hence $t=(2m+1)\pi/(q+1)$. In both cases each of the sequences s_{2n-1} and s_{2n} has a single limit T which satisfies the relation $q^2T\equiv T$ (mod 2π).

(II) For some N, $s_N\equiv t$; and hence $s_n\equiv t,\ \pi-t$, for $n\geqq N$. For, otherwise, the relation

$$(1)\qquad s_{n+2}-t\equiv q^2(s_n-t), \qquad q^2>1,$$

would imply non-convergence for the sequence. Combining (I) and (II) we see that if the sequence $f_n(x)$ converges we have

$$x=\frac{2m}{q^N(q-1)}\quad\text{or}\quad x=\frac{2m+1}{q^N(q+1)},$$

where m and N are any integers, and the limits are respectively

$$\sin\frac{2m\pi}{q-1},\qquad \sin\frac{(2m+1)\pi}{q+1}.$$

Conversely if x has any of these values, $f_n(x)$ evidently converges to the limit stated.

Also solved by J. B. Kelly, J. G. Millar, L. A. Ringenberg, and the Proposer, and partially solved by P. M. Anselone and Chia-shun Yih.

Editorial Note. The above argument applies with little change when q is even. Then, however, the values $x=(2m+1)/q^N(q+1)$ do not satisfy, since

$$s_{n+1}+s_n\equiv 2\pi\ (\text{mod } 2\pi), \qquad n>N.$$

Hence, unless $s_n=0$ or π, $f_{n+1}=-f_n$, contrary to the convergence of f_n.

For those who might desire more detail, the argument at (1) is expanded as follows. Let $s_n-t\equiv 2\pi r_n(0\leqq r_n<1)$. Then

$$s_{n+2k}-t\equiv 2\pi(q^2)^k r_n, \qquad k=1,2,\cdots.$$

If r_n is irrational, the residues (mod 1), r_{n+2k} of $(q^2)^k r_n$ are dense in the interval 0, 1, contrary to convergence. If r_n is rational, its denominator d may be taken prime to q (with a larger value of n if necessary). If $d>2$, the residues r_{n+2k} form a finite set of fractions repeated cyclically infinitely often, contrary to convergence. If $d=2$ we have $s_{n+2k}-t\equiv\pi$, which gives non-convergence unless $t=0$ or π. If $d=1$, then $s_{n+2k}=t$ for all k.

RECENT PUBLICATIONS

EDITED BY E. P. VANCE, Oberlin College

All books for review should be sent directly to the editor of this department, American Mathematical Monthly, 531 West 116 Street, New York 27, N. Y. and not to any of the other editors or officers of the Association.

Primer of College Mathematics. By J. F. Randolph. New York. The Macmillan Company, 1950. 13+545 pages. $4.75.

This book is an interesting and well constructed text, readable, rigorous, discriminating in choice of material, and refreshing in manner of presentation. The format is good, the appearance of the pages uncrowded and attractive.

In the preface the author says, "This book is written to try to restore some of the unity to elementary mathematics." Trigonometry, college algebra, and analytical geometry are well blended throughout. Application is made at appropriate points to simple concepts of investment and statistics. An optional chapter is included on the calculus applied to algebraic and logarithmic functions. Intended for a freshman course, the "Primer" would provide training both thorough and efficient, since the author has seized the advantage of a unified approach by presentation of ideas just when the student is most receptive to them. The ample content and close integration of subject matter imply to the reviewer that the text would probably be used more successfully in a ten-hour than in a six-hour course.

The book opens with permutations and combinations. These lead to a study of integers, and this in turn to further review of topics in algebra. The student thus gets off to an interesting and absorbing start learning new material and recalling old. Throughout the text each new concept is introduced by use of a specific problem skillfully chosen and presented so that the idea which the author wishes to develop stands out clearly. Similarly, definitions and symbols are introduced in so natural a manner and as filling such an obvious need that students will use them without realizing that they are new and therefore probably hard. As important concepts recur in later parts of the text, their significance and power become more apparent. For instance, the introduction of curve plotting in a paragraph entitled *Sets of Ordered Pairs and Sets of Points* emphasizes the functional aspect of a graph and also makes use of parametric representation quite natural, although equations of curves have not yet been mentioned and formal work in graphical analysis also comes later.

This "spiral" approach is correct pedagogically when in the hands of as competent a writer as Professor Randolph. He has made sure that important concepts recur and that problems and exercises give continual cumulative review. By use of this text the student should augment his knowledge by a very sound, interesting, and satisfying process. The instructor should enjoy teaching it, for it abounds with new and fruitful methods of dealing with traditional material.

JULIA W. BOWER

Modern College Geometry. By D. R. Davis. Cambridge. Addison-Wesley Press, Inc., 1949. 9+238 pages. $4.50.

This book seems to strike a happy medium between over-emphasis upon construction as one extreme and classical theorem study as the other. The first six chapters concerning geometric procedures, loci, fundamental theorems, similar figures, auxiliary figures, and harmonic ranges should open up a new world to the average non-mathematical student who later might find himself trying to teach high school geometry.

The last four chapters bring to light many of the modern discoveries concerning the triangle and the circle. While the average reader may not remember for long the distinction between a Steiner point and a Tarry point, or the relation between a Miguel triangle and a Simpson line, yet his vision is enlarged by knowing where to find such interesting modern geometric concepts.

The author of this book has wisely included many of the essential high school plane geometry theorems in the first chapter. This feature is a welcome oasis for both teacher and student of college geometry.

A reasonable number of problems are available in each chapter. The bibliography of twenty-four books listed on page 233 should preclude the reader from believing that geometry is static or died with Euclid.

The author and publisher have rendered a valuable service to present day students and teachers of geometry.

W. R. Hutcherson

First Course in Probability and Statistics, Vol. I. By J. Neyman. New York. Henry Holt and Company, 1950. 9+350 pages. $3.50.

According to the author, this book is intended to provide material for a beginning one-semester course. Three different categories of students are contemplated: (1) students who would like to take just one course in mathematical statistics for purposes of general education (2) prospective future mathematical statisticians and (3) students who specialize or intend to specialize in one of the fields of application.

The purpose of this basic course is to introduce the most fundamental concepts of modern statistical theory and to connect these concepts with as many fields of application as is practicable. It is assumed that the student's mathematical education is limited to high school algebra.

It has long been felt by many statisticians that the basic ideas should be taught at the beginning of any sequence of courses in statistics, but some have expressed doubt that these could be made comprehensible to students with a very limited mathematical background. Because the author has been, and is, so intimately connected with the development of modern statistical theory as well as being a successful teacher in that field, he is eminently qualified to undertake this important task.

There are five chapters in this volume. The first chapter is an introduction and attempts to explain the scope of statistics. Chapter 2 introduces the

basic concepts of probability and the rules concerning the combinations of probabilities. The third chapter is concerned with applications of the theory of probability to genetics. In the fourth chapter the ideas of random variables and frequency distributions are discussed. The final chapter is the capstone of the book. In this chapter the author considers the theory of testing hypotheses and examines the ideas involved (errors of the first and second kind, power of a test, best critical regions, etc.) giving numerous examples to bring out the delicate points involved in these concepts. This is carefully done and will prove very valuable to the conscientious student.

Since this book is written for different classes of students there are starred sections which may be omitted with those students who do not have the mathematical preparation. Some ideas from the calculus are used and for those students who have no acquaintance with that discipline, the instructor will need to elaborate the passages in the book.

There are a number of interesting and novel problems phrased so as to stimulate the student's critical thinking. Emphasis is placed on careful and exact statements, which is so often lacking in many of the "practical" statistics texts.

There are several references given at the ends of Chapters 1, 3 and 5, some of these being fundamental research papers which will prove difficult reading for most of the students to whom this book is addressed. No reference is given at the end of Chapter 2, although there must be some that the student could read profitably.

With the help of a competent and well-trained instructor, this book can be most useful in introducing many basic concepts of statistics to students. For this purpose it is one of the best, if not the best, that has yet been written.

J. R. VATNSDAL

Introduction to the Theory of Statistics. By A. M. Mood. New York. McGraw-Hill, 1950. 9+433 pages. $5.00.

Although mathematical statistics has made great strides in the last twenty-five years, the subject remains, at least in part, an oral one. The principal media for initial orientation remain the lectures and notes of lectures in the relatively few schools where the modern theory is being developed. The result is that students without access to these sources are likely to be misled as to the nature of statistics and the problems to be investigated. Often competent mathematicians who attempt to work in statistics spend their time on unimportant problems.

The situation is little better when it is a question of instructing the people who will apply the theory to practical problems. The books in wide use are "cookbooks," that is, books which lay down dicta and describe rules, without explaining their logic, or the assumptions underlying the rules, or the precise consequences of the rules. The student whose knowledge comes exclusively from such books is at a loss when he encounters a problem not treated in these books

(he often cannot recognize that the problem has not been treated in his textbooks). These books describe at great length good methods of computing statistics and then devote little space to telling the student how to answer the questions which gave rise to the computations. A famous book discusses numerous tests of significance (tests of hypotheses) without once telling the reader what such tests are, the meaning of "null hypothesis," *etc.*

This book by Mood with its modest title is an excellent introduction to statistics. At present writing (May, 1950) only one other such book is known to the reviewer. Mood's book presupposes a knowledge of the calculus, but no prior knowledge of probability or statistics is required; indeed, the first part of the book is devoted to an introduction to probability. Numerous interesting problems carry the theory further, and are a challenge to the student and a blessing to the instructor. Among the chapter headings we may mention the following: point estimation, interval estimation, tests of hypotheses, regression and linear hypotheses, experimental designs and the analysis of variance, sequential tests of hypotheses.

The book, of course, has a few inadequacies. The description of a test of hypothesis on page 245 is very meager and inadequate. The central limit theorem is considerably more than the very special case described on page 136. Formula 6 on page 266 is incorrect. So is the last sentence in the first paragraph on page 157. The reviewer certainly does not agree that the whole problem of point estimation is virtually solved (page 161), nor would he agree with all other value judgments of the author. The reviewer regrets that the author has nowhere described the fact that many important problems in statistics are multi-decision problems for which the classical theory is inadequate. Such a discussion could well have accompanied a more adequate exposition of the notion of testing hypotheses. A few words about a modern approach to these problems and references to other literature would have appealed to the good student.

Some of the above criticisms represent divergences of opinion between the author and the reviewer, on which the informed reader must have his own opinion. Other criticisms refer to errors from which no book, no matter how excellent, can be free, and which could be corrected in subsequent editions. The author is to be highly complimented for a job well and painstakingly done.

JACOB WOLFOWITZ

NEW BOOKS RECEIVED

A Manual for the Slide Rule. By P. E. Machovina. New York, McGraw-Hill, 1950. 73 pp. $.75.

Wo steckt der Fehler? By W. Lietzmann. Leipzig, B. G. Teubner, 1950. 183 pp. No price.

Einführung in die Analytische Geometrie. By L. Bieberbach. Bielefeld, Verlag für Wissenschaft und Fachbuch G. M. B. H., 1950. 168 pp. No price.

Analytic Geometry and Calculus. By L. M. Kells. New York, Prentice-Hall, Inc., 1950. viii+623 pp. $4.75.

Corso di Matematica. Algebra—Parte Prima. By G. Andruetto and A. Corio. Turin, Italy, G. B. Paravia and Co. 1950. No price.

Operational Calculus. By B. Van der Pol and H. Bremmer. New York, Cambridge University Press, 1950. xiii+415 pp. $10.00.

Technological Applications of Statistics. By L. H. C. Tippett. New York, John Wiley and Sons, Inc., 1950. ix+189 pp. $3.50.

Algebraic Technique of Integration. By H. F. MacNeish. Florida, University of Miami Press. viii+109 pp. No price.

Elasticity, Vol. III, Proceedings of Symposia on Applied Mathematics. By Churchill, Reissner and Taub. New York, McGraw-Hill Book Co., 1950. v+233 pp. $6.00.

Fourier Transforms (Annals of Mathematics Studies, No. 19). By S. Bochner and Chandrasekharan. Princeton University Press, 1949. 219 pp. $3.50.

The Nomogram. By H. J. Allcock and J. Reginald Jones. New York, Pitman Publishing Corp., 1950. x+238 pp. $3.75.

Differential and Integral Calculus. By Edmund Landau. New York, Chelsea Publishing Company, 1950. 366 pp. No price.

Elementary Theory of Equations. By Samuel Borofsky. New York, The MacMillan Co., 1950. x+302 pp. $4.25.

Mathematical Engineering Analysis. By Rufus Oldenburger. New York, The MacMillan Co., 1950. xiv+426 pp. $6.00.

Principles of Finance and Investment. By L. G. Whyte. Cambridge, England, Cambridge University Press, 1950. 8+176 pp. $2.50.

Introduction to Linear Algebra and the Theory of Matrices. By Hans Schwerdtfeger. Holland, P. Noordhoff, 1950. 280 pp. 17½ Guilders.

Teirica dell'ammortamento. By Insolera, Filadelfo. Turin, Italy, Giulio Einaudi, 1950. 304 pp. Paper bound, Lire 2400.

Introduction to Mathematical Logic and Its Applications. By Ira Rosenbaum. Florida, University of Miami Press, 1950. 98 pp. No price.

Einführung in die Höhere Algebra. By Gunter Pickert. Gottingen, Vandenhoeck and Ruprecht, 1951. 298 pp. DM 14.80.

Mathematics—Queen and Servant of Science. By E. T. Bell. New York, McGraw-Hill, 1951. xx+437 pp. $5.00.

Life Insurance and Mathematics. By R. E. Larson and Erwin A. Gaumnitz. New York, John Wiley and Sons. vii+184 pp. $3.75.

The Hodograph Method in Gas Dynamics. By Dr. A. G. Ghaffari. Tahran Taban Press, 1950. iv+129 pp. No price.

College Algebra. By H. K. Fulmer and Walter Reynolds. Boston, Ginn and Company, 1951. 204+xiv pp. $2.85.

Demography. By P. R. Cox. New York, Cambridge University Press, 1951. xii+326 pp. $4.00.

Statistics, Vol. II. By N. L. Johnson and H. Tetley. New York, Cambridge University Press, 1951. xii+318 pp. $4.00.

CLUBS AND ALLIED ACTIVITIES

EDITED BY L. F. OLLMANN, Hofstra College

Send reports of all activities, such as club reports, special features, topics with references, student papers, and other material of interest to L. F. Ollmann, Hofstra College, Hempstead, New York.

CLUB REPORTS, 1949–50

EDITORIAL NOTE. Letters have been sent to the Heads of Mathematics Departments at various colleges and universities, requesting that the Sponsor of the Mathematics Club, Society, or Fraternity at that institution record the activities of his organization with the Editor of this department of the MONTHLY. The reports should be sent in immediately, as the publication date will depend on the order in which the records have been received. All mathematics organizations are invited to submit a neatly written summary of their activities, even though no letter has been received. When writing up the activities, please follow the form which is used on these pages.

Mathematics Society, St. John's College

The *Mathematics Society* of St. John's College reports the following activities for 1949–50:

Professor Tolle of St. John's University Graduate School spoke on *Transfinite numbers* at one of the meetings.

The society publishes *The Mathazine*, a magazine for undergraduates, to help publicize the mathematics department and the society. A guidance clinic, to aid those who have not yet decided upon a career, was held at which speakers discussed the various possibilities open to mathematicians and the necessary prerequisites for each. The society made a field trip to Brookhaven National Laboratories at Upton, New York.

The year's activities were climaxed by a banquet at which an award was made to Gennaro Montanino, the winner of the Mathematics Contest in the college. Certificates were also awarded to members elected to the Honors Council, a group of outstanding mathematics students.

Officers elected for the year 1950–51 were: President, Carl Wampole; Secretary, Richard Ziesig.

Mathematics Club, University of Dayton

The *Mathematics Club* of the University of Dayton met bi-weekly throughout the school year 1949–50. The papers presented at these meetings were:

An introduction to group theory, by Joseph Schell

Asymptotes to curves of infinite extent, by Eduardo Mulanovich

The phase rule, by Jorge Nunez

Polynomial divisibility, by Demetriuz Zonars

The sign of the quadratic polynomial, by Aspasia Zonars

Physical interpretation of concepts in vector analysis, by Richard Segers

Numerical solution of Laplace's equation as applied to photoelasticity, by Joseph Gallagher

Servomechanism analysis, by Robert A. Thomson, S.M.

Matrix algebra, by T. David Riney

Bessel functions and their application to cylindrical wave guides, by Albert Windeljohaun

Evaluation of numbers, by Dr. K. C. Schraut.

The club held a joint meeting with the *Chemistry Society* during the winter season at which Dr. Burbage of the Monsanto Chemical Company lectured on *The use of radio-active materials*.

In the spring, several members of the club accompanied Dr. Schraut to the University of Michigan for an undergraduate conference on Mathematics in conjunction with the Michigan Section of the Association. At this meeting the following two papers were presented by members of our group:

The inverse integral of the Laplace transformation, by Christof Neugebauer

The complex fourier series, by Richard Segers.

Demetrius Zonars was presented the Dean of Science award at the annual dinner for the best paper of the first semester and Richard G. Segers received the award for the second semester.

Prof. M. O. Thurston, Acting Head of the Department of Physics of the Air Force Institute of Technology, spoke on *Oscillations, classic and atomic* at the final meeting of the year.

The officers for 1949–50 were: President, Joseph Schell; Vice-President, Gordon Mills; Recording Secretary, T. David Riney; Publicity Secretary, Bro. R. A. Thomson, S.M.; Treasurer, Hilary Allemeier; Faculty Advisor, Dr. K. C. Schraut.

James G. White Mathematics Club, University of Kentucky

The *James G. White Mathematics Club* is open to anyone interested in mathematics and held seven meetings during the 1949–50 academic year. The following programs were presented:

Some applications of Archimedes theories, by D. C. Rose

A few remarks on tan $n\theta$, by M. C. Brown

Committee report on a six months' survey of the occupational opportunities for Mathematics majors, by Allen Wilson, Ernest Steele, Genevieve Snider, and John Wells

Mathematical curiosities, by Dr. H. H. Downing

Matrices, by Dr. J. A. Ward.

Officers serving for the year 1949–50 were: President, Lillian Griffey; Secretary-Treasurer, Ann Wiesman; Faculty Sponsor, Ruric E. Wheeler.

Order of Magnitude, Pasadena City College

The *Mathematical Section* of the *Order of Magnitude*, an organization made up of mathematics students and astronomy students, at Pasadena City College held the following programs during 1949–50:

Magic squares, by Marvin Ridley
Apollonius' problem, by Robert Phelan
Angle trisections, by Ray Kilgrove
Binary numbers, by Marvin Ridley.

The club meets every two weeks, at which times a member of the club or a guest speaker gives a talk on some interesting mathematical topic or a related subject. The Mathematical Section and the Astronomy Section had one joint meeting during the year.

Social activities included a Christmas Party and a trip to the Calico Mountains.

Officers for the first semester were: President, Marvin Ridley; Vice-President, Andy Markell; Secretary-Treasurer, Robert Phelan.

Officers for the second semester were: President, Robert Phelan; Vice-President, Andy Markell; Secretary-Treasurer, Nancy O'Dell.

Pi Mu Epsilon, Hunter College

During 1949–50, the *New York Beta* chapter of *Pi Mu Epsilon*, continued its practice of presenting papers at each meeting.

The topic for the Fall semester was *Vector analysis*, and papers were presented by: Ruth Grabenheimer, Therese Schwarz, Shirley Schneiderman, Mary Shanahan, Yvonne Roach, Doris Cohen, Robert Sanders, Edith Bruton, Goria Bullock, Joan Salatino, Beverly Schiff, and Fay Gorenstein.

The topic for the Spring semester was *Algebraic Geometry*, with the following participating: Doris Matthews, Eloise Solomon, Carla Schmitz, Pearl Goldstein, Hermia Hochberg, Seymour Goldberg, Carol Weisner, Marie Roston, Flora Johns, and Hanna Fischer.

The main event of the year was the 25th Anniversary Dinner held in April. Dr. Tomlinson Fort, Former Chairman of the Department of Mathematics at Hunter College at the time of the founding of the New York Beta chapter of Pi Mu Epsilon, addressed the group. He urged all the women present to take an active part in the mathematical research being conducted at the present time.

Dr. H. M. MacNeille, Executive Director of the American Mathematical Society, spoke on *The International Congress of Mathematics*. Professor Bushey introduced some outstanding alumnae who spoke to the members about their experiences and the work they are engaged in.

The following officers were elected for 1950–51: President, Marie Barbiere; Corresponding Secretary, Ursel Kramer; Recording Secretary, Dorothy Freudenberger; Treasurer, Hannah Stein.

NEWS AND NOTICES

EDITED BY EDITH R. SCHNECKENBURGER, University of Buffalo

Readers are invited to contribute to the general interest of this department by sending news items to Edith R. Schneckenburger, University of Buffalo, Buffalo 14, New York. Items must be submitted at least two months before publication can take place.

INSTITUTE FOR TEACHERS OF MATHEMATICS

The University of California at Los Angeles announces that an Institute for Teachers of Mathematics will be held on its campus, July 9–20, 1951. Professor W. W. Rankin of Duke University will be the Director of the Institute.

Two lectures daily will be given by outstanding men from business, industry, and teaching. The following study groups will be conducted daily by leaders of recognized ability chosen from teachers in the various fields of mathematical instruction:

1. Aids in the teaching of geometry and algebra.
2. The development and coordination of basic ideas in mathematics.
3. Understanding in the teaching of arithmetic.
4. The teaching of applications of mathematics used in industry and other fields.
5. Use of mathematical instruments.
6. Modern trends in the organization of secondary curricula in mathematics.

Certificates of attendance will be issued to those registered in the Institute. It will also be possible for registrants to earn two units of college credit by attending lectures and by participating in a sufficient number of study groups.

Application blanks for the Institute and further details may be obtained by writing to Professor Clifford Bell, Mathematics Department, University of California, Los Angeles 24, California.

ANNUAL SUMMER MEETING OF THE NATIONAL COUNCIL OF TEACHERS OF MATHEMATICS

The National Council of Teachers of Mathematics is holding its annual Summer Meeting at St. Olaf College, Northfield, Minnesota, on August 20–23, 1951. Outstanding teachers and nationally known leaders in the field of mathematics will conduct meetings and present demonstrations at all levels of mathematics. The meetings will include general sessions, sectional meetings, laboratory sessions, discussion periods, and study groups. These study groups will be under the direction of prominent teachers and leaders for a continuing three-day session and will have limited enrollments. The latest in audio-visual aids will be exhibited; films, film-strips, models. Anyone having a model that has proven useful in the teaching of mathematics is asked to exhibit the model at this con ference. Write to Mr. Emil Berger, Monroe High School, St. Paul, Minnesota, for information if you wish to exhibit material.

The college dormitories will provide adequate accommodations; meals may be obtained at the college cafeteria. Reservations for rooms and registration

for study groups should be made as soon as possible. Send reservations or requests for information and registration forms to Professor C. S. Carlson, St. Olaf College, Northfield, Minnesota. A registration and reservation form will appear in the May issue of *The Mathematics Teacher*.

CARE-UNESCO BOOK FUND

The CARE-UNESCO Book Fund is a program which supplies important new technical books to universities, libraries and medical and scientific centers overseas. Its primary purpose is to bring as many professional people as possible up-to-date on current developments in their fields. The Fund has been established in cooperation with UNESCO, the Library of Congress, the American Library Association, medical and scientific groups, and governmental authorities overseas.

By making a general contribution, in any amount, you can share directly in this important program. Your gift will be allocated on the basis of lists compiled by the Fund in consultation with UNESCO and ministries of education. Individuals and organizations giving $10 or more may request books for a specific institution or indicate their choice of country, type of institution and category of books.

Further information may be secured by writing to the CARE-UNESCO Book Fund, 20 Broad Street, New York 5, or your local CARE Office. The Fund will be glad to advise as to where books are most needed and to provide interested groups with additional material, posters and films.

SUMMER COURSES

The following institutions announce advanced courses in mathematics for the summer of 1951:

Columbia University, Teachers College. July 2 to August 10: Mr. Campbell, professionalized subject matter in junior high school mathematics, current problems in teaching arithmetic; Professor Carnahan, teaching algebra in secondary schools, teaching geometry in secondary schools; Professor Clark, teaching arithmetic in the elementary school; Professor Fehr, current problems in teaching secondary school mathematics, professionalized subject matter in advanced secondary school mathematics; Professor Shuster, business mathematics, field work in mathematics; Professor Yates, applications of plane geometry, teaching of elementary college mathematics.

De Paul University. June 25 to July 31: Professor Caton, probability, mathematical foundations of statistics; Professor De Cicco, infinite series, finite differences.

Oklahoma Agricultural and Mechanical College. June 18 to July 14: Professor Ahlfors of Harvard University, conformal mapping. June 18 to July 2: Professor Tarski of University of California at Berkeley, arithmetical classes and types of mathematical systems.

University of California at Berkeley. June 18 to July 28: Professor Davis of Northwestern University, non-linear problems.

University of Tennessee. June 11 to August 24: Staff, ordinary differential equations, theory of equations, modern geometry; Professor Harrold, topology; Professor Snyder, theory of measure (given at Oak Ridge Institute of Nuclear Studies); Professor Wilson, theory of matrices, complex variable (given at Oak Ridge Institute of Nuclear Studies).

PERSONAL ITEMS

Professor W. W. S. Claytor of Howard University was the representative of the Association at the celebration of the One Hundredth Anniversary of Miner Teachers College, Washington, D. C. on March 4–10, 1951.

Professor P. A. Caris of the University of Pennsylvania and Professor C. O. Oakley of Haverford College were the delegates of the Association at the Fifty-fifth Annual Meeting of the American Academy of Political and Social Science, which was held in Philadelphia, Pennsylvania, April 6–7, 1951.

Arkansas Polytechnic College announces that as a result of its transition from a junior college to a four year college conferring degrees academic rank has been granted to its teaching staff as follows: Professor J. R. Abernithy, Associate Professor Maggie B. Davis, Assistant Professor E. H. Ham, Assistant Professor Aleze G. Fullerton, Assistant Professor Maude Moore.

De Paul University announces the promotions of Assistant Professor W. B. Caton to an associate professorship and Lecturer E. P. Merkes to an instructorship.

Trinity College, Hartford, Connecticut, reports that Dr. E. N. Nilson of Trinity College and Mr. S. L. Crossman and Mr. Walter Ramshaw of the United Aircraft Computing Laboratory have developed a course which combines numerical mathematical analysis with the use of IBM punch card computing machinery. Lectures on numerical analysis and machine methods are given at the College, supplemented by a laboratory period at the United Aircraft Computing Laboratory where students work with the latest types of IBM electronic computing equipment.

University of Alberta makes the following announcements: Associate Professor E. S. Keeping has been promoted to a professorship; Assistant Professor Max Wyman has been promoted to an associate professorship; Dr. D. R. Crosby has been appointed to an assistant professorship; Mr. T. M. Fostvedt, Mr. R. C. Jacka, Mr. A. Shaw and Mr. L. A. Fisher have been appointed Lecturers.

Miss Florence R. Anderson of the University of Southern California has accepted a position as research laboratory analyst with Northrop Aircraft, Incorporated.

Assistant Professor Lulu Bechtolsheim of the University of Redlands has been promoted to an associate professorship.

Mr. Jonas Beraru, formerly a research analyst at North American Aviation, Incorporated, is now a mathematician at the Rand Corporation.

Miss Mary P. Burkart, previously a graduate assistant at the University of Detroit, has been appointed to an instructorship at Trinity College, Washington, D. C.

Professor Emeritus W. B. Carver of Cornell University has been teaching at the University during the second semester of the current academic year.

Mr. E. J. Cogan of Pennsylvania State College at Pottsville has been transferred to Pennsylvania State College at State College, where he has a position as part-time instructor.

Mr. K. L. Cooke, previously a graduate student at Stanford University, has been appointed to an instructorship at State College of Washington.

Mr. G. A. Culpepper of the University of Colorado has accepted a position as mathematician at White Sands Proving Ground.

Mrs. Alice B. Dickinson, who was a teaching fellow at the University of Michigan, has been appointed to an instructorship at Pennsylvania State College.

Dr. O. L. Dustheimer has been appointed Pension Consultant for D. Miley Phipps and Associates, Cleveland, Ohio.

Miss Ruth B. Eddy is teaching at Hope High School, Providence, Rhode Island.

Mr. R. L. Eisenman of the University of Connecticut has been appointed to an instructorship at the University of Maryland.

Mr. W. E. Felling, graduate student at St. Louis University, has been appointed to an instructorship at Parks College.

Mr. F. H. Fisher, previously a teaching fellow at West Virginia University, has a position as teacher at McDonald High School, McDonald, Pennsylvania.

Mr. A. G. Hansen of the University of Maryland has accepted a position as aeronautical research scientist for the National Advisory Committee for Aeronautics, Cleveland, Ohio.

Mr. J. J. Hart, previously a graduate student at the University of Alabama, has been appointed to an instructorship at Tennessee Polytechnic Institute.

Dr. R. T. Hood of Beloit College is now Pastor of the Lincoln Baptist Church, Macedon, New York.

Professor L. A. Hopkins of the University of Michigan has retired with the title of Professor Emeritus.

Dr. C. C. Hsiung of the University of Wisconsin has been appointed Lecturer at Northwestern University.

Assistant Professor Fritz John of New York University has a position as mathematician at the Institute for Numerical Analysis.

Mr. L. G. Jones, formerly a graduate assistant at the University of Oregon, has been appointed to the position of mathematician in the Physical Research Unit, Boeing Airplane Company, Seattle, Washington.

Assistant Professor Wilfred Kaplan of the University of Michigan has been promoted to an associate professorship.

Dr. Samuel Karlin of California Institute of Technology has been appointed to an assistant professorship at Princeton University.

Mr. C. E. Kelley of the University of Missouri has accepted a position as teacher in the public school system of Texarkana, Texas.

Dr. P. J. Kelly of the University of Southern California has been appointed to an associate professorship at the University of California, Santa Barbara.

Mr. A. L. Lanckton of Socony Vacuum Oil Company, Athens, Greece, has been transferred to Istanbul, Turkey.

Mr. J. N. P. Lawrence, previously a student at Johns Hopkins University, has an A.E.C. Radiological Physics Fellowship at Vanderbilt University.

Mr. Joseph Levitt of Pratt Institute has been appointed Process Engineer at the M. W. Kellogg Company, Jersey City, New Jersey.

Mr. R. W. Long of New York University has been appointed to an assistant professorship at Washington and Jefferson College.

Mr. W. C. Lowry of Kent State University is now a part-time instructor at Ohio State University.

Mr. Edward McGaughy of Lawrence College has been appointed Lecturer at Columbia University.

Professor B. E. Mitchell, Millsaps College, has been appointed to a professorship at the University of Mississippi.

Mr. Benjamin Mittman, previously a student at Illinois Institute of Technology, has been appointed Teaching Assistant at the University of California at Los Angeles.

Mr. D. S. Park has accepted a position as mathematician at the National Bureau of Standards.

Mr. M. O. Peach, Carnegie Institute of Technology, has been appointed to an associate professorship at the University of Notre Dame.

Assistant Professor C. L. Perry, Jr., University of Arkansas, has a position as mathematician at the Oak Ridge National Laboratory.

Dr. G. W. Petrie, III, has been appointed Government Representative for the International Business Machine Corporation, Washington, D. C.

Associate Professor L. V. Robinson of the University of South Carolina has a position as mathematician at the Aberdeen Proving Ground, Maryland.

Mr. W. G. Rouleau of George Washington University has accepted a position as mathematician at the Army Map Service, Washington, D. C.

Mr. J. W. Sawyer of the University of Missouri has been appointed to an assistant professorship at the University of Georgia, Atlanta Division.

Mr. O. T. Schultz, formerly assistant section head of research at Curtiss-Wright Corporation, Ohio, has been appointed to a position as physicist at the Naval Ordnance Laboratory, Silver Spring, Maryland.

Assistant Professor James Singer of Brooklyn College has been promoted to an associate professorship.

Assistant Professor D. L. Thomsen, Jr., Haverford College, has received an appointment as research fellow at California Institute of Technology.

Mr. P. M. Treuenfels has a position as mathematician in the Ballistics Research Laboratories, Aberdeen Proving Ground, Maryland.

Assistant Professor R. S. Britton of New York University died on February 2, 1951.

Professor Emeritus Daniel Buchanan of the University of British Columbia died on December 1, 1950. He was a charter member of the Association.

Associate Professor Emeritus Otto Dunkel of Washington University died on January 16, 1951. He was a charter member of the Association. A more detailed statement will be given in a forthcoming issue of this journal.

Professor Emeritus N. B. Heller of Temple University died on January 31, 1951.

Assistant Professor J. G. Millar of the Calgary Branch of the University of Alberta died on December 9, 1950.

Professor Emeritus G. A. Miller of the University of Illinois died on February 10, 1951. He was a charter member and honorary life member of the Association. A more detailed statement will be given in a forthcoming issue of this journal.

Professor J. F. Ritt of Columbia University died on January 5, 1951. He had been a member of the Association for twenty-eight years.

Professor Emeritus W. H. Roever of Washington University died on January 31, 1951. He was a charter member of the Association.

Professor Abraham Wald of Columbia University died on December 13, 1950.

Professor Emeritus F. S. Woods of Massachusetts Institute of Technology died on December 1, 1950. He was a charter member of the Association.

THE MATHEMATICAL ASSOCIATION OF AMERICA

Official Reports and Communications

OCTOBER MEETING OF THE MINNESOTA SECTION

The fall meeting of the Minnesota Section of the Mathematical Association of America was held at the University of Minnesota, Duluth Branch, in Duluth, Minnesota, on Saturday, October 7, 1950. Sessions were held in the forenoon, at luncheon, and in the afternoon. Professors W. R. McEwen, F. C. Smith and H. M. Anderson presided at the respective sessions.

Forty-three persons attended the meeting, including the following twenty-seven members of the Association: H. M. Anderson, F. J. Arena, J. E. Bearman, L. E. Bush, H. D. Colson, J. C. Cothran, C. B. Germain, Ruby M. Grimes, A. G. Hill, J. S. Hill, J. E. Hafstrom, W. C. Kalinowski, Karlis Kaufmanis, W. H. McBride, W. R. McEwen, A. G. Montgomery, M. J. Norris, L. W. Sheridan, S. C. Simonson, Sister M. Mercedes, Sister M. Prudentia, F. C. Smith, R. C. Staley, Irwin Stoner, A. G. Swanson, Takaski Terami, Matilda B. Thompson.

The committee on high school contests reported that the committee on contests of the High School Principals' Association of Minnesota had not approved the application of the Minnesota Section to hold a mathematics contest in the high schools of the State, and that the committee felt that without the coopera-

tion of the high school officials it would be futile to make any further attempt to hold a mathematics contest.

By invitation of the executive committee, Professor R. E. Graves of the University of Minnesota delivered an address at the morning session. The title of his address was *Intuitionism versus Formalism.* Abstract of this address follows:

As a science develops it frequently has occasion to re-examine its own foundations, to modify them, or, more rarely, to discard these foundations entirely and to develop a new set in order better to meet certain objectives or subjective criteria of that science. To a large extent, such a critical analysis of foundations may appear to belong to the domain of philosophy, or even psychology, rather than to the specific science under consideration. However, the results of such analysis may have a profound and far-reaching influence upon the entire structure of the science. From these remarks, it should be evident that the borderline between a science and the philosophy of that science may be of a somewhat dubious character. Perhaps as good a way as any to make the distinction is to say that a scientist makes investigations *in* his science, while the philosopher of the science makes investigations *about* the science. Of course, on the basis of such a definition there are many who must be classed as both scientist and philosopher. After these introductory remarks, the speaker mentioned the various schools of thought concerning the foundations of mathematics. The remainder of the address was devoted to a comparison of the main features of the *intuitionism* of Brouwer and the *formalism* of Hilbert. In this comparison, particular emphasis was placed upon the radically different roles assigned to language (natural or formal) in the two doctrines.

The following five short papers were presented:

1. *Recurrence relations for the n-th derivative of the circular functions*, by Professor F. J. Arena, North Dakota State College.

Often in certain Taylor series expansions the successive derivatives of the circular functions are needed. The work of calculating the successive derivatives of some of the circular functions becomes rather complicated. In order to simplify this task the speaker gave recurrence relations for the n-th derivative of the circular functions, and then calculated the first six derivatives of tan $(ax+b)$ and sec $(ax+b)$.

2. *Alternative methods of treating increasing and decreasing annuities*, by Professor F. C. Smith, College of St. Thomas.

In most elementary texts in the mathematics of finance, the formula for the amount of an increasing annuity and the formula for the present value of a decreasing annuity are developed, but no direct methods are given for calculating the present value of an increasing annuity or the amount of a decreasing annuity. In this paper, the author proves that if one takes the increment negative, the formula for the amount of an increasing annuity will give the amount of a decreasing annuity; and if one takes the decrement negative, the formula for the present value of a decreasing annuity will give the present value of an increasing annuity.

3. *A slide rule for steam power plant loading calculations*, by Professor A. G. Montgomery, College of St. Thomas and Northern States Power Company.

Steam-electric generating stations in an interconnected system are assigned their power load so that the derivatives of their total cost-vs-power functions are equal. The calculations previously required were outlined in the case of three stations whose cost functions were inter-related by transmission line losses. It was then shown how this procedure—requiring construction of numerous graphs with various multiplications and divisions with values obtained therefrom—was simplified and greatly shortened by use of a special slide rule. It was noted that the calculations are readily adaptable to the use of amplifiers and associated servo-mechanisms but initial cost may prove prohibitive.

4. *A note on W-surfaces*, by Mr. Dale Woods, North Dakota State College (introduced by Professor A. G. Hill).

After a discussion of surfaces of centers, general W-surfaces and special W-surfaces, the author made use of a theorem about convex W-surfaces to derive some interesting theorems about the euclidean sphere.

5. *Some necessary convergence conditions*, by Professor M. J. Norris, College of St. Thomas.

The obvious generalization, and the analog for integrals, of the following result is proved. If $\sum a_n$ is a convergent series of positive terms and na_n is decreasing, then $a_n n \log n$ approaches zero.

L. E. Bush, *Secretary*

NOVEMBER MEETING OF THE OKLAHOMA SECTION

The annual meeting of the Oklahoma Section of the Mathematical Association of America was held in connection with the convention of the Oklahoma Education Association in Oklahoma City on Monday, November 13, 1950. Professor J. T. Krattiger, Chairman of the Section, presided.

Sixty-six persons attended the meeting, including the following thirty-five members of the Association: E. F. Allen, R. V. Andree, Jean M. Baldwin, Arthur Bernhart, J. C. Brixey, N. A. Court, R. B. Deal, A. H. Diamond, R. C. Dragoo, N. A. Eisen, A. A. Grau, E. V. Greer, L. D. Gregory, Claire A. Harrison, J. O. Hassler, W. N. Huff, S. L. Hull, H. V. Huneke, P. W. M. John, J. T. Krattiger, J. E. LaFon, H. I. Lane, H. W. Linscheid, R. D. McDole, G. E. Meador, R. D. Morrison, R. R. Murphy, C. M. Pirrong, D. P. Richardson, W. E. Roth, Harold Shniad, D. R. Shreve, T. C. Smith, C. E. Springer, J. H. Zant.

At the business session the following officers were elected: Chairman, C. M. Pirrong, Jr., Oklahoma City University; Vice-Chairman, R. R. Murphy, Panhandle A. and M. College; Secretary, J. C. Brixey, University of Oklahoma.

The program consisted of the following seven papers.

1. *The Laplace transform in a boundary value problem*, by Professor S. L. Hull, University of Arkansas.

Attention was centered on the use of the Laplace transform in solving certain partial differential equations.

2. *A new method of computing matric inverses*, by Professor R. V. Andree, University of Oklahoma.

This paper was published in this Monthly, vol. 58, pp. 87–92.

3. *The solution of equations in matrices*, by Professor W. E. Roth, University of Tulsa.

A solution, X, of the unilateral equation

$$A(X) = A_0X^m + A_1X^{m-1} + \cdots + A_m = 0,$$

where the A_i, $i=0, 1, 2, \cdots, m$, are $n\times n$ matrices, must be such that its characteristic function, $\phi(\lambda)$, is a divisor of the determinant $|A(\lambda)|$. This fact makes it possible to determine a second equation, $\phi(X)=0$, which may be solved simultaneously with the given equation, $A(X)=0$, by

eliminating all powers of X higher than the first between them. This procedure was discussed and examples showing its application and limitations were given.

4. *A simple postulational development of the real numbers*, by Mr. R. B. Deal, University of Oklahoma.

Postulates and definitions for the real numbers were given. The positive integers were characterized as a well ordered set S such that: (1) S has no last; (2) every bounded set has a last. This system was proved to be categorical.

5. *A comparison of the classical notions of completeness for the real numbers*, by Professor Casper Goffman, University of Oklahoma, introduced by Professor J. C. Brixey.

This was an expository report on the relation between completeness in the sense of Archimedes and in the Cantor Dedekind sense for ordered abelian groups.

6. *Loxodromes on a general surface*, by Professor C. E. Springer, University of Oklahoma.

In this paper the differential equations of the curves on a general surface which make a constant angle with the curves along which the curvilinear coordinate $u^1=$constant were obtained. These equations were compared with the differential equations of the union curves with respect to a congruence of lines referred to the surface. A unique congruence was exhibited with respect to which the union curves coincide with the loxodromes. In particular, it was shown that the loxodromes of a sphere are union curves with respect to the congruence of lines which have the directions of the radii of the circles of latitude on the sphere.

7. *Pythagorean angles*, by Professor Arthur Bernhart, University of Oklahoma.

Consider the angles in a right triangle with integral sides. These are the angles whose sines, cosines and tangents are all rational, and half such angles have rational tangents; and conversely. The sum (or difference) of two Pythagorean angles is again Pythagorean, so that in any triangle if two angles are Pythagorean, the third is also. Half an angle is still Pythagorean if the subtended leg is even and the hypotenuse is an odd square.

Let $T_n(C_n, S_n)$ be an angle with rational tangent (cosine, sine) but whose other functions involve the square root of n. Then $T_m \pm T_n = T_{mn}$ as in the addition of logarithms. Similarly $C_n+C_n = C_n$, $C_n+S_n=S_n$, and $S_n+S_n=C_n$ as in the addition of even (C) and odd (S) integers.

An arbitrary triangle with integral sides has each angle of type C_n, indeed with the same n. The cosines are rational, but the sines, tangents, altitudes, inradius, circumradius and area are similar surds. Conversely if $C_n \pm C_{n'} = C_{n''}$, then $n=n'=n''$, and these are the angles of a triangle with commensurable sides. The type $C_1=S_1=T_1$ are the Pythagorean angles.

J. C. BRIXEY, *Secretary*

NOVEMBER MEETING OF THE PHILADELPHIA SECTION

The annual meeting of the Philadelphia Section of the Mathematical Association of America was held at Lehigh University, Bethlehem, Pennsylvania, on Saturday, November 25, 1950. In the absence of the Chairman of the Section, Professor G. C. Webber, University of Delaware, the meeting was conducted by the Secretary.

There were thirty-three present, including the following twenty-seven members of the Association: C. B. Allendoerfer, Laura M. Ashbaugh, Joshua Barlaz,

D. W. Blackett, R. K. Brown, J. O. Chellevold, F. E. Clark, C. D. Firestone, H. S. Grant, Theodore Hailperin, S. T. Hu, B. C. Kenny, P. A. Knedler, T. L. Koehler, V. V. Latshaw, L. J. Mordell, C. O. Oakley, W. B. Pitt, G. E. Raynor, H. A. Seebald, C. A. Shook, L. L. Smail, E. P. Starke, R. R. Stoll, Albert Wilansky, R. H. Wilson, Jr., H. J. Zimmerberg.

At the business meeting the following officers were elected for the coming year: Chairman, P. A. Caris, University of Pennsylvania; Secretary, C. O. Oakley, Haverford College. The Program Committee for the next meeting will be E. P. Starke (Chairman) Rutgers University, Alexander Tartler, Lafayette College, and A. W. Tucker, Princeton University. The 1951 meeting of the section will be held at the University of Pennsylvania on November 24.

The program consisted of the following papers:

1. *The essential roughness of mathematical objects*, by Professor Albert Wilansky, Lehigh University.

Given a set and a smoothness property of some of its elements, then most of the elements of the set do not have the property. A smoothness property is one such that if f and g have the property so does $f+g$, or $af+bg$. For example, continuity of functions, rationality of numbers. "Most" can be interpreted in such senses as category, denumerability.

Use is made of classical ideas such as theorems 2, p. 22, and 1, p. 36 of Banach's monograph, *Studia Math.* 1 (1929); p. 51 (Steinhaus), p. 92 (Mazurkiewicz), p. 174 (Banach), *Studia Math.* 3 (1931); art. 8.33, p. 173 of Zygmund's monograph, *Trigonometrical Series*; and also the ideas of Cantor. Some of these can be cast in the language used above. An extra condition (Baire property) on the "smooth subspaces" appears. This cannot be omitted as shown by an example, due to Hausdorff, of a proper subspace of a Banach space which is of the second category.

2. *Systems of axiomatic set theory*, by Professor C. D. Firestone, Rutgers University.

Among the various systems of mathematical logic which are adequate for all or part of classical mathematics, a prominent part is played by those known as systems of "axiomatic set theory." We consider three such systems: the Zermelo set theory (Z), the Zermelo-Skolem set theory (ZS), and the Godel set theory (G).

It is shown that Z is inadequate for certain parts of classical mathematics, and it is indicated that G is probably adequate for all of known classical mathematics. Further, it is proved that Z is weaker than either ZS or G and that ZS and G are of equal strength. We consider also the possibility of constructing systems of set theory which are stronger than G and a simple and natural procedure is indicated for constructing an infinite sequence of such systems each stronger than its predecessor.

3. *The coefficients of Schlicht functions*, by Professor Bernard Epstein, University of Pennsylvania (introduced by the secretary).

This paper was given by title. Let S be the family of functions $f(z)=z+\sum_{n=2}^{\infty} a_n z^n$ which, in the circle $|z|<1$, are regular and schlicht—i.e., if $|z_1|<1$ and $|z_2|<2$ and $f(z_1)=f(z_2)$, then $z_1=z_2$. Similarly let Σ be the family of all functions $g(z)=z+\sum_{n=1}^{\infty} b_n z^{-n}$ meromorphic and schlicht for $|z|>1$. During the past three decades many very remarkable inequalities involving the coefficients a_n and b_n have been discovered. Many of them have been obtained by surprisingly elementary (but by no means obvious) methods. A basic result which has been frequently used is the "area-principle" of Bierberbach, namely $\sum_{n=1}^{\infty} n|b_n|^2 \leq 1$. This leads rather easily to the inequality

$|a_2| \leqq 2$ and, less easily, to $|a_n| \leqq Kn$, K an absolute (and as yet unknown) constant. Attempts to prove the Bieberbach conjecture that $K=1$ have led to the development of more profound methods. Among the most significant results obtained by such methods is that $|a_3| \leqq 3$, i.e., the aforementioned conjecture is correct for $n=3$.

More generally, the derivation of inequalities concerning the coefficients of functions schlicht (and suitably normalized) in a given domain has led to the solution of many interesting external problems closely connected with the theory of conformal mapping of simply and multiply connected domains.

4. *Topological properties of spaces of curves*, by Professor S. T. Hu, Institute for Advanced Study.

An elementary exposition was given of the most fundamental part of the works of M. Morse and E. Pitcher based on which recent developments have been made by various authors on topology as well as calculus of variations in the large.

A path in a given metric space X with metric d is a continuous map $\sigma: I \rightarrow X$ of the closed unit interval I. The totality of paths in X form a metric space W with the metric d defined by

$$d(\sigma, \tau) = \sup_{T \in I} d[\sigma(t), \tau(t)].$$

The Frechet distance d^* was also introduced in the space W, and its vanishing gives an equivalence relation in W. Thus the paths W are divided into disjoint classes called the curves in X. By means of the Morse length and parametrization, a proof of Pitcher's theorem was sketched. This theorem states that the totality of the curves in X with the Frechet distance form a metric space which is homeomorphic with a deformation retract of the space W.

C. O. OAKLEY, *Secretary*

DECEMBER MEETING OF THE MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA SECTION

The regular fall meeting of the Maryland-District of Columbia-Virginia Section of the Mathematical Association of America was held at Catholic University, Washington, D. C., on Saturday, December 9, 1950. Professor O. J. Ramler, Chairman of the Section, presided at both morning and afternoon sessions.

Ninety-seven people registered their attendance, including the following sixty-seven members of the Association: J. C. Abbott, R. P. Bailey, L. F. Boron, C. C. Bramble, B. H. Buikstra, W. E. Byrne, H. H. Campaigne, H. W. Charlesworth, H. J. Cheston, Randolph Church, G. R. Clements, Abraham Cohen, G. F. Cramer, J. A. Duerksen, F. D. Faulkner, W. J. Feeney, E. J. Finan, Joyce B. Friedman, A. M. Gleason, Michael Goldberg, R. A. Good, E. S. Grable, Clem Grabner, E. C. Gras, R. E. Greenwood, D. W. Hall, Marshall Hall, H. G. Hertz, S. B. Jackson, Walter Jennings, L. M. Kells, H. L. Kinsolving, A. E. Landry, B. J. Lockhart, C. G. Maple, M. H. Martin, J. M. McLynn, D. St. C. Melvin, Joseph Milkman, A. K. Mitchell, T. W. Moore, W. K. Morrill, H. Dorothy Mortell, Paul Nesbeda, M. W. Oliphant, Hyman Orlin, Walter Penney, F. M. Pulliam, O. J. Ramler, J. N. Rice, W. G. Rouleau, S. W. Saunders, E. D. Schell, W. T. Sharp, Sister Catherine Marie, E. R. Sleight, C. F. Stephens, H. C. Stotz, Feodor Theilheimer, O. M. Thomas, John Todd, Mary C. Varnhorn, C. H. Wheeler, P. M. Whitman, Mildred M. Wiker, J. W. Wrench, J. W. Wright.

It was voted to hold the next meeting on Saturday, April 28, 1951. This meeting will be at the United States Naval Academy in Annapolis, Maryland. Professor G. R. Clements, regional governor of the Section, reported on the meeting of the Board of Governors at Cambridge in September, during the International Congress. He commented on the medals available as awards for high school mathematics competition, the desire of the Association to reach a larger proportion of the Junior College teachers, and the attempts made to improve the teaching of mathematics in the schools. Mr. Ware Cattell described briefly a new scientific journal entitled *Research Report* devoted to prompt publication (fifteen to thirty days) of brief scientific articles (a thousand words or less). Mr. Cattell is the editor of this journal which is scheduled to begin publication early in 1951. A nominating committee was appointed consisting of Mr. Michael Goldberg, Professor E. J. Finan, and Professor D. W. Hall.

The program for the morning session consisted of four papers as follows.

1. *Finding the needle in the haystack*, by Dr. G. F. Cramer and Dr. H. H. Campaigne, Navy Department. (Presented by Dr. Campaigne)

The problem considered was to order a set of objects in such a way as to maximize the probability that an exceptional object (the needle) will be early in the order. The set of objects was taken to be continuous, and a formula was given for the first objects in the order. The ordering was in terms of a property of the objects. The special cases where this property is distributed normally and Poisson were considered.

2. *A summation process*, by Miss Barbara McGehee, University of Richmond, introduced by Professor E. R. Sleight.

In a paper by A. L. O'Toole entitled *Insights on Trick Methods*, National Mathematics Magazine, vol. XV, Oct. 1940, it was pointed out that if there exists a function $f(x)$ for which $u_x = f(x+1) - f(x)$, the series $\sum u_x$ from $x=a$ to $x=b$ is summed by the formula

$$\sum_{x=a}^{b} u_x = f(b+1) - f(a) = f(x)\Big]_a^{b+1}$$

This paper considered what types of series could be summed by this procedure. The method was used to sum arithmetic progressions, geometric progressions, series of the type $\sum n^k$ for positive integral k, and series $\sum 1/n(n+1)(n+3)$. It was conjectured that the method always applies when the nth term is a rational function of n.

3. *Isotopy and projective planes*, by Professor J. C. Abbott, United States Naval Academy.

Let l be a fixed line of a projective plane π, 0 and ∞ fixed points on l, and l_0 and l_∞ fixed lines through 0 and ∞ such that l, l_0, and l_∞ are all distinct. Let G be the set of points of l distinct from 0 and ∞, e any point in G, and l_e any line through e but not through the intersection of l_0 and l_∞. Then under the classical definition of multiplication given by Veblen and Young, G becomes a loop dependent on l_e as a parameter, whereas principal isotopy is equivalent to the case that the change in parameter leaves its intersection with l_∞ fixed.

4. *Connected subsets of a line*, by Professor D. W. Hall, University of Maryland.

A set A was defined to be connected provided there exists no continuous mapping $f(A) = p+q$, where p and q are distinct points. This new definition of connectedness, which is easily seen to be equivalent to the usual one, yields very simple proofs of the standard theorems on connected sets. Connected subsets of the line are quickly characterized by this procedure.

5. *Effective processes and turing machines*, by Mr. W. W. Boone, Catholic University of America, introduced by the Secretary.

The program for the afternoon session consisted of this invited address by Mr. Boone. The paper was expository. The intuitive notion of an *effective process* for solving a problem was described as follows: a process which yields an answer to the problem in a finite number of steps, and, such that at no stage does the problem solver have a choice as to his next operation. The analogies of a machine and of concrete instructions furnished a clerk were discussed. The concern of algebraists and topologists—as well as logicians—for effective processes was emphasized. Relations between *effective enumerability* and effective processes were described. Various precise formalizations of the concept of an effective process, due to Church, Godel, Kleene, and Post were described. Post's concepts of *degree of unsolvability* and *creative set* were discussed, as were certain results of Kleene. The talk then proceeded with a detailed description of another technical representation of effective processes due to Turing, the *turing machine.* Proofs in the general theory of turing machines were sketched. Post's proof—using turing machines and identifying them with creative processes—that the word problem for semigroups is unsolvable was presented. Turing's analogous result for cancellation semigroups was described.

S. B. JACKSON, *Secretary*

CALENDAR OF FUTURE MEETINGS

Joint meeting with American Society for Engineering Education, Michigan State College, East Lansing, June 25–26, 1951.

Thirty-second Summer Meeting, University of Minnesota, Minneapolis, September 3–4, 1951.

Thirty-fifth Annual Meeting, Brown University, Providence, Rhode Island, December 29, 1951.

The following is a list of the Sections of the Association with dates of future meetings so far as they have been reported to the Associate Secretary.

ALLEGHENY MOUNTAIN, Duquesne University, Pittsburgh, Pennsylvania, May 5, 1951.
ILLINOIS, University of Illinois, Urbana, May 11–12, 1951.
INDIANA, May 5, 1951.
IOWA
KANSAS
KENTUCKY
LOUISIANA-MISSISSIPPI, Northwestern State College, Natchitoches, Louisiana, February 15–16, 1952.
MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA
METROPOLITAN NEW YORK
MICHIGAN
MINNESOTA
MISSOURI
NEBRASKA
NORTHERN CALIFORNIA, University of California, Berkeley, January 26, 1952.
OHIO
OKLAHOMA
PACIFIC NORTHWEST, State College of Washington, Pullman, June 15, 1951.
PHILADELPHIA, University of Pennsylvania, Philadelphia, November 24, 1951.
ROCKY MOUNTAIN
SOUTHEASTERN
SOUTHERN CALIFORNIA
SOUTHWESTERN
TEXAS
UPPER NEW YORK STATE, Hamilton College, Clinton, May 5, 1951.
WISCONSIN, Carroll College, Waukesha, May 12, 1951.

GEORGE BANTA PUBLISHING COMPANY, MENASHA, WISCONSIN

THE AMERICAN MATHEMATICAL MONTHLY

THE OFFICIAL JOURNAL OF

THE MATHEMATICAL ASSOCIATION OF AMERICA, INC.

VOLUME 58 NUMBER 6

CONTENTS

JUNE-JULY 1951

The AMERICAN MATHEMATICAL MONTHLY

(FOUNDED IN 1894 BY BENJAMIN F. FINKEL)

EDITORIAL CORRESPONDENCE should be addressed to the Editor, C. V. NEWSOM, State Education Building, Albany 1, N. Y. After September 1, 1951, papers submitted for publication should be sent to C. B. Allendoerfer, Department of Mathematics, University of Washington, Seattle 5, Washington.

ADVERTISING CORRESPONDENCE should be addressed to J. F. RANDOLPH, Morey Hall, University of Rochester, Rochester 3, N. Y.

NOTICE OF CHANGE OF ADDRESS by members of the Association as well as correspondence regarding subscriptions to the MONTHLY should be sent to the Secretary-Treasurer, H. M. GEHMAN, University of Buffalo, Buffalo 14, N. Y.

THIS IS THE OFFICIAL JOURNAL OF THE

MATHEMATICAL ASSOCIATION OF AMERICA, INC.

(Devoted to the Interests of Collegiate Mathematics)

Entered as second class matter at the post office at Menasha, Wis. Acceptance for mailing at special rate of postage provided for in the Act of February 28, 1925, embodied in Paragraph 4, Section 538, P. L. and R., authorized April 1, 1926.

SUBSCRIPTION PRICE: To Members, $4 a Year, To Others, $5 a Year.

PUBLISHED BY THE ASSOCIATION at Menasha, Wisconsin, and Albany, N. Y. during the months of January, February, March, April, May, June-July, August-September, October, November, December.

OTTO DUNKEL

P. R. RIDER, Washington University

Otto Dunkel, son of Frederick W. and Elizabeth Check Dunkel, was born at Richmond, Virginia, May 25, 1869. He met a tragic death in St. Louis, Missouri, on January 15, 1951, having slipped and fallen down an embankment on the right-of-way of the Wabash Railroad. He lay at the foot of the embankment, unable to get back to the top, until death came from exposure. He was never married and is survived by a nephew and four nieces.

Dunkel was forced by circumstances to work during the years when most boys are in college, and it was not until he was twenty-four that he entered the University of Virginia in 1893. He received three degrees from that institution, M.E. in 1896, A.B. and A.M. both in 1898. During the academic year 1896–97 he was assistant in astronomy at Virginia under Professor Ormond Stone, founder of the Annals of Mathematics. From Virginia he went to Harvard, where he received an A.M. in 1899 and a Ph.D. in 1902. His dissertation, "Regular singular points of a system of homogeneous linear differential equations of the first order," was written under the direction of Professor Maxime Bôcher.

After receiving his doctor's degree he spent two years as instructor in mathematics at Wesleyan University, Hartford, Connecticut. He then went abroad for two years, studying at the University of Göttingen during 1904–05 and at the University of Paris during 1905–06. Upon returning to the United States he became an instructor at the University of Minnesota, which position he held for one year, then going to an instructorship at the University of Missouri. He was at Missouri during the period 1907–16, except for a leave of absence for the academic year 1909–10, which he spent at the University of Pisa. In 1916 he became assistant professor of mathematics at Washington University, St. Louis. He was promoted to an associate professorship in 1919 and became emeritus in 1939.

Dunkel published numerous papers in various journals, including the AMERICAN MATHEMATICAL MONTHLY, the Annals of Mathematics, the Bulletin of the American Mathematical Society, the Proceedings of the American Academy of Arts and Sciences, School Science and Mathematics, and the Washington University Studies. With E. R. Hedrick he translated into English Goursat's *Cours d'Analyse Mathématique.*

He served as an editor of the Problem Department of the MONTHLY from 1918 through 1946, being in charge of the department after 1933. He was zealous, conscientious, and very meticulous in this service. He probably worked through every solution that was submitted. If a solution was incorrect in any particular he would explain the matter to the solver and suggest a correction. Often when no solution was forthcoming to an especially difficult problem he supplied it himself. He was quite ingenious, and seemed to be equally at home in all fields of mathematics. He not only served faithfully as an editor, but remembered the Association generously in his will.

Otto Dunkel was a man of very simple tastes and lived quite frugally. He read a great deal and seemed especially fond of novels in foreign languages, notably French, Spanish, Italian, German. Although quiet and retiring he was friendly and was well liked by his colleagues. Because of his wide reading, extensive travel and a certain dry wit, he was an interesting conversationalist when in the mood. His death is a distinct loss to his friends and to mathematics.

THE INSTITUTE FOR NUMERICAL ANALYSIS OF THE NATIONAL BUREAU OF STANDARDS*

J. H. CURTISS, National Bureau of Standards

Suppose that a research worker in the course of a scientific investigation suddenly found that he had to solve 100 linear equations in 100 unknowns. Suppose further that he was afforded access to a large automatic digital computing machine, a machine with whose coding and operation he was familiar. How should he set about preparing the problem for the machine?

This situation is by no means as unrealistic as it may seem at first glance. Large sets of simultaneous equations are encountered in many data-reduction problems (those arising in geodetic triangulation problems are an example), in mathematical treatments of logistics and economics, and in the problems of engineering and applied physics. In fact, a great many of the numerical problems of applied mathematics, such as the numerical solution of integral and differential equations, can be reduced in one way or another to solving large sets of simultaneous linear equations, although in individual cases this may not be the best procedure.

The reader may say, "But how easy! After all, any high school student knows how to solve simultaneous linear equations. For example, why not write the various solutions down directly as the quotients of pairs of determinants and make the machine evaluate the determinants directly from their definition?"

A beautiful theoretical solution. Let us see how it would work in practice. Each of these determinants has 100 rows and 100 columns. By definition, the value of each determinant is the algebraic sum of $100\times 99\times 98\times\cdots\times 3\times 2\times 1$ signed products, each consisting of 100 factors chosen from among the elements of the determinant so that each row and each column is represented. That is, if the definition of a determinant were used as the basis of the method, for each determinant over 10^{157} products of 100 numbers each would have to be formed and added together algebraically. If a multiplication could be done in 100

* Based on an address given to the Pacific Northwest Section of the Association at Oregon State College, Corvallis, Oregon on March 26, 1949.

microseconds on the automatic computing machine, and an addition in 10 microseconds, and if an infinitely extensive high-speed internal memory could be assumed, then each determinant would be evaluated in something over 10^{147} years of machine time! And if all the solutions to the equations are wanted, 101 of these determinants will have to be evaluated. It looks as if the machine will be busy for some time to come!

So our beautiful method turns out to be ridiculous for a problem of this size. By use of efficient methods, each determinant can be evaluated in something like 3×10^5 multiplications and a comparable number of algebraic additions. For the 101 determinants required, the straight computing time would be something like one hour. Another three to six hours must be allowed for internal logical operations like changing the instructions within the machine. However, the assumption of unlimited memory is pure nonsense; the really high-speed memories in the current crop of machines are limited to about 1,000 numbers of ten to twelve decimal digits each. To carry out the procedure, it would be necessary constantly to bring data in and out of the high-speed memory, a process which goes so slowly that, in comparison with it, computing time can be ignored. Finally, the twin problems of significant digits and round-off errors are so serious as to make it quite unlikely that any ordinary procedure suitable for five or six equations would get even a usable rough approximation to the answer to our problem with 100 equations.

What about other methods of solving the problem? As anyone knows who has gone even superficially into the matter, there is certainly no dearth of them to choose from. Dr. George E. Forsythe* has been working recently on a classification and bibliography of the methods. He distinguishes two main categories: direct methods, like the method of determinants referred to above, and iterative methods, in which the solution is arrived at by successively closer approximations. There is no room here to go into details, but perhaps a mere rapid recapitulation of some of the short names of the methods will suggest the variety of approaches available. In the direct category are the Chio and Aitken determinant methods; the Bingham, the R. Schmidt, the Lanczos,* and the Frame characteristic equation methods; the Gauss elimination method and many abbreviations thereof; the Cholesky (or square-root) and the "escalator" triangularization methods; the Gram-Schmidt, the Fox-Huskey*-Wilkinson, and the Bodewig orthogonalization methods; block elimination methods; the "below-line" device, the R. A. Fisher technique, and other special methods. In the interative category are about a dozen distinct variants of the Wittmeyer type (some of them associated with names like Gauss, Seidel, Jacobi, and von Mises), and a number of versions of the least-squares type of iterative method, including an entire sub-class first characterized as such by Rosser** and Hestenes,† and

* Member of the staff of the Institute for Numerical Analysis of the National Bureau of Standards. (This name will henceforth in these footnotes be abbreviated to NBS-INA.)

** Former Director of Research of NBS-INA.

† Member of the staff of NBS-INA.

the sub-class of gradient methods with several variants, the iterated elimination methods, and "relaxation" methods. Then too, there is a separate category of miscellaneous special methods, such as "Monte-Carlo" methods (*i.e.*, sampling methods based on probability theory), and a method of H. Levy‡ for solving equations when the solution is known to be in integers.

This catalogue is quite probably incomplete, but even as it stands, it presents a truly bewildering problem of selection. D. R. Hartree§ says [1] "It is probably the case that there is no one best method for the evaluation of the solution of a set of simultaneous linear algebraic equations, but that the best method in any particular case depends on the structure of the set of equations concerned. This is certainly true of methods for treating such equations without the use of an automatic machine, and may well still be true when such assistance is available. For example, it may be that most of the coefficients are non-zero, and are not small integers and moreover are known only approximately, either because they are derived from measurements which are subject to experimental error or because they are results of previous calculations and are subject to rounding-off errors; or it may be that each equation involves only a few of the variables, and these with coefficients which are small integers and are known to be exact. It is quite likely the most appropriate methods in the two cases will be different."

But how should the choice be made? It is clear that somehow, somewhere, guide lines must be laid down which will permit an intelligent selection of methods to be made with due regard to the peculiarities of a given system of equations and to the specifications of available computing equipment. The literature is full of expositions of special methods, illustrated in many instances by more or less suitable examples; but the over-all guide book has not yet been written. In fact, there seem to be many cases in which none of the known methods even now gives feasible solutions. Moreover, relatively little is known about how round-off errors and other errors inherent in computational work pile up in the case of many of the methods listed above when they are applied to really large sets of equations. Perhaps the only way to resolve that particular sort of question will be by means of extensive arithmetical experiments carried out on high-speed automatic machines.

Here is an important type of problem, then, which re-occurs over and over again in applied mathematics. A lot is known about it, but for the full and effective exploitation of automatic digital computing machinery, a lot more needs to be known, and what is already known needs to be pulled together in a usable form. In other words, here is an area of the science of numerical analysis in which both background research and foreground research, together with laboratory experimentation, are urgently needed to maximize the nation's return from its multimillion dollar investment in computing machinery.

‡ Consultant of NBS-INA.

§ A former Director of Research.

It is to work on such problems that the Institute for Numerical Analysis of the National Bureau of Standards was established in 1947 with the support of the office of Naval Research of the U. S. Navy Department, and with the cooperation of the University of California. The Institute is located on the campus of the University of California at Los Angeles, and, of course, maintains close liaison with the University. Its permanent personnel complement numbers about 70 and includes some 15 scientists (mathematicians and theoretical physicists) at the post-Ph.D. level. In addition to the permanent staff, a number of well-known mathematicians have worked at the Institute on temporary appointments.

The Institute is a branch of the National Applied Mathematics Laboratories of the Bureau of Standards, which ONR played a fundamental rôle in establishing. The story of the Institute can best be told in a setting of that of the larger organization. This has been set forth at some length in a readily accessible article in Science [2] so only a brief outline is necessary here.

In 1945, a study was made of Navy Department computing requirements by the Office of Research and Invention, now ONR. Subsequently, Rear Admiral H. G. Bowen, then Chief of Naval Research, approached Dr. E. U. Condon, Director of the National Bureau of Standards, with the suggestion that ONR and NBS should jointly undertake to establish a national computing center, which would be equipped with large scale automatic machines and would engage in the development of such machines. In a later formalization of the idea, it was agreed that NBS would be solely responsible for administration of the center, and also that ONR's participation was conditional upon obtaining commitments from other agencies to support the new activity.

It might be mentioned here that ONR's decision to approach the National Bureau of Standards in connection with the operation of the proposed center was a rather natural one, in that ONR had been supporting on a temporary basis the Bureau's famous Mathematical Tables Project, originally a WPA project but financed during the war on OSRD funds. Also, in the spring of 1946, the Census Bureau had requested the NBS to construct a large automatic digital machine suitable for the preparation of census reports. Therefore, the nucleus for the center was already present at NBS early in 1946.

A year of cooperative study ensued, with both ONR and NBS personnel participating in various conferences with possible clients and applied mathematical groups all over the country. During the course of the year, the Air Materiel Command of the U. S. Air Force became interested in the proposed new organization, and made certain commitments in connection with its machine development program. This support, when added to other expressions of interest of a less substantial nature from other agencies, fulfilled the ONR requirements for outside participation. In the summer of 1947, the National Applied Mathematics Laboratories were established as a new division of the Bureau. In accordance with the conclusions reached during the cooperative study, it was organized into four branches: the Institute for Numerical Analysis

in Los Angeles; the Computation Laboratory in Washington (to carry on the work of the Mathematical Tables Project but with more emphasis on problem-solving and less on tables); the Statistical Engineering Laboratory in Washington; and the Machine Development Laboratory in Washington.

The Institute is the focal point in the mathematics division for basic research and training in the types of mathematics which are pertinent to the efficient exploitation and further development of high-speed automatic digital computing equipment. Secondary functions are to provide a computing service for Southern California and to give assistance in the formulation and analytical solution of problems in applied mathematics.

The Institute is divided into two sections, the Research section and the Mathematical Services section. The Research section has from the beginning been supported almost exclusively by ONR, although there is now some research work under way in it for the Air Comptroller of the United States Air Force, and the Bureau itself provides a small subsidy to pay for a part of the necessary experimental computing. The Mathematical Services section operates the computing service mentioned above, and in addition, conducts a modest computing machine development program. (Most of the NBS activity in the development of large-scale computing machinery is located in Washington.) The construction of a large automatic digital computing machine has recently been completed at the Institute under the direction of Dr. Harry D. Huskey. The machine, which is called the National Bureau of Standards Western Automatic Computer (SWAC for short), operates in the parallel mode and uses an electrostatic memory system, employing the so-called Williams principle. The electrostatic memory was originally put into operation with a 256 word capacity, but this is to be increased presently to 512 words. A rotating-drum magnetic memory of 10,000 word capacity is also being added. The machine was financed by the Office of Air Research of the Air Materiel Command. This Office from the start has been an important client of the computing services of the Institute.

The guiding philosophy in staffing the Institute has been that since the program is essentially mathematical, the organization should be staffed by professional mathematicians. At first glance, this idea might not seem to be very novel, but the fact is that in the field of numerical analysis, as in certain other parts of applied mathematics, there seems to be a well-established tradition of amateurism. Such a tradition has its good points, but by and large, it tends to retard an orderly and scholarly development of the discipline. There is, in fact, a popular saying among computers of the older school that nothing new has been discovered in numerical analysis since Gauss. The statement is certainly not true now, and was not quite true ten years ago, either. But there has indeed been a deficiency of professional, creative research in the theory and science of computation during the last 100 years—an era which has seen tremendous progress in many other branches of mathematics.

So a sustained effort has been made to staff the Research Section of the

Institute with competent professional mathematical research workers of established reputation. It has been found desirable to have at least one theoretical physicist and one expert in classical applied mathematics on the staff, so that advice can readily be obtained as to profitable directions in which to work.

Every care has been taken to provide attractive working conditions for the research staff. For example, most of the senior members of the research staff have private offices; experienced typists are on hand for typing mathematical manuscripts; desk computing machines are readily available for those who need them; a conscious effort is made to insulate the scientists from administrative red tape.

The Institute staff members have faculty privileges in the excellent UCLA library nearby; but a book in hand is worth two on somebody's desk in a fraternity house six blocks away, and it was soon clear that a good working collection should be built up in the Institute. The aim has been to make it preeminent in all items bearing directly on numerical analysis, both ancient and modern, and to make it quite complete in up-to-date material in mathematical analysis and applied mathematics. Secondary emphasis has been placed on theoretical physics and certain branches of pure mathematics, such as abstract algebra, although outstanding reference works in such fields are ordered. The Institute library has been fortunate in picking up back editions of the standard mathematical journals; for example, it has about all volumes of Mathematische Annalen back to vol. 1 in 1869.

The computing equipment of the Institute, in addition to the SWAC, consists of hand machines and punched-card machinery. The punched card installation includes an IBM Card Programmed Calculator (this is an electronic computing machine with a substantial internal memory), a 604 Calculating Punch, two 602A multipliers, a tabulator, and various supporting items. The research mathematicians have easy access to the machines at all times (there is no restricted area), and unless an emergency arises they usually get prompt action on any experimental computing.

Considerable emphasis is placed on the educational aspects of the Institute program. Scholars in various fields have been introduced to the facilities of the Institute through the medium of many short-term appointments of senior research workers. There have also been a number of visitors to the Institute whose travel and expenses were paid for by their own institutions. It is now well-known among scientists that they are always welcome at the Institute, and the visitors' book looks like a "Who's Who" of the Mathematical world. The remark has been heard that it is the Institute for Advanced Study in the East and the Institute for Numerical Analysis in the West that are visited by all mathematical travelers in this country.

More formal educational programs are conducted, too. The Institute collaborates with UCLA on various courses pertinent to its work. For example, an ambitious program of graduate mathematical instruction pertinent to research in modern numerical analysis is being given cooperatively by the Institute and

UCLA in the summer of 1951. Fellowships of two types are also offered: short term studentships for summer work aimed at familiarizing graduate students of mathematics and physics with the work of the Institute, and thesis fellowships lasting one or more years, designed to enable a graduate student to complete a Ph.D. thesis in a topic of interest to the Institute. A number of symposia, colloquia, and short, intensive courses on modern automatic computing machinery have also been organized from time to time.

Credit for identifying the fundamental need for the research program of the Institute should go particularly to Dr. Mina Rees, Director of the Mathematical Sciences Division of ONR. She recognized that the great activity in developing automatic digital computing machines in this country was not being adequately paralleled by theoretical investigation aimed at finding out how best to use them once they had been built, and that indeed it seemed that many of them would probably stand idle for inadmissibly long periods while mathematicians catch up with them.

The topics which have received the most attention to date by the INA research staff have been (a) the linear equation problem, mentioned in the opening paragraphs of this article, together with related matrix-inverting techniques; (b) eigenvalue problems for matrices and for continuous linear operators; and (c) numerical solutions of differential equations. Currently the last topic is one of the chief lines of research being pursued. The immediate orientation has been derived from certain difficult problems in non-linear parabolic differential equations brought into the Mathematical Services Section by one of the clients of the computing service.

In the case of linear equations, perhaps the most important task is to evaluate and study the many known methods in their relation to various types of modern automatic computing machinery. In the case of many problems of the second and third types mentioned above, it is by no means clear that appropriate methods have yet been found for really difficult cases. It is therefore interesting to try completely novel approaches.

A fruitful district in which to hunt for new methods is in probability theory. It is known that the distribution functions associated with certain random walks provide exact solutions of certain difference equations and asymptotic solutions of the related differential equations. These distribution functions can be estimated numerically by sampling procedures. Numerical techniques based in this way on probability theory are being called "Monte Carlo Methods" in computation laboratories.* Institute research workers, notably Professor Mark Kac, Dr. W. Wasow, and Dr. Robert Fortet have developed various "Monte Carlo" methods for calculating the eigenvalues of the Schrödinger equation

$$\tfrac{1}{2}\Delta u - Vu + \lambda u = 0,$$

* The reader who wishes to pursue the subject further will find a rather complete introduction and bibliography in [3].

where V is a known function and Δ is the Laplacian operator. Monte Carlo methods for solving partial differential equations and for inverting matrices have also been studied.

But this is no place, of course, for a complete catalogue of the problems worked on so far by the Institute. They are listed and described in the widely available public quarterly report entitled, "Projects and Publications of the National Applied Mathematics Laboratories." The detailed results are constantly being written up in numerous technical articles, which are sent in to the standard specialized scientific journals and the NBS Journal of Research. Over 100 of these papers have been prepared and submitted for publication since January 1, 1948.

The Institute is an example of a successful venture of the Government into the realm of *fundamental* research. It would be inappropriate, however, to bring this article to a close without saying a word or two about the immediate significance of the INA for national security.

It is well-known that automatic digital computing machinery and the associated mathematical techniques are important elements in the defense picture. The research program of the Institute is, of course, aimed quite directly at developing this national resource. More than that, the Institute maintains within its own walls a pool of equipment and a reservoir of talent in applied mathematics and computing technology which could immediately be drawn on for specific tasks in case of an acute national emergency. The organization also has built up over the last few years a substantial body of alumni and ex-staff members who are distributed all over the country, and whose contact with the Institute has made them familiar with, and interested in, modern methods of numerical analysis. Most of these persons are in academic positions, where they are able to attract and develop further talent. So although it is true that most of the research of the Institute has to date been of a very long-range, fundamental character, nevertheless the significance of the program for national security would seem to go well beyond that of the individual published results.

References

1. Calculating Instruments and Machines, D. R. Hartree, The University of Illinois Press, Urbana, 1949.

2. A Federal Program in Applied Mathematics, J. H. Curtiss, Science, 107 (1948) 257–262.

3. Sampling Methods applied to Differential and Difference Equations, J. H. Curtiss, Proceedings of Seminar on Scientific Computation held by IBM Corp. 87 (November 1949).

AN EQUILATERAL DISTANCE*

R. H. BING, University of Wisconsin

1. Introduction. A function $D(p, q)$ on the pairs of points of a topological space is called a distance for the space provided

(1) $D(p, q) \geqq 0$, the equality holding if and only if $p=q$,
(2) $D(p, q)=D(q, p)$, (symmetry),
(3) $D(p, q) \leqq D(p, r)+D(r, q)$, (triangle inequality), and
(4) $D(p, q)$ preserves limit points.

By (4) we mean that p is a limit point of the set M if and only if 0 is a limit of the set of numbers $D(p, x)$, where x is a point of M.

Examples of distance functions for the plane are

$$D_1(p, q) = \sqrt{(x_p - x_q)^2 + (y_p - y_q)^2}, \qquad D_2(p, q) = | x_p - x_q | + | y_p - y_q | ,$$

and

$$D_3(p, q) = \max (| x_p - x_q | , \ | y_p - y_q |).$$

Furthermore, if D is any distance function, then each of the following is a distance:

$$D_4 = cD, \quad D_5 = \sqrt{D}, \quad D_6 = \min (1, D), \quad \text{and} \quad D_7 = D/(1 + D).$$

While any function $F_1(p, q)$ satisfying conditions (2), (3), and (4) in a space without isolated points also satisfies (1),

$F_2(p, q)=2|x_p-x_q|+x_p-x_q+|y_p-y_q|$ does not satisfy (2),
$F_3(p, q)=(x_p-x_q)^2+(y_p-y_q)^2$ does not satisfy (3), and
$F_4(p, q)$ equals 0 or 1, according as p is or is not q, does not satisfy (4).

Although general distances may be quite different from the Euclidean distance, the interiors of spheres are open sets, and spaces with distance functions have a type of uniformity that facilitates their study. Some types of distances are more like the Euclidean distance than others. For example, a distance function $D(p, q)$ is convex provided:

(5) For each pair of points p, q there is a point r different from p and q such that $D(p, q)=D(p, r)+D(r, q)$.

We call $D(p, q)$ an equilateral distance if it satisfies conditions (1), (2), (3), (4), and

(6) For each pair of points p, q there is a point r such that $D(p, q)=D(p, r) =D(r, q)$.

It may be noticed that the point r is not between p and q but forms an equilateral triple with them.

---

* Excerpt from an address "An equilateral distance" before the Wisconsin Section of the Association, held at Lawrence College, Appleton, Wisconsin, on May 14, 1949.

A misprint influenced the starting of this paper. The author became interested in equilateral triples as a result of looking at Gustav Beer's paper *Beweis des Satzes, dass jede im kleinen zusammenhängende Kurve convex metrisiert werden kann.* A typographical error [Fundamenta Mathematicae, vol. 31 (1938), p. 281] in the introduction of that paper caused a distance to be called convex if it satisfied (6) instead of (5). However, the "+" was intended instead of the "=" because the paper considered the assigning a convex metric to the one-dimensional image of a straight line interval under a continuous transformation. One might wonder if this continuum might be assigned a distance satisfying (6). It did not seem intuitively obvious that even an interval could be assigned such a distance. The ruler that is described below was constructed to show that it could be assigned one.

2. Equilateral triples in the plane. The ordinary distance for Euclidean 2-space (the plane) is equilateral because each pair of points lies on the vertices of an equilateral triangle. However, this is not true of Euclidean 1-space (the line).

THEOREM 1. *If M is a bounded plane set with more than 3 points, some pair of points of M does not belong to an equilateral triple in M under the metric $D(p, q) = \sqrt{(x_p-x_q)^2+(y_p-y_q)^2}$.*

Proof. If each pair of points of M belongs to an equilateral triple, then each pair of points of $\overline{M}$ (M plus its limit points) belongs to an equilateral triple in $\overline{M}$, for suppose that p and q are points of $\overline{M}$. Let $p_1, p_2, \cdots$ and $q_1, q_2, \cdots$ be sequences of points of M converging to p and q, respectively. Suppose r_i is a point of M such that p_i, q_i, r_i is an equilateral triple. Some subsequence of $r_1, r_2, \cdots$ converges to a point r of $\overline{M}$, and p, q, r is an equilateral triple.

For some pair of points a, b of $\overline{M}$, $D(a, b)$ is the diameter of $\overline{M}$ (that is, no pair of points of $\overline{M}$ are farther apart than a and b). Suppose c is a point of $\overline{M}$ such that a, b, c is an equilateral triple.

We now show that there is a point d of $\overline{M}$ outside the triangle abc. Suppose d' is a point of $\overline{M}$ other than a, b, or c, and suppose that z is the one of a, b, c that is farthest from d'. If d' is not outside the triangle abc, then the point of $\overline{M}$ that forms an equilateral triple with z and d' is outside.

Without loss of generality, we suppose that the interval ad intersects the interval bc at f and that b and the point e of $\overline{M}$ which forms an equilateral triple with a and d are on opposite sides of the line through a and d. By the laws of cosines and sines,

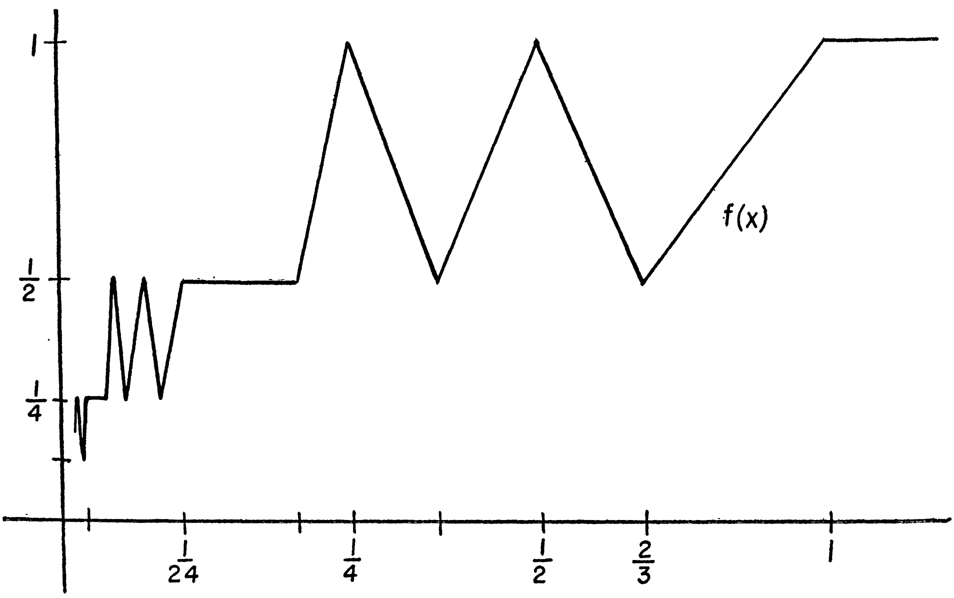
$$
\begin{aligned}
D^2(e, b) &= D^2(a, b) + D^2(a, e) - 2D(a, b)D(a, e) \cos \angle eab \\
&> D^2(a, b) + D(a, e)[D(a, f) - 2D(a, b) \cos \angle eab] \\
&= D^2(a, b) + D(a, e)D(a, b)[\sin \angle abf \\
&\qquad - 2 \cos \angle eab \sin \angle afb]/\sin \angle afb.
\end{aligned} \tag{7}
$$

Since $\angle eab$ and $\angle afb$ are supplementary, the expression in the bracket of the fourth member of (7) is equal to $\sin(\pi/3)-\sin(2\angle eab)$, which is positive. Hence, the assumption that each pair of points of M belongs to an equilateral triple has led us to the contradiction that e and b of $\overline{M}$ are farther apart than any two points of $\overline{M}$.

3. An equilateral ruler. The ruler portrayed in the figure that follows is the graph of a function $f(x)$ satisfying

(8) for each x, either $f(x)=f(2x)$ or $f(x)=f(x/2)$, and

$$\begin{Bmatrix} f(a) \leqq f(b)+f(c) \\ f(b) \leqq f(a)+f(c) \\ f(c) \leqq f(a)+f(b) \end{Bmatrix} \quad \text{provided} \quad \begin{Bmatrix} 0 \leqq a \leqq b+c \\ 0 \leqq b \leqq a+c \\ 0 \leqq c \leqq a+b \end{Bmatrix}. \tag{9}$$



If $E(p, q)$ is a distance, and r_1 is the point halfway between p and q under E and r_2 is the point such that q is halfway between p and r_2, then it follows from (8) that one of the points r_1, r_2 belongs to an equilateral triple with q and p under $D(p, q)=f[E(p, q)]$. We shall now describe $f(x)$ further and show that $D(p, q)=f[E(p, q)]$ is a distance.

The graph of $f(x)$, $(1/2\leqq x\leqq 1)$, is the sum of the interval from $(1, 1)$ to $(2/3, 1/2)$ and the interval from $(2/3, 1/2)$ to $(1/2, 1)$; $f(x)=f(2x)$, $(1/6\leqq x\leqq 1/2)$; $f(x)=1/2$, $(1/24\leqq x\leqq 1/6)$; $f(x)=f(24x)/2$, $(0<x\leqq 1/24)$; $f(0)=0$; $f(x)=1$, $(x\geqq 1)$. It may be noted that

$$f(a)/a \geqq 12^i\cdot 9 \text{ if } 0<a\leqq 1/24^{i+1}, \tag{10}$$

and

(11) $$|f(c)-f(b)|/(c-b) \leqq 12^i \cdot 6 \text{ if } 1/24^{i+1} \leqq b < c.$$

It is necessary that $f(x)$ satisfy condition (9) or else $D(p, q) = f[E(p, q)]$ does not satisfy condition (3) and is not a distance. Suppose $0 < a \leqq b \leqq c$ and $1/24^{i+1} \leqq b \leqq 1/24^i$. Then $f(a) \leqq f(b) + f(c)$ because $f(a)$ is not more than twice as large as either $f(b)$ or $f(c)$. If $a \geqq 1/24^{i+1}$, then (9) holds because $c \leqq 2/24^i$ and each of the numbers $f(a), f(b), f(c)$ lies between $1/2^{i+1}$ and $1/2^i$, inclusive. If $a < 1/24^{i+1}$, then it follows from (10) and (11) that $f(a) > |f(b)-f(c)| \cdot [a/(c-b)]$. Since $a/(c-b) \geqq 1$, $f(a) > f(b) - f(c)$ and $f(a) > f(c) - f(a)$. Hence, condition (9) holds and $D(p, q)$ satisfies condition (3).

Since $f(x)$ is continuous, $f(0) = 0$, and $f(x) > 0$ if $x > 0$, $D(p, q)$ satisfies (1) and (4). It is symmetric because $E(p, q)$ is symmetric. Hence, $D(p, q)$ is a metric topological equivalent to $E(p, q)$.

To find the distance between p and q with this ruler, place it so that p falls at the origin and q is on the positive x-axis. Then $D(p, q)$ is the height of the graph above q. If this ruler is used to measure a straight line interval of length 7/4 or more, an equilateral distance is obtained. Hence,

THEOREM 2. *Each arc has an equilateral distance.*

This ruler also gives an equilateral distance to a Hilbert space and to any Euclidean space.

If n is a positive integer and bc is an interval, a ruler can be constructed in a similar fashion that will cause each pair of points of bc to belong to a collection of n points of bc such that the distance between any two of them is the distance between any other two.

A NEW INTERPOLATION FORMULA*

P. M. HUMMEL and C. L. SEEBECK, JR., University of Alabama

1. Introduction. An interpolation formula is usually a polynomial which has two or more points in common with the function (or table of functional values) in question. The interpolation formula developed here differs from those now in use in that, not only are the functional values used, but as many derivatives can be incorporated in the formula as the user desires. Usually the inclusion of one or more derivatives greatly improves the accuracy. When no derivatives are used, our formula reduces to the well known Lagrange formula. We also give an error function which can be evaluated to give both an upper and lower bound for the error.

---

* Presented as an address "A refinement for linear interpolation applied to approximations of the roots of equations" by C. L. Seebeck, Jr. before the Southeastern Section of the Association, held at the University of Alabama, University, Alabama, March 18, 1949.

2. The general formula. Let $f(x)$, together with its first $m+n+1$ derivatives, be continuous in an interval containing $x_0, x_1, \cdots, x_n$, where $x_i \neq x_j$. We define

$$c_i' = \frac{(m+n-i)!m!}{(m+n)!(m-i)!i!}, \qquad c_i = \frac{(m+n-i)!n!}{(m+n)!(n-i)!i!}.$$

$$S_k = f(x_k) + \sum_{i=1}^{m} c_i' f^{(i)}(x_k)(x-x_k)^i, \qquad k = 0, 1, 2, \cdots, n. \tag{1}$$

$$R_k = (-1)^n \cdot \frac{m!n!(x-x_k)^{m+n+1}}{(m+n)!(m+n+1)!} f^{(m+n+1)}(\theta_k), \tag{2}$$

where θ_k lies somewhere between x and x_k.

$$R = \sum_{i=0}^{n} \prod_{j\neq i} \frac{x_j - x}{x_j - x_i} R_i. \tag{3}$$

We are now ready to prove the

THEOREM. *Under the conditions stated*

$$f(x) = \sum_{i=0}^{n} \prod_{j\neq i} \frac{x_j - x}{x_j - x_i} S_i + R. \tag{4}$$

The interpolation formula is obtained by using the right member of (4) with R omitted, which is the error due to using the formula.

Proof. The function $f(x)$ may be expanded in a generalized Taylor Expansion* as follows

$$f(x) = f(x_k) + \sum_{i=1} \frac{(m+n-i)!}{(m+n)!} \left[{}_mC_i f^{(i)}(x_k) - (-1)^i {}_nC_i f^{(i)}(x)\right](x-x_k)^i + R_k, \tag{5}$$

where the ${}_pC_q$ are binomial coefficients and R_k is given by (2). By rearranging terms, (5) can be written

$$f(x) + \sum_{i=1} c_i f^{(i)}(x)(x_k - x)^i = S_k + R_k, \qquad k = 0, 1, \cdots, n. \tag{6}$$

These may be considered as $n+1$ linear equations in $f(x), f'(x), \cdots, f^{(n)}(x)$. The determinant of the coefficients is

$$D = \begin{vmatrix} 1 & c_1(x_0 - x) & c_2(x_0 - x)^2 & \cdots & c_n(x_0 - x)^n \\ 1 & c_1(x_1 - x) & c_2(x_1 - x)^2 & \cdots & c_n(x_1 - x)^n \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ 1 & c_1(x_n - x) & c_2(x_n - x)^2 & \cdots & c_n(x_n - x)^n \end{vmatrix}.$$

This is a modified form of the Vandermonde determinant, and has the value

* A Generalization of Taylor's Expansion, P. M. Hummel and C. L. Seebeck, Jr. This MONTHLY, Vol. 56, No. 4, 1949, pp. 243–247.

$$D = \left(\prod_{i=1}^{n} c_i\right)\left(\prod_{i>j} (x_i - x_j)\right),$$

which does not vanish since the x's are distinct and the c's are different from zero. Hence we can solve the system of equations (6) for $f(x)$ by Kramer's rule, getting

$$D\cdot f(x) = \begin{vmatrix} S_0 + R_0 & c_1(x_0 - x) & c_2(x_0 - x)^2 \cdots c_n(x_0 - x)^n \\ S_1 + R_1 & c_1(x_1 - x) & c_2(x_1 - x)^2 \cdots c_n(x_1 - x)^n \\ \cdots & \cdots & \cdots \\ S_n + R_n & c_1(x_n - x) & c_2(x_n - x)^2 \cdots c_n(x_n - x)^n \end{vmatrix}.$$

We expand the above determinant according to the elements of the first column. This gives

$$D\cdot f(x) = \sum_{k=0}^{n} (-1)^k (S_k + R_k) M_k, \tag{7}$$

where

$$M_k = \begin{vmatrix} c_1(x_0 - x) & c_2(x_0 - x)^2 \cdots c_n(x_0 - x)^n \\ c_1(x_1 - x) & c_2(x_1 - x)^2 \cdots c_n(x_1 - x)^n \\ & k\text{th row missing} \\ c_1(x_n - x) & c_2(x_n - x)^2 \cdots c_n(x_n - x)^n \end{vmatrix}.$$

This is easily expanded and gives

$$M_k = \left[\prod_{i=0}^{n} c_i\right]\left[\prod_{i\neq k} (x_i - x)\right]\cdot\left[\prod_{i>j;i,j\neq k} (x_i - x_j)\right].$$

If we divide (7) through by D, and cancel out the factors that are common to D and M_k, the coefficient of (S_k+R_k) is

$$\frac{(-1)^k \prod_{i\neq k} (x_i - x)}{\left[\prod_{i>k} (x_i - x_k)\right]\left[\prod_{k>j} (x_k - x_j)\right]}.$$

The product $\prod_{k>j}(x_k - x_j)$ can be written as $(-1)^k \prod_{j<k}(x_j - x_k)$, and the formula given in the theorem readily follows.

Bounds for the error resulting from the use of this formula may be obtained by evaluating R as given by (2) and (3). By proper choice of the θ's, both an upper and a lower bound can be established.

3. A correction for linear interpolation. When $n=m=1$, equation (4) becomes

$$f(x) = I + C + R, \tag{8}$$

where

$$I = \frac{(x_1 - x)f(x_0) + (x - x_0)f(x_1)}{x_1 - x_0} \tag{9}$$

gives the approximate value of $f(x)$ obtained by linear interpolation;

$$C = \frac{(x_1 - x)(x - x_0)}{2(x_1 - x_0)} [f'(x_0) - f'(x_1)] \tag{10}$$

is an additive correction to be applied to the interpolated value, and

$$R = \frac{(x_1 - x)(x - x_0)}{12(x_1 - x_0)} [(x_1 - x)^2 f'''(\theta_1) - (x - x_0)^2 f'''(\theta_0)], \tag{11}$$

with θ_1 between x_1 and x, and θ_0 between x_0 and x, may be used to determine bounds for the error remaining after the correction C has been added to I.

A little tedious algebra makes it possible to determine an upper bound for the absolute value of the largest possible error when x is contained in the interval (x_0, x_1). We will prove the following

THEOREM. *If $f'''(x)$ does not change sign in the interval (x_0, x_1), $R=0$ for some value of x in this interval. Moreover, if M and m are the largest and smallest values of $|f'''(x)|$ in the interval, then*

$$|R| < \left[\frac{M}{124} + \frac{M - m}{144}\right] |x_1 - x_0|^3.$$

Proof. We set $a = |f'''(\theta_0)|^{1/2}$, $b = |f'''(\theta_1)|^{1/2}$, and define

$$F(x) = \frac{(x_1 - x)(x - x_0)}{12(x_1 - x_0)} [(x_1 - x)^2 b^2 - (x - x_0)^2 a^2].$$

We assume that $a \geqq b$ since this involves only the choice of notation. By inspection it is seen that $|R| = |F(x)|$. Since the factor $(x_1 - x)^2 b^2 - (x - x_0)^2 a^2$ has opposite signs for $x = x_0$ and $x = x_1$, it must vanish for some value of x between these values, and the first part of the theorem is established.

To prove the second part of the theorem we transform $F(x)$ into a more convenient form by setting $x - x_0 = (x_1 - x_0)y$. Then as x varies between x_0 and x_1, y varies between 0 and 1. This transformation gives

$$F(x) = \frac{(x_1 - x_0)^3}{12} y(1 - y)[(1 - y)^2 b^2 - y^2 a^2].$$

The right-hand side is readily factored, and we have

$$F(x) = \frac{(x_1 - x_0)^3}{12} y(1 - y)[b - (b + a)y][b - (b - a)y].$$

Now clearly the factor $b-(b-a)y$ varies between b and a so that its maximum value in the interval is a. Therefore

$$|F(x)| \leqq \frac{|x_1 - x_0|^3}{12} ay(1 - y)|b - (b + a)y|.$$

The expression

$$g(y) = y(1 - y)[b - (b + a)y]$$

is a cubic polynomial which vanishes for $y=0$, $y=1$, and $y=b/(a+b)$. The maximum value of $|g(y)|$ is therefore given by one of the solutions of $g'(y)=0$, both of which are in the interval (0, 1). By the usual procedure it is found that the solutions of $g'(y)=0$ are

$$\frac{a + 2b \pm (a^2 + ab + b^2)^{1/2}}{3(a + b)}$$

and hence,

$$\max |g(y)| = \frac{1}{27(a + b)^2}[(a - b)(2a + b)(a + 2b) + 2(a^2 + ab + b^2)^{3/2}].$$

Since a function which is symmetric in two variables takes on its absolute maximum, if it has one, when these variables are equal, it can be discovered, and then proved by direct algebraic methods, that

$$\frac{(2a + b)(a + 2b)}{27(a + b)^2} \leqq \frac{1}{12}, \quad \text{and} \quad \frac{2(a^2 + ab + b^2)^{3/2}}{27(a + b)^2} \leqq \frac{a\sqrt{3}}{18} < \frac{3a}{31}.$$

Using these limits we have

$$|F(x)| \leqq \frac{|x_1 - x_0|^3}{12}\left[\frac{a^2 - ab}{12} + \frac{3a^2}{31}\right] \leqq \left[\frac{a^2}{124} + \frac{a^2 - b^2}{144}\right]|x_1 - x_0|^3,$$

and the theorem follows.

4. Illustrations. We will find the value of sin 35° by the method developed in the preceding section. We take $x_0=\pi/6$ and $x_1=\pi/4$. From (9) and (10), $I=0.569036$ and $C=0.004624$. Thus

$$\sin 35° = I + C = 0.57366 \text{ approximately.}$$

To judge the accuracy of this result we evaluate (11), using the worst possible

values for the θ's, and get $|R| \leqq 0.00014$. Actually $\sin 35° = 0.57358$, so that the error is 0.00008. It is interesting to note that if we evaluate (11) for the function $\sin x$ when the interval (x_0, x_1) is $1°$, we find that $|R| \leqq 0.00000005$, so that $I+C$ gives at least seven decimal-place accuracy.

The method of the preceding section may be used for solving equations of the type $f(x)=0$. For let $y=f(x)$. The value of x which makes $y=0$, is first found by linear interpolation after which the correction C is computed and applied. Since the inverse function is encountered here,

$$C = \frac{-y_0 y_1}{2(y_1 - y_0)}\left[\frac{1}{y_0'} - \frac{1}{y_1'}\right], \quad \text{where} \quad y_i = f(x_i), \text{ and } y_i' = f'(x_i).$$

We will illustrate the procedure by solving* $\cos x = 10x$. We set $y = \cos x - 10x$ and take $x_0 = 0$, $x_1 = \pi/6$. Then $I = 0.09751$ and $C = 0.00194$, giving $x = 0.09945$. Here $|R| \leqq 0.000104$, whereas the actual error is about 0.00006. Clearly the procedure may be repeated for greater accuracy.

As a final illustration, we use the formula for $n = m = 2$, and compute the value of the compound interest factor $(1.031)^{50}$. Here we take $f(x) = (1+x)^{50}$, $x_0 = 0.0275$, $x_1 = 0.03$, $x_2 = 0.035$, and $x = 0.031$. In this case equation (4) reduces to

$$(1.031)^{50} = -\frac{16}{75}S_0 + \frac{84}{75}S_1 + \frac{7}{75}S_2 + R,$$

where $S_0 = 4.22213016$, $S_1 = 4.49115519$, and $S_2 = 5.06235147$. Whence

$$(1.031)^{50} = 4.6018588 + R.$$

Using equations (2) and (3) it is found that

$$-0.00000032 \leqq R \leqq -0.00000026$$

so that we have $(1.031)^{50} = 4.6018585$, accurate to seven decimal places. When this interest factor is evaluated by linear interpolation between the rates 0.03 and 0.035, the error exceeds 2 in the second decimal place.

5. Concluding remarks. It is well known that Lagrange's interpolation formula is, except for form, identical with various other formulas using successive differences. When the x_i are equally spaced and computing machines are not available, these formulas are more convenient to use than that of Lagrange. It will be observed that formula (4) not only reduces to the Lagrange formula as a special case, but also is formally identical with it when S_i is used to replace $f(x_i)$. Consequently, with a suitable choice of origin and scale, the various formulas of the calculus of finite differences may be used to evaluate (4). For example, (4) may be written in the Newton-Gregory form

---

* For the usual method of solving this type of problem, see Love's Calculus, 4th Ed., page 216.

$$f(x) = S_0 + \Delta S_0 x + \frac{\Delta^2 S_0}{2!} x(x-1) + \cdots$$
$$+ \frac{\Delta^n S_0}{n!} x(x-1) \cdots (x-n-1) + R.$$

It should be observed that the S_k as defined by (1) need not all contain the same number of derivatives. Thus S_0 could contain m_0 derivatives, S_1 include m_1 derivatives, etc. This generalization however does not affect the form of equation (4).

Finally we wish to point out that in actual practice it is usually possible to estimate the accuracy of the formula by computing one or two of the R_i as indicated by the coefficients in equation (3).

SOME CHARACTERIZATIONS OF COMPACTNESS

V. L. KLEE, JR., University of Virginia

1. Introduction. A distance-function ρ on a topological space M will be called an allowable metric for M if the topology generated by ρ coincides with M's original topology. Space M is called metrisable if it has at least one allowable metric. (For definitions of terms used but not defined here, and for proofs of results used here, see [**6**; pp. 1–10, 24–27].) Niemytzki and Tychonoff [**3**] showed by an adaptation of the Alexandroff-Urysohn metrisation proof that a metrisable space is compact if and only if it is complete in each of its allowable metrics. Hausdorff [**2**] proved that each allowable metric on a closed subset of a metrisable space can be extended to an allowable metric on the whole space. By use of this fact he gave a simple proof of the above result and showed that "complete" can be replaced by "bounded." Vaughan [**5**] improved on these characterizations by further use of Hausdorff's extension theorem and Bing [**1**] gave a new proof of the latter.

In this note we establish an elementary embedding theorem and apply this to give rather simple and pictorial proofs for Vaughan's principal results as well as to establish two new characterizations of compactness which seem not to be directly obtainable from the extension theorem.

2. The embedding theorem. It is well-known (and easy to prove) that if M is a topological space then the collection $C(M)$ of all bounded single-valued real continuous functions on M, with the distance between f and g defined as $\|f-g\|$ (where $\|h\| = \sup_{x \in M} |h(x)|$), is a metric space. (By metric space we mean

a metrisable space associated with a particular one of its allowable metrics.) If (M_1, ρ_1) and (M_2, ρ_2) are metric spaces (ρ_i being an allowable metric for M_i), they are said to be isometric provided there is a bi-unique transformation $f(M_1)=M_2$ such that $\rho_1(x, y)=\rho_2(f(x), f(y))$ whenever $x, y \in M_1$. The transformation f is called an isometry.

Now suppose that (M, ρ) is a metric space and for each $x \in M$, let f_x be the function $\rho(x, y) \mid y \in M$. For some fixed $z \in M$, define $F(x)=f_x-f_z$ for all $x \in M$. It is not hard to verify that $F(M) \subset C(M)$, and, in fact, that F is an isometry. This result is frequently stated as follows: "(M, ρ) can be isometrically embedded in $C(M)$." Here we shall need the following slightly different embedding theorem.

Theorem. *Suppose that M is a metrisable space and ρ is an allowable metric for M. Then there is an allowable metric δ for M and a number $n>0$ such that*

(i) if either $\rho(x, y)<n$ or $\delta(x, y)<n$, then $\rho(x, y)=\delta(x, y)$, and (ii) (M, δ) is isometric with a subset of the boundary B of the sphere $S(\phi, n)$ in $C(M)$.

(Here ϕ is the function identically zero on M and $S(f, r)$ denotes the set of all points at distance $\leqq r$ from f.)

Proof. We may suppose without loss of generality that the diameter Δ of (M, ρ) is positive. Now define $\delta=\min(n, \rho)$, where $n=\Delta/3$ if Δ is finite and $n=1$ otherwise. It is easy to verify that δ is a distance-function and clearly (i) is satisfied so δ is allowable. Now for each $p, x \in M$ let $f_p(x)=\delta(p, x)$. Consider an arbitrary $p \in M$. Clearly $f_p \in C(M)$. And since $n<\Delta/2$ there are points x_1 and x_2 of M such that $\rho(x_1, x_2)>2n$. But then for at least one value of i, we have $\rho(p, x_i)>n$ and hence $f_p(x_i)=\delta(p, x_i)=n$. Thus $\|f_p\|=n$ and $f_p \in B$. By further use of the triangle inequality, we see that $\|f_p-f_q\|=\delta(p, q)$ whenever $p, q \in M$, so $f: p \rightarrow f_p$ is an isometry on (M, δ) and the proof is complete.

Notice that the proofs in the sequel are merely the analogues for subsets of B (or of $n^{-1}B$) of those which immediately suggest themselves for subsets of the square $\{(x, y) \mid \max(|x|, |y|)=1\}$ in the plane.

3. The characterizations. A metrisable space is called *compact* if each of its infinite subsets has a limit point in the space. A metric space is called *bounded* if its diameter is finite; *totally bounded* if for each positive ϵ, it is the sum of a finite number of sets each of diameter $<\epsilon$; *complete* if each Cauchy sequence of its points converges to a point in the space; and *totally complete* if each of its bounded closed subsets is compact. It is well-known that, in each of its allowable metrics, a compact metrisable space must be complete and totally bounded. Hence in Theorems 1 to 3 below (which are the same as Theorems 1 to 3 of [5]) we need merely establish the "if" part.

Theorem 1. *A metrisable space is compact if and only if in each allowable metric it is either complete or bounded.*

THEOREM 2. *A metrisable space is compact if and only if in each allowable metric in which it is bounded it is also either totally bounded or complete.*

THEOREM 3. *A metrisable space is compact if and only if it has an allowable metric in which it is complete, and in each such metric is either totally complete or bounded.*

Proofs of Theorems 1 to 3. Let M be a non-compact metrisable space. We see from the embedding theorem that M has a homeomorph H in the subset $U=\{f \mid \|f\|=1\}$ of the space $C(M)$ and, that if M allows a complete metric, then we may take H to be complete. Since H is non-compact, there is a denumerably infinite set $\{x_1, x_2, \cdots\}$ of points of H and a set $\{\epsilon_1, \epsilon_2, \cdots\}$ of positive numbers such that $S(x_i, \epsilon_i)$ and $S(x_j, \epsilon_j)$ are disjoint for $i \neq j$ and such that, if $y_i \epsilon S(x_i, \epsilon_i)$ for each i, then the set $\{y_1, y_2, \cdots\}$ has no limit point in H.

Now whenever $x \epsilon U$, $\epsilon > 0$, and $t \geqq -1$, let $F(x, \epsilon, t)$ be the image of $S(x, \epsilon) \cdot H$ under the transformation which takes a point y at distance η from x into the point $y + (\epsilon - \eta)\epsilon^{-1} t y$. (This is a homeomorphism which takes x into $x + tx$ and is the identity on the set of points at distance ϵ from x.)

To prove Theorem 1 we observe that the set

$$\left[H - \sum_i S(x_i, \epsilon_i)\right] + \sum_{i \text{ odd}} F(x_i, \epsilon_i, i) + \sum_{i \text{ even}} F(x_i, \epsilon_i, -1 + i^{-1})$$

is (under the metric of $C(M)$) a homeomorph of M which is neither bounded nor complete.

To prove Theorem 2 consider this same set with the metric ρ defined by $\rho(x, y) = \min\ (1, \|x-y\|)$.

For Theorem 3 we take the set

$$\left[H - \sum_{i \text{ odd}} S(x_i, \epsilon_i)\right] + \sum_{i \text{ odd}} F(x_i, \epsilon_i, i)$$

in the metric of $C(M)$.

A slight elaboration of this method will establish the following result, which is Lemma 1 of [4].

THEOREM 4. *A metrisable space is locally compact if and only if in each allowable metric it is locally complete.*

A proper subset of a metric space will be called a *distance-set* if to each point in the complement there is a (not necessarily unique) nearest point in the set.

THEOREM 5. *A proper subset of a metrisable space is compact if and only if in each allowable metric it is a distance-set.*

Proof. The "only if" part is well-known. Now let X be a non-compact proper subset of a metrisable space M, let H be as above, and let Y be the part of H

which corresponds to X. We may suppose that Y is closed in H, for otherwise it certainly is not a distance-set. Now since Y is non-compact there are points x_i and positive numbers $\epsilon_i (0 \leqq i < \infty)$ such that $x_i \epsilon Y$ for $i \geqq 1$, $x_0 \epsilon H - Y$, $S(x_i, \epsilon_i) \cdot S(x_j, \epsilon_j) = 0$ for $i \neq j$, and such that if $y_i \epsilon S(x_i, \epsilon_i)$ for each i then the set $\{y_1, y_2, \cdots\}$ has no limit points in $Y + S(x_0, \epsilon_0)$. Now let

$$H' = \left[H - \sum_{i=0}^{\infty} S(x_i, \epsilon_i) \right] + F(x_0, \epsilon_0, -1) + \sum_{i=1}^{\infty} F(x_i, \epsilon_i, -\tfrac{1}{2} + i^{-1}).$$

Then (in the metric of $C(M)$) H' is a homeomorph of M. Let ρ be the metric assigned to M by its natural correspondence with H' and let p be the point of M which corresponds to ϕ (in H' or to x_0 in H). Then $\rho(p, X) = \frac{1}{2}$ although $\rho(p, x) > \frac{1}{2}$ for each $x \epsilon X$. This completes the proof.

Theorem 5 acquires added interest in the light of the following remark.

THEOREM 6. *A proper subset of a metrisable space is closed if and only if in some allowable metric it is a distance-set.*

Proof. The "if" part is obvious. Now suppose conversely that X is an arbitrary closed subset of the metrisable space M and let ρ be an allowable metric for M. For each two points x and y of M define

$$\delta(x, y) = \max (\rho(x, y), 2 \mid \rho(x, X) - \rho(y, X) \mid).$$

Clearly δ is an allowable metric for M. Now consider an arbitrary point y of $M - X$. For each $x \epsilon X$ we have $\delta(y, x) \geqq 2\rho(y, X)$. However, there is a point $\bar{x} \epsilon M$ for which $\rho(y, \bar{x}) < 2\rho(y, X)$ and we then have $\delta(y, \bar{x}) = 2\rho(y, X) = \delta(y, X)$, so the proof is complete.

It is well-known that the diameter of a compact metric space must actually be attained as the distance between two of its points. Thus Theorem 7 provides another characterization of compactness.

THEOREM 7. *Each non-compact metrisable space M allows a metric ρ such that for each $p \epsilon M$, $\sup\{\rho(p, x) \mid x \epsilon M\}$ is less than the diameter of (M, ρ).*

Proof. Let H, the x_i's, and the ϵ_i's be as in the proofs of Theorems 1 to 3 above. Let g be the function identically 1 on M and let H' be the image of H under the transformation $x' = \frac{1}{2}(x + g)$. Then $H' \subset U$. Since this transformation merely halves distances, H' is a homeomorph of M. Also, the points x_i' and numbers $\epsilon_i' = \epsilon_i/2$ have the properties with respect to H' that the x_i's and ϵ_i's had with respect to H. Now let

$$H'' = \left[H' - \sum_{i} S(x_i', \epsilon_i') \right] + \sum_{i \text{ odd}} F(x_i', \epsilon_i', 1 - i^{-1}) + \sum_{i \text{ even}} F(x_i, \epsilon_i - 1 + i^{-1}).$$

H'' is a homeomorph of M, and each function in H'' has values between 0 and 2 everywhere on M. The images for odd i's of the x_i''s have norms approaching 0, and for even i's approaching 2, so the diameter (in the metric

of $C(M)$) of H'' is 2. And since each function in H'' is bounded away from both 0 and 2 the theorem is established.

4. An additional illustration. A metrisable space is called topologically complete if it allows a metric in which it is complete. A G_δ-set is the intersection of a countable number of open sets. As an additional application of the embedding theorem, we prove the following well-known result.

THEOREM 8. *Every G_δ-set in a complete space M is topologically complete.*

Proof. Consider first the case of an open subset G of M. By the embedding theorem, there is a homeomorphism h of M into the boundary of the unit sphere of $C(M)$ such that $h(M)$ is complete. Now for each point $x \epsilon h(G)$, let $d(x)$ denote the distance from x to the set $h(M)-h(G)$. Then each point of G has a neighborhood on which d is bounded away from zero. Hence the transformation $f(x)=d(x)^{-1}x (x \epsilon G)$ is a homeomorphism taking $h(G)$ onto a complete subset of $C(M)$.

Now suppose that $G_1, G_2, \cdots$, are open subsets of M and $P=\pi_i G_i$. Each G_i allows a metric ρ_i in which it is complete and has diameter $\leqq 1$. Now for x, $y' \epsilon P$, let $\rho(x, y) = \sum_i 2^{-i}\rho_i(x, y)$. It is easily verified that ρ is an allowable metric for P. Every Cauchy sequence in P relative to ρ must also be one relative to each ρ_i, and hence converges for each ρ_i to a point of G_i. But then it must converge (relative to ρ) to a point of P and the proof is complete.

It is possible, of course, to give still further applications of the embedding theorem. For example, every locally complete metric space M can be embedded by means of a homeomorphism which preserves Cauchy sequences in a complete metric space which contains at most one point in addition to the image of M.

References

1. R. H. Bing, Extending a metric, Duke Mathematical Journal, vol. 14 (1947) pp. 511–519.

2. F. Hausdorff, Erweiterung einer Homöomorphie, Fundamenta Mathematicae, vol. 16 (1930) pp. 352–360.

3. V. Niemytzki und A. Tychonoff, Beweis des Satzes, dass ein metrisierbar Raum dann und nur dann kompact ist, wenn er in jeder Metrik vollständig ist, Fundamenta Mathematicae, vol. 12 (1928) pp. 118–120.

4. H. E. Vaughan, On locally compact metrisable spaces, Bulletin of the American Mathematical Society, vol. 43 (1937) pp. 532–535.

5. H. E. Vaughan, On the class of metrics defining a metric space, Bulletin of the American Mathematical Society, vol. 44 (1938) pp. 557–561.

6. G. T. Whyburn, Analytic Topology, American Mathematical Society Colloquium Publications, Volume XXVII, New York, 1942.

THE TWO-AREA COVERING PROBLEM*

B. M. STEWART, Michigan State College

1. Introduction. Given two regions A_1 and A_2 bounded respectively by the simple continuous closed plane curves C_1 and C_2, let us seek the boundary C of a region A which is minimal in area and which will, when properly placed, cover either A_1 or A_2. An equivalent problem is to fix A_2 in position on a coordinate system, to establish a reference point $P:(x, y)$ and a reference line $L:(\theta)$ in A_1, and to determine the position and orientation of A_1 for which $A(x, y, \theta)$, the area covered by A_1 and A_2, will be a minimum. Since, using point set notation, $A = A_1 + A_2 - B$, where $B = A_1A_2$ is the area covered by both A_1 and A_2, it follows that an equivalent problem is to seek the maximum of $B(x, y, \theta)$.†

In the first part of this paper we derive expressions for A_x, A_y, A_θ, under hypotheses which guarantee that A_x, A_y, A_θ are continuous, and with the aid of these expressions we characterize in a simple way the stationary values of A; namely, those positions of A_1 with respect to A_2 for which A_x, A_y, A_θ are simultaneously zero, and these values, by the usual calculus criteria, will include, of course, the minimal values of A.

In the second part of the paper we apply these results to the particular case where A_2 is a given triangle and A_1 is a circle with its center as the reference point P. Letting P^* indicate a position of P which makes A a minimum, we study the locus of P^* for circles of varying radius, and we are thus led to some interesting geometry of the triangle.

2. Partial derivatives of $A(x, y, \theta)$. Let us restrict our problem to the case where the boundary curves C_1 and C_2 can intersect in only a finite number of points, for in the sequel it will appear that otherwise A_x, A_y, A_θ may not be continuous. In counting the points of intersection let us agree to count only "crossing" points, not "touching" points. As a consequence of the Jordan theorem the number of crossing points must be even and may be designated, in order, starting from any convenient point P_1 at which C_1 enters A_2 and tracing C_1 in counterclockwise sense, as $P_i:(x_i, y_i)$, $i = 1, 2, \cdots, 2n$. If the notation arises, we are to understand that $P_{i+2n} = P_i$. Let r_i indicate the length PP_i. In what follows all summations, unless otherwise indicated, run from $i = 1$ to $i = 2n$. With these agreements we can establish the following formulas:

---

* We are led to this problem by reading of the Besicovitch minimal area problem in W. W. R. Ball, Mathematical Recreations and Essays, 11th edition, New York, 1939, p. 99.

† That A actually has a minimum value is shown by a combination of facts: first, A is continuous in x, y, θ; a curve N can be found, encircling A_2 at a sufficient distance, so that, regardless of the value of θ, if P is on N, then $B = 0$ and $A = A_1 + A_2$; but for any fixed θ, A has a smaller value when P is located, as it can obviously be, so that $B \neq 0$, and then, by the usual continuity argument, A must attain an absolute minimum $A^*(\theta)$ for at least one $P^*(\theta)$ within N; but $A^*(\theta)$ is continuous in θ and is periodic of period 2π and hence attains an absolute minimum value for at least one θ^*, $0 \leqq \theta^* \leqq 2\pi$; $A^*(\theta^*)$ is the desired minimal value of A.

If C_1 and C_2 have no crossing points, then

$$A_x = 0, \tag{1'}$$

$$A_y = 0, \tag{2'}$$

$$A_\theta = 0. \tag{3'}$$

If C_1 and C_2 have $2n$ crossing points $\{P_i\}$, then

$$A_x = -\sum(-1)^i y_i, \tag{1}$$

$$A_y = +\sum(-1)^i x_i, \tag{2}$$

$$A_\theta = +\tfrac{1}{2}\sum(-1)^i r_i^2. \tag{3}$$

Once these formulas have been established it is clear that they are independent of the "entering" point chosen as P_1, and should P_1 be chosen as a "leaving" point or should the direction of tracing C_1 be reversed, then the signs preceding the summations in (1), (2), (3) should be changed.

To establish (1) let us consider the change ΔB in $B(x, y, \theta)$ produced by giving x the increment Δx, holding y and θ constant. Consider first the case that C_1 and C_2 have just two crossing points, $U:(x_u, y_u)$ and $V:(x_v, y_v)$, with C_1 entering A_2 at U. For the simplest case, suppose that the boundary curve of A_1 is given by $x = C_1(y)$ which is single-valued from U to V, as in Figure 1.

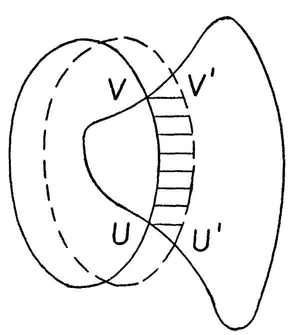


FIG. 1

Let U' and V' be the crossing points corresponding to the translated position, where the boundary curve of A_1 has the equation $x = C_1'(y) = C_1(y) + \Delta x$. Let m_u and M_u (m_v and M_v) be, respectively, the minimum and maximum values of y on C_2 between U and U' (V and V'). Since $\int\{C_1'(y) - C_1(y)\}dy = \Delta x \int dy$, it follows that ΔB satisfies the inequalities,

$$\Delta x(m_v - M_u) \leqq \Delta B \leqq \Delta x(M_v - m_u), \quad \text{when} \quad \Delta x > 0;$$

or the same inequalities, with both signs reversed, when $\Delta x < 0$. In either case, in the limit, $B_x = y_v - y_u$, hence $A_x = -B_x = -(y_v - y_u)$.

If $x = C_1(y)$ is not single-valued from U to V, as in Figure 2, every extra contribution to ΔB resulting from a portion of C_1 where y is decreasing is balanced by a contribution where y is increasing, and the final result is the same as in the simple case.

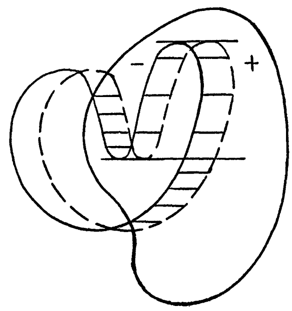


FIG. 2

Furthermore, the same type of argument shows that a "touching" point, as in Figure 3, makes no contribution to B_x or A_x. Consequently the case where there are no crossing points is one for which $B_x = 0$ and hence $A_x = 0$, which establishes (1′).

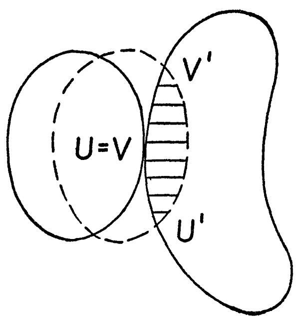


FIG. 3

If C_1 and C_2 have $2n$ crossing points, with C_1 entering A_2 at P_1 then for each pair of crossing points, P_{2i-1} and P_{2i}, we have the contribution $-(y_{2i} - y_{2i-1})$ to A_x, and it follows that (1) is correct.

By almost the same argument it is possible to establish (2′) and (2). The change in sign, comparing (1) and (2), is not surprising if one recalls the similar situation in Green's theorem.

To establish (3) we consider the variation ΔB in $B(x, y, \theta)$ produced by giving θ the increment $\Delta\theta$, holding x and y constant. This means fixing the point P and rotating A_1 around P through the angle $\Delta\theta$, a natural situation for polar coordinates, using P as a pole and any convenient line through P as an

axis. Consider first the case that C_1 and C_2 have just two crossing points, $U\colon(r_u, \theta_u)$ and $V\colon(r_v, \theta_v)$, with C_1 entering A_2 at U. For the simplest case suppose that the boundary curve of A_1 is given by $\theta = C_1(r)$ which is single-valued from U to V as in Figure 4. Let U' and V' be the crossing points corresponding to the rotated position, where the boundary curve of A_1 has the equation

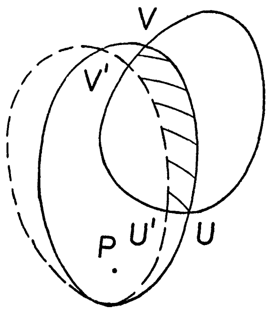


FIG. 4

$\theta = C_1'(r) = C_1(r) + \Delta\theta$. Let m_u and M_u (m_v and M_v) be, respectively, the minimum and maximum values of r on C_2 between U and U' (V and V').

Since $\int\{(C_1'(r) - C_1(r)\}rdr = \Delta\theta\int rdr$ it follows that ΔB satisfies the inequalities

$$\frac{\Delta\theta}{2}(m_v^2 - M_u^2) \leqq -\Delta B \leqq \frac{\Delta\theta}{2}(M_v^2 - m_u^2), \quad \text{when} \quad \Delta\theta > 0;$$

or the same inequalities, with both signs reversed, when $\Delta\theta < 0$. In either case, in the limit, $B_\theta = -\frac{1}{2}(r_v^2 - r_u^2)$, and $A_\theta = -B_\theta$.

If $\theta = C_1(r)$ is not single-valued from U to V, changes in ΔB resulting from a portion of C_1 where r is decreasing are balanced by changes where r is increasing, and the net result is the same as in the simple case. The same type of argument shows that touching points make no contribution to B_θ, and with this possible difficulty out of the way, it follows that when there are no crossing points then $B_\theta = 0$ and hence $A_\theta = 0$ which establishes (3'). If C_1 and C_2 have $2n$ crossing points, with C_1 entering A_2 at P_1, for each pair of crossing points, P_{2i-1} and P_{2i}, we have the contribution $+\frac{1}{2}(r_{2i}^2 - r_{2i-1}^2)$ to A_θ, and it follows that (3) is correct.

3. Stationary values of $A(x, y, \theta)$. With the formulas (1), (2), (3) in mind it is clear that the hypothesis that C_1 and C_2 can have only a finite number of points in common will guarantee the continuity of A_x, A_y, A_θ, for the continuity of the boundary curves C_1 and C_2 carries with it the continuity of x_i, y_i, r_i. But if C_1 and C_2 were allowed to have, say, in one position, a segment of a curve in common, then in this position, A_x, for example, might have a saltus, equal to the difference in the y-coordinates of the end points of the segment. Then by

the usual theorem from calculus, the minimum values of A will be included among the stationary values of A for which $A_x=0$, $A_y=0$, $A_\theta=0$, simultaneously.

According to (1'), (2'), (3') stationary values of $A(x, y, \theta)$ result when C_1 and C_2 have no crossing points. According to (1), (2), (3) stationary values of $A(x, y, \theta)$ result when C_1 and C_2 have $2n$ crossing points $\{P_i\}$, satisfying the following conditions:

$$\sum (-1)^i x_i = 0, \tag{4}$$

$$\sum (-1)^i y_i = 0, \tag{5}$$

$$\sum (-1)^i r_i^2 = 0. \tag{6}$$

A few consequences of this result are worth noting. Instead of the reference point $P:(x, y)$, suppose that the reference point $P':(x', y')$ is chosen with $x'=x+h$, $y'=y+k$. Let R_i be the length of $P'P_i$. Since

$$\begin{aligned} R_i^2 &= (x + h - x_i)^2 + (y + k - y_i)^2 \\ &= r_i^2 - 2hx_i - 2ky_i + [h^2 + 2hx + k^2 + 2ky], \end{aligned}$$

then in the following alternating sum, the bracketed terms cancel, and

$$\sum (-1)^i R_i^2 = \sum (-1)^i r_i^2 - 2h\sum (-1)^i x_i - 2k\sum (-1)^i y_i.$$

Then from (4), (5), (6) we find that for a stationary value of A, $\sum(-1)^i R_i^2=0$. This shows that the set of conditions (4), (5), (6) is independent of the choice of the reference point $P:(x, y)$.

Let T_i indicate the vector $P_{i-1}P_i$. Let L_a be a line making an angle a with the x-axis. If T_{ia} is the (signed) projection of T_i on L_a, we find $T_{ia}=(x_i-x_{i-1})\cos a+(y_i-y_{i-1})\sin a$. Then

$$\sum_{i=1}^{n} T_{2i,a} = \cos a\sum (-1)^i x_i + \sin a\sum (-1)^i y_i.$$

Hence for a stationary value of A, it follows from (4), (5) that

$$\sum_{i=1}^{n} T_{2i,a} = 0. \tag{7}$$

Similarly,

$$\sum_{i=1}^{n} T_{2i-1,a} = 0. \tag{7'}$$

Conversely, if (7) is satisfied for each of two non-parallel lines, L_a and L_b, it is easy to prove that (4) and (5) are satisfied.

In summary, if a position of A_1 with respect to A_2 determines a stationary

value of A, then the set of crossing points $\{P_i\}$ of C_1 and C_2 must either be empty, or such that the sum of the signed projections of alternate sides of the figure $\{P_i\}$ upon any line and the alternating sum of the squares of radial distances of the $\{P_i\}$ from any point must vanish.

It can be observed directly that the case of no crossing points always yields a maximum value of A when A_1 and A_2 are in such a position that $B=0$, or a minimum value of A when either A_1 or A_2 can be so placed that one completely contains the other. The case of 2 crossing points P_1, P_2 can never yield a stationary value of A, for (7) fails to hold when L_a is parallel to P_1P_2. The case of four crossing points, P_1, P_2, P_3, P_4 leads to a stationary value of A if and only if $P_1P_2P_3P_4$ forms a rectangle.* On the one hand taking L_b and L_a parallel and perpendicular, respectively, to P_1P_2 leads by (7) to the requirement that $P_1P_2P_3P_4$ be a parallelogram; but taking P to the point of bisection of the diagonals of the parallelogram, we see by (6) that the diagonals must be equal in length, hence the parallelogram must be a rectangle. Conversely, by the remark preceding the summary, when $P_1P_2P_3P_4$ is a rectangle, all of (4), (5), (6) are satisfied.

4. The circle-triangle covering problem for 0 and 4 crossing points. Let A_2 be a given triangle and let A_1 be a given circle of radius a with its center P as the reference point. Because of the perfect symmetry of A_1 it is clear that condition (6) is now trivially satisfied, and it remains to discuss the minimal values for A for the cases of 0, 4, and 6 crossing points in the light of conditions (4), (5), and (7). A position of P leading to a minimal value of A will be designated as P^*.

The case of no crossing points leads to a minimal value of A if and only if one of the figures is sufficiently large to completely contain the other. If the triangle is sufficiently large to contain the circle we must have $a \geqq r$, where r is the radius of the inscribed circle, and we can then locate P^* at any point within a triangle which is within the given triangle and which has its sides parallel to, and at a distance a from, the sides of the given triangle. To discuss the situation when the circle contains the triangle we need to make a division into cases. If the triangle is acute, this circumstance arises when $a \geqq R$, where R is the radius of the circumscribed circle, and we can then locate P^* at any point which is common to the three circles, each having a vertex of the given triangle as a center and each of radius a. If the triangle is obtuse, the circle can contain the triangle if $a \geqq m$, where m is one half the length of the longest side, with P^* located as in the previous case.

The case of two crossing points is ruled out by the discussion concluding §3. The same discussion shows that in the case of four crossing points, $P_1P_2P_3P_4$ must form a rectangle. This is impossible unless two of the points, say P_2 and P_3, are on one side, with P_2 the projection of P_1, and P_3 the projection of P_4,

* Compare this result with C. S. Ogilvy's remark in solving E 814, this MONTHLY, vol. 56, 1949, p. 37.

and P_1P_4 parallel to P_2P_3. P^* must be at the center of the rectangle with $P^*P_1 = P^*P_4 = a$. Furthermore, since there are to be just four crossing points, we must have $P^*V_1 \leqq a$, where V_1 is the vertex of A_1 opposite the edge containing P_2 and P_3. This requires the angle at V_1 to be obtuse, hence this case of four crossing points can arise only for an obtuse triangle. The locus of P^* (for circles of varying radius) considered only as the center of the inscribed rectangle $P_1P_2P_3P_4$ is the straight line segment joining the midpoint M_1 of V_2V_3 with the midpoint H_1 of the altitude drawn from V_1. But if, see Figure 5, we note that triangle $P_1P_4V_1$ is homothetic to the given triangle $V_2V_3V_1$, we see that the circumcenter of $P_1P_4V_1$ is on the line joining V_1 to O, the circumcenter of $V_2V_3V_1$, and then the restriction $P^*V_1 \leqq a$ means that we have this case of four crossing points only when a is in the range $p \leqq a < m$, where $p = O_1V_1$

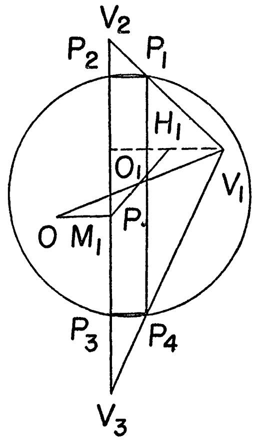


FIG. 5

and O_1 is the intersection of the lines M_1H_1 and OV_1. In case the triangle $V_1V_2V_3$ is isosceles, O_1 is not defined by this construction. An alternative definition of O_1 covering this exception is noted in the next section.

5. The circle-triangle covering problem for 6 crossing points. The remaining discussion will show that there are no other stationary values of A for $a \geqq R$ or $a \leqq r$ for an acute triangle, or for $a \geqq p$ or $a \leqq r$ for an obtuse triangle, and this leaves only the following, but most interesting, cases that there should be six crossing points with $r < a < R$ for an acute triangle and $r < a < p$ for an obtuse triangle. Before proceeding further we should note that the case of the right triangle is easily included in the discussion of either of the above cases, for it is the in-between case for which $p = m = R$.

Let (d_1, d_2, d_3) indicate the signed distances of a point $P:(x, y)$ from the respective sides V_2V_3, V_3V_1, V_1V_2 of the given triangle, with the agreement that d_1, d_2, d_3 are all positive for points interior to the triangle. Let (x_1, x_2, x_3) be the trilinear coordinates of P, with $x_1 = kd_1$, $x_2 = kd_2$, $x_3 = kd_3$, $k \neq 0$. Let

s_1, s_2, s_3 and c_1, c_2, c_3 indicate the sines and cosines, respectively, of the interior angles of the triangle at V_1, V_2, V_3, respectively. If we suppose that the circle of radius a and center P intersects the triangle in six crossing points, necessarily two on a side, and if, for the moment, we neglect the fact that these two points are supposed to have real coordinates and be on the side, not the side extended, then condition (7), applied to lines L_1, L_2, L_3 which are the respective altitudes drawn from V_1, V_2, V_3, gives the following conditions for $P:(x, y)$ to be P^*—the second set of conditions being a more easily identified form of the first set:

$$s_3\sqrt{a^2 - d_2^2} = s_2\sqrt{a^2 - d_3^2}, \tag{8.1}$$

$$s_1\sqrt{a^2 - d_3^2} = s_3\sqrt{a^2 - d_1^2}, \tag{8.2}$$

$$s_2\sqrt{a^2 - d_2} = s_1\sqrt{a^2 - d^2}; \tag{8.3}$$

$$s_3^2d_2^2 - s_2^2d_3^2 = (s_3^2 - s_2^2)a^2, \tag{9.1}$$

$$s_1^2d_3^2 - s_3^2d_1^2 = (s_1^2 - s_3^2)a^2, \tag{9.2}$$

$$s_2^2d_1^2 - s_1^2d_2^2 = (s_2^2 - s_1^2)a^2. \tag{9.3}$$

Of course the third equation in each set is redundant. To find the locus of P^* we eliminate a by adding the equations (9.1), (9.2), (9.3) and we obtain, after changing to trilinear coordinates,

$$(s_2^2 - s_3^2)x_1^2 + (s_3^2 - s_1^2)x_2^2 + (s_1^2 - s_2^2)x_3^2 = 0. \tag{10}$$

Equation (10) in general represents a conic E with respect to which the given triangle is self-polar. If the triangle is isosceles, but not equilateral, E is determined by (10) but is degenerate. If the triangle is equilateral, (10) is meaningless, but equations (8) show that the locus of P^* is a single point, the incenter. It is easily checked that E passes through incenter I and the excenters I_1, I_2, I_3 of triangle V_1, V_2, V_3, for these points have the following trilinear coordinates: $I:(1, 1, 1)$; $I_1:(-1, 1, 1)$; $I_2:(1, -1, 1)$; $I_3:(1, 1, -1)$. But I, I_1, I_2, I_3 form an orthocentric system, and every conic through an orthocentric system is known to be an equilateral hyperbola with its center Q on the circumcircle of the related diagonal triangle,* which in this case is $V_1V_2V_3$; hence this description fits the conic E. Furthermore, it is easily seen that the symmedian point S (which is the intersection of the symmedian lines, which are isogonal conjugates of the familiar median lines) and the exsymmedian points S_1, S_2, S_3 (which are the vertices of a triangle formed by the tangent lines drawn to the circumcircle at V_1, V_2, V_3) are points of E because these points have the following trilinear coordinates: $S:(s_1, s_2, s_3)$; $S_1:(-s_1, s_2, s_3)$; $S_2:(s_1, -s_2, s_3)$; $S_3:(s_1, s_2, -s_3)$. From the well-known identity $c_i^2 = 1 - s_i^2$, it then follows readily that the points $O:(c_1, c_2, c_3)$; $O_1:(-c_1, c_2, c_3)$; $O_2:(c_1, -c_2, c_3)$; $O_3:(c_1, c_2, -c_3)$ are on E; but O, the circumcenter of $V_1V_2V_3$ and O_1, O_2, O_3, may be determined

* For example, see T. F. Holgate, Projective Pure Geometry, New York, 1930, p. 207.

as intersections of lines, one through each vertex and harmonic conjugate to the circumradial line drawn to that vertex with respect to the sides of the triangle meeting at that vertex; O_1 is exactly the O_1 of the preceding section—namely, the intersection of M_1H_1 and OV_1, when this construction is valid; and similarly, for O_2 and O_3.

The restriction that the six crossing points neither have non-real coordinates nor be on the extended sides of triangle $V_1V_2V_3$ is readily investigated, for E is determined in terms of the parameter a by functions that are continuous. For an acute triangle the restriction is that P^* be on that segment of E between I and O for which $r<a<R$. For an obtuse triangle the restriction is that P^* be on that segment of E between I and O_1 for which $r<a<p$.

This completes the discussion of the locus of P^*, except for a few remarks concerning the hyperbola E and the meaning of equations (9).

Equation (9.1) represents an hyperbola E_{a1} with its center at V_1 and with the symmedian and exsymmedian through V_1 as its asymptotes. Four points on E_{a1} which are easily constructed are those for which $d_2=\pm a$, $d_3=\pm a$. In the same manner (9.2) and (9.3) represent hyperbolas E_{a2} and E_{a3}, respectively. The set of four hyperbolas E, E_{a1}, E_{a2}, E_{a3} intersect not only at $P_a^*:(x_1, x_2, x_3)$ but also at $P_{a1}:(-x_1, x_2, x_3)$; $P_{a2}:(x_1, -x_2, x_3)$; $P_{a3}:(x_1, x_2, -x_3)$. Since the construction of the points of intersection of two conics is, in general, a fourth order problem, it will not be possible, in general, to construct P_a^* in the six point case, with ruler and compass alone.

The center Q of E is the pole, with respect to E, of the line at infinity whose equation is $s_1x_2+s_2x_2+s_3x_3=0$. Hence we find that Q is the point $(2s_1^2c_1-s_2s_3, 2s_2^2c_2-s_3s_1, 2s_3^2c_3-s_1s_2)$. But the point W (which is the center of the Brocard circle) midway between S and O has the coordinates $(s_1^2c_1+s_2s_3, s_2^2c_2+s_3s_1, s_3^2c_3+s_1s_2)$ and the median point M is given by (s_2s_3, s_3s_1, s_1s_2). Hence $Q=2W-3M$ is on the line WM. Furthermore, it can be shown by reverting to metric coordinates that M is always between W and Q. Thus a neat construction for Q is afforded by finding the intersection of the circumcircle of the given triangle with the ray drawn from W through M.

There does not seem to be, in general, any simple construction for the asymptotes of E, but in the particular case of a right triangle the following construction is of interest. Let V_1 be the vertex opposite the hypotenuse. With V_2 as a center construct a circle s_2 with radius v_2, where v_2 is the length of the side opposite V_2. Let S_2 intersect the line V_1V_2 in the points X_2 and Y_2 with V_2 between X_2 and V_1. With V_3 as a center construct a circle S_3 with radius v_3 where v_3 is the length of the side opposite V_3. Let S_3 intersect V_1V_3 in the points X_3 and Y_3 with V_3 between X_3 and V_1. Let S_2 and S_3 intersect in the points Q and Q' with Q on the same side of V_2V_3 as V_1. Then the hyperbola E has Q as center, QV_1 and QQ' as asymptotes, and QX_2Y_3 and QX_3Y_2 as axes.

By way of summary, we describe an interesting model obtained from the circle-triangle problem using (x, y, a) rectangular coordinates and plotting the locus of $P^*:(x, y)$ against a. For values $0\leqq a\leqq r$ the locus is a solid tetrahedron

with the points V_1, V_2, V_3 in the $a=0$ plane as a base and with (x_I, y_I, r) as a vertex. For an acute triangle and for values $r<a<R$ the locus is a curved filament, extending from (x_I, y_I, r) to (x_0, y_0, R), the projection of this filament being a segment of the equilateral hyperbola E. For an obtuse triangle and for values $r<a<p$ the locus is a curved filament, extending from (x_I, y_I, r) to (x_{0_1}, y_{0_1}, p), the projection of this filament being a segment of E, but for values $p\leqq a\leqq m$, the locus is a straight line filament extending from (x_{0_1}, y_{0_1}, p) to (x_{M_1}, y_{M_1}, m). The remainder of the locus, in each case, is the solid common to three right-angled circular cones with vertical axes located at V_1, V_2, V_3, respectively. The model for an obtuse triangle is shown in Figure 6.

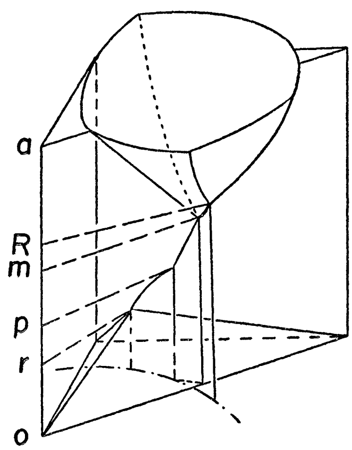


FIG. 6

6. Generalizations. The results given in equations (1)–(7) carry over to the case that A_1 and A_2 are finitely multiply-connected, providing that in setting up these equations a consistent direction of traversing C_1 is maintained with the interior of A_1 always on the left. The locus of P^* for a circle and polygons, other than a triangle, may be considered, but a cursory examination reveals here a situation too complicated with special cases to be of much interest. The problem of the minimal covering for three or more areas is a challenging one involving many variables. Finally, the generalizations to more dimensions may be considered, in particular, the three-dimensional problem. Here topological considerations enter as to the nature of the surfaces bounding the solids and the nature and orientation of the curves of intersection. Analogues to (7) are apparent, but analogues to (6) seem more difficult.

MATHEMATICAL NOTES

EDITED BY F. A. FICKEN, *New York University*

Material for this department should be sent to F. A. Ficken, Institute for Mathematics and Mechanics, New York University, 45 Fourth Ave., New York 3, New York.

ITERATED SUMS OF POWERS OF THE BINOMIAL COEFFICIENTS*

E. B. SHANKS, Vanderbilt University

Since James Bernoulli's formulas for "Summa Potestatum" were given in *Ars Conjectandi*, considerable interest has been manifested by mathematicians in this problem. It is our purpose to give a two parameter generalization of these formulas by giving a formula (equation (6) below) for the pth iterated sum of the kth powers of the $(i+1)$th binomial coefficient which we denote by $S_i^k(n, p)$.

For x any integer (positive, negative, or zero) and y a positive integer, let

$$\binom{x}{y} = \frac{x(x-1)\cdots(x-y+1)}{y!}. \tag{1}$$

It is well known (and can be easily proved by a simple induction) that

$$\sum_{i=1}^{n}\binom{i}{y} = \binom{n+1}{y+1}. \tag{2}$$

This latter equation is valid for all positive integers n and y, where the symbols are defined by (1).

LEMMA 1. *For n any integer, k and i any positive integers,*

$$\begin{aligned}\binom{n}{i}^k = \binom{n+ik-i}{ik} &+ A_2\binom{n+ik-i-1}{ik} + \cdots \\ &+ A_{ik-i}\binom{n+1}{ik} + \binom{n}{ik},\end{aligned} \tag{3}$$

where A_j for $j=2, 3, \cdots, ik-i$ is independent of n.

Proof. Each side of (3) is a polynomial in n of degree at most ik so that the lemma will be proved if it is shown that these polynomials are equal for $ik+1$ distinct values of n. For $n=-1, 0, 1, 2, \cdots, i$ the polynomials are equal irrespective of the values of the A_j as may be verified by substitution and the use of the definition (1). For $n=i+1, i+2, \cdots, ik-1$ the values of $A_2, A_3, \cdots, A_{ik-i}$ are determined successively in a unique manner as positive integers independent of n, again by substitution and the use of (1). For these uniquely determined values of the A_j, the polynomials are equal for the $ik+1$ distinct values $-1, 0, 1, 2, \cdots, ik-1$ and the lemma follows.

* Presented to the Southeastern Section of the Association at Gainesville, Florida, April 8, 1950.

Note that equation (3) is a two parameter generalization of the fact that any sum of two consecutive triangular numbers is a perfect square.

LEMMA 2. *In equation* (3), *for* $j=2, 3, \cdots, ik-i$, *the coefficients* A_j *have the property that*

$$A_j = A_{ik-i-j+2} \tag{4}$$

and are given explicitly by

$$A_j = \begin{vmatrix} 1 & 0 & 0 & \cdots & 0 & \binom{i}{i}^k \\ \binom{ik+1}{ik} & 1 & 0 & \cdots & 0 & \binom{i+1}{i}^k \\ \binom{ik+2}{ik} & \binom{ik+1}{ik} & 1 & \cdots & 0 & \binom{i+2}{i}^k \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ \binom{ik+j-1}{ik} & \binom{ik+j-2}{ik} & \binom{ik+j-3}{ik} & \cdots & \binom{ik+1}{ik} & \binom{i+j-1}{i}^k \end{vmatrix}. \tag{5}$$

Proof. In (3) let n take successively the values $i+j-1$ and $-j$, and we get, respectively,

$$\binom{i+j-1}{i}^k = \binom{ik+j-1}{ik} + A_2\binom{ik+j-2}{ik} + \cdots + A_{j-1}\binom{ik+1}{ik} + A_j,$$

$$\binom{i+j-1}{i}^k = \binom{ik+j-1}{ik} + A_{ik-i}\binom{ik+j-2}{ik} + \cdots + A_{ik-i-j+3}\binom{ik+1}{ik} + A_{ik-i-j+2},$$

where we have used the identity $\binom{x}{y} = (-1)^y\binom{y-x-1}{y}$ in deducing the second equation. Now let $j=2, 3, \cdots, ik-i$ in succession in these equations, and we get $A_j = A_{ik-i-j+2}$ as stated in the lemma. Consider $A_1=1$, the coefficient of $\binom{ik+j-1}{ik}$ in the first equation, and let j assume the values $1, 2, \cdots, j$. The resulting system of j equations in j unknowns has the unique solution (5) by Cramer's rule, the determinant of coefficients having the value $+1$.

Let $S_i^k(n, o) = \binom{n}{i}^k$ and

$$S_i^k(n, p) = S_i^k(1, p-1) + S_i^k(2, p-1) + \cdots + S_i^k(n, p-1),$$

where i, k, n and p are positive integers. We then may state the following

THEOREM. *It is identically true that*

$$
S_i^k(n, p) = \binom{n + ik - i + p}{ik + p} + A_2\binom{n + ik - i + p - 1}{ik + p} + \cdots + A_{ik-i}\binom{n + p + 1}{ik + p} + \binom{n + p}{ik + p}, \tag{6}
$$

where the A_j *are determined by* (4) *and* (5).

Proof. This follows immediately from Lemmas 1 and 2 using (2) in succession p times, starting with the identity (3).

It is an immediate consequence that $S_i^k(n, p)$ is a polynomial in n of degree $ik+p$.

If we set $A_1 = A_{ik-i+1} = 1$ and equate the coefficients of the highest power of n in equation (3), we obtain

$$
\sum_{j=1}^{ik-i+1} A_j = \frac{(ik)!}{(i!)^k}.
$$

In the formula for $S_i^k(n, p)$, Bernoulli's results are given by taking $i = p = 1$. The case $i = p = 1$ was discussed also by E. E. Witmer in an article, "The Sums of Powers of Integers," this MONTHLY, Vol. 42, No. 9, Part I, November, 1935, pp. 540–548. The case $i = 1$, $k = 2$ was the subject of Problem 22, *Mathematics Magazine*, Vol. XXII, No. 1, p. 51, while $i = 1$, $k = 3, 4, 5$ was the subject of Problem 4380, this MONTHLY, Vol. 57, No. 2, February, 1950, where the statement is made, "If possible, determine the form for general k." The case $i = 2$, $k = 1, 2, 3, 4, 5$ was investigated by P. A. Piza and presented at the meeting of the Southeastern section of the Association. His paper included a method for determining the formula in a recursive manner for $i = 2$ and k any positive integer. For a review of the previous work in this field the reader is referred to the references given in Witmer's article.

If in equation (3) we take $i = 1$, we get

$$
n^k = \binom{n + k - 1}{k} + A_2\binom{n + k - 2}{k} + \cdots + A_{k-1}\binom{n + 1}{k} + \binom{n}{k},
$$

which gives a formula for the kth power of any integer n in terms of a linear combination of k successive $(k+1)$th binomial coefficients, beginning with the $(k+1)$th of order n (reversing the order given in the equation above). The A_j are positive integers depending only on k. For example,

$$
n^3 = \binom{n}{3} + 4\binom{n + 1}{3} + \binom{n + 2}{3},
$$

$$
n^4 = \binom{n}{4} + 11\binom{n + 1}{4} + 11\binom{n + 2}{4} + \binom{n + 3}{4}.
$$

From a set of equations such as the above or by calculating the A_j, we obtain the following table:

$$S_1^1(n, 1) = \binom{n+1}{2}$$

$$S_1^2(n, 1) = \binom{n+2}{3} + \binom{n+1}{3}$$

$$S_1^3(n, 1) = \binom{n+3}{4} + 4\binom{n+2}{4} + \binom{n+1}{4}$$

$$S_1^4(n, 1) = \binom{n+4}{5} + 11\binom{n+3}{5} + 11\binom{n+2}{5} + \binom{n+1}{5}$$

. . . .

$$\begin{aligned} S_1^{10}(n, 1) = {} & \binom{n+10}{11} + 1013\binom{n+9}{11} + 47840\binom{n+8}{11} + 455192\binom{n+7}{11} \\ & + 1310354\binom{n+6}{11} + 1310354\binom{n+5}{11} + 455192\binom{n+4}{11} \\ & + 47840\binom{n+3}{11} + 1013\binom{n+2}{11} + \binom{n+1}{11}. \end{aligned}$$

THE DISTANCE SET FOR THE CANTOR DISCONTINUUM

W. R. UTZ, University of Missouri

In this note there is given what seems to be a new and simple proof of a slight generalization of the known theorem (cf. J. F. Randolph, *Distances between points of the Cantor set*, this MONTHLY, vol. 47 (1940), pp. 549–551; *Some properties of sets of the Cantor type*, Journal of the London Mathematical Society, vol. 16 (1941), pp. 38–42) that the distance set for the Cantor middle third set based on the unit interval is the set of all real numbers c such that $0 \leqq c \leqq 1$.

THEOREM. *Given any real numbers c and m such that* $0 \leqq c \leqq 1$ and $\frac{1}{3} \leqq |m| \leqq 3$, *there exist numbers x and y representing points on the Cantor middle third set based on the unit interval* $[0, 1]$ *such that* $y - mx = c$.

Proof. Consider the Cantor set on the unit interval of the x-axis and on the unit interval of the y-axis of a rectangular cartesian coordinate system. Let M be the product of these two sets. That is, M consists of all points of the unit square (0, 0), (0, 1), (1, 1), (1, 0) having coordinates (x, y), each of which is a coordinate of a point of the Cantor set on $[0, 1]$. To prove the theorem we will show that for an admissible c and m, the line $y - mx = c$ meets the set M in at least one point (x, y).

Let S be any closed square in the unit square of the x, y-plane and having sides parallel to the coordinate axes. Divide S into three equal vertical strips by inserting two vertical line segments in S. Delete from S the interior of the middle strip as well as its top and bottom open segments. Now, delete the interior of a similar horizontal "middle third" strip so that there remain four closed squares, each of area one ninth the area of the original square S and separated by a region in the shape of a cross. It is easily seen that if $y=mx+a$ meets S and if $\frac{1}{3}\leqq|m|\leqq 3$, then the line meets at least one of the four smaller squares.

If $0\leqq c\leqq 1$, the line $y=mx+c$ meets the closed unit square and if $\frac{1}{3}\leqq|m|\leqq 3$, the line meets at least one of the four squares that remain after the first cross is deleted corresponding to the first step in the formation of the Cantor sets on the two axes. Fix the attention on one of the squares A, having a point in common with $y=mx+c$. After deletions corresponding to the second step in the formation of the Cantor sets on the two axes the line will still have a point in common with at least one of the smaller squares $B\subset A$. The proof is completed by induction and an application of the theorem that a monotone decreasing sequence of non-empty compact sets is non-empty.

This theorem yields Randolph's Theorems 1 and 2 (of the first paper cited) in case $|m|=1$.

A THEOREM CONCERNING FUNCTIONS DISCONTINUOUS ON A DENSE SET

M. K. FORT, JR., University of Illinois

1. Introduction. It is well known that a function can be discontinuous at points of an everywhere dense set, and yet be continuous at points of a residual set. One might wonder whether or not it is possible for such functions to be differentiable at points of a residual set. The theorem proved in this note shows that this is not possible.

In the problem section of a recent issue of this MONTHLY, C. D. Olds proposed the problem of finding an example of a function which is discontinuous in an everywhere dense set and which is also differentiable in an everywhere dense set.* It follows from our theorem that functions of this type have a somewhat surprising property; namely, at "most" points (*i.e.* on a residual set) the function is continuous but not differentiable.

We let f be a real valued function which is defined for all real numbers. It will be convenient to let A be the set of points at which f is discontinuous, B be the set of points at which f is differentiable and C be the set of points at which f is continuous but not differentiable.

2. Theorem. We shall prove the following

THEOREM. *If A is everywhere dense, then B is a set of the first category.*

* Vol. 57 (1950), p. 557.

Proof. For each positive integer n, we define D_n to be the set of all points p which have the property that there exist points x and y in the $1/n$-neighborhood of p for which

$$\frac{f(x) - f(p)}{x - p} - \frac{f(y) - f(p)}{y - p} > 1.$$

Next we define $D = \bigcap_{n=1}^{\infty} D_n$.

It is clear that f is not differentiable at any point of D. If we can show that D is a residual set, it follows that B must be a set of the first category. Hence, in order to establish our theorem it is sufficient to prove that D is a residual set.

In order to prove that D is a residual set, it is sufficient to prove that for each n the set of interior points of D_n form an everywhere dense set. Let I be an open interval of real numbers. Since A is everywhere dense, there exists $q \epsilon I \cap A$. We now choose real numbers h and k such that

$$\max\,(f(q),\ \limsup_{t\to q} f(t)) > h > k > \min\,(f(q),\ \liminf_{t\to q} f(t)).$$

Now suppose $p \in I$, $q - 1/n < p < q$ and $h - k > q - p$. We obtain

$$\frac{h - f(p)}{q - p} - \frac{k - f(p)}{q - p} = \frac{h - k}{q - p} > 1.$$

It is possible to find x and y arbitrarily close to q for which $f(x) > h$ and $f(y) < k$. Hence it is possible to find x and y in the $1/n$-neighborhood of p for which

$$\frac{f(x) - f(p)}{x - p} - \frac{f(y) - f(p)}{y - p} > 1.$$

This proves that $p \in D_n$. Since the set of all $p \in I$ for which $q - 1/n < p < q$ and $h - k > q - p$ forms an open set, each such p is interior to D_n. It follows that the interior of D_n is everywhere dense.

The following corollary is of interest in connection with functions of the type suggested by Olds in his problem.

COROLLARY. *If A and B are both everywhere dense, then C is a residual set (and hence also everywhere dense).*

Proof. Since f is differentiable at points of an everywhere dense set, f is certainly continuous at points of an everywhere dense set. It follows from a well known theorem that A must be a set of the first category. The above theorem proves that B is a set of the first category. Since C is the complement of $A \cup B$, C must be a residual set.

3. An example. Let $x_1, x_2, x_3, \cdots$ be an enumeration of the set R of rational numbers (or any other countable everywhere dense set). We define $f(x_n) = 1/n^3$ for each positive integer n, and define $f(t) = 0$ if $t \notin R$. It is obvious that f is discontinuous at each point of R, and continuous at every other point.

We let S_n be the open interval $(x_n - 1/n^2, x_n + 1/n^2)$. It is easy to see that the set

$$S = \bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} S_k$$

has measure zero, and hence that $S \cup R$ has measure zero. If $q \notin S \cup R$ then $p \notin (x_n - 1/n^2, x_n + 1/n^2)$ for sufficiently large n, and hence

$$\left| \frac{f(x_n) - f(p)}{x_n - p} \right| \leqq \frac{1}{n}$$

for all sufficiently large n. It follows that f is differentiable at each point of the complement of $S \cup R$. Since $S \cup R$ has measure zero, f is differentiable not only at points of an everywhere dense set but almost everywhere.

It is interesting to note that although our theorem proves that C has "more" points than B in the sense of category, this example shows that B can be "larger" than C in the sense of measure theory.

CLASSROOM NOTES

Edited by C. B. Allendoerfer, Haverford College

All material for this department should be sent to G. B. Thomas, Department of Mathematics, Massachusetts Institute of Technology, Cambridge 39, Mass.

FURTHER NOTE ON THE REMAINDER IN COMPUTING BY SERIES

R. K. Morley, Worcester Polytechnic Institute

This note is a supplement to the author's paper on this topic in the October, 1950, number of the Monthly.

As there we denote a series of positive terms by $\sum_{n=1}^{\infty} u_n$, and we suppose that $u_n = f(n)$ is such that $f(n)$ can be changed to $f(x)$ with x a continuous variable. We suppose that the sum of m terms of the series has been computed and we assume that $\int_m^\infty f(x)dx$ is finite and can be found, and that for $m \leqq x$, $f'(x)$ is negative and $f''(x)$ is positive. We let $R = \sum_{n=m+1}^{\infty} u_n$.

In the previous paper lower and upper bounds for R were found as follows:

$$R > \int_m^\infty f(x)dx - \frac{1}{2} u_m \tag{1}$$

$$R < \int_m^\infty f(x)dx - \frac{1}{2} u_{m+1}. \tag{2}$$

The purpose of the present note is to obtain somewhat closer bounds.

As before we draw the curve $y=f(x)$ and represent the terms u_n by rectangles of base 1 under the curve whose right-hand sides are $f(n)$, and also by rectangles of base 1 above the curve whose left-hand sides are $f(n)$. We denote the points $[m+1, f(m+1)]$, $[m+2, f(m+2)]$, $[m, f(m+1)]$, $[m+2, f(m+1)]$ by B, C, D, G respectively. Draw chord BC. Also draw a tangent to the curve at B and produce it till it cuts $x=m$ at H.

For the lower bound for R we consider $u_{m+1}=$ the rectangle under BG.

$$u_{m+1} > \int_{m+1}^{m+2} f(x)dx + \triangle BCG, \qquad \triangle BCG = \frac{1}{2}(u_{m+1} - u_{m+2})$$

and we have similar expressions for the other terms of R.

Hence

$$R > \int_{m+1}^{\infty} f(x)dx + \frac{1}{2}\left[(u_{m+1} - u_{m+2}) + (u_{m+2} - u_{m+3}) \cdots \right]$$

or

$$R > \int_{m+1}^{\infty} f(x)dx + \frac{1}{2} u_{m+1}. \tag{3}$$

It is evident that this is the same as adding u_{m+1} to the lower bound for $\sum_{m+2}^{\infty} u_n$ given by (1). (3) gives a slightly closer lower bound for R than (1).

For the upper bound for R we consider $u_{m+1}=$ the rectangle under DB. With the restrictions assumed for $f(x)$ the tangent BH is wholly below the curve. Therefore

$$u_{m+1} < \int_{m}^{m+1} f(x)dx - \triangle DBH \tag{4}$$

and we have similar expressions for the other terms of R.

$$\triangle DBH = \frac{1}{2}\left[-f'(m+1)\right].$$

Denote triangles like DBH under tangents at $m+1$, $m+2$, etc. by T_{m+1} etc. Then $T_i = -\frac{1}{2}f'(i)$. By (3) we have

$$\sum_{i=m+1}^{\infty} T_i > \frac{1}{2}\left\{\int_{m+1}^{\infty}[-f'(x)]dx + \frac{1}{2}\left[-f'(m+1)\right]\right\}$$

or

$$\sum_{i=m+1}^{\infty} T_i > \frac{1}{2} f(m+1) + \frac{1}{4}[-f'(m+1)]$$

$$(5) \qquad > \frac{1}{2} u_{m+1} + \frac{1}{4}[-f'(m+1)].$$

From (4)

$$R < \int_m^\infty f(x)dx - \sum_{i=m+1}^{\infty} T_i$$

and applying (5) to this we have for the wanted lower bound for R

$$R < \int_m^\infty f(x)dx - \frac{1}{2} u_{m+1} - \frac{1}{4}[-f'(m+1)]$$

It will be noted that this is the same as the bound that was given by (2) less $\frac{1}{4}[-f'(m+1)]$. It therefore gives a closer bound. In some cases the improvement is considerable.

ALGEBRAIC NUMBERS AND A POINT OF VIEW

R. M. REDHEFFER, University of California at Los Angeles

Introduction. The rational numbers are closed under the arithmetical operations; one cannot get a new number by addition, subtraction, multiplication, or division. On the other hand, if one adjoins to these arithmetical operations the "analytical" operation of taking a limit, then the rationals are no longer closed. There arises a larger set (the real numbers) which includes $\sqrt{2}$, among others.

Consider that new set under the five operations, addition, subtraction, multiplication, division, and taking limits. Is it closed? Offhand, one might be inclined to think not. To be sure, the limit of every *rational* sequence will be real, since the reals were obtained that way. But why should the limit of every real sequence be real? As in the theory of Baire classes, one might expect an infinite hierarchy of more and more complicated numbers.

In spite of one's doubts this does not happen; the reals are closed under the five operations, and we must turn elsewhere to obtain anything new. *Such is the meaning of the Dedekind theorem.* As soon as the rationals are thought of as points on a line, rather than all jumbled up together in a set with no obvious geometrical structure—as soon as this is done, our failure to get anything new becomes perfectly understandable.

Inverse. The notion of adjoining an operation gives an interesting way of looking at algebraic numbers. Instead of "limit," this time, we adjoin the operation of "inverse." If the polynomial $P(z)$ belongs to a field K, in the sense that all its coefficients are in K, then $w = P(z)$ will be in K whenever z is. Thus, given z we can find w. On the other hand, for w in our set there may or may not be a z such that $P(z) = w$; given w, we can *sometimes* find z. On other occasions

we shall get something new. If by chance at least one root of $P(z)=w$ is in K, for all $w\in K$ and all $P\in K$, then we say that the field K has *inversive closure.* Similarly when K is only a set, not a field. Instead of polynomials $P(z)$ we shall sometimes consider the most general function that can be constructed with the operations available.

Since we speak here of complex numbers, this notion of inversive closure surely makes sense—it is known what we mean by the roots of $P(z)=w$, whether or not they are in K. On the other hand, to hunt for the inverse of a given element is a new operation, different from the field operations that we had to start with.

Rational numbers. On the most naive level it is exactly thus that the number system developed. With *counting* as fundamental, one readily defines "+." It is possible then to form a function as complicated as $a+z$, both a and z being integers. If $w=a+z$, then w is in the set whenever z is. On the other hand given w we may or may not be able to find z; the new things that arise thus are the negative integers. Similarly, the solution of $az-b=0$ gives the fractions, while the solution of $z^2+1=0$ gives the imaginaries.

Addition allows one to form not only $a+z$ but also $a+z+z+\cdots+z=a+bz$. If now we hunt for a solution z to the equation $a+bz=w$, we get all the positive and negative rationals. Hence the set becomes a field, and there is inversive closure, in that the set contains all inverses of the most complicated functions that can be formed.

Linear equations. These remarks presuppose that the integers $1, 2, 3, \cdots$ are known to be in the set. What can be said when inversive closure alone is postulated? We confine ourselves at first to the linear function $a+bz$, and make the following observations:

I. *Suppose given a set of complex numbers, at least one of which is different from zero. If the root z of $a+bz=0$ is in the set whenever a and b are, then the set need not form a field. But if $z+1$ is in the set whenever z is, then the set must be a field. And hence if the root of $a+bz=w$ is in the set whenever a, b, w are, then the set is a field.*

To see this, note first that the set consisting of -1 alone satisfies the first closure condition, as does also the set consisting of -1 and 1. For the second part of the assertion, the choice $a=b\neq 0$ shows that -1 is in the set, whence by the $z+1$ condition we know that 0 and 1 are. It is possible then to form $a+z=0$, so that $-a$ is in the set whenever a is. Since negatives are in the set we can consider $az-b=0$ to see that b/a is in the set whenever b and $a\neq 0$ are. In particular $1/b$ is in the set, and hence the root ab of $(1/b)z-a=0$ is in the set. Finally, the presence of b/a insures that of $b/a+1$, by our special condition $z+1$; and product closure shows then that $a(b/a+1)=b+a$ is in the set too.

Turning now to the third part of the statement, we note that $az+a=a$ makes the set contain 0. Then $az+0=a$ shows that it contains 1, while az

$+a=0$ shows that it contains -1. We can now form $z-1=a$, to get the $z+1$ condition needed to complete the proof.

Since the solution of $a+bz=w$ is rational whenever a, b and w are, it is evident that the rational field has inversive closure in our present sense of the word (addition alone being permissible when forming functions of z). The result just established says that no smaller set can be closed in this sense.

Polynomial equations. The function $a+z$ involves a single addition and gives us the negatives. The function $a+bz$ involves repeated addition for integer b, or addition and one multiplication in general. It gives us the positive and negative rationals, and this whether or not the presence of the integers is postulated beforehand. Similarly one might consider az, formed by a single multiplication.

Leaving these special functions, let us proceed at once to full use of both addition and multiplication, with repeats. One obtains the set of all polynomials $P(z)$ with coefficients in the set, and one is led to $P(z)=w$ as the most general equation requiring an inverse. It is evident that the rationals do not have inversive closure in this extended sense, and yet the set of all complex numbers has it. Such is the meaning, for us, of the Gauss theorem on polynomial equations.

Since the rationals are not closed while the set of all complex numbers is closed, one might expect that a process of adjoining suitably chosen numbers to the rationals should eventually lead to a minimal closed set. It will be found that this is the case. There is a set of numbers which is a proper subset of all the complex numbers, which is closed under polynomial inversion, and which is minimal in that every closed set must contain this one as a subset.

Algebraic numbers. Let us see, now, how this minimal closed set (under polynomial inversion) is to be constructed. By I, the special case in which $P(z)$ is linear gives us the positive and negative rationals, at least. Hence we can form all polynomials with rational coefficients, and in particular, we can form the equations $P(z)=0$ where $P(z)$ has integer coefficients. The "inverses" z under this transformation are called *algebraic numbers*. It is clear that they are a notable enrichment of the number system; an equation as simple as $z^2-2=0$ already implies the idea inherent in *limit*, while $z^2+1=0$ gives something strange indeed.

Though we postulate only that $P(z)=0$ shall have one root in the set, it is easy to see that the equation must in fact have all roots in the set. For the set is known, by I, to be closed under the field operations. Hence the coefficients of $Q(z)=P(z)/(z-r)$ will be in the set whenever r is and $P(r)=0$. Therefore $Q(z)$ has a root in the set, and the result follows by repetition. Thus it is that any set closed under polynomial inversion must contain all the algebraic numbers, at least (provided always it is not the set 0 alone). If the algebraic numbers are themselves closed, then our task is finished.

By repeated use of Sylvester's dialytic procedure* we can show rather simply that an equation $P(z)=0$ with algebraic numbers for coefficients has roots which are again algebraic numbers. Moreover if z is algebraic, so is $z+1$; in fact $z+1$ is a root of $Q(z)=P(z-1)=0$ whenever $P(z)=0$, and $Q(z)$ obviously has integer coefficients. Therefore by I we see at once that the algebraic numbers form a field. In particular the sum of two algebraic numbers is algebraic, and hence the equation $P(z)=w$, with algebraic coefficients in P, reduces to $Q(z)=0$ with algebraic coefficients in Q. The result just quoted shows that all the roots are algebraic, and closure is thus established. All this can be summarized as follows:

II. Given a set of complex numbers at least one of which is different from zero. Suppose a root of $az^n+bz^{n-1}+\cdots+k=w$ is in the set whenever $a, b, \cdots, k, w$ are. Then the set contains not only one but all roots of these equations. Also it contains the rationals, hence the algebraic numbers. These latter have the closure property, so that the algebraic numbers could be defined as the unique minimal set closed under polynomial inversion. The algebraic numbers form a field, hence are closed under the arithmetic operations as well.

Dedekind analogue. Starting with the positive integers under addition and multiplication one forms the polynomials in z, that is, the most general functions that addition and multiplication allow. To the operations of addition and multiplication one now adjoins the further operation of polynomial inversion. In this way one obtains the algebraic numbers. If new polynomial equations are formed with algebraic numbers for coefficients, it might be expected that the roots would lead to still other numbers. The result just stated says that this does not happen, but that the set of algebraic numbers is closed under the operations of addition, multiplication, subtraction, division, *and inverse.* Here we have the algebraic analogue of Dedekind's theorem as discussed previously. The closure and minimal property which Dedekind theory establishes for the continuum corresponds to the closure and minimal property established now for the algebraic numbers.

Inverse images of zero. The algebraic numbers appear as solutions of the equations $P(z)=0$, and do not require the more general equations $P(z)=w$. By I it is known that these two kinds of inverse are not equivalent when $P(z)$ is linear, and we shall see how they are related in the general case.

III. Suppose given a set of complex numbers closed in the sense that a root of $az^n+bz^{n-1}+\cdots+k=0$ is in the set whenever $a, b, \cdots, k$ are. Then the set need not be a field. But if $z+1$ is in the set whenever z is, then the set is a field. If the set is a field then it will be closed in the sense $az^n+bz^{n-1}+\cdots+k=w$, and conversely closure in this sense always implies that the set is a field. If the set is a field with a non-zero element then it must contain the algebraic numbers as subfield.

* Philip Franklin, Treatise on Advanced Calculus, John Wiley and Sons, 1940, Problem 30. This problem sequence suggested the present discussion.

To see this, note that the roots of $z^n+bz^{n-1}+\cdots+k=0$ will all be algebraic numbers, integers or units according as the coefficients $b, \cdots, k$ are all algebraic numbers, integers, or units. Again, the quotient of two units is a unit. Hence the set of algebraic units is closed in the sense that $P(z)=0$ has all its roots in the set, and yet it is not a field. The other assertions in *III* follow from *I*.

The generality of the function. Speaking of fields we have available the operation of division. Hence the meaningful functions are the rational functions $R(z)$, not just the polynomials, and the most general equation that makes sense, with the operations at our disposal, is $R(z)=w$. Now, even the set of all complex numbers need not contain solutions to an equation of this form, as is shown by the example $1/z=0$. Since inversive closure in the previous sense is out of the question, we ask instead that $R(z)=w$ have a solution *for all but one* w. The exceptional value happens to be identifiable as $w=0$; in other cases the equation $R(z)=w$ reduces to a polynomial equation in z. Thus closure still holds, both for the complex numbers and for the algebraic numbers, an exceptional w being admitted.

Closure with these "most general constructible" functions $R(z)$ is the strongest type of closure that one could reasonably ask of a field, and the algebraic numbers have it. We use the field operations in forming functions $R(z)$; we ask for a minimal field such that $R(z)=w$ always has a solution, except perhaps for one w; and we find the algebraic numbers as unique set with these properties.

By using inversion one can construct still more general functions (the algebraic functions, among others). Also for complex fields there is defined another arithmetic operation, raising to a power. Thus 2^z has a meaning. Using both exponentiation and inversion one is led to a large class of functions, for which the algebraic numbers fall far short of inversive closure. The Gelfond theorem concerning transcendence of α^β shows that an equation as simple as $2^z=w$ fails unless w is a rational power of 2.

Whether or not all the complex numbers are closed (or in some sense nearly closed) under these functions seems difficult to determine. It is true that the singularities are isolated, as can be shown by induction; and hence for all single-valued functions and for all but one w the Picard theorem guarantees existence of a z such that $F(z)=w$. But the elliptic modular function and other examples show that the assumption of single-valuedness is essential, if we wish to avoid detailed study.

One can bypass the question by seeking a minimal set that contains a root of $F(z)=w$ whenever the set of all complex numbers does so. Here $F(z)$ is any function formed by use of a finite number of additions, subtractions, multiplications, divisions, exponentiations, and inversions upon the indeterminate z and the numbers in the minimal set.

The set of roots of $F(z)=w$ is countable. Also if the set of numbers at hand is countable, then the set of functions F is likewise countable; the proof resembles

that used to show that there are only countably many algebraic numbers. Start with the set E_1 of integers, and add to this the roots z of all $F(z)=w$ formed by the above operations on members of E_1 and on z. There arises a new countable set, E_2. Now form all equations $F(z)=w$ by use of members of E_2 and the basic operations, and adjoin their roots to E_2. This gives a countable set E_3, and so on. The minimal set such that it contains every root of $F(z)=w$ is $E_1+E_2+E_3+\cdots$, whilst the minimum set such that it contains at least one root of $F(z)=w$ is $E_1+E_2+E_3+\cdots$ or a subset of it. In either case the set is countable.

What does all this show? It shows that the operation of inversion is essentially different from, and in some sense weaker than, the operation of limit. Even when carried to extremes, *inversion* gives an enumerable set at most, while *limit* gives the continuum.

ELEMENTARY PROBLEMS AND SOLUTIONS

EDITED BY HOWARD EVES, Oregon State College

Send all communications concerning Elementary Problems and Solutions to Howard Eves, Mathematics Department, Oregon State College, Corvallis, Oregon. This department welcomes problems believed to be new, and demanding no tools beyond those ordinarily furnished in the first two years of college mathematics. To facilitate their consideration, solutions should be submitted on separate, signed sheets, within three months after publication of problems.

PROBLEMS FOR SOLUTION

E 971. *Proposed by A. W. Willcox, Washington, D. C.*

Reconstruct the division problem:

```
* * ) * * * * 2 * ( * * * * *
      * *
      -----
      * * *
        * *
        -----
          * * *
          * * *
          -----
              * *
              * *
              ---
```

E 972. *Proposed by Michael Goldberg, Washington, D. C.*

Dissect a regular pentagon, by straight cuts, into six pieces which can be put together to form an equilateral triangle. (See Ball-Coxeter, *Mathematical Recreations*, top of p. 93.)

E 973. *Proposed by Leo Moser, Texas Technological College*

Show that if

$$x = -(b/a)\cos^2\{(1/4)\cos^{-1}(\{b^2 - 8ac\}/b^2)\}$$

exists, then it is a solution of $ax^2+bx+c=0$.

E 974. *Proposed by Alan Wayne, Flushing, N. Y*

Let L, M, N divide the sides BC, CA, AB of triangle ABC in the ratios $BL/LC=r$, $CM/MA=s$, $AN/NB=t$. Let R, S, T be the intersections of the pairs of cevians BM, CN; CN, AL; AL, BM. Show that the cross-ratios (AL, ST), (BM, TR), (CN, RS) are all equal to rst.

E 975. *Proposed by M. R. Spiegel, Rensselaer Polytechnic Institute*

Prove that

$$\int_0^1 dx/x^x = \sum_{i=1}^{\infty} 1/i^i.$$

SOLUTIONS

Concurrent Spheres

E 940 [1950, 632]. *Proposed by Victor Thébault, Tennie, Sarthe, France.*

Being given a tetrahedron $ABCD$, an arbitrary point P, and four coplanar points A', B', C', D' lying in the planes of the faces BCD, CDA, DAB, ABC, show that the lines PA', PB', PC', PD' intersect the spheres $PBCD$, $PCDA$, $PDAB$, $PABC$ in four points cospherical with P.

Solution by Howard Eves, Oregon State College. Let PA', PB', PC', PD' intersect the spheres $PBCD$, $PCDA$, $PDAB$, $PABC$ in the points A'', B'', C'', D''. Let (S) denote the sphere through A, B, C, D and (S'') the sphere through P, A'', B'', C''. Since A' lies on the radical plane of (S) and $(PBCD)$ and also on the radical plane of (S'') and $(PBCD)$, it lies on the radical plane of (S) and (S''). Similarly, B' and C' lie on the radical plane of (S) and (S''). It follows that D' is on the radical plane of (S) and (S''). But D' is on the radical plane of (S) and $(PABC)$. It is therefore on the radical plane of (S'') and $(PABC)$, and D'' is on (S''). Thus P, A'', B'', C'', D'' are cospherical.

The analogous theorem in the plane may be similarly established.

Also solved by Joseph Langr and Roger Lessard.

Periodic Properties of the Digits in the Powers of 2

E 942 [1950, 686]. *Proposed by L. A. Ringenberg, Eastern Illinois State College*

In the sequence of powers of 2: 2, 4, 8, 16, 32, $\cdots$, the units digits form a sequence of period four. What periodic properties do the other digits have?

Solution by Aaron Buchman, Buffalo, New York. Let the units digit be called the *first* digit; the tens digit, the *second* digit; *etc.* Then the period, k, of the periodic part of the sequence formed by the nth digits will be the smallest value of k which satisfies the congruence

$$2^{s+k} \equiv 2^s (\mathrm{mod}\ 10^n),$$

or

$$2^{s+k-n} \equiv 2^{s-n} (\mathrm{mod}\ 5^n).$$

Putting s equal to its minimum value of n, we have

$$2^k \equiv 1 (\mathrm{mod}\ 5^n).$$

Hence, by Euler's theorem,

$$k = \phi(5^n) = 4(5^{n-1}).$$

It is evident that the non-periodic pre-period of the sequence formed by the nth digits consists of $n-1$ zeros.

By a similar proof we may show that the period, k, of the nth digits of the powers of an integer a expressed in the number system with base $b>a$ is

$$k = \phi(\{b/g\}^n),$$

where $g=(a, b)$. Again, the periodic part of the sequence is preceded by $n-1$ non-periodic digits.

Also solved by J. H. Braun, D. H. Browne, R. E. Gettig, Roger Lessard, Theodore Lindquist, Fred Marer, Prasert Na Nagara, C. S. Ogilvy, and the proposer.

Central Motion

E 943 [1950, 686]. *Proposed by S. H. Gould, Purdue University.*

By analogy with the motion of the planets, it seems natural to assume for any central motion that, at least if the particle never passes through the center, (a) the distance of the particle from the center is a maximum only when the velocity of the particle is a minimum, (b) the velocity is a maximum only when the distance is a minimum. Prove (a) and give a counter-example of (b).

Solution by H. D. Block and C. E. Langenhop, Iowa State College. We shall not only prove what is asked but shall obtain the conditions under which the "natural assumptions" are valid and those under which the anomalous behavior described in (b) can occur. We do not assume that the force is derivable from a potential or is constant in time.

Using polar coordinates let $f(r, \theta, t)$ be the force toward the center so that $f>0$ is an attraction and $f<0$ a repulsion. Then, using the radial and trans-

verse components of acceleration and Newton's law, we have:

$$(1) \qquad -f(r, \theta, t) = m(\ddot{r} - r\dot{\theta}^2),$$

$$(2) \qquad m(r\ddot{\theta} + 2\dot{r}\dot{\theta}) = (m/r)d(r^2\dot{\theta})/dt = 0.$$

From (2) it follows that $r^2\dot{\theta} = h$, where h is constant during the motion. Then (1) becomes

$$(3) \qquad -f(r, \theta, t) = m(\ddot{r} - h^2/r^3).$$

Now $v^2 = \dot{r}^2 + r^2\dot{\theta}^2 = \dot{r}^2 + h^2/r^2$, so that

$$(4) \qquad d(v^2)/dt = 2\dot{r}\ddot{r} - 2h^2\dot{r}/r^3 = 2\dot{r}(\ddot{r} - h^2/r^3).$$

Comparing (3) and (4) we see that we have

$$(5) \qquad d(v^2)/dt = -(2\dot{r}/m)f(r, \theta, t) = -(2/m)f(r, \theta, t)(dr/dt),$$

which might also be obtained directly from the theorem that the change in kinetic energy equals the work done.

We assume that the time parameter in the orbit can always be extended in both directions, and that the motion is sufficiently smooth so that the conditions of maxima and minima are given by a change in sign of the time derivative. Inspection of (5) then leads immediately to

THEOREM I (a) *If* $f(r_1, \theta_1, t_1) > 0$, *then at* (r_1, θ_1, t_1) (i) *v is a minimum if and only if r is a maximum*, (ii) *v is a maximum if and only if r is a minimum.* (b) *If* $f(r_1, \theta_1, t_1) < 0$, *then at* (r_1, θ_1, t_1) (iii) *v is a minimum if and only if r is a minimum*, (iv) *v is a maximum if and only if r is a maximum.*

The condition (iv), however, cannot occur, as one sees from

THEOREM II. *If* $f(r_1, \theta_1, t_1) \leqq 0$, *then r is not a maximum at* (r_1, θ_1, t_1).

This is easily seen by rewriting (3) as $\ddot{r} = h^2/r^2 - f(r, \theta, t)/m$, so that $\ddot{r} > 0$ in this case. Then, if $\dot{r} = 0$, r must be a minimum.

To establish the results asked for in the problem we note that if r is a maximum then, by Theorem II, $f > 0$ and, by Theorem I (a), v is a minimum. As for a counterexample for the assertion (b) we can take the zero force field. Here the particle travels in a straight line with constant velocity, and with respect to any center not lying on the line of motion v is always a maximum or a minimum while r, in general, is neither. A second counterexample which is less trivial is that of a mass on a spring given an initial velocity in the direction of the spring, with the center of attraction considered to be a point on the extension of the line of the spring. Here the maximum v occurs at an intermediate value of r.

In the two counterexamples considered it may be observed that $f = 0$ at the point where the anomalous behavior occurs. That this must be the case follows from our results. For if v is a maximum (r_1, θ_1, t_1), then $f(r_1, \theta_1, t_1) \geqq 0$. But if $f(r_1, \theta_1, t_1) > 0$ then, by Theorem I (a)–(ii), r is a minimum. Thus the

condition that v be a maximum and r not a minimum at (r_1, θ_1, t_1) can occur only if $f(r_1, \theta_1, t_1)=0$.

Also solved by M. S. Klamkin, W. W. Zaring, and the proposer.

A Maximum Truncated Cone

E 944 [1950, 686]. *Proposed by E. R. Bowersox, Chicago, Ill.*

A truncated cone with one end open is to be formed from a circular sheet, cutting out a central portion for the base, and using the remaining annular ring for the curved area. Solve for the truncated cone of maximum volume.

Solution by Frank Herlihy, Comstock, Mich. Assume the given sheet has unit radius and let r denote the radius of the piece to be cut out. Then the dimensions of the truncated cone are easily found to be: radius of lower base $=r$, radius of upper base $=r^2$, slant height $=1-r$, altitude $=(1-r)(1-r^2)^{1/2}$. We then find the volume to be

$$V = (\pi r^2/3)(1 - r^3)(1 - r^2)^{1/2}.$$

Equating dV/dr to zero we get

$$r(r - 1)(6r^4 + 6r^3 + r^2 - 2r - 2) = 0.$$

The roots $r=0$ and $r=1$ lead to minimum volumes. The quartic factor has only one positive root, which is the value of r leading to the maximum volume. We find that $r=0.66148+$, yielding $V_{\max}=0.2242-$.

Also solved by J. H. Braun, D. H. Browne, David Mandelbaum, and C. S. Ogilvy.

Trihedral Angles of a Polyhedron

E 945 [1950, 687]. *Proposed by Leo Moser, University of North Carolina*

If all the faces of a convex polyhedron have central symmetry show that there are at least eight vertices where exactly three edges meet. (The cube has exactly eight such vertices.)

Solution by E. A. Nordhaus, Michigan State College. Consider any polyhedron for which the Euler-Descartes formula $F+V=E+2$ is valid, where F, V, and E are respectively the number of faces, vertices, and edges. Convexity and central symmetry will not be employed, and the result stated will be shown true under the weaker assumption that no face is triangular. Let $F_i(i\geqq 3)$ denote the number of faces having i sides, and $V_j(j\geqq 3)$ the number of vertices where exactly j edges meet, so $F=\Sigma F_i$, $V=\Sigma V_j$. Since each edge is common to two faces and terminates in two vertices, $2E=\Sigma iF_i=\Sigma jV_j$. If we multiply the Euler-Descartes relation by 4 and insert the above values, then

$$V_3 + F_3 = 8 + (F_5 + V_5) + 2(F_6 + V_6) + \cdots .$$

The desired result follows when $F_3=0$. We also notice a duality between faces and vertices, since $F_3\geqq 8$ if $V_3=0$.

ADVANCED PROBLEMS AND SOLUTIONS

EDITED BY E. P. STARKE, Rutgers University

Send all communications concerning Advanced Problems and Solutions to E. P. Starke, Rutgers University, New Brunswick, New Jersey. All manuscripts should be typewritten, with double spacing and margins at least one inch wide. Problems containing results believed to be new or extensions of old results are especially sought. Proposers of problems should also enclose any solutions or information that will assist the editor. In general, problems in well known textbooks or results in readily accessible sources should not be proposed for this department.

PROBLEMS FOR SOLUTION

4434. [1951, 266]. *Correction*

Replace the upper limit 1 in the integral by π. Note also that r and n are positive integers.

4443. *Proposed by R. H. Breusch, Amherst College, Massachusetts*

If

$$P_n(z) \equiv \sum_{p=0}^{[n/2]} (-1)^p \frac{z^{n-2p}}{(n-2p)!} = \frac{z^n}{n!} - \frac{z^{n-2}}{(n-2)!} + \cdots,$$

prove that the number of real zeros of $P_n(z)$ is

$$N_n = \frac{2n}{e\pi} + \frac{\log n}{e\pi} + O(1).$$

4444. *Proposed by J. H. Braun, Illinois Institute of Technology*

Prove that no three of the diagonals of a regular polygon of odd order are concurrent at any point other than the vertices.

4445. *Proposed by Paul Erdös, University of Aberdeen, Scotland*

Split the set of primes $p_1 < p_2 < \cdots$ into two classes q_i and r_i so that $\sum 1/q_i = \sum 1/r_i = \infty$. Define $\mu'(k) = 0$ if k is a multiple of one of the r's, otherwise $\mu'(k) = \mu(k)$, where $\mu(k)$ is the Moebius symbol (0 if k has a square factor, $+1$ if k has an even number of distinct prime factors, -1 if it has an odd number). Prove that

$$\sum_{k=1}^{\infty} \frac{\mu'(k)}{k} = 0.$$

4446. *Proposed by Victor Thébault, Tennie, Sarthe, France*

If the lines which join the vertices of a tetrahedron to the circumcenter (or the orthocenter) of the opposite faces form a hyperbolic group, the tetrahedron is equifacial and conversely. If the lines are concurrent, the tetrahedron is regular.

4447. *Proposed by D. J. Newman, New York University*

Let $f(z)$ be analytic in $|z| \leqq 1$. Suppose $|f(z)| \leqq 2$ there and $f(0)=1$. Show that $f(z)$ has no zeros for $|z| \leqq \frac{1}{2}$.

SOLUTIONS

Determinant Evaluation

4367 [1949, 637]. *Proposed by F. E. Wood, University of Oregon.*

Evaluate the determinant

$$\begin{vmatrix} {}_nP_n & {}_nC_1 & {}_nC_2 \cdots & {}_nC_n \\ {}_{n-1}P_{n-1} & {}_{n-1}C_0 & {}_{n-1}C_1 \cdots & {}_{n-1}C_{n-1} \\ {}_{n-2}P_{n-2} & 0 & {}_{n-2}C_0 \cdots & {}_{n-2}C_{n-2} \\ \cdot & \cdot & \cdot & \cdot \\ {}_0P_0 & 0 & \cdots\quad 0 & {}_0C_0 \end{vmatrix}$$

of the $(n+1)$th order where ${}_nP_r$ and ${}_nC_r$ are the usual permutation and combination symbols.

I. *Solution by G. Y. Cherlin, Rutgers University.* Noting that ${}_rC_0=1$ for all r, and denoting the given $(n+1)$th order determinant by D_n we obtain by expanding with respect to the elements of the first row,

$$D_n = {}_nP_n - \sum_{i=1}^{n} {}_nC_i \cdot D_{n-i}$$

or

$${}_nP_n = \sum_{i=0}^{n} {}_nC_i \cdot D_{n-i}.$$

Using this it is readily proved by induction that

$$D_n = \sum_{s=0}^{n} (-1)^{n-s}{}_nP_s = n! \sum_{i=0}^{n} (-1)^i/i!.$$

II. *Solution by Koichi Yamamoto, Kanazawa University, Japan.* Subtract the $(r+1)$th row from the rth row $(r=1, 2, \cdots, n)$ to obtain a determinant whose last column is filled with zeros except the last element which is 1. We may then delete its last column and last row to get

$$\begin{vmatrix} \Delta(n-1)! & {}_{n-1}C_1 & {}_{n-1}C_2 \cdots & {}_{n-1}C_{n-1} \\ \Delta(n-2)! & {}_{n-2}C_0 & {}_{n-2}C_1 \cdots & {}_{n-2}C_{n-2} \\ \cdot & \cdot & \cdot & \cdot \\ \Delta 0! & 0 & 0 \cdots & {}_0C_0 \end{vmatrix}$$

where $\Delta r! = (r+1)! - r!$. This has the same form as D_{n-1} except for the first column which is symbolically multiplied by the difference operator Δ. We repeat this process n times to arrive at the 1-rowed determinant

$$D_n = \Delta^n 0! = n! - {}_nC_1(n-1)! + {}_nC_2(n-2)! - \cdots + (-1)^n {}_nC_n 0!$$
$$= n!\left\{1 - \frac{1}{1!} + \frac{1}{2!} - \cdots + (-1)^n \frac{1}{n!}\right\}.$$

This is the "rencontre number" and is asymptotic to $n!/e$.

Also solved by J. A. Bullard, R. H. Cole, J. C. Eaves, N. J. Fine, G. P. Henderson, P. M. Hummel, N. D. Lane, Roger Lessard, W. V. Parker and A. C. Cohen, Jr., C. L. Perry, Azriel Rosenfeld, N. T. Seely, Jr., O. E. Stanaitis, H. E. Stelson, Ernst Trost, Louise A. Wolf, and the Proposer.

Editorial Note. Several solvers noted the recurrence relations

$$D_n = nD_{n-1} + (-1)^n, \qquad D_{n+1} = n\{D_n + D_{n-1}\}.$$

Noted also was the fact that the method of evaluation is independent of the first column. If the element in the first column and jth row is a_j, the determinant has the value

$$\sum_{j=0}^{n} (-1)^j {}_nC_j a_{j+1}.$$

Stelson referred to Weihrauch, *Zeitschrift f. Math. u. Phys.* XIX, 1874, pp. 354–360 and Muir, *Theory of Determinants*, v. 3, p. 465. The Proposer hit upon the determinant while working with a problem analogous to no. 4146 [1946, 107].

Miquel Point, Direct Circular Transformation

4370 [1950, 420]. *Proposed by H. F. Sandham, Trinity College, Ireland*

A', B', C' are points on the opposite sides of a triangle ABC. The circles through $B'C'A$, $C'A'B$, $A'B'C$ intersect in M, the Miquel point, whose isogonal conjugate is M'. Prove that M', M are corresponding points under the direct circular transformation set up by A, A'; B, B'; C, C'.

Solution by R. Deaux, Faculté Polytechnique, Mons, Belgium. The pedal triangles $A_1B_1C_1$, $A_1'B_1'C_1'$ of M, M' have a common circumcircle ω whose center is the midpoint of MM', and it is well known that M is the center of similitude of the triangles $A'B'C'$, $A_1B_1C_1$; hence, in the Gaussian plane,

$$(A'B'C'M) = (A_1B_1C_1M)$$

and if A_1'', B_1'', C_1'' are diametrically opposite to A_1, B_1, C_1 on ω

$$(A_1B_1C_1M) = (A_1''B_1''C_1''M').$$

Denoting by ∞ the point at infinity and by $\overline{X}$ the conjugate of a number X, the inversion with center M' and which leaves ω unchanged gives

$$(A_1''B_1''C_1''M') = \overline{(A_1'B_1'C_1'\infty)}.$$

Using finally Schick's theorem* we have

$$\overline{(A_1'B_1'C_1'\infty)} = (ABCM'),$$

whence

$$(A'B'C'M) = (ABCM').$$

Also solved by Howard Eves and the Proposer.

Editorial Note. Eves shows also that if O is the circumcenter of triangle $A'B'C'$, then M, O are corresponding points in the same direct circular transformation.

Odd Primes and Certain Powers of Consecutive Integers

4373 [1949, 695]. *Proposed by Victor Thébault, Tennie, Sarthe, France.*

In every system of numeration having a base B which is the product of distinct prime factors, each to the first power, and which is not divisible by a given odd prime number m nor by any prime number of the form $2km+1$, the mth powers of integers which terminate in the digits 0 to $B-1$ all have distinct units digits.

Editorial Note. This problem and the Proposer's solution, together with a generalization, are given in *Mathesis*, v. LIX, 1950, pp. 10–12. No other solutions have been received.

Divergent Series

4374 [1949, 695]. *Proposed by Paul Erdös, University of Aberdeen, Scotland.*

Let $d_n>0$, $D_n=\sum_{k=1}^n d_k$, $D_n\to\infty$. The well known theorem of Abel-Dini states that $\sum_{n=1}^\infty d_n/D_n=\infty$. Now let $f(n)>0$ be any increasing function which satisfies $\sum_{n=1}^\infty d_n/f(n)=\infty$, then

$$\sum_{n=1}^{\infty} \frac{d_n}{\max\{D_n, f(n)\}} = \infty.$$

I. *Solution by N. J. Fine, University of Pennsylvania.* The result is trivial unless $D_n>f(n)$ infinitely often. Assuming the latter, for any M we can find an $N>M$ such that $D_N>\max\{f(N), 2D_M\}$. Then, since $\max\{D_n, f(n)\}$ increases,

$$\sum_{n=M+1}^{N} \frac{d_n}{\max\{D_n, f(n)\}} \geqq \frac{1}{\max\{D_N, f(N)\}} \sum_{n=M+1}^{N} d_n = \frac{1}{D_N}(D_N - D_M) > \frac{1}{2}\cdot$$

Hence the series diverges.

* See R. Deaux, *Introduction à la Géometrie des nombres complexes*, p. 154, or *Mathesis*, LVIII, p. 142.

II. *Solution by J. B. Kelly, University of Wisconsin.* Put $D(n)$ for D_n. Let $v_n = \max\{D(n), f(n)\}$. We may suppose that infinitely many of the v's are D's. Put $D(n_0)=1$, and from the infinite sequence of those D's which are v's we extract a subsequence $D(n_1), D(n_2), \cdots$, such that $\sum_{k=0}^{\infty} D(n_k)/D(n_{k+1}) < \infty$. Let s be the sum for this convergent series.

Now let n vary from n_k+1 to n_{k+1}. The sequences $D(n)$ and $f(n)$ are increasing. If $v_n = D(n)$, then $v_n < D(n_{k+1})$. If $v_n = f(n)$, then $v_n \leqq f(n_{k+1}) \leqq D(n_{k+1})$. It follows that

$$\sum_{n=n_k+1}^{n_{k+1}} \frac{d_n}{v_n} \geqq \sum_{n=n_k+1}^{n_{k+1}} \frac{d_n}{D(n_{k+1})} = \frac{D(n_{k+1}) - D(n_k)}{D(n_{k+1})} = 1 - \frac{D(n_k)}{D(n_{k+1})}.$$

Therefore

$$\sum_{n=1}^{n_{k+1}} \frac{d_n}{v_n} \geqq \sum_{j=0}^{k}\left(1 - \frac{D(n_j)}{D(n_{j+1})}\right) \geqq k - s.$$

Hence the theorem. If always $D(n) \geqq f(n)$, then the above becomes a proof of the Abel-Dini theorem itself.

Also solved by Joshua Barlaz, J. G. Millar, Norman Miller, and T. A. Newton.

Editorial Note. The Proposer calls attention to the fact that the above result is the best possible in the following sense: Assume that $g(n) \to \infty$ and $\sum_{n=1}^{\infty} d_n / D_n g(n) = \infty$. Then it can be shown that there exists an $f(n)$ with $f(n+1) \geqq f(n)$ for which

$$\sum_{n=1}^{\infty} \frac{d_n}{f(n)} = \infty, \qquad \sum_{n=1}^{\infty} \frac{d_n}{\max\{D_n g(n), f(n)\}} < \infty.$$

These results should be compared with problem 4278, [1949, 423].

Uniformly Bounded Sums

4375 [1950, 41]. *Proposed by N. J. Fine, University of Pennsylvania.*

Let $((x)) = x - [x] - \frac{1}{2}$. Prove that the sums

$$\sum_{n=1}^{m} ((2^n x + \tfrac{1}{2}))$$

are uniformly bounded.

Solution by T. M. Apostol, California Institute of Technology. From the relation

$$[2y] = [y] + [y + \tfrac{1}{2}]$$

we obtain

$$((2y)) = ((y)) + ((y + \tfrac{1}{2})).$$

Taking $y=2^n x$, we have

$$((2^n x + \tfrac{1}{2})) = ((2^{n+1}x)) - ((2^n x)),$$

so that the sum in question telescopes to

$$((2^{m+1}x)) - ((2x)).$$

Since $|((y))| \leqq \frac{1}{2}$, the uniform bound required is unity.

Also solved by C. E. Buell, J. B. Kelly, M. S. Klamkin, Robert Steinberg, H. S. Zuckerman, and the Proposer.

Editorial Note. The Proposer met the above problem in connection with a study of expansions in series of Walsh functions. It would be of great interest to obtain information about

$$S_m(\alpha, x) = \sum_{n=1}^{m} ((2^n x + \alpha)), \qquad 0 < \alpha < 1.$$

Note also the following theorem due to M. Kac: *Let $E_m(\alpha, \beta)$ be the set of numbers x in* (0, 1) *for which*

$$\alpha < \frac{\sqrt{2}}{\sqrt{m}} \sum_{n=1}^{m} ((2^n x)) < \beta.$$

Then the measure of $E_m(\alpha, \beta)$ tends to

$$\frac{1}{\sqrt{\pi}} \int_\alpha^\beta e^{-y^2} dy.$$

Tetrahedron and Hyperbolic Group of Lines

4376 [1950, 41]. *Proposed by Victor Thébault, Tennie, Sarthe, France*

For a tetrahedron $ABCD$ let (A) be the sphere with center A and radius equal to the altitude AA', and let the tangent planes to (A) through the edges BC, CD, DB intersect in the point A_1. Let B_1, C_1, D_1 be analogously defined. If $ABCD$ is orthocentric, show that AA_1, BB_1, CC_1, DD_1 are concurrent at the isogonal conjugate of the orthocenter, and that otherwise AA_1, BB_1, CC_1, DD_1 constitute a hyperbolic group of lines.

Solution by L. M. Kelly, Michigan State College. The tangent planes through the edges BC, CD, DB are in fact the reflections of the base plane BCD in the respective side faces through A. It follows that the isogonal conjugate point of A lies on the three planes through the three edges BC, CD, DB which are reflections of the side faces through A in the base plane. That is, the isogonal point of A is the reflection of the vertex A in the base plane. Thus the altitude through A and the line A_1A are isogonal rays relative to the vertex A. Similarly the lines B_1B, C_1C, D_1D are isogonal rays of their respective altitudes. It follows from established theorems that if the altitudes meet in a point so do their

isogonal rays and that since the altitudes in general constitute a hyperbolic group so do their isogonal rays.

Also solved by the Proposer.

Editorial Note. The corresponding theorem for the plane is also true and reads: For a triangle ABC let (A) be the circle with center A and radius equal to the altitude AA', and let the tangents to (A) through B and C intersect in the point A_1. Let B_1, C_1 be analogously defined. Then AA_1, BB_1, CC_1 are concurrent at the circumcenter of ABC.

Jensen's Inequality

4379 [1950, 42]. *Proposed by W. B. Fulks, University of Minnesota*

If

(1) $\phi(t)$ is defined in $I: m \leqq t \leqq M$,

(2) for any a, b in I, $a \neq b$

$$\phi\left(\frac{a+b}{2}\right) < \frac{\phi(a)+\phi(b)}{2},$$

(3) p_i are non-negative numbers, $i = 1, 2, \cdots, n$,

(4) $t_i \in I$, not all t_i equal,

it has been shown that

$$\phi\left[\frac{\sum_{i=1}^{n} p_i t_i}{\sum_{i=1}^{n} p_i}\right] \leqq \frac{\sum_{i=1}^{n} p_i \phi(t_i)}{\sum_{i=1}^{n} p_i}.$$

See Polya and Szegö, *Aufgaben und Ersätzen aus der Analysis*, vol. 1, p. 53, problem 74.

Show that the relation is true without the equality sign.

Solution by W. S. Gustin, Indiana University. Clearly we may take: $n \geqq 2$, all t_i distinct, all $p_i > 0$, and $\sum_i p_i = 1$; whence $\sum_{ij} p_i p_j = 1$. The $<$ relation may be deduced from the $\leqq$ relation as follows:

$$\begin{aligned}
\phi\left(\sum_i p_i t_i\right) &= \phi\left(\sum_{ij} p_i p_j \left(\tfrac{1}{2} t_i + \tfrac{1}{2} t_j\right)\right) \\
&\leqq \sum_{ij} p_i p_j \phi\left(\tfrac{1}{2} t_i + \tfrac{1}{2} t_j\right) \\
&< \sum_{ij} p_i p_j \left(\tfrac{1}{2}\phi(t_i) + \tfrac{1}{2}\phi(t_j)\right) \\
&= \sum_i p_i \phi(t_i).
\end{aligned}$$

We remark that the conclusion holds if and only if ϕ is continuous in addition to satisfying (1) and (2). For example: let f be a continuous solution of (2) and let g be a discontinuous solution of (2) wherein $=$ replaces $<$ (such solutions are easily constructed by use of a Hamel basis); then the function $\phi=f+g$ satisfies (2) but not the conclusion of the problem.

Also solved by Robert Steinberg and the Proposer.

Iterated Summation

4380 [1950, 119]. *Proposed by P. A. Pizá, San Juan, Puerto Rico*

For arbitrary positive integers n and k let

$$S_1(n, k) = 1^k + 2^k + \cdots + n^k,$$

and put

$$S_{p+1}(n, k) = S_p(1, k) + S_p(2, k) + \cdots + S_p(n, k),$$

with $p=1, 2, \cdots$. Thus $S_p(n, k)$ is the pth iterated sum, the sum of the sum of $\cdots$ the sum of the first n perfect kth powers.

Show that $S_p(n, k)$ is a polynomial in n of degree $p+k$ and determine the form of the polynomial for $k=3, 4, 5$. If possible, determine the form for general k. (The case $k=2$ is the subject of problem 22, *Mathematics Magazine*, vol. XXII, no. 1, p. 51.)

Solution by Roger Lessard, École Polytechnique, Montreal. Let b_i and c_i be arbitrary constants to be specified later, and let

$$x^k = \sum_{i=0}^{\infty} a_i \binom{x + b_i}{c_i} \qquad k = 0, 1, 2, \cdots.$$

If it is possible to solve this system of equations to obtain the values of the a_i, we will have

$$S_1(n, k) = \sum_{x=1}^{n} \sum_{i=0}^{\infty} a_i \binom{x + b_i}{c_i} = \sum_{i=0}^{\infty} a_i \sum_{x=1}^{n} \binom{x + b_i}{c_i}$$
$$= \sum_{i=0}^{\infty} a_i \binom{n + b_i + 1}{c_i + 1},$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$S_p(n, k) = \sum_{i=0}^{\infty} a_i \binom{n + b_i + p}{c_i + p}.$$

The simplest form is given with $b_i=0$, $c_i=i$, whence

$$a_i = \sum_{j=0}^{i} (-1)^{i+j} \binom{i}{j} j^k = \Delta^i 0^k = i! G_k^i,$$

$$x^k = \sum_{i=0}^{k} \binom{x}{i} \Delta^i 0^k = \sum_{i=0}^{k} G_k^i (x)^{(i)}.$$

In the above, $a_i = 0$ for $i > k$. The coefficients $\Delta^i 0^k$ are the differences of zero, and the $G_k{}^i$ are the Stirling's numbers of the second kind. (See problem no. 4276 [1949, 347] where the notation ${}_nK_j$ is used for $\Delta^j 0^n$.) We have then

$$S_p(n, k) = \sum_{i=0}^{k} a_i \binom{n+p}{p+i} = \sum_{i=0}^{k} \binom{n+p}{p+i} \sum_{j=0}^{i} (-1)^{i+j} \binom{i}{j} j^k.$$

For $k=3$; $a_1=1$, $a_2=6$, $a_3=6$. For $k=4$; $a_1=1$, $a_2=14$, $a_3=36$, $a_4=24$. For $k=5$; $a_1=1$, $a_2=30$, $a_3=150$, $a_4=240$, $a_5=120$. It is easy to see that the ith term is a polynomial of degree $p+i$ in n, so that consequently $S_p(n, k)$ is a polynomial in n of degree $p+k$.

Another form which has some advantages may be obtained by taking $b_i = i-1$, $c_i = k$, so that

$$a_i = \sum_{j=0}^{i} (-1)^{i+j} \binom{k+1}{i-j} j^k = a_{k+1-i}, \qquad a_i = 0 \text{ for } i > k,$$

$$x^k = \sum_{i=0}^{k} a_i \binom{x+i-1}{k}.$$

We have then

$$S_p(n, k) = \sum_{i=0}^{k} \binom{n+p+i-1}{k+p} \sum_{j=0}^{i} (-1)^{i+j} \binom{k+1}{i-j} j^k.$$

Many other forms are possible but do not seem to provide anything more useful. The use of "central notation," as usual, cuts the number of terms in the expansion in half, but requires different formulas for k even, odd.

Also solved by M. S. Klamkin, Kovina Milosevich, N. T. Seely, and the Proposer.

Editorial Note. It is not difficult to show that, if k is even

$$S_p(n, k) = \frac{(n+p)!(2n+p)}{(p+k)!(n-1)!} \phi(n^2 + np),$$

with the same form when k is odd except that the factor $(2n+p)$ is suppressed. ϕ is a polynomial of degree $[(k-1)/2]$ with integral coefficients.

RECENT PUBLICATIONS

EDITED BY E. P. VANCE, Oberlin College

All books for review should be sent directly to the editor of this department, American Mathematical Monthly, 531 West 116th Street, New York 27, N. Y. and not to any of the other editors or officers of the Association.

Fundamentals of the Calculus. By D. E. Richmond. McGraw-Hill Book Company, Inc., New York. 1950. 11+233 pages. $3.00.

It is a pleasant surprise to find a freshman textbook in which the author has dared to abandon the conventional in order to present an organized, unified, rigorous, and logical mathematical development suitable for the liberal arts student who elects only one year of college mathematics. By keeping algebraic requirements simple, and by carefully selecting and organizing topics, the author carries the student in a systematic manner from the integer to the integral. In the process he develops in detail the number system, inequalities, sequences, and limits. This development is beautiful and clear, involving many ideas infrequently met in elementary texts, for example, the notion of deleted neighborhood. In addition, analytic trigonometry is developed from the complex numbers, while logarithmic and exponential functions, and the analytic geometry of the conics are presented as applications of the derivative.

The readability and clarity of the writing should enable the student to understand the text with minimum assistance. The problems are carefully integrated with the text to promote the understanding of it. While many teachers might prefer that standard notations had been used throughout, the author avoids some on the ground that the notations make unnecessary difficulties for the student.

In no sense does the book pretend to be a standard calculus, since many topics, including the integral as the limit of a sum, are omitted. It might well be used, however, in any course where only an introduction to calculus is desired. The title may cause the book to be overlooked by teachers hunting a first year mathematics text. That would be unfortunate. Teachers who are weary of the many current texts, revised and enlarged, containing abundant material which is readily adaptable, but which may be omitted, should welcome this closely knit volume.

ROTHWELL STEPHENS

The Theory of Algebraic Numbers. By Harry Pollard. The Carus Mathematical Monographs, Number 9. Published by the Mathematical Association of America. Distributed by John Wiley and Sons, Inc., New York. 143 pages. $3.00.

Professor Pollard's book is a most welcome addition to the literature on Algebraic Number Theory in that it partially fulfills a long felt need for a read-

able English text on the subject. On the whole the book is well written and it is almost self-contained. The theorems are precisely stated and very complete cross references to earlier theorems make the proofs easy to follow. However, the definitions are occasionally not precisely stated and are often hidden in the midst of a discussion paragraph. (e.g. integral domain.) Efforts at motivation are frequent and usually successful. Practically the only prerequisites for the reader are a facility with the vocabulary of algebra and sufficient mathematical maturity to appreciate the problems raised.

The author begins by proving the unique factorization theorem of rational integers. He then raises the question as to whether the concept of integers can be generalized and if so, whether the factorization theorem continues to be valid. As examples he discusses the integers of $R(i)$, $R(\sqrt{-3})$, and $R(\sqrt{-5})$. In answering these questions he follows the classical approach using ideal theory, and the proofs are usually to be found in the works of Landau, Hecke, and Ore.

Chapters II through VII are devoted to introducing and developing such elementary facts of algebraic number theory as polynomial domains, field extensions, bases, discriminants, algebraic integers, units, primes, integral bases, and ideals. Also, the Liouville numbers are developed to demonstrate that not all complex numbers are algebraic. Throughout these chapters the reader is given a feeling for the subject by means of many fine examples taken from the quadratic and cyclotomic fields. However, the motivation of the definition of algebraic integers is weak and might be enhanced by the concepts of quotient field of rings and integral closure. Also, the author's aversion to the use of symmetric polynomials causes undue difficulties.

In Chapters VIII and IX the fundamental theorem on factorization of ideals is proved and a few of its consequences, such as the ramification of prime ideals, are studied. The unique factorization of integers into primes in an algebraic number field is shown to be equivalent to having every ideal in its ring of integers principal. Class numbers and Fermat's last problem are discussed in Chapter X. It is shown how parts of algebraic number theory were developed in an attempt to solve this famous problem. A theorem of Kummer is established, namely, if p is a regular odd prime, then $x^p+y^p=z^p$ has no solution in rational integers. Minkowski's theorem on lattice points and his theory of units comprise the final chapter. Unfortunately no examples are given in the last four chapters and as a consequence much of the theory remains sterile. For example, the theorems on ideals might have been considerably clarified by exhibiting the prime ideals of the quadratic fields.

All in all, one can be certain that the book will awaken in many of its readers an interest in this beautiful theory and thus the purpose of its authorship is fulfilled.

D. J. LEWIS

Calculus and Analytic Geometry. By C. T. Holmes. McGraw-Hill, New York, 1950. 10+416 pp. \$4.75.

The first five chapters of this book present the basic material of plane analytic geometry, the fundamentals of the calculus, both differential and integral, and applications, all involving polynomial functions. At least this much of the book is designed to be covered in one semester, either as a terminal course or as an introduction to the rest of the book.

Chapters 6–15, not all of which can be taken in one semester, include (in order) a discussion of curvilinear motion, the logarithmic, exponential, and trigonometric functions with applications, infinite series, differential equations solid analytic geometry, and the calculus for functions of several variables.

In the preface the author states that he makes free use of geometric intuition, preserving clear distinctions between what is proved and what is assumed; and he remarks that "we think that the book has as much rigor as most beginners can stand."

Some interesting questions are raised by a consideration of the structure of the book. Can a really satisfactory one-semester course in college mathematics for non-science students be devised, and, if so, what should be its subject matter? Is it possible that we underestimate the ability of beginners to absorb the punishment of rigor, and need rigor be punishing? There is no place in a short review for a discussion of questions of this sort, so that the author's premises with respect to the type of course for which he is writing will be accepted in what follows.

On the whole the book may be characterized as being conservative, with a number of novel features. The "point of division" formula, to give an example of conservatism, is presented in the unsymmetrical form, $x=(x_1+rx_2)/(1+r)$. On the other hand, the development of $\lim_{n\to\infty}(1+1/n)^n$ is straightforward and elegant, a heuristic discussion of tangents at the origin to the graph of $P(x, y)=0$ with constant term zero is excellent, and a modification of the second derivative test for extremes is really useful. The problems are well-chosen, carefully arranged, and contain instructive references and remarks.

An analysis of the author's style reveals at once his best feature and his greatest weakness. He has succeeded in transplanting the informality of a good class room teacher into the pages of a book. The text reads smoothly, points are brought up easily and naturally, the arguments are not labored. However, there is a lack of precision in some statements, which is at least annoying and may be confusing to the student. One would expect more care in the phrasing of statements which are formally labelled "Definition" or "Theorem" than is evidenced in the following example: "We say that $f(x)$ becomes infinite as x approaches $a \cdots$ if the value of $f(x)$ can be made to exceed any given number by taking x close enough to a." The fundamental theorem of the integral calculus is introduced by a consideration of area, but there is no mention of the

fact that the notion of area is as sophisticated as that of instantaneous speed, and that the area in question is *defined* as the limit of a sum. In 100 pages the reviewer encountered 9 instances of what he considers to be logical inaccuracies.

If more attention had been given to the phrasing of formal statements, this book would be an outstanding one of its kind.

R. A. ROSENBAUM

NEW BOOKS RECEIVED

Borderlands of Science. By Alfred Still. New York, Philosophical Library, Inc. ix+424 pp.

Quantum Mechanics. By Alfred Lande. New York, Pitman Publishing Company. x+307 pp. $5.50.

College Trigonometry. By William L. Hart. Boston, D. C. Heath and Company, 1951. viii+130 pp. $3.50.

Theory of Probability. By M. E. Munroe. New York, McGraw-Hill Book Company, 1951. viii+213 pp. $4.50.

Foundations of Analysis. By Edmund Landau. Chelsea Publishing Company, 1951. xiv+134 pp. No price given.

The Kernel Function and Conformal Mapping. Mathematical Surveys No. 5. By S. Bergman. American Mathematical Society, New York. vii+161 pp. No price given.

Algebra I and II. I—Die Grundlagen, viii+300 pp. DM 16. II—Die Theorie der Algebraischen Gleichungen, 3rd ed. viii+260 pp. DM 14. By O. Perron. Walter de Gruyter and Company, 1951.

Analyse Mathematique. Tome I, Analyse des Courbes, Surfaces et Fonctions Usuelles, Integrales Simples. ix+408 pp. 2000 fr. Port. 100 fr. Tome II, Equations differentielles, Developpements en series, Nombres complexes. Integrales multiples, Probabilities, Determinants, Exercises. 434 pp. 2200 fr. Port 110 fr. By Paul Appell. Paris, Gauthier-Villars, 1951.

Exercices de Mecanique. By H. Beghin, and G. Julia, Tome I, Fascicule 1. 2nd ed. Paris, Gauthier-Villars, 1946, vii+336 pp. No price given.

Problemes de Propagations guidees des Ondes Electromagnetiques, 2nd ed. By Louis de Broglie. Paris, Gauthier-Villars, 1951. vii+118 pp. $3.38.

Integraltafel, Zweiter Teil. Bestimmte Integrale, VI. By W. Grobner and N. Hofreiter. Springer-Verlag in Wien, 1950. $5.80.

Handbuch der Laplace-Transformation, Band 1. Theorie der Laplace-Transformation. By G. Doetsch. Basel, Verlag Birkhauser, 1950. 581 pp. Bound—78 fr.; unbound 74 fr.

Die Zweidimensionale Laplace-Transformation. By D. Voehlker and G. Doetsch. Basel, Verlag Birkhauser, 1950. 259 pp. Bound 43 fr.; unbound 35 fr.

The Main Street of Mathematics. By Edna E. Kramer. Oxford University Press, 1951. xii+321 pp. $5.00.

Tables d'Interets et D'Annuites. Editees Par le Credit Communal de Belgique, Brussels, 1950. 163 pp. No price given.

CLUBS AND ALLIED ACTIVITIES

EDITED BY L. F. OLLMANN, Hofstra College

Send reports of all activities, such as club reports, special features, topics with references, student papers, and other material of interest to L. F. Ollmann, Hofstra College, Hempstead, New York.

CLUB REPORTS, 1949–50

Pythagoreans, City College of the City of New York

The 1949–50 program of the mathematics club, *Pythagoreans*, of the City College of the City of New York included the following talks by undergraduates:

Concurrency and colinearity, by Milton Halem

Summation of series, by Jack Bricker

Basic considerations of topology, by Karl Brunell

Transformations, by Donald Newman of New York University

Partitions, by Bernard Weitzer

Elementary proof of a theorem by Hardy and Ramanujian, by Donald Newman of New York University

Integration in groups, by Carl Engelman

Extension of a function, by L. Rubel.

The following talks by members of the faculty were also given:

A problem in probability, by Prof. Henry Malan

The 4-color problem, by Prof. B. P. Gill

Cremona transformations, by Prof. Shelbourne Barber

Summation and integration, by Prof. W. H. Ingram.

The officers for 1950–51 include: President, Howard Young; Faculty Representative, Mr. Ingram.

Mathematics Society, Massachusetts Institute of Technology

The Mathematics Society of Massachusetts Institute of Technology held weekly meetings which featured lectures by various members of the faculty as well as by student members of the group. Among the lectures given were:

Exterior differential forms, by Prof. C. B. Allendoerfer

Finite difference equations and applications, by Prof. F. B. Hildebrand

The mathematics of Descartes, by Prof. D. J. Struik

Problems of topology, by Prof. G. W. Whitehead

The existence of continuous functions without derivatives, by Prof. W. Ambrose

Infinite systems of linear equations, by Dr. E. A. Coddington

What is dimension?, by Prof. W. Hurewicz

The isoperimetric problem, by Prof. I. S. Cohen

Calculus of variations, by Prof. G. B. Thomas

Boolean algebra as applied to the design of electrical switching circuits, by Mr. H. L. Reed, Jr.

Inversion of algebraic curves, by Mr. D. G. Aronson

Geometric inversion, by Mr. R. W. Preisendorfer.

The Society continued to publish its *Bulletin* with Mr. C. W. Bostick as editor.

The mathematics department exhibits at the M.I.T. Open House were constructed and staffed by the members of the Society under the supervision of Prof. R. D. Douglass and the Executive Committee of the Society.

The officers for the year were: President, D. G. Aronson; Vice-President, H. L. Reed, Jr.; Program Manager, R. Silver; Secretary-Treasurer, R. B. Kellogg.

Pi Mu Epsilon, Northwestern University

The following talks were given at the *Illinois Beta* Chapter of *Pi Mu Epsilon* meetings in 1949–50:

The development of logic, by Prof. Paul Henle of the Philosophy department

Why have cash?, by Prof. Jacob Marschak of the Cowles Commission, University of Chicago

A larger role for polar coordinates in analytics, by Prof. H. A. Simmons.

The annual initiation banquet was held in May, when 38 new members were initiated. Prof. E. H. C. Hildebrandt spoke on *The history and organization of Pi Mu Epsilon.*

The winners of the Pi Mu Epsilon mathematics contest held under the direction of Prof. Simmons were Richard Goldberg, Edmund Cohler, and Ann Evans, in that order.

Mathematics Society, The Cooper Union

The *Mathematics Society* of The Cooper Union heard papers on the following topics during 1949–50:

Matrices, by Irving Lowe

Nomographs, by Walter Kahn

Projective geometry, by Peter Redmond

Graphical solution of non-linear differential equations, by Harry Hochstadt

Group theory, by Mr. P. Chessin

Metric geometry, by Prof. J. N. Eastham

Imaginary curves, by Prof. F. H. Miller

Number theory, by Alan Berndt.

Officers for 1950–51 are: President, Peter J. Redmond; Vice-President and Treasurer, Irving J. Lowe; Secretary, Harry Schwarzlander; Faculty Adviser, Prof. J. N. Eastham.

Kappa Mu Epsilon, Mount Mary College

The *Wisconsin Alpha* chapter of *Kappa Mu Epsilon* held regular monthly meetings, 1949–50, at which the following talks were given by the members:

Non-Euclidean geometry, by Joan Daley

The fourth dimension, by Mary Kilkelly

History of the calculus, by Mary Hunt

Relativity, by Kathleen Hanley
Number systems, by Wanda Kropp
The slide rule, by Janet Haig
The planetarium, by Betty Prossen
Radar, by Bernadine Spitznogle
Aristotle, by Norma Harding.

Other activities included movies on the solar system, a Christmas party, a radio skit given by the initiates, and a dinner given for the initiates.

Two faculty members and two students attended the convention of the National Council of Mathematics Teachers held in Chicago. Miss Dorothy Karner, one of the students, read a paper in the Kappa Mu Epsilon section of the convention. Her paper was entitled *Geometric inversion* and was later printed in the Fall '50 issue of the *Pentagon*.

Officers for 1949–50 were: President, Dorothy Karner; Vice-President, Rosemary White; Secretary, Mary Hunt; Treasurer, Mary Kilkelly; Corresponding Secretary, Sister Mary Petronia; Faculty Sponsor, Sister Mary Felice.

Mathematics Club, Boston University

The Boston University *Mathematics Club* held its meetings twice a month, at which time various talks were given by the faculty and student members. Of the many discussions, two were outstanding. They were *Non-linear differential equations* and *Boolean algebra*.

Boston University was host to the annual meeting of the Greater Boston Intercollegiate Mathematics Club which was held in December, 1949. Prof. Elmer B. Mode, Chairman of the Mathematics Department, gave a humorous talk on *My mathematical scrapbook*. At this meeting Tufts College volunteered to be host in 1951.

A successful picnic was held in the Spring.

New officers for 1950–51 are: President, William Donovan; Vice-President, Barbara Torrey; Secretary, Elliot Croft; Treasurer, William Fitzgerald.

The Mathematics and Physics Club, College of St. Thomas

Papers presented to the *Mathematics and Physics Club* of the College of St. Thomas for 1949–50 included:

The brachistochrone problem and related topics, by E. J. Camp, Professor of Mathematics, Macalester College

Some early mathematical and astronomical tables, by Mr. W. D. Morgan, noted collector of early tables

A perpetual calendar, by Dr. Charles Hatfield, Jr., Professor of Mathematics, University of Minnesota.

Phenomena at the temperature of liquid helium, by Dr. J. R. Feldmeier, Professor of Physics

Some of the fallacies in mathematics, by Dr. W. S. Loud, Professor of Mathematics, University of Minnesota.

The *Mendel Forum Club* of St. Catherine's College attended the meeting at which Dr. Hatfield spoke, while the local club attended a meeting of the *Forum Club* at which Dr. J. F. Briggs spoke on *Your heart beat*.

The officers for the year 1950–51 are: President, Clarence B. Germain; Vice-President, R. P. Goblirsch; Secretary, Eugene Heath; Moderator, Dr. L. W. Sheridan.

Mathematics Society, University of Miami

The following talks were presented during the school year of 1949–50:

Hindu mathematics, by H. D. Sprinkle

Fallacies, by John Maecher

String models, by William Franzen

Magic squares, by Mabel Pauley

Computing machines, by Mr. John Kelley

Peano's space filling curve, by D. J. R. Foulis

Needle problem of Count Buffon, by D. J. R. Foulis

Integration of elliptic integrals, by Dr. H. F. MacNeish.

In addition to its regular meetings, the Mathematics Society held a Christmas Party and its Annual Spring Banquet.

The society has applied for membership in *Pi Mu Epsilon*, National Honorary Mathematics Fraternity, and has been accepted. The installation ceremonies are to be held in the Spring of 1951.

The officers for 1949–50 were: President, D. J. R. Foulis; Vice-President, John Maecher; Secretary, Mabel Pauley; Treasurer, Marjorie Stern (first semester) and Carolyn Palmer (second semester); Faculty Advisor, Prof. Georgia Del Franco.

NEWS AND NOTICES

EDITED BY EDITH R. SCHNECKENBURGER, University of Buffalo

Readers are invited to contribute to the general interest of this department by sending news items to Edith R. Schneckenburger, University of Buffalo, Buffalo 14, New York. Items must be submitted at least two months before publication can take place.

DIVISION OF MATHEMATICS OF THE NATIONAL RESEARCH COUNCIL

The National Research Council has announced the formation of a new division to be known as the Division of Mathematics. Professor Marston Morse of the Institute for Advanced Study has been appointed Chairman. Professor Marshall Stone, chairman of the Department of Mathematics of the University

of Chicago, will serve as Vice-Chairman. Professor J. R. Kline of the University of Pennsylvania has been appointed Executive Secretary. The new division will be composed of mathematicians who were formerly members of the Division of Mathematical and Physical Sciences, now the Division of Physical Sciences, and others to be appointed by the Chairman of the National Research Council.

PACIFIC JOURNAL OF MATHEMATICS

The Pacific Journal of Mathematics, a nonprofit corporation, announces a new journal of the same name, to be devoted to the publication of basic research articles in the various branches of mathematics. Publication of the *Pacific Journal of Mathematics* is sponsored by the following West Coast universities: the University of British Columbia; the California Institute of Technology; the University of California at Berkeley, Davis, Los Angeles and Santa Barbara; Oregon State College; the University of Oregon; the University of Southern California; Stanford University; Washington State College; and the University of Washington. The Institute for Numerical Analysis of the National Bureau of Standards is providing valuable clerical assistance and space for the *Journal*, as far as the managing editor is concerned. In addition, the American Mathematical Society is furnishing financial aid to the *Journal* during the initial stages of its existence.

A minimum of 600 pages per yearly volume of four issues is planned. The first issue of 160 pages is scheduled for March, 1951.

Subscription price for the *Journal* is $8.00 per year, with a reduction of $4.00 to individual faculty members of the above supporting institutions and to members of the American Mathematical Society. Subscriptions should be sent to the *Pacific Journal of Mathematics*, University of California Press, Berkeley 4, California.

Manuscripts for publication in the *Journal* should be submitted to any of the three editors: E. F. Beckenbach, University of California, Los Angeles 24, California; Herbert Busemann, University of Southern California, Los Angeles, California; R. M. Robinson, University of California, Berkeley 4, California.

SUMMER COURSES

The Department of Mathematics Education of the School of Education of New York University announces the following courses for its summer session: Dr. A. D. Bradley, teaching and curricular problems of college mathematics, applications of mathematics; Dr. J. J. Kinsella, research investigations in mathematics education.

The Statistical Laboratory of the University of California at Berkeley announces the following for the first summer session, June 18th to July 28th: Professor M. G. Kendall of the London School of Economics and Political Science, recent developments in the theory of statistics, seminar on time series and related problems; Professor Jerzy Neyman will be available for consultation on work leading to higher degrees.

PERSONAL ITEMS

Professor C. B. Allendoerfer of Haverford College has been appointed to a professorship at the University of Washington in Seattle where he will hold the position of Executive Officer of the Department of Mathematics. Professor R. M. Winger is retiring as Executive Officer but will continue on the active faculty. As was announced in an earlier issue Professor Allendoerfer will become Editor of the MONTHLY on January 1, 1952. After September 1, 1951 papers submitted for publication in the MONTHLY should be sent directly to Professor C. B. Allendoerfer, Department of Mathematics, University of Washington, Seattle 5, Washington and not to the present editor, Dr. C. V. Newsom.

Professor H. A. Robinson of Agnes Scott College was the representative of the Association at the inauguration of President O. C. Aderhold of the University of Georgia on May 9, 1951.

Professor Emeritus Mary E. Sinclair of Oberlin College represented the Association at the inauguration of President A. S. Knowles of the University of Toledo on May 9, 1951.

Kent State University reports that a Mathematics Workshop for Secondary School Teachers was conducted by its Mathematics Department on April 14, 1951. The following members of the Department served on the Workshop Committee: Emalou Brumfield as chairman, Carl Lowry and E. T. Stapleford.

University of Rochester announces: Dr. Eduardo Caianiello, formerly of the University of Naples, has been appointed to an assistant professorship; Mr. David Barton, formerly instructor at Roberts College, has been appointed Graduate Instructor; Mr. Donald Smith, previously of the University of Rochester, Mr. Charles Younger, a graduate of West Texas Teachers College, and Mr. Deiter Gaier of the Technische Hochschule at Stuttgart have received appointments as teaching assistants; Assistant Professor Dorothy L. Bernstein has a year's leave of absence which she is spending at the Institute for Advanced Study; Instructor Trevor McMinn is now a graduate student at the University of California at Berkeley; Mr. Warren Stenberg who taught at the University during the first semester is now a graduate assistant at the University of California at Berkeley; Teaching Assistant Eugene Trabka has accepted a position at the Cornell Aeronautical Laboratory, Buffalo, New York.

Miss Janet E. Abbey, formerly an instructor at the University of Buffalo, has been appointed Head of the Department of Mathematics of Griffith Institute and Central School, Springville, New York.

Mr. A. D. Anderson has received an appointment as mathematician in the Naval Research Laboratory, Washington, D. C.

Professor H. C. Ayres of Jersey City Junior College is now in military service.

Assistant Professor W. H. Badgley, Jr., of Florence State Teachers College, Alabama, is a graduate student at Vanderbilt University.

Miss Jean M. Baldwin, formerly a student at Carleton College, has been appointed to a graduate assistantship at the University of Oklahoma.

Assistant Professor W. E. Barnes of the College of William and Mary has accepted a position as mathematician at the United States Naval Proving Ground, Dahlgren, Virginia.

Dr. F. A. Beeler, previously a graduate student at the University of Michigan, has been appointed to a professorship at Western Michigan College of Education.

Mr. P. R. Beesack, formerly a student at McMaster University, has been appointed to a graduate assistantship at Washington University.

Mr. D. C. Benson, a student at Pomona College, has been appointed to a teaching assistantship at Stanford University.

Mr. C. R. Bonnell of the School of Mines and Metallurgy of the University of Missouri has been appointed Research Engineer at the Minneapolis-Honeywell Company, Minneapolis, Minnesota.

Mr. L. F. Boron of the University of Kentucky has a position as mathematician in the Navy Department, Washington, D. C.

Mr. J. E. Brown, previously a student at the University of Georgia, has been appointed acting instructor at Florida State University.

Mr. R. A. Burrows, formerly a student at Albion College, is employed by General Motors Corporation, Detroit, Michigan.

Mr. L. G. Campbell of New Jersey State Teachers College has been appointed to an instructorship at Teachers College, Columbia University.

Associate Professor L. Virginia Carlton of Northwestern State College has been appointed Head of the Department of Mathematics of Wesleyan College.

Mr. H. C. Carter of David Lipscomb College is now in military service.

Mr. B. B. Clark of Grinnell College is in the United States Air Force.

Mr. M. J. Cleveland, part-time instructor at the University of Florida, is in the United States Navy.

Mr. J. J. Fischer of Morgan Park Junior College has accepted a position as research engineer in the Aerophysics Laboratory, North American Aviation, Incorporated, Downey, California.

Mr. H. C. Griffith, formerly assistant instructor at the University of Missouri, has been appointed to a teaching assistantship at the University of Tennessee.

Mr. Ralph Hafner of the University of Dayton has a position as mathematician in the Naval Ordnance Plant, Indianapolis, Indiana.

Mr. W. C. Hobbs of Hampton Institute has been appointed Head of the Department of Mathematics of Morris Brown College.

Mr. Julius Honig, previously research assistant at Long Island College of Medicine, has accepted a position as mathematician with the Raytheon Electric Corporation, Waltham, Massachusetts.

Assistant Professor S. T. Hu of Tulane University has been promoted to an associate professorship; he will be on leave of absence during the year 1951—52 and will spend this year at the Institute for Advanced Study.

Dr. H. D. Huskey, chief of the Machine Development Unit, Institute for

Numerical Analysis, National Bureau of Standards, has been promoted to the position of Assistant Director.

Mr. T. C. Hutchison, who has been an electrical engineer at Haller, Raymond and Brown, Incorporated, State College, Pennsylvania, has accepted a position as project engineer at the Sperry Gyroscope Company, Great Neck, New York.

Mr. R. G. Ingle of Sandia Corporation, Albuquerque, New Mexico, has a position as vibration engineer at Chance Vought Aircraft, Dallas, Texas.

Mr. O. C. Juelich, formerly a student at Hofstra College, has been appointed to an assistantship at Ohio State University.

Professor Mark Kac of Cornell University will serve as visiting lecturer at Massachusetts Institute of Technology during the first part of the summer session.

Professor Emeritus Edward Kasner of Columbia University was a teacher in the New School for Social Research during the spring term.

Mr. C. E. Kerr of LaSalle College has been appointed to a graduate assistantship at the University of Delaware.

Mr. W. S. Knight, previously a student at the University of Georgia, is now in the United States Army.

Mr. B. W. Marks of Midland High School, Midland, Texas, is in the United States Army.

Dr. W. H. Marlow, who has been a graduate assistant at the State University of Iowa, has a position as research associate, Logistics Research Projects, George Washington University.

Professor Karl Menger of Illinois Institute of Technology delivered a series of lectures on metric geometry at the Sorbonne, University of Paris, during April-May, 1951.

Instructor R. H. Oehmke of the University of Detroit is now a graduate student at the University of Chicago.

Instructor F. R. Olson, Kent State University, has been appointed to a part-time instructorship at Duke University.

D. H. A. Palmer, Head of the Physical Science Department of Midwestern University, is now a chemist in the Research Department of the United Gas Corporation, Shreveport, Louisiana.

Mr. J. D. Riley, previously a mathematician at the Naval Research Laboratory, is a graduate student at the University of Kansas.

Auxiliary Professor Margarita Rodriguez of Instituto del Vedado is now a mathematics teacher at Vedado High School, Vedado, Havana.

Dean Evelyn C. Rusk of Wells College will resign from her position as dean on July 1, 1951; after a year's leave of absence she will return to the college as professor of mathematics.

Mr. J. D. Rutledge, previously a student at Swarthmore College, is now a Programmer Trainee at Eckert-Mauchly Computer Corporation, Philadelphia, Pennsylvania.

Mr. L. R. Schlauch, who has been a graduate student at the University of Virginia, has received an appointment as analyst for the Department of Defense, Armed Forces Security Agency, Washington, D. C.

Mr. R. J. Semple has been appointed teaching fellow at the University of Toronto.

Colonel W. E. Sewell is Professor of Military Science and Tactics at the State University of Iowa.

Sister M. Laurine of Mount St. Joseph Junior College has been appointed Registrar and Instructor at Brescia College, Owensboro, Kentucky.

Graduate Assistant O. D. Smith of Oregon State College is now a teaching assistant at the University of Southern California.

Mr. B. R. Snyder, formerly of Johns Hopkins University, has accepted a position with the law firm of K. F. Steinmann, Baltimore, Maryland.

Graduate Assistant J. C. Sorenson of the University of Oregon has been appointed to a graduate assistantship at Utah State Agricultural College.

Mr. S. P. Spaulding, formerly a student at Boston University, has received an appointment as ordnance engineer at Naval Torpedo Station, Newport, Rhode Island.

Dr. M. D. Springer has accepted a position as mathematical statistician with United States Naval Ordnance, Indianapolis, Indiana.

Mr. J. A. Standerfer, graduate assistant at North Texas State College, has been called to active duty in the United States Navy.

Mr. G. W. Starch, formerly a student at Oklahoma Agricultural and Mechanical College, has a position as junior mining geologist, New Cornelia Mine, Phelps Dodge Corporation, Ajo, Arizona.

Associate Professor J. K. Sterrett of Marshall College is now a mathematician at Ballistics Research Laboratory, Aberdeen Proving Ground, Maryland.

Mr. W. B. Stovall, Jr., statistician at the Bureau of Vital Statistics, State Board of Health, Jacksonville, Florida, is serving in the United States Navy.

Associate Professor H. L. Turrittin of the University of Minnesota has been appointed to an associate professorship at Princeton University.

Associate Professor W. R. Van Voorhis is on leave of absence from Fenn College and is serving as Director of Training for the Military and Civil Defense Commission of Pennsylvania.

Dr. S. S. Walters, who has been a mathematician at the Rand Corporation, has been appointed to an instructorship at the University of California at Los Angeles.

Mr. Chih-Yi Wang has been appointed to a teaching assistantship at the University of Minnesota.

Instructor E. H. Wang of the University of Cincinnati has established a Mathematical Engineering Service in Cincinnati.

Mr. A. W. Wortham, instructor and research assistant at Oklahoma Agricultural and Mechanical College, has a position as senior project analytical engineer in Dynamics Analysis Section, Chance Vought Aircraft, Dallas, Texas.

Assistant Professor G. C. Zader of The Citadel is on leave of absence and is serving in the United States Navy.

Mr. L. H. Cutting of Kansas City High School died on February 13, 1951. He had been a member of the Association for thirty years.

Professor Emeritus R. M. McDill of Hastings College died on March 19, 1951. He had been a member of the Association for thirty years.

Dean Emeritus H. L. Slobin of the University of New Hampshire died February 22, 1951. He was a charter member of the Association.

THE MATHEMATICAL ASSOCIATION OF AMERICA

Official Reports and Communications

NEW MEMBERS

Professor H. M. Gehman, Secretary-Treasurer, announces that the following eighty-four persons have been elected to membership by the Board of Governors on applications duly certified:

J. C. Abbott, Ph.D. (Notre Dame) Asso. Professor, U. S. Naval Academy, Annapolis, Md.

J. T. Ahlin, M.A. (Southern California) Research Engineer, Douglas Aircraft Company, Santa Monica, Calif.

W. A. Al-Salam, B.S. (California) Grad. Student, University of California, Berkeley, Calif.

W. E. Andrus, Jr., A.B. (Syracuse) Technical Engineer, International Business Machines Corporation, Endicott, N. Y.

Mrs. Edith W. Annis, A.B. (Hunter) Chairman, Department of Mathematics, Mamaroneck Senior High School, N. Y.

E. S. Ashcraft, M.A. (North Carolina) Asst. Professor, Stetson University, DeLand, Fla.

A. V. Banes, M.S. (Oklahoma A and M) Asst. Professor, Trinity University, San Antonio, Tex.

J. H. Banks, Ph.D. (George Peabody) Asso. Professor, George Peabody College for Teachers, Nashville, Tenn.

T. J. Benac, Ph.D. (Yale) Asso. Professor, U. S. Naval Academy, Annapolis, Md.

J. W. Blattner, Student, Central College, Fayette, Mo.

R. G. Brown, M.A. (Yale) Research Engineer, Willow Run Research Center of the University of Michigan, Ypsilanti, Mich.

J. P. Carberry, B.S. (Roanoke) Grad. Student, Carnegie Institute of Technology, Pittsburgh, Pa.

A. J. Carlan, B. A. (Brooklyn C.) 214 Avenue N, Brooklyn, N. Y.

R. J. Cary, Student, Harpur College, Endicott, N. Y.

Y. W. Chen, Dr. (Göttingen) Asso. Professor, University of Oklahoma, Norman, Okla.

K. J. Cohen, Student, Reed College, Portland, Ore.

J. I. Derr, B.S. (Southern Methodist) Grad. Student, Southern Methodist University, Dallas, Tex.

Angelo DiBella, B.S. (New Mexico A and M) Research Assistant, Sandia Corporation, Albuquerque, N. M.

R. E. Dowds, Student, Kent State University, Ohio

Myrtle Edwards, M.A. (Georgia) Teacher, Mars Hill College, N. C.

G. E. Forsythe, Ph.D. (Brown) Mathematician, National Bureau of Standards, Los Angeles, Calif.

Barbara A. Futral, Student, Agnes Scott College, Decatur, Ga.

Joaquin Galvan P., Student, St. Mary's University, San Antonio, Tex.

G. H. Gleissner, M.A. (Columbia) Mathematician, U. S. Naval Proving Ground, Dahlgren, Va.

F. S. GOEPPER, JR., M.A. (Cincinnati) Grad. Student, Harvard University, Cambridge, Mass.

A. J. GOLDMAN, Student, Brooklyn College, N. Y.

LOUISA S. GRINSTEIN, B.A. (Buffalo) Grad. Student, University of Buffalo, N. Y.

M. P. GUHSE, B.S. (Hamilton) Grad. Student University of Massachusetts, Amherst, Mass.

ROBERT HARTRANFT, B.S. (Stanford) Mathematician, Naval Air Missile Test Center, Pt. Mugu, Calif.

E. Y. HILL, Student, University of British Columbia, Vancouver, B. C. Canada.

MR. JIRO ISHIHARA, M. S. (Northwestern) Grad. Student, Northwestern University, Evanston, Ill.

P. F. IVERSON, A.B. (Hastings) Head Department of Mathematics, Potomac State School of West Virginia University, Keyser, W. Va.

M. W. JONES, Ph.D. (Colorado) Asso. Professor, Adams State College, Alamosa, Colo.

T. L. JORDAN, JR., M.A. (Vanderbilt) Teaching Fellow, Vanderbilt University, Nashville, Tenn.

T. K. KAWATA, B.A. (Southwestern C.) Grad. Student, DePaul University, Chicago, Ill.

MRS. HELEN S. KEEFER, Student, Iowa Wesleyan College, Mount Pleasant, Iowa

S. R. KNOX, M.A. (Mississippi) Asst. Professor, Millsaps College, Jackson, Miss.

I. H. KRAL, B.S.R.E. (Indiana Tech.) Junior Engineer, Bendix Aviation Research Laboratories, Detroit, Mich.

R. A. C. LANE, A.B. (Indiana) Grad. Assistant, Lehigh University, Bethlehem, Pa.

L. I. LOWELL, B.S.M.E. (Michigan State C.) Grad. Student, Michigan State College, East Lansing, Mich.

W. A. LUCAS, M.E. (Stevens Tech.) Faculty Instructor, Stevens Institute of Technology, Hoboken, N. J.

M. T. MACNEIL, Student, University of Detroit, Mich.

J. E. MADDEN, M.S. (Brown) Instructor, Portsmouth Priory School, Rhode Island

L. E. MAHURON, B.A. (Hardin-Simmons) Supervisor, I. B. M. Section, Sandia Corporation, Albuquerque, N. M.

P. L. MARSHALL, Student, Hofstra College, Hempstead, N. Y.

J. E. MAXFIELD, M.S. (Wisconsin) Instructor, University of Oregon, Eugene, Ore.

AUDREY L. MICHAELS, Student, Brooklyn College, N. Y.

E. R. MICHALIK, M.A. (Pittsburgh) Asst. Professor, University of Pittsburgh, Pa.

R. W. MOLLER, Ph.D. (Catholic U.) Instructor, Catholic University of America, Washington, D. C.

C. B. MORREY, JR., Ph.D. (Harvard) Professor, University of California, Berkeley, Calif.

C. R. MORRIS, M.A. (Kentucky) Mathematician, U. S. Navy Hydrographic Office, Suitland, Md.

G. R. MOTT, Student, Hofstra College, Hempstead, N. Y.

MARY L. MURPHY, M.S. (Boston C.) Secretary, Mathematics Department, Boston College, Chestnut Hill, Mass.

THEODORA S. NELSON, M.S. (Illinois) Asst. Professor, Nebraska State Teacher's College, Kearney, Neb.

C. D. PARKER, Student, Michigan State College, East Lansing, Mich.

R. E. PINGRY, Ph.D. (Cornell) Asst. Professor, University of Illinois, Urbana, Ill.

LILY B. POWELL, M.A. (Colorado) Instructor, University of Colorado, Boulder, Colo.

J. B. PRYZIE, B.S. (C.C.N.Y.) Consolidated Edison Company, New York, N. Y.

V. ELISE QUALLS, A.B. (Berea) Instructor, Tennessee Polytechnic Institute, Cookeville, Tenn.

B. D. RAMSDELL, M.S. (Michigan) Instructor, Orlando Junior College, Fla.

R. M. REDHEFFER, Ph.D. (M.I.T.) Instructory, University of California at Los Angeles, Calif.

H. L. RICE, Student, St. Mary's University, San Antonio, Tex.

J. W. RIDLEY, JR., Junior Chemist, Allied Chemical and Dye Corporation, Philadelphia, Pa.

P. D. RITGER, A.M. (Pennsylvania) Grad. Student, New York University, N. Y.

T. J. RIVLIN, M.A. (Harvard) Grad. Student, Harvard University, Cambridge, Mass.

F. M. ROLAND, Student, St. Mary's University, San Antonio, Tex.

MRS. ANNIE G. H. SASSER, M.S. (Prairie View A. and M.) Asst. Professor, Tennessee Agricultural and Industrial State College, Nashville, Tenn.

HERBERT SCARF, Student, Temple University, Philadelphia, Pa.

SISTER CHARLES ANNE, M.A. (Ohio State) Dean of Studies, College of St. Mary of the Springs, Columbus, Ohio

GUSTAVE SOLOMON, Student, Yeshiva College, New York, N. Y.

MARVIN STERN, M.S. (N.Y.U.) Mathematics Consultant, Republic Aviation Corporation, Farmingdale, N. Y.

J. A. SULLIVAN, Ph.D. (Indiana) Instructor, University of Notre Dame, Ind.

ANNA L. TAYLOR, A.B. (Southwest Missouri) Asst. Instructor, University of Missouri, Columbia, Mo.

R. J. TEED, M.A. (Illinois) Instructor, Fournier Institute of Technology, Lemont, Ill.

W. E. TIMON, JR., B.S. (Northwestern S. C. of La.) Grad. Fellow, Tulane University, New Orleans, La.

MRS. HALSEY K. TRUAN, B.S. (Pittsburgh) Laboratory Technician, Atomic Power Division, Westinghouse Electric Corporation, Pittsburgh, Pa.

B. O. VANHOOK, M.A. (Vanderbilt) Head, Department of Mathematics, Mississippi Southern College, Hattiesburg, Miss.

E. H. WEISS, Sc.M. (Brown) Mathematician, U. S. Naval Proving Ground, Dahlgren, Va.

H. E. WEISSLER, B.Sc. (St. Mary's) Instructor, St. Mary's University, San Antonio, Tex.

MARGARET M. WELCH, Student, Texas Christian University, Ft. Worth, Tex.

NATHAN WETROGAN, B.A. (Brooklyn C.) Grad. Student, Brooklyn College, N. Y.

LOIS B. WHITMAN, B.A. (U.C.L.A.) Teacher, Fairfax High School, Los Angeles, Calif.

R. G. WOODLE, JR., B.S. (Drury) Instructor, Missouri School of Mines and Metallurgy, Rolla, Mo.

B. K. YOUSE, B.S. (Alabama Poly.) Grad. Assistant, University of Georgia, Athens, Ga.

CALENDAR OF FUTURE MEETINGS

Joint meeting with American Society for Engineering Education, Michigan State College, East Lansing, June 25–26, 1951.

Thirty-second Summer Meeting, University of Minnesota, Minneapolis, September 3–4, 1951.

Thirty-fifth Annual Meeting, Brown University, Providence, Rhode Island, December 29, 1951.

The following is a list of the Sections of the Association with dates of future meetings so far as they have been reported to the Associate Secretary.

ALLEGHENY MOUNTAIN

ILLINOIS

INDIANA

IOWA

KANSAS

KENTUCKY

LOUISIANA-MISSISSIPPI, Northwestern State College, Natchitoches, Louisiana, February 15–16, 1952.

MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA

METROPOLITAN NEW YORK

MICHIGAN

MINNESOTA, North Dakota Agricultural College, Fargo, October 6, 1951.

MISSOURI

NEBRASKA

NORTHERN CALIFORNIA, University of California, Berkeley, January 26, 1952.

OHIO

OKLAHOMA

PACIFIC NORTHWEST, State College of Washington, Pullman, June 15, 1951.

PHILADELPHIA, University of Pennsylvania, Philadelphia, November 24, 1951.

ROCKY MOUNTAIN

SOUTHEASTERN, Georgia Institute of Technology and Agnes Scott College, Atlanta, March 21–22, 1952.

SOUTHERN CALIFORNIA, Occidental College, Los Angeles, March 8, 1952.

SOUTHWESTERN, University of Arizona, Tucson, April 11–12, 1952.

TEXAS

UPPER NEW YORK STATE

WISCONSIN

GEORGE BANTA PUBLISHING COMPANY, MENASHA, WISCONSIN

THE AMERICAN MATHEMATICAL MONTHLY

THE OFFICIAL JOURNAL OF

THE MATHEMATICAL ASSOCIATION OF AMERICA, INC.

VOLUME 58 NUMBER 7

CONTENTS

AUGUST-SEPTEMBER 1951

The AMERICAN MATHEMATICAL MONTHLY

(Founded in 1894 by Benjamin F. Finkel)

EDITORIAL CORRESPONDENCE should be addressed to the Editor, C. V. Newsom, State Education Building, Albany 1, N. Y. Papers submitted for publication should be sent to C. B. Allendoerfer, Department of Mathematics, University of Washington, Seattle 5, Washington.

ADVERTISING CORRESPONDENCE should be addressed to J. F. Randolph, Morey Hall, University of Rochester, Rochester 3, N. Y.

NOTICE OF CHANGE OF ADDRESS by members of the Association as well as correspondence regarding subscriptions to the Monthly should be sent to the Secretary-Treasurer, H. M. Gehman, University of Buffalo, Buffalo 14, N. Y.

Entered as second class matter at the post office at Menasha, Wis. Acceptance for mailing at special rate of postage provided for in the Act of February 28, 1925, embodied in Paragraph 4, Section 538, P. L. and R., authorized April 1, 1926.

Subscription Price: To Members, $4 a Year, To Others, $5 a Year.

Published by the Association at Menasha, Wisconsin, and Albany, N. Y. during the months of January, February, March, April, May, June-July, August-September, October, November, December.

GEORGE ABRAM MILLER

H. R. BRAHANA, University of Illinois

George Abram Miller was born on a farm near Lynville, Pennsylvania, on July 31, 1863, and died at Urbana, Illinois, on February 10, 1951. He had long been established in the front rank of the mathematicians of the world when he assisted at the founding of the Mathematical Association of America on December 30, 1915. He was one of the two vice-presidents in 1916, and was president in 1921. He was made an honorary life member in 1937. His connection with the MONTHLY began in its earliest days and the earliest days of his mathematical career. In volumes 2 and 3, in 1895 and 1896, he published in a series of fourteen installments some forty pages of exposition of substitution groups. This set of papers constitutes the simplest and most complete account of the substitution groups of low degree in the literature. Thereafter he contributed seventy papers to the MONTHLY, the last in 1933 when he was seventy years old. In 1909 the University of Illinois joined the University of Chicago in support of the MONTHLY, and Miller became one of the editors; he remained on the editorial board until 1915. The last issue of the MONTHLY in 1909 carried the announcement of Editor Miller's marriage to Miss Cassandra Boggs of Urbana; she was an understanding and cherished companion for forty years.

His family spoke German at home; the schools were conducted in English. He became a teacher at seventeen to earn money to prepare for and to attend college. He attended Franklin and Marshall Academy, a subdivision of the college of that name at Lancaster, in 1882–1883, and Muhlenberg College from 1884 to 1887. He was graduated from Muhlenberg with honorable mention, ranking third in his class. The next year he was Principal in the public schools of Greeley, Kansas, and then he became Professor of Mathematics at Eureka College at Eureka, Illinois. His preparation for this position was meager, but the requirements were modest. The curriculum at Muhlenberg offered calculus in the third year and astronomy and meteorology in the fourth; Eureka required a knowledge of calculus for the Bachelor of Arts degree. After the first year he spent part of the summer of 1889 at the Johns Hopkins University, and he spent the summer of 1890 at the University of Michigan; neither university was in session during those summers. He was made a Master of Arts by Muhlenberg in 1890, but the degree was offered to every Bachelor of Arts of three years' standing who was of good moral character and had been engaged in liberal or professional pursuits. In the meantime he had started the collection of a mathematics library at Eureka and in 1890 he offered a set of advanced courses leading to the degree of Doctor of Philosophy on the completion of two years of satisfactory work. In the year 1891–1892 he was enrolled by correspondence as a graduate student in Cumberland University, where he was granted the degree of Doctor of Philosophy in 1892, thus becoming the only Ph.D. on the staff at Eureka. In 1890 the staff of the University of Illinois had four Ph.D.'s and one D.Sc.; Eureka had about 300 students and Illinois 351.

In 1893 Miller went to the University of Michigan as an Instructor. With the encouragement of F. N. Cole he started upon the study of finite groups which he followed for the rest of his life. Cole had spent two years in Germany where he had studied under Lie and Klein; he had returned to receive his doctor's degree at Harvard; he had lectured on groups at Harvard in 1886, and those lectures have been said to have inaugurated a new era in instruction in advanced mathematics there. Miller's first two papers were published in volume 3 (1894) of the Bulletin of the New York Mathematical Society. They were the result of the redetermination of the substitution groups of degrees eight and nine. Cayley had given a list of the former in 1892 and Cole had corrected it; Cole had listed the groups of degree nine. Miller added two groups to each list, bringing the totals to 200 and 258, respectively, and in the latter case he corrected Jordan's list of the primitive groups of degree nine published in 1872. In the same year he published his own list of 994 intransitive groups of degree ten. These lists have not been questioned since that time.

This successful measuring of his powers against Cole and Jordan and Cayley was the encouragement he needed. In 1895–1897 he attended the lectures of Lie in Leipzig and Jordan in Paris, but his time was spent very much as it would have been spent if he had remained at Ann Arbor; he had published 27 papers by the end of the year of his return. By 1900 he was acknowledged to be in the front rank of students of groups in the country, and by 1910 he was unsurpassed in the world.

His publications on abstract groups began in 1896. In his first paper he determined for the first time the 15 groups of order 24 and the 51 groups of order 32. In 1930 he determined the 294 groups of order 64. The fact that in the meantime there had been only one attempt, and that unsuccessful, to determine the groups of order 64 gives an indication of the difficulty of the problem. Throughout his life he worked directly on and very close to the problem of the determination of the groups of order n. Very few of the results which he obtained have been improved upon.

After Miller's return from Europe in 1897, he spent four years at Cornell, five years at Leland Stanford, and then nearly forty-five years at Illinois. He was a member of the mathematical societies of England, Germany, Spain, India, and America. He was elected to high office in the American Mathematical Society and the Association for the Advancement of Science, and he held membership in the National Academy of Sciences and the American Academy of Arts and Sciences. He was made an honorary Doctor of Letters by Muhlenburg in 1936.

After his retirement in 1931 he followed the program he had been following before. He came to his office morning, afternoon, and evening regardless of the calendar, the weather, or those minor ills that plague us all and occasionally keep some of us at home. In 1935 he published 18 papers and in 1944, when he was eighty-one, he published ten. His last publications were two short papers in 1947. He came regularly to his office until Christmas, 1950.

He published two books, *Determinants* (1892), and *Historical Introduction to Mathematical Literature* (1916). He wrote parts of two more, *The Algebraic Equation* in *Monographs on Topics of Modern Mathematics* (1911) and the first part of *Finite Groups* (1916) by Miller, Blichfeldt, and Dickson. He published some 820 papers in the educational, scientific, and mathematical journals of eleven countries. His non-technical papers included expositions of groups for non-specialists, expositions of mathematics for non-mathematicians, and studies on the history of mathematics.

His 450 (approximately) technical papers on groups constitute a permanent addition to the knowledge of finite groups. A knowledge of the substitution groups of low degrees and the abstract groups of low orders has a value in situations far removed from the obvious ones. He was the first to prove many of the things that every student of the subject uses. In some directions he carried his investigations far enough to show that lines of development which looked promising are not feasible.

G. A. Miller was a man of power which he directed to worthy ends; he hastened the development of his subject; and he added largely to the prestige of American mathematics.

THE STORY OF TANGENTS

J. L. COOLIDGE, Harvard University

1. The Greeks. Whoever has given the least thought to the subject of plane curves has given some consideration to tangents. But what are tangents? To the uninstructed, a plane curve, not a straight line, is the path traced by a moving point whose motion changes its direction at each instant. Had the point at any instant decided not to change the direction of motion, the line through that point in the direction of instantaneous motion would be the tangent. All of this is perfectly clear to any observant and unsophisticated person who has not endeavored to go below the surface, it is only when some one suggests awkward questions like "What do you mean by direction?" "What is meant by instantaneous motion?" that difficulties begin to appear.

Perhaps a beginner starts with a static approach. Here is an arc of a curve. We take a point on the arc and draw through it a line which meets the curve there and nowhere else in the vicinity. This line we may call the tangent and prove, if we can, that it is unique. In the case of the circle it is perpendicular to the radius to that point. But suppose we start with the conic sections and ask whether in the case of a hyperbola a line parallel to an asymptote, which certainly does not meet the curve elsewhere, is really a tangent. There is clearly no

suggestion of touching in such a case. It is clear that we can not be too naïve in answering such questions, it is the purpose of the present article to tell the story of how men have attempted to answer them [1].

I can not make out that before Euclid anyone was much concerned with tangents. I turn therefore to Euclid Book III, Definition I [2].

A straight line is said to touch a circle which meeting the circle and, being produced, does not cut the circle.

I am no Greek scholar so I follow what Heath says on the following page of his commentary. He draws a distinction between ἅπτεσθαι which means to meet, and ἐφάπτεσθαι which means to touch. He suggests that this distinction was common with the Greeks, although he cites exceptions one way or the other, one being in the work of Archimedes, though the exact reference to the passage is not given. In the work of Archimedes on spirals, to which I shall presently return, the word used is επιψαυη which Heath also translates "touch" [3]. I think there is no difficulty in seeing what Euclid means. He expands the idea further, for in Euclid III (16) we have

"The straight line drawn at right angles to the diameter of a circle from its extremity will fall outside the circle, and into the space between the straight line and the circumference another straight line can not be interposed." The tangent meets the circle once only and lies wholly outside, and no other line through the point of contact has this property.

When we come to the conic sections, we learn from Apollonius I, (17) and (32) *"If a straight line be drawn through the extremity of the diameter of any conic parallel to the ordinates to that diameter, the straight line will touch the conic, and no other straight line can fall between it and the conic."* This theorem can not, however, have been original with Apollonius, for he speaks of his first four books as having been worked out by his predecessors. Archimedes in the first proposition on the quadrature of the parabola remarks, "These propositions are proved in the elements of the conics," referring, probably, to the works of Euclid and Aristaeus. I think it safe to say that Euclid knew the Apollonian theorems which I have given.

It is time to turn to Archimedes, especially to his work on spirals. An Archimedian spiral is the curve traced by a point advancing uniformly on a line which turns uniformly about a fixed point. Here we run across the very curious fact that although he speaks frequently of tangents to spirals, he does not define them. I think it is evident that he means by a tangent a straight line which meets the spiral at a point but does not cross it there; a tangent meets the curve at one point but nowhere else in the vicinity. He could doubtless have proved that no other line lies between the tangent and the curve.

Archimedes shows next that as the radius vector and angle both increase uniformly, presumably with the time, if a series of angles form an arithmetical progression, the same is true of the corresponding radii vectores. If thus we take the radii vectores to two points of the same turn of the spiral, the radius vector which bisects the angle between them is equal in length to one half their sum.

This leads to the rather curious Proposition 13, *If a straight line touch a spiral, it will touch it once only.* This means at only one point of that turn, as use is made of the preceding theorem about arithmetical series. Let O be the origin, and suppose a certain tangent touches the curve at both P and Q. Let OR be the radius vector which bisects the angle. Then $OP+OQ=2OR$. But we can show, though Archimedes does not do so, the distance to PQ along the bisector is less than this, so there is a point on PQ inside the curve, but this is contrary to the hypothesis that the line PQ was a tangent. As a matter of fact the line PQ meets the curve an infinite number of times, so there are plenty of points on both sides of it but not in the POQ sector. The whole treatment seems to me much looser than we have a right to expect from this wonderful geometer.

Archimedes' great accomplishment is to give an actual construction for the tangent. This is done by giving the polar sub-tangent, that is to say, the distance along the perpendicular at the origin to the radius vector from the origin to the tangent, the distance which we should write $r^2d\theta/dr$. Here he uses the method of exhaustion and the method of verging, which consists in placing a line segment of given length with its ends on given curves while its line passes through given points. His fundamental theorem is that if P be a point of the nth turn and $OP=r$, the polar sub-tangent will be $a=2\pi(n-1)r+\text{arc } AP$ where A is the starting point. The proof I find difficult to follow. It consists in showing that if the theorem were not true we could find a point outside the spiral which should be analytically inside. I can not see where the actual tangency comes in, the proof merely shows that in any other case an impossible construction would result. An analytical expansion of the method will be found in [4].

How did Archimedes discover this result? We do not know, but we can make a shrewd guess. In his *Method*, Archimedes shows how he was first led to theorems about areas and volumes, which he proved rigorously by the method of exhaustion, by cutting his figures into slices and then comparing their turning moments in different positions. We have here the concept of infinitely thin slices. Is it not likely that he also had the idea of a tangent as the line of an infinitely short chord, a perfectly familar concept to the mathematicians of the seventeenth century. If, then, a be the polar sub-tangent corresponding to the infinitely small advances dr, $rd\theta$, we have by similar triangles

$$\frac{a}{r} = \frac{rd\theta}{dr},$$

but in the case of the spiral of Archimedes

$$r = k\theta, \qquad a = r\theta$$

and this is Archimedes rule.

2. Fermat and Descartes. The idea of a tangent as the limiting position of a secant when two intersections with the curve tend to fall together was slow in attaining mathematical acceptance. The fact is that the idea of a limiting posi-

tion of any sort came to birth slowly and painfully. But the idea of two intersections falling together was well understood before the middle of the seventeenth century. I think that here we should lay great stress on the work of Fermat, who certainly had it clearly in mind even though certain modern writers are very insistent that he did not understand at all the limit idea.

The ideas of Fermat were first set forth in 1629 in a letter to a certain M. Despagnet [**5**]. The method of finding tangents seems to have been a bi-product of his method of finding maxima and minima. Whether this is really an anticipation of the method of differentiation is disputable, an elaborate discussion by Wieleitner is found in [**6**]. The best explanation of what is found occurs much later in a letter of 1643 to Brulart de St. Martin [**7**].

Suppose that we seek a maximum or a minimum for a function f. This will appear on the graph as the top or bottom of an arch at a specific point A. The functions $f(A+E)$ and $f(A-E)$ will both be greater than or less than $f(A)$. Suppose, then, that we write

$$f(A \pm E) = f(A) \pm Ef'(A) + \frac{E^2}{2}f''(A) + \cdots .$$

The function f is supposed to be a polynomial, and so the higher powers of E may be neglected. We have then

$$f(A + E) = f(A - E); \qquad f'(A) = 0.$$

The particular example that he takes is $f(A) \equiv BA^2 - A^3$. A similar process is applied to the problem of finding the tangent to a parabola; finding the tangent does not mean finding its equation but finding the sub-tangent [**8**]. He ends the article with the claim "Nec unquam fallit methodus." A somewhat different explanation is found on page 147 of Vol. I of [**5**], and Wieleitner in [**6**] makes much of the difference; neither account strikes me as particularly convincing.

Fermat makes what I think is a much better use of his method on page 158 ff. of Vol. I of [**5**]. Let the equation of the curve be $F(x, y) = 0$. We seek the subtangent at the point (x, y). A very near point shall be $(x+e)$: The ordinate up to the tangent is found by similar triangles to be $y(1+e/a)$ and this we treat as if it were also on the curve, so that

$$F(x, y) = F\left(x + e, y\left(1 + \frac{e}{a}\right)\right) = 0.$$

He takes the "cissoid" of Diocles and the "conchoid" of Nicomedes, finding a by this limiting process. He points out, incidentally, that to find an inflexion, we must find a maximum or minimum of the angle which a tangent makes with a given direction, and this means finding a maximum or a minimum of its cotangent, and that means to maximize or minimize a/y, where a is the subtangent.

A good example of Fermat's method is found in the case of the "folium" of Descartes, as here we do not have y as an explicit function of x:

$$x^3 + y^3 = nxy,$$

$$(x + e)^3 + y^3\left(1 + \frac{e}{a}\right)^3 - ny(x + e)\left(1 + \frac{e}{a}\right) = 0,$$

$$e\left(3x^2 + \frac{3y^3}{a} - \frac{nxy}{a} - ny\right) + e^2\left(3x + \frac{3y^3}{a^2} - \frac{ny}{a}\right) + e^3\left(1 + \frac{y}{a^2}\right) = 0.$$

This holds for all values of e. We therefore divide e out, and for the tangent assume $e = 0$, then

$$a = -\frac{3y^3 - nxy}{3x^2 - ny}.$$

I will point out that this amounts to putting $a = -yF_y/F_x$ but this general formula did not appear before the work of de Sluse, which we shall see later.

An interesting example is Fermat's oval, which may be written

$$y = K\sqrt{\cos\frac{x}{b}}.$$

Here we introduce an auxiliary circle of radius b, and x becomes the length of an arc on this circle. The process of assuming the identity of two near points is used with regard to this, rather than the original curve. Immediately following the foregoing are examples of finding the tangent to one curve from the tangent to another, a process also followed by Barrow. Corresponding abscissas are equal, and the ordinate to one equals the arc-length to the other. He shows later that the slope of one is equal to the secant of the slope-angle of the other. He solves the problem in the case of the parabola twice, once by his own methods, once by what he calls the "Ancients" method where a tangent is defined as a line meeting a curve but once in a certain region. A preliminary theorem tells us that if we take a point of an arc whose tangent turns continually in one way, and proceed to a nearby ordinate, the distance along the tangent is less than that along the curve, if this be the direction of decreasing ordinates, but greater if it be the direction of increasing ordinates. The proof consists in applying the principle that an arc is longer than its chord, but less than the sum of two tangents from an external point to its ends. This is preliminary to an elaborate study of the lengths of curves.

I can not leave Fermat without expressing admiration for his method, which is essentially that of the infinitesimal calculus, even if he did not see all that was involved. He had, I think, a better grasp of the essential principles than his contemporaries, and was certainly early, perhaps the earliest, in the field.

It is time to pass to Fermat's great rival in this matter, René Descartes. He first attacked the problem of tangents in 1637, which was later than Fermat's

letter to Despagnet, but before he had heard directly on the subject from Fermat. It is very curious that Descartes began by seeking to draw a normal to a curve, that is to say, a perpendicular to a tangent at its point of contact [10]. Which straight line cuts a curve at right angles at a given point, or, as he says, cuts the "contingent" at right angles? If (x, y) be the given point, and if $(x, 0)$ be where the normal meets the X-axis, it is the center of a circle, two of whose intersections with the curve fall together. Thus, if we write,

$$(X - x_1)^2 + Y^2 = (x - x_1)^2 + y^2, \qquad f(X, Y) = 0,$$

and eliminate Y, a necessary condition for a normal is that two of the roots of this equation in X should fall together. If we take the case of the parabola,

$$\begin{aligned} Y^2 &= 2mX, \\ (X - x_1)^2 + 2mX &= (x - x_1)^2 + 2mx, \\ X^2 + 2X(m - x_1) &= x^2 + 2x(m - x_1). \end{aligned}$$

The two roots will be equal if $x_1 = m + x$.

In the examples which Descartes worked out, he did not have available a general method of handling the case of equal roots. He wrote

$$\Phi(X) \equiv (x - x_1)^2(c_0 X^m + c_1 X^{m-1} + \cdots + c_m)$$

and identified the coefficients on the two sides. The general rule was first worked out by Hudde in 1683 [11].

In 1638, when Descartes first heard of Fermat's method for maxima and minima, he was not a little stirred up, and expressed his disgust in letters to Mersenne, Hardy and others. He disliked especially the quantity e, which was divided out, because it was not equal to 0, and then equated to 0. He attacked Fermat's method of tangents as though it involved considerations of maxima and minima. In this Descartes was wrong. Fermat did not say that finding a tangent was a maximum-minimum problem, but that the methods developed for one case, also fitted the other, and this Descartes finally saw. He put his own method in a letter to Hardy of June 1638 [12].

Let

$$y^3 = lx.$$

Let us find two points (x, y), (x_1, y_1) so that $y_1 = ky$.

$$x_1 = x + e, \qquad y_1 = y\left(1 + \frac{e}{a}\right),$$

$$\frac{y^3}{x} = \frac{y^3\left(1 + \dfrac{e}{a}\right)^3}{x + e} = \frac{k^3 y^3}{x + e},$$

$$a^3 = 3a^2x + 3aex + e^2x.$$

But when $k=1$, $e=0$ and $a=3x$; this does not seem to me essentially different from Fermat's method, even in the reasoning. Nor does it seem to justify the long correspondence involving Descartes, Fermat, Mersenne, Hardy and others. Descartes had, however, a third method, applied at least to the cycloid. The problem of finding a tangent to this curve had occupied various geometers. Fermat had solved it, rather clumsily. Roberval had found a solution which I shall return to later, but Descartes gave an absurdly simple solution which is as follows. Suppose [**13**] that a polygon rolls along a straight line, a chosen vertex will trace a succession of circular arcs whose centers are successive vertices which lie on the line. The line from the tracing vertex to the corresponding fixed vertex on the line will be perpendicular to the tangent to the arc. Now, if we consider a circle, as presently became popular, as a regular polygon of an infinite number of sides, we see that the instantaneous center is the point of contact with the given line, and the line thence to the tracing point is the normal to the cycloid. The tangent will thus always go from the point of contact to the highest point of the rolling curve.

Descartes puts the matter somewhat differently. He draws through the point of contact a line parallel to the fixed line to meet the rolling circle when it has rolled one half the distance, or when the tracing point has reached its highest position. The line from the intersection to the point of contact of the new circle is parallel to the normal sought. For any rolling curve, the tangent is perpendicular to the line from the point of contact to the instantaneous center.

3. Roberval and Torricelli. In discussing the work of Fermat and Descartes, we have been to a certain degree running around in circles, explaining more or less similar methods of handling infinitesimals. It was the era in which the infinitesimal calculus was struggling towards birth. Let us take a vacation from this sport by looking at a totally different approach to the subject of tangents by two more or less rival geometers whose relative merits I shall certainly not try to evaluate. There is an elaborate weighing of their comparative worth in a recent work by Evelyn Walker [**22**]. I, therefore, take up first Giles Persone de Roberval. He seems to have first been occupied with the classical question of the tangent to the cycloid, a curve called to his attention by Mersenne, but the best exposition of his work with tangents is found in [**23**].

Roberval's idea was simplicity itself. A curve is traced by a moving point; the tangent anywhere is the line of instantaneous motion of that point. The real philosophical difficulty, to define what is meant by instantaneous motion, was veiled in the future, to bedevil those of his successors who occupied themselves with the foundations of the calculus. To the unspoiled eye of common sense there was no difficulty. Here is the Axiome or principe d'invention that he gives on pp. 24 and 25c of [**23**]. *La direction du mouvement d'un point qui décrit une ligne courbe, est la touchante de la ligne courbe en chaque position de ce point là.* This seems to be more or less tautologous, a tangent is a line which touches, but he goes on at once to explain himself. *Par les propriëtez spécifiques, (qui vous seront*

données) examinez les divers mouvements qu'a le point à l'endroit ou vous voulez mener la touchante; de tous ces mouvements composez en un seul et tirez la ligne de direction du mouvement composé, vous aurez la touchante de la courbe.

The meaning is this. Determine two measurements which connect the moving point with two fixed elements. Determine the vector velocities of the changes of these two. Their vector sum will give the instantaneous velocity. Gomes de Carvalho points out on page 53 of [1] that Roberval is rather cavalier in his reasoning about infinitesimal triangles; the parallelogram whose sides are dx/dt, dy/dt is not the same as that whose sides are dr/dt, $rd\theta/dt$. However, in each case, the diagonal gives the direction of instantaneous motion.

Let us take some examples of Roberval's method. First, we take the parabola. The two motions are away from the focus and away from the directrix. As these two distances are always equal, the instantaneous velocities are equal. Hence the tangent makes equal angles with the axis on the focal radius. He shows carefully that this is the result given by Apollonius.

The central conics are handled in analogous fashion. We have distances from the two foci whose sum or difference is constant. Hence the differences or sums of the instantaneous velocities are constant and so the tangent makes equal angles with the focal radii, or with one radius and the other produced.

Roberval next considers the family of conchoids. Take the conchoid of Nicomedes. Lines radiate from a fixed origin to meet a directrix, a fixed line not through the origin. Each radiating line is produced a fixed distance beyond the directrix. The two motions are a radial one away from the origin and a circular one about the origin. The distance out from the directrix to the curve is independent of the choice of radius vector. If the chosen angle θ gives us ρ for the directrix, and r for the curve, the corresponding rotational velocities are $rd\theta/dt$ and $\rho d\theta/dt$; the stretching velocities are dr/dt, $d\rho/dt$. Hence the tangent of the angle which the tangent to the conchoid makes with the radius vector is ρ/r times the tangent of the angle made with the directrix. The other conchoids come from another choice of directrices.

Roberval gives a simple enough construction for the tangent to the spiral if we assume, as does the Master, that we can draw a straight line equal in length to the circumference of a given circle. He becomes, however, rather deeply involved when he comes to finding the tangents to the quadratrix, or the cissoid. I confess that his analysis of the infinitesimal motions is not convincing to me. He has much better success when he comes to the cycloid. He even allows his rolling wheel to slip a bit, so that the length of the track covered in one complete turn is not necessarily equal to the circumference of the wheel. Assuming that both the sliding and turning motions are uniform, we have merely to draw, through a point on the curve, vectors parallel to the track and tangent to the wheel proportional to the distance slid and the distance turned and find their sum. In reading [23] it is easy to forget that he allows for slipping and one wonders why he does not give the simpler construction of Descartes.

Anything one says about Roberval brings to mind the name of the rival

inventor of the method of determining tangents by means of instantaneous velocities, Evangelista Torricelli. In 1644, he published his *Opera Geometrica* where, in the second section of Part 1 entitled "De motu gravium," we find expressed the technique which I will now describe. It will be found in [24]. He starts with some propositions of Galileo about falling bodies. Suppose that we start with a weighted point that falls a certain distance, then is shot off at right angles and thereafter is also allowed to fall naturally. The path will then be a parabola, for the distance slid will be proportional to the time, and the distance fallen to the square of the time. The falling velocity will also be proportional to the time, and, consequently, to the distance slid. If the parabola be $x^2=2my$ and we take the sliding velocity as the constant x, the ratio of dropping to sliding velocity, which will give the direction of the tangent, will be x/m.

We thus get Torricelli's construction for the tangent. We connect the point of contact with the reflection in the vertex of the foot of the ordinate. He adds on page 123, "Haec demonstratio peculiaris est pro parabola, sed universalem habemus pro qualibet sectione conica, consideratis aequalibus velocitatibus unius puncti, quod aequaliter movetur in utraque linea quae ex focis procedit." This is certainly very suggestive of Roberval's procedure for the central conics. He goes on to state, "Eadem ratione Demonstratur Propositio 18 de lineis spiralibus Archimedis unica brevique demonstratione, . . . Immo et hac ratione etiam unico Theoremata tangentes quarundam curvarum, inter quas, omnium linearum Cycloidalum."

Torricelli claims that he has a general method applicable not only to the parabola, but the central conics, the Archimedian spiral, and all cycloids. What is his general method? Presumably it is the composition of velocities, but he carries it out only in the one case. I have an unpleasant feeling that those who have written on the subject have not bothered to think the matter through. Jacobi writes in [25], on page 268, a direct transcript of the original with the statement "progressivi impetus ad lateralem ratio ut *ad* ad *bf* per praecedentem Propositionem," and that's all. Walker [22], page 138, states "The ratio of the progressive impetus to the lateral is as $AD:BF$. These are the ordinates of the given point and the focus, with no hint of where he gets this important fact. "Impetus" does not mean acceleration, but instantaneous velocity.

There has arisen a good deal of discussion among historians of mathematics as to which of the geometers, Roberval or Torricelli, first thought of determining tangents by instantaneous velocities. There is an elaborate discussion of the subject with dates in [22], [23], and [24]. I will not go further into the matter, but I should like to insist on the originality of the method. It is a great step forward, to pass from considering a tangent as a line meeting a curve but once, at least in a small region, to that of treating it as the limit of a secant whose intersections fall together. It was equally bold to consider it as the line of instantaneous advance.

4. DeSluse and Barrow. I return to the general line originated by Fermat and Descartes. An admirable extension of this was first put into words by

René François Walter, Baron deSluse. Suppose we have a curve whose equation is $f(x, y)=0$, the function being a polynomial. Reject all constant terms. Let all terms in y be placed on the right with signs changed, and let each term be multiplied by the exponent of y. Let each term in x be placed on the left, and multiplied by the exponent of x, and let one x in each term be replaced by a. If a term involve both x and y, it should appear with the proper sign on both sides and handled appropriately. We then solve for a, the subtangent [14] this will give

$$a = -\frac{yf_y}{f_x}\cdot$$

The question arises at once, where did deSluse get this method? One's first idea is to credit it to Newton who had essentially the same technique, but I think the discovery must have been independent. LePaige, in the carefully written article [15], says that in 1652 deSluse had some sort of a method of drawing tangents. We find him in 1658 writing to Pascal [16], after explaining his method of drawing tangents to certain curves called "perles," "Cette méthode est tirée d'une plus universelle laquelle comprend toutes les ellipsoides, mesme avec peu de changement, les paraboloides et les hyperboloides." Newton did not begin to think about the method of fluxions before 1665. He published nothing on the subject before his paper of 1669, dictated to Collins, "De analysi per aequationes numero terminorum imfinitos." In 1673, he finally gave priority to deSluse [17].

But how did deSluse happen to hit on this technique? LePaige suggests that it comes from the formula for expanding $(x^2-y^2)/(x-y)$. I am afraid I do not see the connection. It seems to me more likely that he reflected on the work of Fermat, and noted the relation of exponents and coefficients in simple differentiation, then stepped from this to partial differentiation. It may be significant that deSluse, like Fermat, used a for the subtangent in [14], though he used η and ω, where we should use x and y.

In connection with deSluse, it is necessary to say something about Barrow. He gives deSluse' method [14] in his [18], stating, modestly enough, that he gives his method at the request of a friend (presumably Newton), "Though I scarcely perceive the use of doing so, considering the several methods of this Nature now in use." His rule, as stated, means very little, but, worked out, it amounts to that of deSluse. In his first example, he commits what seems to me to be the most heinous possible mathematical sin. He uses the same letter to mean two different things in the same problem, writing the equation

$$qq - 2qe + mm - 2ma = BLq,$$

where the right side means BL^2.

Barrow's most interesting example is his fifth where $y=\tan x$. He takes an auxiliary circle of radius r, and finds the point with Cartesian coordinates

(r cos x, r sin x). He then gives x a small increment dx and this he puts equal to $\frac{1}{2}$ sin 2 dx. If then we compare an infinitesimal triangle with a finite one, we get the fundamental formula

$$d \cos x = -\sin x dx.$$

He finally reaches

$$\frac{dy}{dx} = 1 + y^2.$$

J. M. Child, whose admiration for Barrow seems to me a bit excessive, says in [19] "If $y=\tan x$, $dy/dx=\sec^2 x$." He must have known (for it is in itself evident) that the same two diagrams can be used for any of the trigonometric ratios. Therefore, Barrow must be credited with the differentiation of the circular functions. This is possible, but not at all certain. Eight years later James Bernoulli gave, as known, the formulae for the derivatives of tan x and sec x. The credit for differentiating all six functions is usually assigned to Roger Cotes and his *Harmonia Mensurarum* of 1722. Child even assigns to Barrow the credit for inventing the process of differentiation.

I should mention, in connection with Barrow, two other theorems. The first is in his fourth lecture, and consists in a proof of Fermat's sub-tangent formula $a/y=dx/dy$. His proof, like the rest of his work, I find obscurely written. He works, not with infinitesimals, but with velocities. Next, let us take a convex arc and a point M where the slope of the tangent is l. An ordinate shall move at a constant rate, cutting the curve at O and the tangent at K. When K and O are above G, where the moving ordinate cuts the horizontal line through M, if O is further from this horizontal line than is K, it must go further in the same time than does K. Its dropping velocity $dy/dt>l\, dx/dt$, where dx/dt is the constant velocity of the ordinate. But, when the dropping velocity on the tangent is less than that on the curve, we have $dy/dt<l(dx/dt)$. Barrow concludes that at M, $dy/dt=l(dx/dt)$. I have naturally shortened the proof by using modern notation. Fermat's formula will come at once from this. I should mention that on page 61 of [19] we have the fundamental statement: *If the arc MN is assumed indefinitely small we may safely substitute instead of it a small bit of the tangent.* This is, of course, the basis of Fermat's process of "adequation" and, in fact, of most seventeenth century work with tangents, except that of the school of Roberval and Torricelli.

5. Newton and Leibniz. We have seen that Newton's method of drawing tangents was the same as that of deSluse. He probably discovered it in 1664 or 1665, when he was first thinking through the methods of the infinitesimal calculus. He published nothing before 1669, and then only in a letter. A complete discussion giving nine different examples appeared under Problem IV of [26]. He begins like deSluse. If we take two near points of a curve, and connect them by a straight line, which we treat as if it were the tangent, we have

two similar triangles. The one is bounded by the tangent, the sub-tangent and the ordinate; the other by the element of arc and what he calls the "moments" of the two coordinates, which are in the language of Leibniz the differences, and are proportional to their "fluxions." He would write Fermat's first formula

$$\frac{e}{y} = \frac{\dot{x}}{\dot{y}} \cdot$$

The first example is

$$x^3 - ax^2y + axy - y^3 = 0,$$

$$3\dot{x}x^2 - 2a\dot{x}x + a\dot{x}y + ax\dot{y} - 3\dot{y}y^2 = 0,$$

$$l = \frac{3y^3 - axy}{3x^2 - 2ax + ay} \cdot$$

He gives other examples, sometimes introducing other variables, but nothing essentially different. Some of his methods show the influence of Roberval, or a like-thinking geometer. In the third memoir, he assumes that a point is known by r_1 and r_2, its distances from two fixed points. These are connected by a known equation, from which we find $\dot{r}_1/\dot{r}_2$. We draw a perpendicular to the second radius vector at its origin, and extend it till it meets the tangent, and thence drop a perpendicular on the first radius vector. If the distance from this last intersection to the point of contact be S, we have, by similar trapezoids, $s/r_2 = \dot{r}_1/\dot{r}_1$. Hence s is known, and we can find a point on the tangent. In the same way, he handles tangents given in polar coordinates, finding the polar sub-tangent.

When it comes to Leibniz, I move with extreme caution, not wishing to express any opinion on the Newton-Leibniz priority question. Newton wrote in a letter to Oldenburg, October 24, 1676, intended for Leibniz, a long description of his methods, including a reference to deSluse's method of finding tangents. Leibniz answered [27] at once, maintaining that deSluse's methods are not in themselves sufficient, and that he himself had discovered a method. He adds, "Clarissimi Slusii Methodum Tangentium nodum esse absolutam celeberrimo Newtono assentior. Et jam a multo tempore rem Tangentium longe generalius tractavi, scilicet per differentia Ordinaturum."

The essential part is found here in the words, "Longe generalius tractavi." He says that for a long time he had treated tangents by the differences of ordinates, the differences of abscissas were treated as constants. He gives various examples. If we disregard higher infinitesimals, we have

$$dy^2 = 2ydy, \qquad dy^2x = 2xydy + y^2dx,$$

given

$$a + by + cx + dyx + ey^2 + fx^2 + \cdots = 0,$$

$$-\frac{dy}{dx}=\frac{c+dy+2fx+\cdots}{b+dx+2ey+\cdots}.$$

"Quod coincidit cum Regular Slussiana." He writes as a universally accepted principle; "Si sit aliqua potentia aut radix x^z erit $dx^z=zx^{z-1}dx$.

To find $d\ \sqrt[3]{a+by+cy^2+\cdots}$ he puts

$$z=1/3;\qquad x=a+by+cy^2+\cdots$$

$$dx^z=\frac{dx}{3x^{2/3}}=\frac{bdy+2cydy+\cdots}{3(a+by+cy^2+\cdots)^{2/3}}.$$

Newton had written to Collins that he did not wish to show his method of tangents to Leibniz. The letter ended with the following, "Arbitrior quae celare voluit Newtonus de Tangentibus ducendis, ab his non abludere." I see no point in going further into the famous Newton-Leibniz priority controversy.

References

1. An excellent discussion of the whole matter is to be found in Gomes' de Carvalho A teoria des tangentes antes de Invencio do Calculo Differencial, Coimbra, 1919.
2. Heath, The thirteen books of Euclid's Elements, Second Ed., Cambridge, 1926, Vol. II, p. 1.
3. The works of Archimedes, Cambridge, 1897, p. 167.
4. Ueber Spiralen von Archimedes, Dr. Arthur Czwalina-Allenstein, Leipzig, 1922.
5. Fermat, Oeuvres, Ed. Tannery and Henry, Paris, 1891–1922, Vol. 2 p. 71.
6. Bemerkungen zu Fermats Methoden der Aufsuchung von Extremwerthen Jahresbericht der deutschen Mathematikervereinigung, Vol. 38, 1929.
7. Fermat, Oeuvres, Vol. V, pp. 120 ff.
8. Fermat, Oeuvres, Vol. I, p. 135.
9. Fermat, Oeuvres, Vol. II, p. 172. The equation is given in a footnote. Fermat gives merely a figure, and I follow Gomes de Carvalho (pp. 38, 39).
10. Descartes, La Geométrie, Nouvelle Edition, Paris, 1927, pp. 34 ff.
11. De maximis et minimis, in Geometria a Renato Descartes, Amsterdam, 1683.
12. Descartes, Oeuvres, Cousin Ed., Paris, 1824, Vol. 7, pp. 62, 63.
13. Descartes, Oeuvres, Cousin Ed., Vol. 7, p. 89.
14. de Sluse, A method of drawing tangents to all geometrical curves. Phil. Trans. Abridged Vol. 2, for years 1672–83. London, 1809, pp. 38 ff.
15. Correspondence de René, François de Sluse, Bolletino di Bibliografia e di Storia dell Scienze. Matematiche e Fisiche, Vol. 17, 1884, p. 477.
16. Pascal, Oeuvres, Ed. Brunschwicg, Boutroux and Gazier, Vol. 8, Paris, 1914, p. 8.
17. Commercium Epistolicum, published by Biot and Leffort, Paris, 1850, pp. 54, 85.
18. Barrow, Geometrical Lectures, London, 1735, pp. 173 ff.
19. The Geometrical Lectures of Isaac Barrow, London, 1916.
20. Jacobi Bernoulli, Opera, Vol. I, Geneva, 1744, pp. 436.
21. See [19], p. 94 Note.
22. A study of the Traité des Indivisibles of Roberval, Teachers College, New York, 1932.
23. Observations sur la composition des mouvements, Mémoires de l'Académie Royale des Sciences, Vol. VI, 1730.
24. Evangelista Torricelli, Opere Loria and Vassuro, Ed., Vol. II, Faenza, 1919, p. 12.
25. Ferdinando Jacoli, Evangelista Torricelli ed il metodo delle tange nti, Volletino di Bibliografia e di Storia delle Scienze Matematiche e fisiche, Vol. 8, 1875.

26. The Method of Fluxions and Infinite Series, translated from the Author's original, not yet made public, by John Colson, London, 1736.

27. Leibniz, Mathematische Schriften. Gerhardt Ed., Vol. I, pp. 154–162. Also Sloman, The Claims of Leibniz to the invention of the Differential Calculus, Cambridge, 1860.

A NEW ABSOLUTE GEOMETRIC CONSTANT?

ROBIN ROBINSON, Dartmouth College

In Whitworth, *Choice and Chance*, appears the following problem: If there be n straight lines in one plane, no three of which meet in a point, the number of groups of n of their points of intersection, in each of which no three points lie in one of the straight lines, is $\frac{1}{2}(n-1)!$

We shall assume either that no two lines are parallel, or that the problem refers to the extended plane. To prove that there are just two points on each line, note that since each of the n points lies on two lines, there are $2n$ incidences in the set, which is an average of 2 incidences per line. But since no line may contain as many as 3 incidences, each must contain just 2.

The problem may now be regarded as requiring the number of polygons of n sides formed by a plane network of n lines, no three concurrent. Whitworth's answer is correct only if no polygon is allowed to consist of two or more polygons of fewer sides, *e.g.*, a composite polygon consisting of two polygons of k and $n-k$ sides, respectively. There is, however, nothing in his statement of the problem to rule out such cases.

Let g_n be the number of n-gons formed by a network of n lines. We shall show that g_n becomes infinite like $\sqrt{n}(n-1)!$, which will correct Whitworth's answer. To be more precise, we shall show that g_n becomes infinite like $n^n e^{-n}$, in other words, that

$$\lim_{n\to\infty} \frac{g_n}{n^n e^{-n}} = B,$$

where B is a constant.

Let us proceed by induction. Suppose we raise the number of lines from $(n-1)$ to n by adding one line to $(n-1)$ already given. Each $(n-1)$-gon can be converted into an n-gon by the following operation: Replace any one vertex by the two points where the two sides through it meet the new line. Conversely,

an n-gon which can be so obtained can be evolved by this operation from only one $(n-1)$-gon, and by replacement of only one vertex, since the operation is reversible uniquely. For example, in Figure 1, the 4-gon $ABCD$ may be converted into the 5-gon $ABCD_1D_2$, when the fifth line D_1D_2 is added; this is the only way that the 5-gon $ABCD_1D_2$ may be obtained in this manner from a 4-gon formed by the four lines originally given.

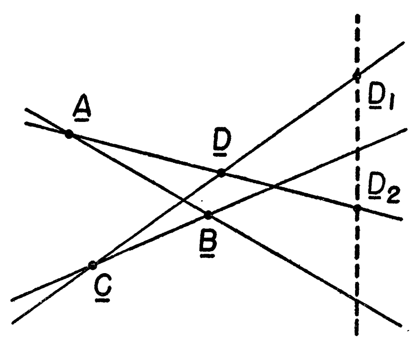


FIG. 1

If these n-gons were the only ones in the new network of n lines, then we should have $g_n=(n-1)g_{n-1}$, since one n-gon corresponds to each vertex of each $(n-1)$-gon. Since obviously $g_3=1$, we should have

$$g_n = (n-1)g_{n-1} = (n-1)(n-2)g_{n-2} = \cdots$$
$$= (n-1)(n-2)\cdots 3g_3 = \tfrac{1}{2}(n-1)!,$$

which is Whitworth's answer. Unfortunately for Whitworth, there are n-gons which cannot be obtained by this operation, namely, every composite n-gon one of whose parts is a triangle having the nth line as one side. For example, in Figure 2, the 6-gon $ABCDEF$ consists of the triangles ABC and DEF. If the line EF is removed, allowing E and F to be replaced by D in the reverse of the operation defined above, the result consists of the triangle ABC and the vertex D counted twice, which is not a 5-gon at all.

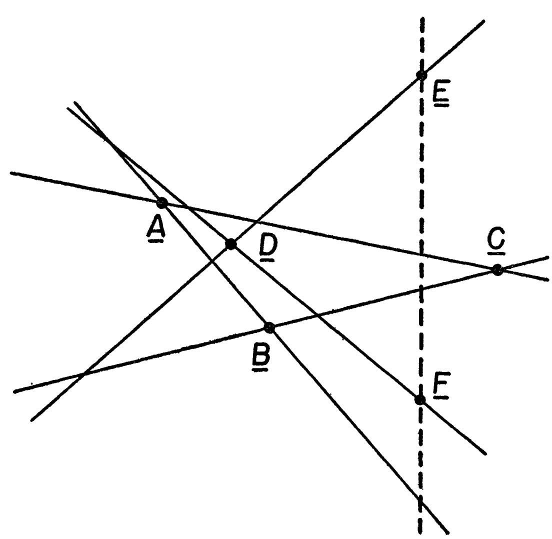


FIG. 2

These n-gons are, however, the only exceptions, and can be counted. On the nth line there are just $(n-1)$ possible vertices, a pair of which can be chosen in C_2^{n-1} ways. When a pair is chosen, the triangle is determined, and the remainder of the composite n-gon may be chosen from the $(n-3)$-gons on the remaining $(n-3)$ lines in g_{n-3} ways. Hence,

$$g_n = (n-1)g_{n-1} + C_2^{n-1}g_{n-3},$$

or, more conveniently,

$$g_{n+1} = ng_n + C_2^n g_{n-2}. \tag{1}$$

We may simplify our work by introducing u_n, the ratio of g_n to Whitworth's answer for non-composite polygons:

$$g_n = u_n \cdot \tfrac{1}{2}(n-1)!$$

Making this change in (1), and dividing by $\frac{1}{2}n!$,

$$u_{n+1} = u_n + \frac{1}{2(n-2)} u_{n-2},$$

or,

$$u_{n+3} = u_{n+2} + \frac{1}{2n} u_n. \tag{2}$$

Since no composite polygons exist with less than 6 sides, $u_3 = u_4 = u_5 = 1$. Necessarily $n \geqq 3$.

Although this recursion formula is not readily solved for u_n explicitly in terms of n, we can nevertheless determine quite a bit about the behavior of u_n as n becomes large.

LEMMA 1. $u_{n+1} > u_n$, $n > 4$.

This is obvious from (2).

THEOREM 1.

The number $b = \lim_{n\to\infty} \dfrac{u_n^2}{n - \frac{9}{4}}$ *exists, and* $b \geqq 0$.

Since $u_n < u_{n+1} < u_{n+2}$,

$$u_{n+3} = u_{n+2} + \frac{1}{2n} u_n < u_{n+2} + \frac{1}{2n} u_{n+2},$$

$$u_{n+3}^2 < u_{n+2}^2\left(1 + \frac{1}{2n}\right)^2 = u_{n+2}^2\left[1 + \frac{1}{n}\left(1 + \frac{1}{4n}\right)\right],$$

$$u_{n+3}^2 < u_{n+2}^2\left[1 + \frac{1}{n}\left(1 + \frac{1}{4n} + \frac{1}{(4n)^2} + \cdots\right)\right],$$

$$u_{n+3}^2 < u_{n+2}^2\left[1 + \frac{1}{n}\cdot\frac{1}{1 - \dfrac{1}{4n}}\right] = u_{n+2}^2\,\frac{n + \frac{3}{4}}{n - \frac{1}{4}},$$

$$0 < \frac{u_{n+3}^2}{n + \frac{3}{4}} < \frac{u_{n+2}^2}{n - \frac{1}{4}} < \frac{u_n^2}{n - \frac{9}{4}}.$$

Since $u_n^2/(n-\frac{9}{4})$ decreases as n increases, and is always positive, it has a limit, positive or zero.

THEOREM 2.

$$\lim_{n\to\infty} \frac{u_n^2}{n-i} = b,$$

where i is any constant.

This follows from

$$\lim_{n\to\infty} \frac{n-\frac{9}{4}}{n-i} = 1,$$

and Theorem 1.

LEMMA 2.

$$u_n > u_{n+2}\left(1 - \frac{1}{n-\frac{9}{4}}\right).$$

In proving Theorem 1, we showed that

$$\frac{u_{n+2}^2}{n-\frac{1}{4}} < \frac{u_n^2}{n-\frac{9}{4}}.$$

$$u_n^2 > u_{n+2}^2 \frac{n-\frac{9}{4}}{n-\frac{1}{4}} = u_{n+2}^2\left(1 - \frac{2}{n-\frac{1}{4}}\right).$$

Since $n \geqq 3$, $2/(n-\frac{1}{4}) < 1$, we may write

$$u_n > u_{n+2}\left(1 - \frac{2}{n-\frac{1}{4}}\right)^{1/2}$$

$$u_n > u_{n+2}\left[1 - \frac{1}{2}\left(\frac{2}{n-\frac{1}{4}}\right) - \frac{1}{8}\left(\frac{2}{n-\frac{1}{4}}\right)^2 - \frac{1}{16}\left(\frac{2}{n-\frac{1}{4}}\right)^3 - \cdots\right].$$

Note that all coefficients after the second term of the series are numerically less than $\frac{1}{2}$, and negative:

$$\pm \frac{\frac{1}{2}(-\frac{1}{2})(-\frac{3}{2})(-\frac{5}{2})\cdots}{1\cdot 2\cdot 3\cdot 4\cdots}.$$

So

$$u_n > u_{n+2}\left\{1 - \frac{1}{n-\frac{1}{4}}\left[1 + \frac{1}{4}\left(\frac{2}{n-\frac{1}{4}}\right) + \frac{1}{8}\left(\frac{2}{n-\frac{1}{4}}\right)^2 + \cdots\right]\right\}.$$

Now all the coefficients in the brackets are positive and $\leqq 1$, so

$$u_n > u_{n+2}\left\{1 - \frac{1}{n-\frac{1}{4}}\left[1 + \left(\frac{2}{n-\frac{1}{4}}\right) + \left(\frac{2}{n-\frac{1}{4}}\right)^2 + \cdots\right]\right\},$$

$$u_n > u_{n+2}\left(1 - \frac{1}{n-\frac{1}{4}} \cdot \frac{1}{1 - \dfrac{2}{n-\frac{1}{4}}}\right) = u_{n+2}\left(1 - \frac{1}{n-\frac{9}{4}}\right).$$

LEMMA 3. *If* $n \geqq 13$, then

$$\frac{1}{n} - \frac{\frac{5}{4}}{n(n-\frac{9}{4})} > \frac{1}{n+\frac{7}{4}} \cdot$$

$$n \geqq 13 > \tfrac{49}{4},$$

$$7n - 5n = 2n > \tfrac{98}{4} = \tfrac{63}{4} + \tfrac{35}{4}.$$

$$7n - \tfrac{63}{4} > 5n + \tfrac{35}{4},$$

$$7(n - \tfrac{9}{4}) > 5(n + \tfrac{7}{4}),$$

$$\frac{7}{n+\frac{7}{4}} > \frac{5}{n-\frac{9}{4}},$$

$$\frac{1}{n} - \frac{1}{n+\frac{7}{4}} = \frac{\frac{7}{4}}{n(n+\frac{7}{4})} > \frac{\frac{5}{4}}{n(n-\frac{9}{4})},$$

$$\frac{1}{n} - \frac{\frac{5}{4}}{n(n-\frac{9}{4})} > \frac{1}{n+\frac{7}{4}} \cdot$$

THEOREM 3. $b > 0$.

By Lemma 2,

$$u_{n+3} = u_{n+2} + \frac{1}{2n}u_n > u_{n+2} + \frac{1}{2n}u_{n+2}\left(1 - \frac{1}{n-\frac{9}{4}}\right),$$

$$u_{n+3} > u_{n+2}\left[\left(1 + \frac{1}{2n}\right) - \frac{1}{2n(n-\frac{9}{4})}\right],$$

$$u_{n+3}^2 > u_{n+2}^2\left[\left(1 + \frac{1}{n} + \frac{1}{4n^2}\right) - \frac{1 + \dfrac{1}{2n}}{n(n-\frac{9}{4})} + \frac{1}{4n^2(n-\frac{9}{4})^2}\right].$$

Since $n > 2$, $1/2n < \frac{1}{4}$, and

$$u_{n+3}^2 > u_{n+2}^2\left[1 + \frac{1}{n} - \frac{\frac{5}{4}}{n(n-\frac{9}{4})}\right].$$

By Lemma 3,

$$u_{n+3}^2 > u_{n+2}^2\left[1 + \frac{1}{n+\frac{7}{4}}\right] = u_{n+2}^2\,\frac{n+\frac{11}{4}}{n+\frac{7}{4}}, \qquad n \geqq 13.$$

$$\cdots > \frac{u_{n+3}^2}{n+\frac{11}{4}} > \frac{u_{n+2}^2}{n+\frac{7}{4}} > \frac{u_n^2}{n-\frac{1}{4}} > 0, \qquad n \geqq 15.$$

Hence

$$b = \lim_{n\to\infty} \frac{u_n^2}{n-\frac{1}{4}} > 0.$$

THEOREM 4.

$$\lim_{n\to\infty} (u_{n+1}^2 - u_n^2) = b.$$

$$\begin{aligned} u_{n+3}^2 - u_{n+2}^2 &= (u_{n+3} - u_{n+2})(u_{n+3} + u_{n+2}) \\ &= \frac{u_n}{2n}(u_{n+3} + u_{n+2}) \\ &= \frac{1}{2}\frac{u_n}{\sqrt{n}}\left(\frac{u_{n+3}}{\sqrt{n}} + \frac{u_{n+2}}{\sqrt{n}}\right). \end{aligned}$$

$$\lim_{n\to\infty} (u_{n+3}^2 - u_{n+2}^2) = \tfrac{1}{2}\sqrt{b}(\sqrt{b} + \sqrt{b}) = b.$$

Incidentally, we have shown that $u_n^2/(n-\frac{9}{4})$ always decreases, while $u_n^2/(n-\frac{1}{4})$ always increases beyond a certain point. The author has strengthened these theorems considerably in work not included here, and from these results and a study of numerical tables, it seems virtually certain that $u_n^2/(n-i)$ always decreases, $(n>6)$ when $i \geqq \frac{5}{4}$, but always increases beyond a certain point $(n>N_i)$ when $u<\frac{5}{4}$. However, apparently, $\lim_{i\to\frac{5}{4}} N_i=\infty$. If these facts are true, it can be shown that

$$\frac{u_n^2}{n-\frac{5}{4}} > u_{n+1}^2 - u_n^2 > \frac{u_n^2}{n-i}, \qquad i < \tfrac{5}{4},$$

the first inequality being true for $n>6$, the second for $n>N_i$.

These conjectures are valuable chiefly because they aid in the numerical computation of the limit b. Suppose we tabulate values of $u_{n+1}^2-u_n^2$ and $u_n^2/(n-\frac{5}{4})$. The following is an excerpt from such a table:

n	$u_{n+1}^2-u_n^2$	$u_n^2/(n-\frac{5}{4})$
21	0.283,797	0.284,405
22	0.283,825	0.284,376
23	0.283,850	0.284,350
24	0.283,871	0.284,328
25	0.283,890	0.284,309
26	0.283,907	0.284,292
27	0.283,921	0.284,277
28	0.283,934	0.284,264
29	0.283,946	0.284,252
30	0.283,956	0.284,241

The tabular differences of these values are:

Δ_1	Δ_2
33	34
28	29
25	26
21	22
19	19
17	17
14	15
13	13
12	12
10	11

Since one of the quantities is decreasing toward b, the other increasing, the similarity of the tabular differences suggests that the average of the values in the two columns constitutes a good first approximation to b. This value for $n=30$ is 0.284,098.

We are now in a position to write as an asymptotic expression,

$$u_n \cong \sqrt{b(n - \tfrac{5}{4})}.$$

Using this with Stirling's formula, that is,

$$n! \cong n^n e^{-n}\sqrt{2\pi(n + \tfrac{1}{6})},$$

we may also write

$$g_{n+1} = u_{n+1}\cdot\tfrac{1}{2}n! \cong n^n e^{-n}\sqrt{\tfrac{1}{2}\pi b(n + \tfrac{1}{6})(n - \tfrac{1}{4})}.$$

Only slightly less accurately, but more simply,

$$g_{n+1} \cong n^{n+1}e^{-n}\sqrt{\tfrac{1}{2}\pi b}.$$

From this,

$$g_n \cong (n-1)^n e^{-(n-1)}\sqrt{\tfrac{1}{2}\pi b} = n^n e^{-n}\sqrt{\tfrac{1}{2}\pi b}\cdot e\left(1 - \frac{1}{n}\right)^n.$$

Since

$$\lim_{n\to\infty} e\left(1 - \frac{1}{n}\right)^n = 1,$$

we have, setting $B=\sqrt{\tfrac{1}{2}\pi b}$,

$$g_n \cong B\cdot n^n e^{-n},$$

which was the contention at the start of the paper. The numerical value of B is 0.668,027, but this is no more illuminating than the numerical value of b.

By improving the asymptotic expression by series expansion corrections,

the author has been able to improve the numerical value for b by several decimal places; but this greater accuracy seems to be of slight use, unless perchance it may suggest further avenues of approach to the true nature of b.

Thus b assumes the role of an absolute geometric constant. The author is interested in discovering whether it is an algebraic number, which seems unlikely, whether it is expressible in terms of π and e, or whether it is entirely new. Perhaps some reader of this discussion will be able to throw light on the subject by use of other methods.

HYPERBOLIC TRIGONOMETRY DERIVED FROM THE POINCARÉ MODEL*

HOWARD EVES and V. E. HOGGATT, JR., Oregon State College

1. Introduction. The formulas of hyperbolic trigonometry have been derived in a number of ingenious ways. As early as 1766 Lambert [1] suggested that the geometry of the "third hypothesis" could be verified on a sphere of imaginary radius. The historical process, developed by Bolyai and Lobachewsky, made use of the fact that the geometry of horocycles on the horosphere is euclidean in nature [2]. Sommerville has given an excellent elementary treatment along these lines [3]. Some early writers, however, regretted any appeal to solid geometry in the derivation of formulas for a plane trigonometry. Clever methods were devised to remedy the defect, one of the neatest being due to Liebmann [4], and subsequently reproduced by such textbook writers as Carslaw [5] and Wolfe [6]. Other ingenious methods were furnished by Gérard [7], Young [8], and, more recently, by Fulton [9]. Coolidge [10] supplied a careful treatment based upon the fact that hyperbolic geometry is euclidean in the small.

In order to establish the relative consistency of the hyperbolic and euclidean geometries, it suffices to devise a model in euclidean space containing elements, with appropriate connecting relations, which, when substituted for the undefined elements and relations of a postulate set for hyperbolic geometry, will interpret those postulates as true theorems of euclidean geometry. Many such models have been devised, the most famous ones being due to Beltrami, Cayley, Klein, and Poincaré [11]. Once such a model has been formed it is conceivable that some theorems of hyperbolic geometry might be more readily established by demonstrating the euclidean analogues in the model rather than the originals directly from the accepted postulate set. It is the purpose of this paper to de-

* A fuller development of the ideas of this paper may be found in V. E. Hoggatt's Oregon State College Masters Thesis, 1951.

velop the formulas of hyperbolic plane trigonometry in such a manner, employing the elementary model usually referred to as the *Poincaré model* [12]. Carslaw has shown the utility of this particular model by very simply establishing from it several difficult theorems of hyperbolic geometry [13].

2. Description of the Poincaré model. As an acceptable postulate set for plane hyperbolic geometry one could select that of Hilbert [14]. The primitive terms of this postulate set are *point, line, on* (applied to a point and a line), *between* (applied to three points on a line), and *congruent* (applied to segments and angles). The postulates are statements concerning these primitive terms. A euclidean model of plane hyperbolic geometry must, then, be a system of geometrical elements and relations which, when substituted for the primitive terms, convert the Hilbert postulates into true theorems in euclidean geometry. The Poincaré model accomplishes this as follows.

A fixed circle Σ is selected and called the *fundamental circle.* A point of the hyperbolic plane is represented by a point interior to Σ, hereafter called a *nominal point.* A line of the hyperbolic plane is represented by the arc interior to Σ of any circle orthogonal to Σ, hereafter called a *nominal line.* The relationships of a point on a line, and a point between two points, have the obvious interpretations. To suitably interpret congruence of segments and congruence of angles two definitions are made. The (positive) *nominal length* of a nominal segment AB is defined as

$$\overline{AB} = \log_a (AB, TS) = k \ln (AB, TS), \qquad k = -\ln a,$$

where S and T are the points where the nominal line AB meets Σ, A lying between S and B, and (AB, TS) denotes the cross-ratio $(AT/BT)/(AS/BS)$ of the circular range A, B, T, S. And the *nominal measure* of the angle between two intersecting nominal lines is defined as the ordinary radian measure of the angle between the two circles on which the nominal lines lie. Two nominal segments are congruent if and only if they have equal nominal lengths, and two nominal angles are congruent if and only if they have equal nominal measures.

It can be shown that with the above interpretations the Hilbert postulates for plane hyperbolic geometry become true theorems in euclidean geometry. For every theorem in hyperbolic geometry there is the euclidean counterpart in the Poincaré model, and the establishment of the latter carries with it that of the former. We now proceed to establish hyperbolic trigonometry by obtaining the necessary counterparts in the Poincaré model.

3. Preliminary theorems. Consider any nominal right triangle $O'P'Q'$, right angled at Q' (see Fig. 1), and let the circles determined by the nominal lines $O'P'$ and $O'Q'$ intersect again in C. Invert the figure with respect to C as center and with a power that carries Σ into itself. Since inversion is a conformal transformation, the circles $CP'O'$, $CQ'O'$, being orthogonal to Σ and passing through the center of inversion C, invert into two diametral lines of Σ. Thus, by the

inversion, the right triangle $O'P'Q'$ is carried into the right triangle OPQ, where OP and OQ are radial lines of Σ. Since both angles and cross-ratios are preserved under inversion, it follows that nominal triangles $O'P'Q'$ and OPQ are nominally congruent, and, to obtain the fundamental formulas of hyperbolic plane trigonometry, it suffices to study the relations connecting the nominal lengths of the sides and the nominal measures of the angles of the specially placed right triangle OPQ. We shall consistently distinguish euclidean lengths from nominal lengths by placing bars over the latter. Since angles have the same nominal and euclidean measures, no bars are here needed.

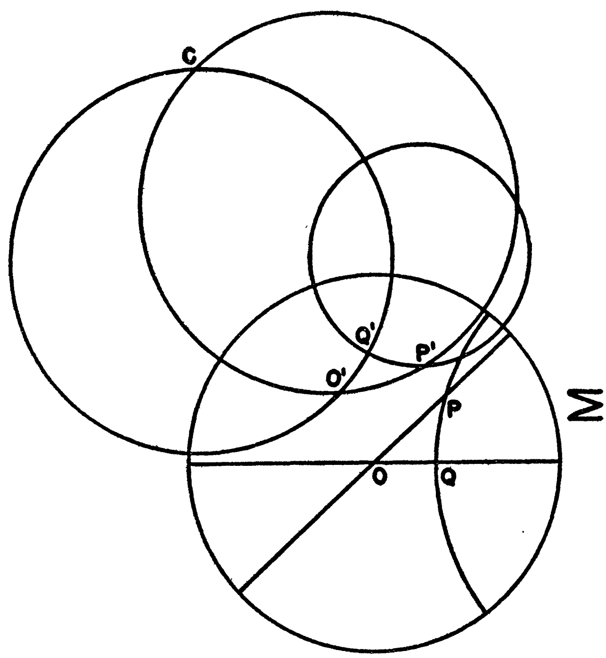


FIG. 1

Let the circle Π determined by the nominal line QP cut Σ in S and T, Q lying between S and P (see Fig. 2), and let $IOQJ$ be a diameter of Σ, cutting Π again in W. We now establish a short chain of theorems connected with Figure 2.

THEOREM 3.1. *If WS and WT cut Σ again in U and V, then UV is the diameter of Σ perpendicular to diameter IJ.*

Select W as center of inversion, and choose a power such that Σ inverts into itself. Then S inverts into U, and T into V. Since Π is orthogonal to both Σ and IJ, it follows that UV is the diameter of Σ perpendicular to diameter IJ.

THEOREM 3.2. *Let WP cut UV in R, and designate the lengths of OW and OR*

by m and n, and the radius of Σ by r. Let K be the center of Π *and let M and N be the feet of the perpendiculars dropped from P on OW and OR respectively. Then*

(a) $KP = (m^2 - r^2)/2m$, (b) $OM = m(n^2 + r^2)/(m^2 + n^2)$,

(c) $OP = (m^2n^2 + r^4)^{1/2}/(m^2 + n^2)^{1/2}$, (d) $OQ = r^2/m$.

Since $KP = OW - OK = m - (r^2 + KP^2)^{1/2}$, it follows that

$$KP = (m^2 - r^2)/2m.$$

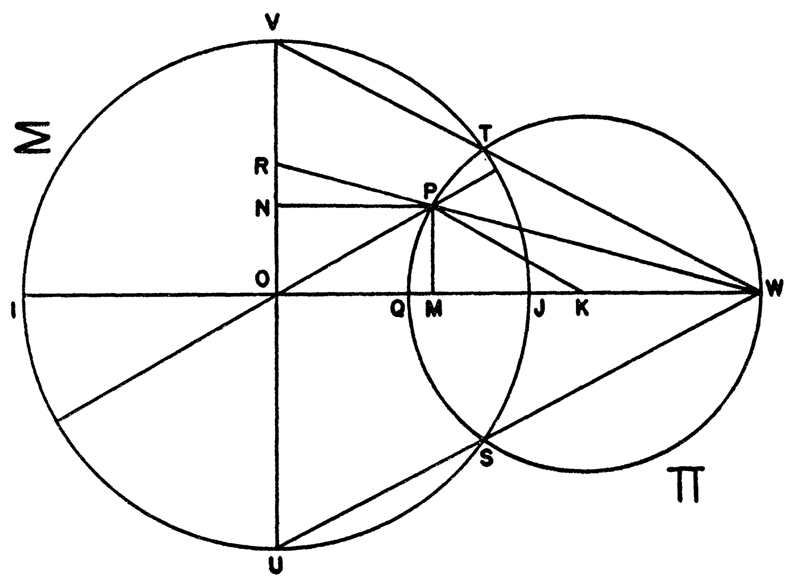


FIG. 2

Also, since $\tan PWO = n/m$, and since $\sphericalangle PKO = 2\sphericalangle PWO$, it follows that

$$\tan PKO = 2mn/(m^2 - n^2), \qquad \sin PKO = 2mn/(m^2 + n^2).$$

Therefore

$$MP = KP \sin PKO = n(m^2 - r^2)/(m^2 + n^2).$$

And, from similar triangles RNP and ROW,

$$OM = NP = (OW)(NR)/OR = m(n - MP)/n = m(n^2 + r^2)/(m^2 + n^2).$$

Then

$$OP^2 = MP^2 + OM^2 = (m^2n^2 + r^4)/(m^2 + n^2).$$

Finally,

$$OQ = OW - 2KP = m - 2(m^2 - r^2)/2m = r^2/m.$$

THEOREM 3.3. *The segments OP, OQ, OR are connected by the relation*

$$\frac{r^2 + OP^2}{r^2 - OP^2} = \frac{r^2 + OR^2}{r^2 - OR^2} \cdot \frac{r^2 + OQ^2}{r^2 - OQ^2}.$$

For, by theorem 3.2 (c),

$$\frac{r^2 + OP^2}{r^2 - OP^2} = \frac{r^2(m^2 + n^2) + m^2n^2 + r^4}{r^2(m^2 + n^2) - m^2n^2 - r^4} = \frac{(r^2 + n^2)(m^2 + r^2)}{(r^2 - n^2)(m^2 - r^2)}$$

$$= \frac{r^2 + n^2}{r^2 - n^2} \cdot \frac{r^2 + r^4/m^2}{r^2 - r^4/m^2} = \frac{r^2 + OR^2}{r^2 - OR^2} \cdot \frac{r^2 + OQ^2}{r^2 - OQ^2},$$

since $OR = n$ and, by theorem 3.2 (d), $OQ = r^2/m$.

THEOREM 3.4. *If OQ is any radial segment of* Σ, *then*

(a) $$\cosh (\overline{OQ}/k) = (r^2 + OQ^2)/(r^2 - OQ^2),$$

(b) $$\sinh (\overline{OQ}/k) = 2rOQ/(r^2 - OQ^2),$$

(c) $$\tanh (\overline{OQ}/k) = 2rOQ/(r^2 + OQ^2).$$

For, since $\overline{OQ}/k = ln(OQ, IJ)$, we have

$$\begin{aligned} \cosh (\overline{OQ}/k) &= [\exp (\overline{OQ}/k) + \exp (-\overline{OQ}/k)]/2 \\ &= [(OQ, IJ) + (OQ, JI)]/2 \\ &= [(OI/QI)/(OJ/QJ) + (OJ/QJ)/(OI/QI)]/2 \\ &= [QJ/IQ + IQ/QJ]/2 \\ &= [(r - OQ)/(r + OQ) + (r + OQ)/(r - OQ)]/2 \\ &= (r^2 + OQ^2)/(r^2 - OQ^2), \end{aligned}$$

and relation (a) is established. Relations (b) and (c) follow in a similar manner, or from the identities $\sinh^2 x = \cosh^2 x - 1$ and $\tanh x = (\sinh x)/(\cosh x)$.

4. Hyperbolic plane trigonometry. We are now ready to derive the formulas of hyperbolic plane trigonometry. It is well known that the formulas for the general hyperbolic triangle are readily derived from those for the hyperbolic right triangle. Let us be given such a right triangle ABC, right angled at C, and designate the lengths of the sides opposite A, B, C by a, b, c, and let k be the parameter of hyperbolic geometry. Then it is easily shown that all the remaining formulas for the right triangle can be obtained by purely analytical procedures from the following two:

$$\cosh c/k = \cosh a/k \cosh b/k,$$

$$\cos A = (\tanh b/k)/(\tanh c/k).$$

It therefore remains for us to establish these two formulas, and we accomplish this by proving the following two theorems.

THEOREM 4.1. *In figure* 2, $\cosh(\overline{OP}/k) = \cosh(\overline{OQ}/k)\cosh(\overline{QP}/k)$.

As an immediate consequence of Theorems 3.3 and 3.4 (a) we have

$$\cosh(\overline{OP}/k) = \cosh(\overline{OQ}/k)\cosh(\overline{OR}/k).$$

But, by Theorem 3.1, $(QP, ST) = W(QP, ST) = (OR, UV)$, whence $\overline{OR} = \overline{QP}$, and the theorem is established.

THEOREM 4.2. *In Figure* 2, $\cos QOP = \tanh(\overline{OQ}/k)/\tanh(\overline{OP}/k)$.

For, by theorem 3.4 (c),

$$\tanh(\overline{OQ}/k)/\tanh(\overline{OP}/k) = OQ(r^2 + OP^2)/OP(r^2 + OQ^2).$$

Substituting the expressions for OP and OQ as given by Theorem 3.2 (c) and (d), and simplifying, we find

$$\begin{aligned}\tanh(\overline{OQ}/k)/\tanh(\overline{OP}/k) &= m(r^2 + n^2)/(m^2n^2 + r^4)^{1/2}(m^2 + n^2)^{1/2}\\ &= [m(r^2 + n^2)/(m^2 + n^2)]/[(m^2n^2 + r^4)^{1/2}/(m^2 + n^2)^{1/2}]\\ &= OM/OP \qquad \text{(Theorems 3.2 (b) and (c))}\\ &= \cos QOP,\end{aligned}$$

and the theorem is established.

References

1. J. H. Lambert, Theorie der Parallellinien, 1766. Reproduced in Engel and Stäckel, Die Theorie der Parallellinien von Euklid bis auf Gauss, Leipzig, 1895.
2. D. E. Smith, Source Book in Mathematics, pp. 360–375.
3. D. M. Y. Sommerville, The Elements of Non-Euclidean Geometry, pp. 56 ff. and 84.
4. H. Liebmann, Nichteuklidische Geometrie, 2nd ed., 1912, chapt. 3.
5. H. S. Carslaw, Elements of Non-Euclidean Plane Geometry and Trigonometry, chapt. 4.
6. H. E. Wolfe, Introduction to Non-Euclidean Geometry, chapt. 5.
7. M. Gérard, Nouvelles Annales de Mathématiques, 1893, pp. 74–84.
8. W. H. Young, American Journal of Mathematics, vol. 33, 1911, pp. 249–286.
9. C. M. Fulton, Mathematics Magazine, vol. 22, no. 5, 1949, pp. 255–262.
10. J. L. Coolidge, The Elements of Non-Euclidean Geometry, chapt. 4.
11. H. S. M. Coxeter, Non-Euclidean Geometry, chapt. 14.
12. H. Poincaré, Science and Hypothesis, translated by W. J. Greenstreet.
13. H. S. Carslaw, *ibid*, chapt. 8, and Proceedings of the Edinburgh Mathematical Society, vol. XXVIII, 1910, pp. 95–120.
14. D. Hilbert, Grundlagen der Geometrie, 7th ed., 1930.

THE BETA FUNCTION

C. S. OGILVY, Larchmont, New York

The First Eulerian Integral, called the Beta Function, is defined by $B(x, y) = \int_0^1 t^{x-1}(1-t)^{y-1}dt$, which converges for $x>0$ and $y>0$. The well-known equation connecting the Beta and Gamma functions, $B(x, y) = \Gamma(x)\Gamma(y)/\Gamma(x+y)$, is therefore valid only for positive real x and y. However, this relation is commonly used as a definition, to extend $B(x, y)$ so that the function has meaning for values other than positive real x, y. It is the purpose of this paper to discuss the behavior of $B(x, y)$ over all real x and y for which the function is thus defined.

The discussion is greatly facilitated by graphic aids. Since $B(x, y)$ is continuous (with isolated exceptions) over the regions of definition, it can be represented by a Cartesian surface, $z = B(x, y)$. In particular, z is continuous for all $x>0$, $y>0$, and so is represented by a smooth unbroken surface throughout the first octant (Fig. 1). Since it is obvious from the extended definition that $B(x, y)$

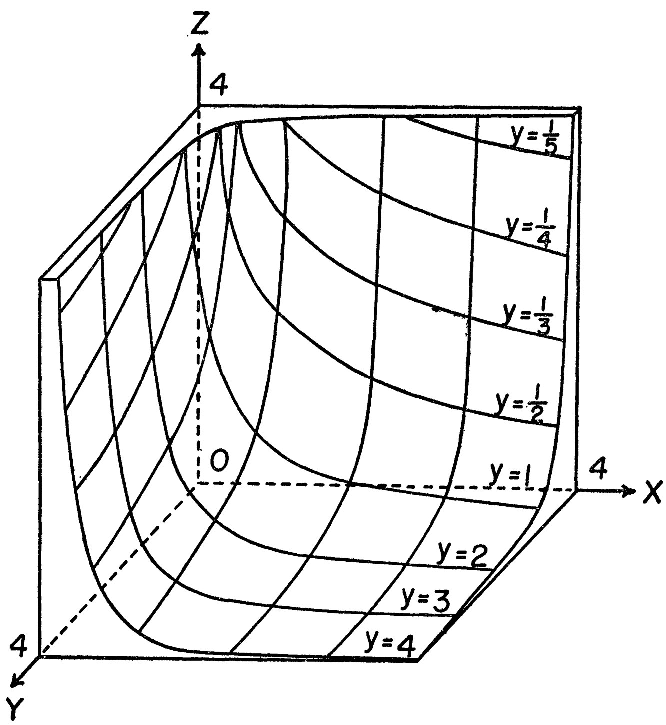


FIG. 1

$= B(y, x)$, we have symmetry with respect to the plane $y=x$, a property which also holds elsewhere than in the first octant. The surface passes through the point (1, 1, 1) but its asymptotic approach to the plane $z=0$ is much more

rapid than its approach to the planes $x=0$ and $y=0$. The value of $B(4, 4)$ is only 1/140. It may be noted in passing that $B(\frac{1}{2}, \frac{1}{2})=\pi$.

When either or both of (x, y) are allowed to assume negative values, the tracing of the surface becomes a much more lively affair. The surface is no longer everywhere continuous. Indeed there are points where it is not even defined. $\Gamma(x)$ oscillates between finite positive and negative values for all $x<0$, except that $\Gamma(-n)=\infty$ for all $n=1, 2, 3, \cdots$. Hence a vigorous oscillation of Beta, a combination of two Gammas, is to be expected in the negative range.

Let us first determine the points, if any, where $z=0$. $B(x, y)=0$ whenever $\Gamma(x+y)=\infty$ but $\Gamma(x)$ and $\Gamma(y)$ are both finite. Obviously this requires that the

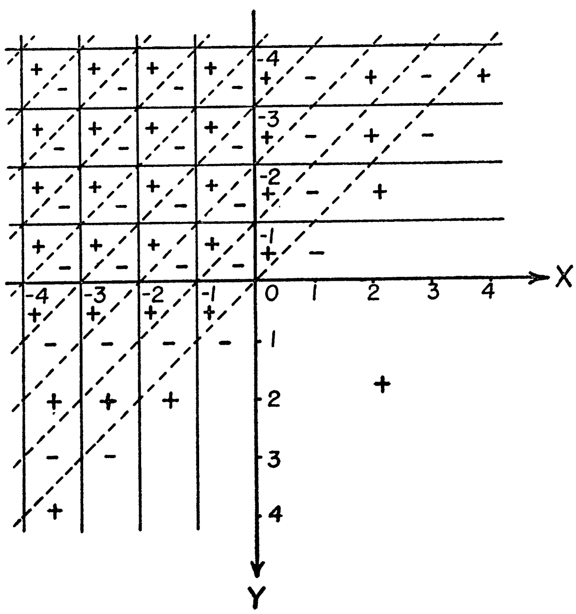


FIG. 2

sum $x+y$ be a negative integer but that neither x nor y be a negative integer. Samples are $B(-1/3, -2/3)=0$ and $B(3/2, -7/2)=0$. To indicate the locus of all such points we need another diagram. Figure 2 represents the plane $z=0$. Y is taken positive downward to conform with the orientation of Figure 1. The plus and minus signs indicate whether the surface z is above or below the plane of the paper in the region in question. Thus the whole first quadrant is marked $+$. The dotted diagonals represent the entirety of points where $z=0$. That is, the surface passes through the plane of the paper along every diagonal.

If $\Gamma(x)$ or $\Gamma(y)$ is infinite but $\Gamma(x+y)$ is finite, then $z=\infty$. This will occur when either x or y is a negative integer, but not both. Therefore along every horizontal and vertical line of Figure 2 the surface $z=B(x, y)\to\infty$ asymptotically. Whether it approaches infinity through positive or negative values of z is governed by the sign of the region. Thus the planes $x=0, -1, -2, \cdots,$

and $y=0, -1, -2, \cdots$, divide the entire coordinate 3-space into cells, whose walls the surface cannot cross continuously. In the octants where x and y are both negative, each cell is a unit square column, closed on four sides and extending to infinity in the z direction only; in the other octants the cells have one side open as well.

If both x and y are negative integers, or if both are integers and one is negative with absolute value $\geqq$ the other, then $\Gamma(x)\Gamma(y)=\infty$ and also $\Gamma(x+y)=\infty$. These are the only points where $B(x, y)$ is not defined, and they are the points in Figure 2 where any diagonal line crosses a horizontal or vertical line, inasmuch as the function cannot equal zero and infinity simultaneously. Thus the function is defined everywhere except at each corner of every cell.

At the risk of undue repetition, let us note again that z is never discontinuous within any one cell. In order to preserve continuity in the long open cells of the lower left octants of Figure 2, the surface undergoes various twists and contortions. These cells are reproduced in mirror image across the plane $y=x$, and we shall therefore describe both sets by describing one. Each diamond-shaped region (of the lower left set), if extended in the z direction perpendicular to the plane of the paper, contains a saddle, which approaches infinity along its right and left edges ($+$ or $-$ as indicated), but merges smoothly into the next saddle along the dotted lines.

At each point of non-definition, the surface contains the line through that point perpendicular to the plane $z=0$. Thus the geometric interpretation of the lack of definition is that z takes on all values from $+\infty$ to $-\infty$ at once.

The behavior of $B(x, y)$ in the octants where both x and y are negative is relatively simple. In half of every cell z is plus, in the other half minus, as marked.

Finally an examination of a few plane sections $y=k$ may be instructive. The relation

$$B(x, y) = \frac{(y-1)!}{x(x+1)(x+2)\cdots(x+y-1)}$$

holds whenever y is a positive integer, x unrestricted. This emphasizes that the traces in Figure 1 found by setting $y=k$ are each part of some curve which has another branch or branches to the left of $x=0$. This last observation applies whether k is integral or not.

If $y=1$ we have $B(x, 1)=z=1/x$, the equilateral hyperbola of Figure 3(a), the upper branch of which appears as a trace in Figure 1. The reader may wonder how the lower branch, a continuous curve, can successfully pierce all the barriers of discontinuity to the left of $x=0$. It accomplishes this feat by the simple expedient of meeting each barrier exactly at a cell corner, where the function is not defined. Thus the lower branch of Figure 3(a) does not everywhere represent the function $B(x, 1)$. At $x=-1, -2, \cdots$, as well as at $x=0$, it fails to do so.

If $y=2$, $z=1/x(x+1)$, (Fig. 3(b)). This trace is symmetric with respect to

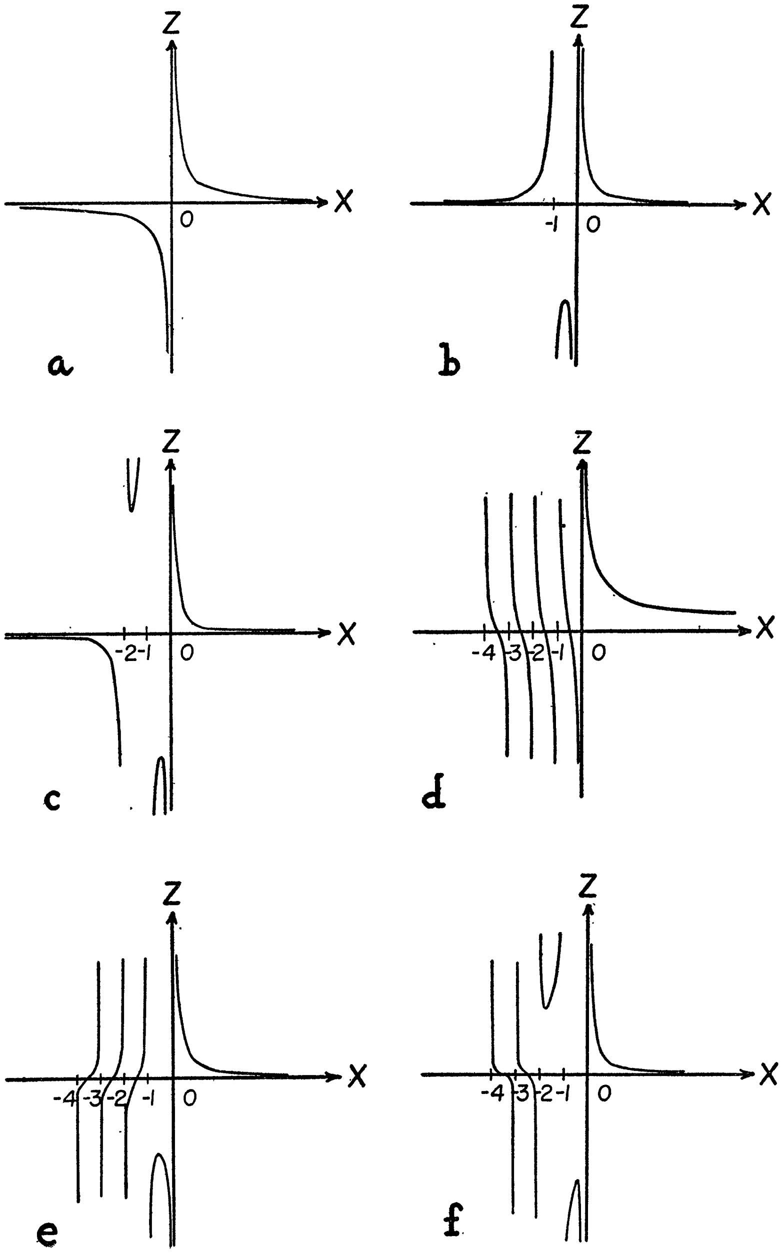


FIG. 3

the line $x=-\frac{1}{2}$. Figure 3(c) shows $y=3$, $z=1/x(x+1)(x+2)$, symmetric with respect to the point $(-1, 0)$. Sections where $y=4, 5, \cdots$, continue the trend thus indicated.

If $y=1/2$, $3/2$, $5/2$ in turn, we obtain the very different traces shown in Figure 3(d, e, f) respectively. Note that all of these have an infinite number of vertical asymptotes, because there is no way for them to get through any of the cell walls to the left of $x=0$. In fact any trace $y=k$, k non-integral, will have an infinite number of asymptotes. If on the other hand k is a positive integer, then the number of vertical asymptotes equals k.

Space does not permit analysis of traces $y=-k$, k fractional, which throw further light on the behavior of the surface. Attempts actually to sketch some of the saddle cells require ingenuity in handling perspective and produce some intriguing and often amusing results.

References

The writer has been unable to find a pictorial representation of the Beta Function in any of the standard texts, although it would seem that the notion of such a representation, particularly in the first octant, could scarcely be new. Nor is the function generally tabulated. Jahncke-Emde has a sketch of $\Gamma(x)$, and there is a careful sketch of $\Gamma(x)$ for positive x on p. 109, Vol. 2, of Edwards' *A Treatise on the Integral Calculus*, (1922). The extension of the definition to negative values of the arguments of $B(x, y)$ is mentioned often; for example, see Franklin, *A Treatise on Advanced Calculus*, p. 562 (1940). Many books of mathematical tables contain useful tabulations of $\Gamma(x)$.

THE WILLIAM LOWELL PUTNAM MATHEMATICAL COMPETITION

L. E. BUSH,* College of St. Thomas

The following results of the eleventh annual William Lowell Putnam Mathematical Competition held March 31, 1951, have been determined in accordance with the constitution of the Competition. This Competition is supported by the William Lowell Putnam Intercollegiate Memorial Fund left by Mrs. Putnam in memory of her husband and is held under the auspices of The Mathematical Association of America.

The first prize, four hundred dollars, is awarded to the Department of Mathematics of Cornell University, Ithaca, New York. The members of the

* Director of the Competition.

team were M. Cohen, J. H. Gay, M. Horowitz; to each of these a prize of forty dollars is awarded.

The second prize, three hundred dollars, is awarded to the Department of Mathematics of Harvard University, Cambridge, Massachusetts. The members of the team were P. Martin, G. Rayna, A. Zemach; to each of these a prize of thirty dollars is awarded.

The third prize, two hundred dollars, is awarded to the Department of Mathematics of The Cooper Union, New York, New York. The members of the team were A. F. Berndt, I. J. Lowe, P. J. Redmond; to each of these a prize of twenty dollars is awarded.

The fourth prize, one hundred dollars, is awarded to the Department of Mathematics of the City College of New York, New York, New York. The members of the team were A. Benson, H. Hanisch, H. Widom; to each of these a prize of ten dollars is awarded.

The five persons ranking highest in the examination, named in alphabetical order, were A. P. Dempster, University of Toronto; J. B. Herreshoff IV, University of California; Herbert Kranzer, New York University; P. J. Redmond, The Cooper Union; Harold Widom, The City College of New York. Each of these will receive a prize of fifty dollars.

The six succeeding persons ranking highest in the examination, named in alphabetical order, were Paul Cohen, Brooklyn College; Michael Horowitz, Cornell University; P. H. Lord, Princeton University; Arthur Mattuck, Swarthmore College, Gerhard Rayna, Harvard University; Herbert Scarf, Temple University. To each of these a prize of twenty dollars is awarded.

The following teams, named in alphabetical order, won honorable mention: Columbia College, New York, New York, the members of the team being R. Feldman, H. Gordon, C. J. Kaufman; Massachusetts Institute of Technology, Cambridge, Massachusetts, the members of the team being W. L. Baily, L. deBranges III, J. E. Kimber, Jr.; University of British Columbia, Vancouver, British Columbia, the members of the team being E. Critoph, C. A. Swanson, D. A. Trumpler; University of California, Berkeley, California, the members of the team being J. B. Herreshoff IV, R. R. Kissling, Keiichi Nishimura.

Nine individuals were given honorable mention. The names are listed in alphabetical order: Michael Cohen, Cornell University; J. H. Gay, Cornell University; George Gioumousis, Polytechnic Institute of Brooklyn; J. R. Jamieson, University of Toronto; R. Lehman, Stanford University; Hans Stetter, Colorado A & M College; H. F. Trotter, Queen's University; D. A. Trumpler, University of British Columbia; Ariel Zemach, Harvard University.

The following is a list of all colleges and universities which entered teams in the Competition. The list, in alphabetical order, is: Brooklyn College, Carleton College, Carnegie Institute of Technology, City College of New York, College of the Holy Cross, College of St. Thomas, Colorado A & M College, Columbia College, The Cooper Union, Cornell University, Gustavus Adolphus College, Harvard University, Kent State University, Knox College, Laval University,

Loyola College (Montreal), Massachusetts Institute of Technology, McGill University, McMaster University, New York University, Polytechnic Institute of Brooklyn, Princeton University, Queen's University, Reed College, Rutgers University, St. Joseph's College (Connecticut), St. Joseph's College for Women (Brooklyn), Stanford University, State Teachers College (Nebraska), Swarthmore College, University of Arkansas, University of British Columbia, University of Buffalo, University of California (Berkeley), University of Detroit, University of Illinois, University of Miami (Florida), University of Minnesota, University of North Carolina, University of Toronto, Ursinus College, Yale University.

The following additional colleges and universities entered individual contestants only: Brown University, Case Institute of Technology, Central College, Clark University, Clemson Agricultural College, Haverford College, Iowa State College, Loyola University (New Orleans), Pomona College, Purdue University, Temple University, University of Colorado, University of Kentucky, Wayne University.

A total of 209 undergraduates, representing 56 institutions, took part in the Competition.

Participants in the Competition were given the following lists of problems:

Part I. Three Hours

(Answer the questions in any order and by any method. Show all of your work in logical sequence and indicate your answers clearly. No tables or other books may be used. Use the right hand pages of your examination booklet for your solutions, use the left hand pages for scratch work. Cross out any work which you do not wish to have considered. Partial credit may be given on a question, even when the solution is not completed.)

Omit one question. You must indicate which question is omitted.

1. Show that the determinant:

$$\begin{vmatrix} 0 & a & b & c \\ -a & 0 & d & e \\ -b & -d & 0 & f \\ -c & -e & -f & 0 \end{vmatrix}$$

is non-negative, if its elements a, b, c, etc., are real.

2. In the plane, what is the locus of points the sum of the squares of whose distances from n fixed points is a constant? What restrictions, stated in geometric terms, must be put on the constant so that the locus is non-null?

3. Find the sum to infinity of the series:

$$1 - \frac{1}{4} + \frac{1}{7} - \frac{1}{10} + \cdots + \frac{(-1)^{n+1}}{3n-2} + \cdots .$$

4. Trace the curve whose equation is:

$$y^4 - x^4 - 96y^2 + 100x^2 = 0.$$

5. Consider in the plane the network of points having integral coordinates. For lines having rational slope show that:
   (a) the line passes through no points of the network or through infinitely many.
   (b) there exists for each line a positive number d having the property that no point of the network, except such as may be on the line, is closer to the line than the distance d.
6. Determine the position of a normal chord of a parabola such that it cuts off of the parabola a segment of minimum area.
7. Show that if the series $a_1+a_2+a_3+ \cdots +a_n+ \cdots$ converges, then the series $a_1+a_2/2+a_3/3+ \cdots +a_n/n+ \cdots$ converges also.

PART II. THREE HOURS

(Answer the questions in any order and by any method. Show all of your work in logical sequence and indicate your answers clearly. No tables or other books may be used. Use the right hand pages of your examination booklet for your solutions, use the left hand pages for scratch work. Cross out any work which you do not wish to have considered. Partial credit may be given on a question, even when the solution is not completed.)

Omit one question. You must indicate which question is omitted.

1. Find the condition that the functions $M(x, y)$ and $N(x, y)$ must satisfy, in order that the differential equation $Mdx+Ndy=0$ shall have an integrating factor of the form $f(xy)$. You may assume that M and N have continuous partial derivatives of all orders.

2. Two functions of x are differentiable, and not identically zero. Find an example of two such functions having the property that the derivative of their quotient is the quotient of their derivatives.

3. Show that if x is positive, then

$$\log_e (1 + 1/x) > 1/(1 + x).$$

4. Investigate in any way which yields significant results, the existence, in the plane, of the configuration consisting of an ellipse simultaneously tangent to four distinct concentric circles.

5. A plane through the center of a torus is tangent to the torus. Prove that the intersection of the plane and the torus consists of two circles.

6. Assuming that all the roots of the cubic equation $x^3+ax^2+bx+c=0$ are real, show that the difference between the greatest and the least roots is not less than $(a^2-3b)^{1/2}$ nor greater than $2(a^2-3b)^{1/2}/3^{1/2}$.

7. Find the volume of the four-dimensional hypersphere $x^2+y^2+z^2+t^2=r^2$, and also the hypervolume of its interior $x^2+y^2+z^2+t^2<r^2$.

EDITED BY F. A. FICKEN, *New York University*

Material for this department should be sent to F. A. Ficken, Institute for Mathematics and Mechanics, New York University, 45 Fourth Ave., New York 3, New York.

A SIMPLIFIED TECHNIQUE FOR A TSCHIRNHAUS TRANSFORMATION

H. S. THURSTON, University of Alabama

1. Introduction. If ρ is a root of an irreducible equation $f(x)=0$ with coefficients in a field $\mathfrak{F}$, and η, an integral rational function of ρ, is a primitive number in the field $\mathfrak{F}(\rho)$, then ρ can be expressed as an integral rational function of η with coefficients in $\mathfrak{F}$. The classical proof* of this theorem involves the use both of symmetric function theory and the irreducible equation with coefficients in $\mathfrak{F}$ satisfied by η, and expresses ρ as a rational but nonintegral function of η. The purpose of this note is to demonstrate a proof of the theorem which expresses ρ directly as a polynomial in η with coefficients in $\mathfrak{F}$. The method introduces a function of the coefficients of both η and $f(x)$, whose vanishing is a necessary and sufficient condition that η be imprimitive.

Although the theory is applicable to any field, it will be understood in what follows that $\mathfrak{F}$ represents the field of rational numbers and that all coefficients are in this field.

2. The Tschirnhaus transformation. Let ρ be a root of the irreducible equation $f(x)=\sum_{i=0}^{n} c_i x^i=0$. Let $\eta=\sum_{j=0}^{n-1} a_{1j}\rho^j$ be primitive and express successive powers of η in the form

$$\eta^i = \sum_{j=0}^{n-1} a_{ij}\rho^j \qquad (i = 1, 2, \cdots, n-1). \tag{1}$$

Let B denote the value of the determinant of the coefficients of $\rho, \rho^2, \cdots, \rho^{n-1}$, in the system of equations (1). On multiplying these equations respectively by A_{i1}, the cofactors of the elements of the first column of the matrix (a_{ij}), we obtain an equation

$$\sum_{i=1}^{n-1} A_{i1}\eta^i = A + B\rho \tag{2}$$

where A and B are rational. Since η, being primitive, satisfies no equation of degree less than n with rational coefficients, B must be different from zero. We can solve (2) for ρ obtaining the desired result.

3. Imprimitive numbers. It will be observed that the method of the preceding section determines whether a given number η is primitive or not. If $B=0$, η satisfies an equation of degree less than n and hence is imprimitive. Conversely, if η is imprimitive, satisfying an equation of degree $k<n$, then B must be zero

* See, for instance, C. C. MacDuffee, Introduction to Abstract Algebra (1940), p. 87.

for if not we could solve for ρ as a polynomial in η as above. Then ρ, being in the field $\mathfrak{F}(\eta)$, will satisfy an equation of degree $k<n$, contrary to the hypothesis. We may therefore state the following theorem.

THEOREM. *A necessary and sufficient condition that η be imprimitive is that* $B=0$.

For the reduced cubic $f(x)=x^3+px+q$ and $\eta=a+b\rho+c\rho^2$, we find that $B=b^3+pbc^2+qc^3=c^3f(b/c)$. If $B=0$, then since for b and c rational and c not zero, $f(b/c)$ cannot be zero, we must have $b=c=0$. Hence, in a cubic field, the only imprimitive numbers are the rationals, as is well known from other considerations.

For $n\geqq 4$ and the general number η, both the computation and the expression for B assume formidable proportions. Some special cases may be of interest. For example, if $f(x)=x^4+px^2+qx+r$ and $\eta=a+b\rho+c\rho^2$, we find that

$$B = b^6 + 2pc^2b^4 + (p^2c^4 - 4rc^4)b^2 - q^2c^6. \tag{3}$$

Since, to be imprimitive, η must satisfy an equation of degree two when $n=4$, the coefficient of ρ^3 in η^2 must be zero. This coefficient is $2bc$. The conditions $bc=0$, $B=0$ lead to the conclusion that η is imprimitive only when $b=c=0$ or when $b=q=0$.

On the other hand, if $B=0$, and η is found to satisfy an equation of degree three but not of degree two, we must conclude that $f(x)$ is not irreducible. For example, if $f(x)=x^4-12x-5$ and $\eta=2+2\rho+\rho^2$, then B, given by (3), is zero and we find that $\eta^3-11\eta^2-25\eta+51=0$. But $x^3-11x^2-25x+51$ has factors $x+3$ and $x^2-14x+17$ neither of which vanish for $x=\eta$. Then $f(x)$ is reducible and is, in fact, identical with $(x^2+2x+5)(x^2-2x-1)$.

A CHARACTERISTIC PROPERTY OF THE CIRCLE IN THE MINKOWSKI PLANE

F. A. VALENTINE, University of California, Los Angeles

1. Introduction. Let M_2 be a two-dimensional Minkowski space. The unit sphere in M_2 is a centrally symmetric convex set having interior points [2, p. 23]. By a circle of radius r and center a we mean the set of points $x\in M_2$ for which the distance function satisfies $\|x, a\|\leqq r$. The circle having the segment xy as a diameter which passes through its center of symmetry is denoted by $K(x,y)$, and the boundary of $K(x, y)$ by $A(x, y)$. A semicircular arc of $A(x, y)$ having xy as a diameter is denoted by $C(x, y)$. If we wish to discuss both semicircular arcs of $A(x, y)$ having xy as a diameter, we do so by means of superscripts. The arc $C(x, y)$ always lies in one of the closed half-planes determined by the line $L(x, y)$ containing xy. The reader should realize that semicircular arcs $C(x, y)$ vary in shape in accordance with the direction of xy. A point is called an extreme point of $A(x, y)$ if it is *not* an interior point of a line segment of $A(x, y)$.

A set $S\subset M_2$ is said to be arcwise convex if each pair of points x and y in S can be joined by a convex arc $P(x, y)\subset S$. If $P(x, y)=xy$, then S is convex. If $P(x, y)$ is a polygonal arc containing at most two segments, then S is called an L set [4]. We are interested here in investigating the situation when $P(x, y)=C(x, y)$, a semicircular arc. The following theorem has interest from a pedagogical as well as from a mathematical point of view.

THEOREM 1. *A necessary and sufficient condition that a bounded closed set $S\subset M_2$ be a Minkowski circle is that for each pair of points $x\in S$, $y\in S$, there exists a semicircular arc $C(x, y)$ in S having x and y as endpoints.*

Proof: In order to prove the sufficiency, the following remark is helpful.

Remark A. There exist two points $a\in S$, $b\in S$ such that $\|a, b\|=$diameter of S, and such that a and b are extreme points of $A(a, b)$.

To prove this let the Minkowski diameter of S be D, and let M denote the collection of point pairs $[(x, y)]$, $x\in S$, $y\in S$ such that the Minkowski distance $\|x, y\|$ satisfies $\|x, y\|=D$. Let $|x, y|$ denote the Euclidean distance determined by x and y. Since S is bounded and closed, there must exist a point pair $(a, b)\in M$ which has a maximum Euclidean distance $|a, b|$ relative to the set M. The points a and b must be extreme points of $A(a, b)$; for if this were not so, there would exist two points $u\in C(a, b)\subset S$, $v\in C(a, b)\subset S$ such that $|u, v|>|a, b|$, $\|u, v\|=D$. This proves Remark A. The reader should realize that any two points in S which have a maximum Euclidean distance in S do not, in general, have a maximum Minkowski distance in S. It is also possible to prove Remark A by an argument which does not refer to the Euclidean distance; however, it is considerably longer, and is therefore not included.

To prove Theorem 1, consider the points a and b in Remark A. Let $R(a, b)$ be the closed half-plane containing the arc $C(a, b)\subset S$. Let $L(a)$ and $L(b)$ be two parallel lines of support to $K(a, b)$ at a and b respectively. Since a and b are extreme points of $A(a, b)$, we have either $C(a, b)\cdot L(a)=a$, or $C(a, b)\cdot L(b)=b$, or both. Suppose $C(a, b)\cdot L(a)=a$. Choose a sequence of points $a_i\in C(a, b)$ so that $a_i\to a(a_i\neq a)$ as $i\to\infty$. Let $R(a_i, b)$ be the closed half-plane which is bounded by $L(a_i, b)$, and which contains a. Since $a_i\cdot L(a)=0$, and since the diameter of S is equal to $\|a, b\|$, it follows that the arcs $C(a_i, b)\subset S$ are such that $C(a_i, b)\subset R(a_i, b)$. Hence $C(a_i, b)\to C^*(a, b)$, a semicircular arc in S, since S is closed. Moreover, $C(a, b)+C^*(a, b)=A(a, b)\subset S$. Since the diameter of S is equal to $\|a, b\|$, we have shown that the outer boundary of S is the circular circumference $A(a, b)$.

To complete the proof we merely need to show that S is simply connected. Let θ be the center of $K(a, b)$, and suppose $K(a,b)-S\neq 0$. Choose a point $p\in K(a, b)-S$. Let $L(\theta)$ be a line through θ which intersects $A(a, b)$ at points u and v at which unique lines of support $T(u)$ and $T(v)$ to $K(a, b)$ exist. Let $R(u, v)$ be a closed half-plane bounded by $L(\theta)$ and containing p. Choose $r\in K(a, b)\cdot R(u, v)$ so that a line of support $L(r)$ to $K(a, b)$ at r exists which is parallel to $L(\theta)$. By a continuity argument, there exists a point $c\in pr$ such that the line $L(c)$

through c parallel to $L(\theta)$ has the following property. There exist points $s \in L(c) \cdot A(a, b)$, $t \in L(c) \cdot A(a, b)$ such that $\|m, p\| = \|m, s\| = \|m, t\|$, where m is the midpoint of ts. Hence $c \neq p$.

Case 1. If $s \in T(u)$, $t \in T(v)$, then the circle $K(s, t)$ has the same diameter as $K(u, v)$. Hence any semicircular arc $C(s, t)$ either contains $p \notin S$, or it intersects the complement of $K(a, b)$. Hence in this case no semicircular arc exists in S joining s and t.

Case 2. We can suppose, without loss of generality, that $s \notin T(u)$, and that s is nearer to $L(c) \cdot T(u)$ than is t. Since $T(u)$ is a unique line of support to $K(a, b)$ at u, each line of support to $K(a, b)$ at s must intersect $T(u)$ at a point $q \neq u$. Since st is parallel to uv, there is a unique line of support to $K(s, t)$ at s, which is moreover parallel to $T(u)$. Hence the semicircular arc $C(s, t)$ not containing p intersects the complement of $K(a, b)$. Hence no semicircular arc in S exists joining s and t.

Thus S is simply connected, and, by the previous results, we have shown that $S = K(a, b)$. This completes the sufficiency proof. The proof of the necessity is simple, and will be omitted.

2. Generalizations. Theorem 1 does not hold in higher dimensions. For instance, the *solid hemisphere in* E_3, the three-dimensional Euclidean space, is a *counterexample*. Each pair of points in the hemisphere can be joined by a semicircular arc lying in the hemisphere. It would be of interest to determine in E_3 all of the sets which are arcwise convex via semicircular arcs. If we replace semicircular arcs by hemispherical surfaces, then we can generalize Theorem 1 as follows.

THEOREM 2. *A necessary and sufficient condition that a bounded closed set* $S \subset M_n$ *be a sphere is that for each pair of points* $x \in S$, $y \in S$ *there exists an* $(n-1)$-*dimensional hemispherical surface in* S *having* xy *as a diameter through its center.**

For the sufficiency, let (a, b) be a pair of points $a \in S$, $b \in S$ whose Minkowski distance satisfies $\|a, b\| = D$, where D is the diameter of S, such that (a, b) has a maximum Euclidean distance $|a, b|$ among the set of pairs $[(x, y)]$, $x \in S$, $y \in S$ for which $\|x, y\| = D$. Let P be a two-dimensional plane containing the line $L(a, b)$. The two-dimensional plane P_0 which is parallel to P, and which passes through the center of the unit sphere U, intersects U in a two-dimensional Minkowski circle $P_0 \cdot U$. Theorem 2 follows from Theorem 1 since each two-dimensional plane section $P \cdot S$ containing a and b is a circle which is similar to $P_0 \cdot U$, and which has ab as a diameter through its center.

If each pair of points of a set S can be joined by a *circular* arc in S, then S belongs to quite a large class of sets. Such a set need not be simply connected. The circumference of a circle is an example. Otto Haupt has investigated sets which he calls "Kreiskonvex" [3]. In such a set S for every *three* points in S

* The center of the sphere determined by a hemispherical surface is also called the center of the hemispherical surface.

there exists a circular arc or a segment in S which contains the three points. Haupt has given an interesting classification of such sets. It would also be of interest to classify circularly arcwise convex sets, but this appears to be a difficult problem.

References

1. E. Blanc, Les ensembles surconvexes plans, Ann. Sci. École Norm. Sup. (3) vol. 60 (1943), pp. 215–246.

2. Bonnesen and Fenchel, Theorie der Konvexen Körper, Chelsea Publishing Co., 1948.

3. Otto Haupt, Über eine Kennzeichnung der Kugel, Jber. Deutsch. Math. Verein, vol. 50 (1940), pp. 113–120.

4. Alfred Horn and F. A. Valentine, Some properties of L sets in the plane, Duke Mathematical Journal, vol. 16 (1949), pp. 131–140.

5. A. E. Mayer, Eine Überkonvexität, Math. Zeit. vol. 39 (1935), pp. 511–531.

6. F. A. Valentine, Arcwise convex sets, Proc. Amer. Math. Soc., vol. 2 (1951) pp. 159–165.

CLASSROOM NOTES

Edited by C. B. ALLENDOERFER, University of Washington

All material for this department should be sent to G. B. Thomas, Department of Mathematics, Massachusetts Institute of Technology, Cambridge, 39, Mass.

A THEOREM CONCERNING EXACT DIFFERENTIAL EQUATIONS

R. S. UNDERWOOD, Texas Technological College

The purpose of this note is to call attention to: (a), the simple but unused (at least in most textbooks) generalization of the method of solving exact differential equations in two variables based upon the evaluation of a line integral; and (b) a theorem providing a second method which also applies to the case of n variables, and of which Euler's theorem is a special case.

It is known* that, if

$$Mdx + Ndy = 0 \tag{1}$$

is an exact differential equation whose solution is $F=C$, we may find F by evaluating the line integral $\int_{(a,b)}^{(x,y)}(Mdx+Ndy)$ along any convenient path and from any convenient starting point (a, b). The latter is usually best taken at the origin unless M or N is undefined there; but in any case the effect of a change in (a, b) is merely to change the additive constant in F. According to the writer's experience, this method is usually the shortest and simplest one for a given problem, especially when it must be used in generalized form to deal with more than two variables.

* Woods: Advanced Calculus, Revised Edition, page 186.

The generalization is direct and obvious. For example, when three variables are involved we may integrate along the broken line path from (0, 0, 0) to $(x, 0, 0)$ to $(x, y, 0)$ to (x, y, z), or as will be better on occasion, along some different but similar path such as one involving, say, the point $(0, y, 0)$. For any specified number of variables the path of integration may be actually drawn if one uses the writer's "wheel-spoke" system of axes.† Of course the n-space concept yields the same result, aside from the picture; but the latter is useful to students in visualizing different paths, each with its separate integration pattern and special order of brevity or difficulty.

Again, we may integrate along the path from the origin to the point, $P(x_1, x_2, \cdots, x_n)$, by use of the parametric equations $y_i = x_i t$ $(i = 1, 2, \cdots, n)$, where the coordinates x_i of P are treated as constants. Evidently the coordinate y_i of a point on the path ranges from 0 to x_i as t ranges from 0 to 1. (On the n-axes plane of the writer's system† this path is a straight line.) Then, denoting $\partial F/\partial y_i$ by F_i, we have

$$\sum_{i=1}^{n} \int_0^{x_i} F_i(y_1, y_2, \cdots, y_n) dy_i = \int_0^1 \sum_{i=1}^{n} F_i(x_1 t, x_2 t, \cdots, x_n t) x_i dt.$$

The theorem below follows.

THEOREM: *If $\sum_{i=0}^{n} F_i(x_1, x_2, \cdots, x_n) dx_i$ is the exact differential of a function F, and if the F_i are defined at the origin, then we may find F, aside from the arbitrary constant, by replacing x_i and dx_i by $x_i t$ and $x_i dt$ respectively, and then evaluating the definite integral in t between the limits 0 and 1, the x_i being treated as constants.*

COROLLARY: *If the F_i of the theorem are homogeneous, of degree k, and defined at the origin, then*

$$F = \frac{\sum_{i=1}^{n} x_i F_i}{k + 1}.$$

It should be noted that this potent corollary is merely Euler's theorem "in reverse" as compared with its usual presentation. The typical text book approach is illustrated by the following problem: "Verify Euler's theorem for $f(x, y, z) = x^2 y + xy^2 + 2xyz$." The student, noting that $(2xy + y^2 + 2yz)x + (x^2 + 2xy + 2xz)y + (2xy)z = 3f$, has duly verified what seems to be an interesting but pointless result. But probably both he and his teacher have failed to notice that if he *starts with* the differential equation $dF \equiv (2xy + y^2 + 2yz)dx + (x^2 + 2xy + 2xz)dy + (2xy)dz = 0$, checks its exactness by the usual test, and observes that the quantities in parentheses are homogeneous and of degree 2, then he need only replace dx, dy, and dz by x, y, and z respectively to get $3F$ at once and thus solve the equation. While of course the relative efficiency of the

† Underwood: An Analytic Geometry for N Variables, this MONTHLY, May, 1945.

method increases with the number of variables, in the example cited the result is obtained about as quickly by the line integral method, which yields $F=\int_0^x 0dx + \int_0^y (x^2+2xy)dy$ (with x constant) $+\int_0^z 2xydz$ (x and y constant) $=x^2y+xy^2+2xyz$.

Editorial Note. A general form of Underwood's theorem has been proved by H. Cartan in connection with exterior differential forms. For the simple case of $Mdx+Ndy$, let $F(x, y)=\int_0^1[M(xt, yt)x+N(xt, yt)y]dt$. Then Cartan's result is:

$$\frac{\partial F}{\partial x} = M(x, y) + \int_0^1 \left[\frac{\partial N(xt, yt)}{\partial(xt)} - \frac{\partial M(xt, yt)}{\partial(yt)}\right] ytdt;$$

$$\frac{\partial F}{\partial y} = N(x, y) - \int_0^1 \left[\frac{\partial N(xt, yt)}{\partial(xt)} - \frac{\partial M(xt, yt)}{\partial(yt)}\right] xtdt.$$

This proves Underwood's theorem without resorting to other theorems from advanced calculus. It also proves that $Mdx+Ndy$ is exact whenever

$$\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} = 0;$$

and thus at a single stroke one can prove all the important facts about $Mdx+Ndy$.

This method can be extended to more dimensions and to differential forms of higher degree. When applied to the form: $Pdx+Qdy+Rdz$, it proves the theorem: "If a vector has zero curl, then it is a gradient." Application of a similar technique proves the theorem: "If a vector has zero divergence, then it is the curl of another vector." The trick is to set:

$$X(x, y, z) = \int_0^1 [Q(xt, yt, zt)zt - R(xt, yt, zt)yt]dt$$

$$Y(x, y, z) = \int_0^1 [R(xt, yt, zt)xt - P(xt, yt, zt)zt]dt$$

$$Z(x, y, z) = \int_0^1 [P(xt, yt, zt)yt - Q(xt, yt, zt)xt]dt$$

Then curl $(X, Y, Z)=(P, Q, R)$ provided that div $(P, Q, R)=0$.

THE BETA FUNCTION

M. R. Spiegel, Rensselaer Polytechnic Institute

In all of the books on Advanced Calculus which the author has seen, the evaluation of the Beta function

$$(1) \qquad \int_0^1 x^{m-1}(1-x)^{n-1}dx = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)} \qquad m, n > 0$$

is performed with the aid of a suitable double integral. If the Laplace transform is used, this evaluation is immediate.

Consider

$$F(t, m, n) = \int_0^t x^{m-1}(t-x)^{n-1}dx. \tag{2}$$

Taking the Laplace transform with respect to t we have

$$\mathcal{L}_t F(t, m, n) = \mathcal{L}_t(t^{m-1})\mathcal{L}_t(t^{n-1}) \tag{3}$$

by the convolution theorem.

Since

$$\mathcal{L}_t(t^p) = \frac{\Gamma(p+1)}{s^{p+1}}$$

(3) becomes

$$\mathcal{L}_t F(t, m, n) = \frac{\Gamma(m)\Gamma(n)}{s^{m+n}}. \tag{4}$$

Inverting now with respect to t we have

$$F(t, m, n) = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)} t^{m+n-1}$$

since

$$\mathcal{L}^{-1}\left(\frac{1}{s^p}\right) = \frac{t^{p-1}}{\Gamma(p)}.$$

If, now, we put $t=1$ the result (1) follows.

ELEMENTARY PROBLEMS AND SOLUTIONS

EDITED BY HOWARD EVES, Oregon State College

Send all communications concerning Elementary Problems and Solutions to Howard Eves, Mathematics Department, Oregon State College, Corvallis, Oregon. This department welcomes problems believed to be new, and demanding no tools beyond those ordinarily furnished in the first two years of college mathematics. To facilitate their consideration, solutions should be submitted on separate, signed sheets, within three months after publication of problems.

PROBLEMS FOR SOLUTION

E 976. *Proposed by R. D. Stalley, University of Arizona*

What is the least area of a right triangle which can be placed on a rectangular coordinate system so that no side is parallel to a coordinate axis and so that the coordinates of its vertices and the lengths of its sides are integral?

E 977. *Proposed by José Gallego-Diaz, Madrid, Spain*

Let OA be a radius of a circle with center O. Produce OA its own length to B. Through A draw a horizontal line and on it mark off, in each direction from A, $AM=AN=OA$. If OB and MN are taken as the equal conjugate diameters of an ellipse, find the loci of the vertices and the foci of this ellipse as the radius OA varies. Also find the envelope of the major axis of the ellipse as OA varies.

E 978. *Proposed by J. M. Kingston, University of Washington*

If $[x]$ denotes the integral part of x and if j is any positive integer, show that

$$\sum_{i=0}^{[j/3]} [(j-3i)/2] = [(j^2+2j+4)/12].$$

E 979. *Proposed by E. S. Keeping, University of Alberta*

A student is asked on a test to match n given historical events against n different given dates, assigning a date to each event. He is sure of k dates, but is completely ignorant of the others and assigns them at random. Show that a fair method of scoring would be to deduct 1 from the number of correct matches.

SOLUTIONS

A Quadratic Diophantine Equation

E 941 [1950, 686]. *Proposed by P. A. Piza, San Juan, Puerto Rico*

Find positive integers a, b, c such that $a^2+b^2+3c^2=(a+b+c)^2$.

Solution by E. P. Starke, Rutgers University. Given any two relatively prime positive integers x, y, such that $x<y<2x$, and an arbitrary positive integer k, then the complete solution is

(1) $$c = kxy, \qquad a = kx(2x-y), \qquad b = ky(y-x),$$

where the values of a and b may be interchanged.

That these numbers satisfy is seen immediately upon substituting them in the given equation. Conversely, if a, b, c are positive integers which satisfy the given equation, or its equivalent

$$(2) \qquad 2c^2 = (c + a)(c + b),$$

then there exist integers x, y such that (1) is satisfied. To show this, suppose that a, b, c have a greatest common divisor k. Then $a' = a/k$, $b' = b/k$, $c' = c/k$ have no common factor and satisfy

$$(2') \qquad 2c'^2 = (c' + a')(c' + b').$$

Then $c' + a'$ and $c' + b'$ are relatively prime and, therefore, from (2′),

$$(3) \qquad c' + a' = 2x^2, \qquad c' + b' = y^2, \qquad c' = xy,$$

or the same with a' and b' interchanged. Now, from (2′) and (3), we have $c' < y^2 < 2c'$, whence $x < y < 2x$, which establishes the proposition.

Also solved by Furio Alberti and N. C. Scholomiti (jointly), R. V. Andree, Norman Anning, W. E. Baxter, Alan Berndt, Daniel Block, J. L. Botsford, W. G. Brady, D. H. Browne, R. L. Caskey, P. L. Chessin, F. L. Dennis, F. J. Duarte, C. C. Foster, Jr., R. E. Gettig, A. L. Gilmore, Jr., Vern Hoggatt, M. S. Klamkin, Joseph Klein, Sam Kravitz, Max LeLeiko, Roger Lessard, Theodore Lindquist, David Mandelbaum, D. V. Mardle and R. J. Taylor (jointly), Fred Marer, D. C. B. Marsh, Eric Mickalup, Leo Moser, Prasert Na Nagara, C. S. Ogilvy, M. Paul, L. A. Ringenberg, J. E. Sanders, C. M. Sandwick, Jr., J. W. Sawyer, O. E. Stanaitis, D. Strebe, Elijah Swift, R. S. Underwood, N. D. Vlachos, J. E. Weidlich, and the proposer. Many of these solutions were incomplete.

Maximinimal Configuration of Five Points on a Sphere

E 946 [1951, 36]. *Proposed by C. S. Ogilvy, Columbia University*

It is well known that the distribution of six points on the surface of a given sphere which makes the least distance between any pair a maximum is that of the vertices of the regular octahedron. What is the corresponding distribution for five points?

Solution by Leo Moser, Texas Technological College. There is no unique distribution which maximizes the minimum distance between five points on a sphere. A set of maximinimal configurations is given by taking two points at the north and south poles and the remaining three on the equator, subject only to the condition that the distance between any two of these be at least 90°.

To show that we cannot have each distance greater than 90° note that if this be the case then two points will eliminate the possibility of the remaining three lying in two hemispheres whose poles are more than 90° apart. Thus these three points must lie in a lune of angle less than 90°, or at least two of them must lie

in or on one half this lune, *i.e.*, in or on a triangle having two sides each equal to 90° and the third side less than 90°. But this is impossible.

Also solved by the proposer.

It is interesting that five points can be spaced no farther apart than six.

A Locus Problem

E 947 [1951, 36]. *Proposed by Victor Thébault, Tennie, Sarthe, France*

Find the locus of a point in the plane of a given triangle such that the pedal triangle of its isogonal conjugate is right angled.

I. *Solution by K. W. Crain, Purdue University.* Let the given triangle be $A_1A_2A_3$, the considered point P', and its isogonal conjugate P, whose pedal triangle is $P_1P_2P_3$. Since A_iP' is perpendicular to P_jP_k (*cf.* Johnson, Modern Geometry, sec. 237, page 156), the requirement that the pedal triangle be right angled at P_k is equivalent to making the angle $A_iP'A_j=90°$. Thus the locus of P' consists of the three circles having the sides of the given triangle as diameters.

II. *Solution by Roscoe Woods, State University of Iowa.* In this solution the pedal triangle of a point P in the plane of the given triangle ABC is defined as the triangle LMN, where L, M, N are, respectively, the traces of AP, BP, CP on the sides BC, CA, AB (*cf.* Altshiller-Court, College Geometry, page 128).

If the trilinear (normal) coordinates of P are (x, y, z), referred to triangle ABC as reference triangle, the coordinates of P', the isogonal conjugate of P, are (yz, zx, xy). The coordinates of the vertices of the pedal triangle $L'M'N'$ of P' are readily found to be: $L'(0, z, y)$, $M'(z, 0, x)$, $N'(y, x, 0)$. The condition that triangle $L'M'N'$ be right angled at vertex L', say, is found to be

$$x^2 - y^2 - z^2 - 2yz \cos A = 0. \tag{1}$$

Hence if the point P lies on the conic whose equation is (1), the pedal triangle $L'M'N'$ is right angled at the vertex L'.

The locus of (1) is never an ellipse. This conic cuts the side AC in the same points as do the internal and external bisectors of angle B. A similar statement holds for side AB. It is readily seen that the points $(a, -b, c)$ and $(a, b, -c)$ lie on the conic. These points are the intersections of the tangents to the circle ABC at B and C with the tangent at A. The center of this conic is at the point $(-\sin^2 A, \cos C, \cos B)$, which lies on the altitude h_a.

The locus of (1) is a parabola when $A=90°$. Its focus is the vertex A, its directrix is the side BC, and its point of contact with the ideal line is the point $(-a, b, c)$. The vertex of the parabola is the midpoint (a, b, c) of h_a, and is the symmedian point of the triangle ABC.

The locus of (1) is a rectangular hyperbola when $A=45°$ or 135°. The centers of these two hyperbolas lie on the altitude h_a and have the coordinates $(\pm 1, 2\cos C, 2\cos B)$.

Also solved by A. Sisk (as in II) and the proposer (as in I).

The Inverse of a Certain Matrix

E 948 [1951, 36]. *Proposed by Roy Dubisch, Fresno State College*

Find the inverse of the general even ordered skew-symmetric matrix all of whose elements above the principal diagonal are equal to 1.

I. *Solution by C. S. Ogilvy, Columbia University.* If a skew-symmetric matrix A has elements $a_{ij}=1$ for all $j>i$, then $a_{ij}=-1$ for all $j<i$, and $a_{ii}=0$. Form the matrix $C=AB$, where B has elements b_{ij} such that $b_{ij}=(-1)^{i+j}$ for all $j>i$, $b_{ij}=(-1)^{i+j-1}$ for all $j<i$, and $b_{ii}=0$. Then C has elements c_{ij} such that

$$c_{ij} = (n/2-1)(1)+(n/2-1)(-1)+(2)(0) = 0, \qquad \text{all } i \neq j,$$
$$c_{ii} = (n/2)(1)+(n/2-1)(-1)+(1)(0) = 1.$$

That is, $C=I$, or $B=A^{-1}$.

II. *Solution by D. R. Morrison, Tulane University.* Let $2n$ be the order of the given matrix A and consider the matrix X, also of order $2n$,

$$X = \begin{bmatrix} 0 & 1 & 0 & 0\cdots 0 \\ 0 & 0 & 1 & 0\cdots 0 \\ 0 & 0 & 0 & 1\cdots 0 \\ \cdot & \cdot & \cdot & \cdot\cdot\cdot\cdot\cdot \\ 0 & 0 & 0 & 0\cdots 1 \\ -1 & 0 & 0 & 0\cdots 0 \end{bmatrix}.$$

One may easily verify that

$$X^{2n} = -I,$$

where I is the identity matrix of order $2n$, and

$$A = f(X) = X + X^2 + X^3 + \cdots + X^{2n-1}.$$

It follows that

$$(I - X)f(X) = X - X^{2n} = I + X.$$

Now $(I-X)$ has determinant 2, so it has an inverse, and

$$f(X) = (I+X)(I-X)^{-1}.$$

Also $(I+X)$ has determinant 2, so it has an inverse, and

$$[f(X)]^{-1} = (I-X)(I+X)^{-1} = f(-X).$$

Thus

$$A^{-1} = f(-X) = -X + X^2 - X^3 + \cdots - X^{2n-1}.$$

This is the skew-symmetric matrix of order $2n$ whose elements above the main

diagonal are alternately $+1$ and -1 in each row and column, with a -1 in the upper right-hand corner.

Also solved by E. V. Haynsworth, A. E. Livingston, Leo Moser, W. V. Parker, R. E. Shear, W. L. Stamey, O. E. Stanaitis, and the proposer.

Several solvers computed the inverse by the standard formula $A^{-1} = (\text{adj } A)/|A|$, or by the method of elementary transformations as explained, *e.g.*, in A. A. Albert, *A rule for computing the inverse of a matrix*, this MONTHLY, 1941, pp. 198–99. One solver found the inverse by the method given by R. V. Andree in *Computation of the inverse of a matrix*, this MONTHLY, 1951, pp. 87–92.

ADVANCED PROBLEMS AND SOLUTIONS

EDITED BY E. P. STARKE, Rutgers University

Send all communications concerning Advanced Problems and Solutions to E. P. Starke, Rutgers University, New Brunswick, New Jersey. All manuscripts should be typewritten, with double spacing and margins at least one inch wide. Problems containing results believed to be new or extensions of old results are especially sought. Proposers of problems should also enclose any solutions or information that will assist the editor. In general, problems in well known textbooks or results in readily accessible sources should not be proposed for this department.

PROBLEMS FOR SOLUTION

4448. *Proposed by Jekuthiel Ginsburg, Yeshiva University, New York City*

Solve in positive integers

$$z^4 = \frac{ax^2 + by^2}{a + b},$$

where a and b are given integers.

4449. *Proposed by D. J. Newman, New York University*

The series $\sum z^n/n$ converges everywhere on the unit circle except at $z = +1$. Can $\pm$ signs be introduced so that $\sum \pm z^n/n$ will converge everywhere on the unit circle with no exceptions?

4450. *Proposed by R. M. Redheffer, University of California at Los Angeles*

Let $g(x)$ be a polynomial of degree $\leqq 2n+1$. Show that a necessary and sufficient condition that the integral

$$\int e^{x^2} g(x)dx$$

is an elementary function is the vanishing of the determinant

$$\begin{vmatrix} g(0) & 1 & 0 & 0 \cdots & 0 & 0 \\ g^{(2)}(0) & 4 & 1 & 0 \cdots & 0 & 0 \\ g^{(4)}(0) & 0 & 8 & 1 \cdots & 0 & 0 \\ g^{(6)}(0) & 0 & 0 & 12 \cdots & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ g^{(2n-2)}(0) & 0 & 0 & 0 \cdots & 4n-4 & 1 \\ g^{(2n)}(0) & 0 & 0 & 0 \cdots & 0 & 4n \end{vmatrix}.$$

4451. *Proposed by Paul Erdös, University of Aberdeen, Scotland*

Let

$$f(z) = \prod_{j=1}^{n} (z - z_j), \qquad |z_j| = 1.$$

Prove that one can connect the origin with a point of the unit circle, so that everywhere on this path, except at the origin, $|f(z)| < 1$.

4452. *Proposed by Leo Moser, Texas Technological College*

Prove that

$$\sum_{r=1}^{n} \mu(r) \left[\frac{n}{r}\right] \{3 - 2\omega(r)\}^2 = 9 + 8\{\pi_1^*(n) + \pi_2^*(n)\},$$

where $\mu(r)$ is the Möbius function, $[n/r]$ denotes the largest integer not exceeding n/r, $\omega(r)$ is the number of distinct prime divisors of r (i.e., $\omega(1)=0$, $\omega(p_1^{a_1}p_2^{a_2} \cdots p_k^{a_k}) = k$), and $\Pi_k^*(n)$ is the number of integers t not exceeding n for which $\omega(t) = k$.

SOLUTIONS

Sequence of Integers No One of which Divides Another

4363 [1949, 557]. *Proposed by Paul Erdös, University of Aberdeen, Scotland*

Let $a_1 < a_2 < \cdots$ be an infinite sequence of positive upper density (i.e., $\underline{\lim}\, a_k/k < \infty$). Then there exists an infinite subsequence such that no element divides another. In fact, there exists an infinite subsequence $a_{i_1}, a_{i_2}, \cdots$ such that $\sum 1/a_{i_k} = \infty$ and no a_{i_k} divides any other.

Solution by the Proposer. First we show that there exists an infinite sequence $a_{i_1} < a_{i_2} < \cdots$ such that no one divides another. Put $\underline{\lim}\, a_k/k = 1/\alpha$, $0 < \alpha \leqq 1$. Then clearly α is the upper density of the a's. Let now a_{i_1} be the least $a > 4/\alpha$, a_{i_2} the least $a > 2a_{i_1}$ which is not a multiple of a_{i_1}, a_{i_3} the least $a > 2a_{i_2}$ which is not a multiple of a_{i_1} or a_{i_2}, and so on. Suppose, if possible, that this

process stops after k steps. Then every sufficiently large a is a multiple of one of the numbers $a_{i_1}, a_{i_2}, \cdots, a_{i_k}$. By definition $a_{i_j} > 2^j \cdot 4/\alpha$. Thus the density of the integers which are multiples of one of the numbers $a_{i_1}, a_{i_2}, \cdots, a_{i_k}$ is not greater than

$$\frac{1}{a_{i_1}} + \frac{1}{a_{i_2}} + \cdots + \frac{1}{a_{i_k}} < \frac{\alpha}{4} \sum_{j=1}^{\infty} \frac{1}{2^j} = \frac{\alpha}{2},$$

or the upper density of the a's is less than $\alpha/2$, a contradiction.

To prove the stronger result, denote by d_x the density of integers which have a divisor between x and $2x$. It has been shown that* as $x \to \infty$, $\lim d_x = 0$. Now we define a sequence $n_1 < n_2 < \cdots$ as follows:

$$d_{n_r} < \alpha/10^r, \tag{1}$$

$$\text{the number of } a\text{'s in } (n_r, 2n_r) \text{ is } > \tfrac{1}{2}\alpha n_r. \tag{2}$$

(This condition can be satisfied since the upper density of the a's is α.) Define our subsequence of the a's as follows: Consider the a's in $(n_r, 2n_r)$ which are not multiples of any integer in $(n_j, 2n_j)$, $j = 1, 2, \cdots, r-1$. Thus we obtain the sequence $a_{i_1}, a_{i_2}, \cdots$. Clearly no one of these divides any other. Also the number of a_{i_k}'s in $(n_r, 2n_r)$ is not less than

$$\frac{\alpha}{2} n_r - \sum \frac{\alpha n_r}{10^j} > \frac{\alpha}{4} n_r.$$

(The number of the a_{i_k}'s in $(n_r, 2n_r)$ is greater than the number of a's in $(n_r, 2n_r)$ minus the number of integers in $(n_r, 2n_r)$ which have a divisor in $(n_j, 2n_j)$, $j = 1, 2, \cdots, r-1$). Hence the upper density of the a's is >0 and thus $\sum 1/a_{i_k} = \infty$.

It is not hard to see that these results are the best possible. In other words if $f(x) \to \infty$ arbitrarily slowly there exists a sequence $a_1 < a_2 < \cdots$ such that for all x the number of a_i's $\leqq x$ exceeds $x/f(x)$, but there does not exist an infinite subsequence $a_{i_1} < a_{i_2} < \cdots$ no one of which divides another.

Regular Polygon

4364 [1949, 557]. *Proposed by Joseph Rosenbaum, The Milford School, Connecticut*

On the sides A_iA_{i+1} of an n-gon $A_1A_2 \cdots A_n$ as bases, isosceles triangles $A_iA_{i+1}B_i$ are constructed, either all exteriorly or all interiorly, with the vertex angle $B_i = 360°/n$. Prove

(a) If $A_1A_2 \cdots A_n$ is a projection of a regular n-gon, then $B_1B_2 \cdots B_n$ is regular.

(b) The problem of locating the points A_i when the points B_i are given is a porism.

---

* Erdös, Note on sequences of integers no one of which divides the other, *London, Math. Soc. Journal*, 1935.

Solution by J. B. Kelly, Institute for Advanced Study. (a) Let the vertices $A_1, \cdots, A_n$ lie in the complex plane with coördinates $z_1, \cdots, z_n$. Let the coördinates of $B_1, \cdots, B_n$ be $z_1', \cdots, z_n'$. Let $\phi = 2\pi/n$. Then clearly

$$z_{i+1} - z_i' = \omega(z_i' - z_i), \qquad i = 1, \cdots, n,$$

where $\omega = e^{i\phi}$. From this it follows that

$$z_i' = \frac{-\omega}{1-\omega} z_i + \frac{z_{i+1}}{1-\omega}, \qquad i = 1, \cdots, n, \tag{1}$$

in which $n+1$ is to be identified with 1, and so on.

If the n-gon $A_1A_2 \cdots A_n$ is the projection of $C_1 \cdots C_n$, where the coördinates of the vertices $C_1, \cdots, C_n$ in their plane are $z_1^*, \cdots, z_n^*$, then there exist constants p, q, r such that

$$z_i = pz_i^* + q\bar{z}_i^* + r, \tag{2}$$

If $C_1 \cdots C_n$ is a regular n-gon, then either

$$z_{i+1}^* - z_i^* = \omega(z_i^* - z_{i-1}^*) \tag{3}$$

or

$$z_{i+1}^* - z_i^* = \bar{\omega}(z_i^* - z_{i-1}^*). \tag{4}$$

Substituting (2) and (3) in (1) we find, after some simplifications,

$$z_{i+1}' - z_i' = \bar{\omega}(z_i' - z_{i-1}'), \tag{5}$$

while, substituting (2) and (4) in (1) gives the same as (5) except that $\bar{\omega}$ is replaced by ω. In either case $B_1 \cdots B_n$ is regular.

(b) The determinant of the system of linear equations (1) is

$$(-1)^n(\omega^n - 1)/(1-\omega)^n = 0.$$

Hence the problem of locating the A's when the B's are given has an infinite number of solutions, if any, and is a porism.

In (a) and (b) it has been assumed that $A_iA_{i+1}B_i$ was constructed exterior to the polygon, and that the vertices $A_1, \cdots, A_n$ were taken clockwise.

Limit of a Ratio of Sums

4378 [1950, 41]. *Proposed by H. W. Smith, Oklahoma Agricultural and Mechanical College*

Determine

$$\lim_{n\to\infty} \frac{\sum_{k=0}^{n} {}_nC_k/(2k+1)}{\sum_{k=0}^{n} {}_{n+1}C_k/(2k+1)},$$

where ${}_nC_k$ are the binomial coefficients.

Solution by Joshua Barlaz, Rutgers University. Let

$$I_n = \int_0^1 (1 + x^2)^n dx = \sum_{k=0}^{n} {}_nC_k/(2k + 1).$$

We shall prove

$$2^n/n < I_n < 2^n/(n - 1) \tag{1}$$

for $n \geqq 3$. By integration by parts we obtain

$$I_{n+1} = \frac{(2n + 2)I_n}{2n + 3} + \frac{2^{n+1}}{2n + 3}\,.$$

Now if (1) is true, we can write

$$\frac{(2n + 2)2^n}{n(2n + 3)} + \frac{2^{n+1}}{2n + 3} < \frac{(2n + 2)I_n}{2n + 3} + \frac{2^{n+1}}{2n + 3} < \frac{(2n + 2)2^n}{(n - 1)(2n + 3)} + \frac{2^{n+1}}{2n + 3},$$

$$\frac{2^{n+1}}{n + 1} < \frac{2^{n+1}(2n + 1)}{n(2n + 3)} < I_{n+1} < \frac{2^{n+1}(2n)}{(n - 1)(2n + 3)} \leqq \frac{2^{n+1}}{n}$$

provided $n \geqq 3$. But (1) is true for $n = 3$, whence by induction (1) is true for all $n \geqq 3$. From (1) it follows that I_n is asymptotic to $2^n/n$ and therefore $I_n/I_{n+1} \to \frac{1}{2}$ as $n \to \infty$ which is the desired result.

Also solved by D. F. Barrow, N. J. Fine, M. S. Klamkin, J. G. Millar, C. E. Reid, O. E. Stanaitis, Robert Steinberg, H. E. Stelson, and the proposer.

Editorial Note. Klamkin shows that the desired result follows directly from Polya and Szegö, *Aufgaben und Lehrsätze*, vol. 1, p. 78, theorem 199. By the same argument it is true that

$$\lim_{n\to\infty} I_{r,n}/I_{r,n+1} = \tfrac{1}{2}, \qquad I_{r,n} = \sum_{k=0}^{n} {}_nC_k/(rk + 1).$$

An Infinite Series of Successive Integrals

4382 [1950, 119]. *Proposed by E. P. Starke, Rutgers University*

Let $f_1(x)$ be Riemann integrable in the interval $0 \leqq x \leqq M$ and let

$$f_{n+1}(x) = \int_0^x f_n(x)dx, \qquad n = 1, 2, \cdots .$$

Show that

$$\phi(x) = \sum_{n=1}^{\infty} f_n(x)$$

is defined and continuous in the interval except perhaps at discontinuities of

RECENT PUBLICATIONS

EDITED BY E. P. VANCE, Oberlin College

All books for review should be sent directly to the editor of this department, American Mathematical Monthly 531 West 116th Street, New York 27, N. Y., and not to any of the other editors or officers of the Association.

The Main Stream of Mathematics. By Edna E. Kramer. Oxford University Press, New York. 1951. 321 pages.

The book was written, primarily as a popular exposition of the fundamental concepts of mathematics, their historical development, and their significance in the modern world. It will, however, prove of equal value to the student, and even to the teacher of mathematics. In broad sweeps it delineates the main features of the entire structure of mathematics, from the primitive attempts at counting to the emergence of transfinite numbers, from early Egyptian geometry to the modern theories of non-Euclidean and multi-dimensional geometry and the structure of relativistic space.

While the book does not purport to be a connected history of mathematics, the author manages to impress the reader with the essential continuity of mathematical ideas. She has quite an original way of passing, almost imperceptively, from a discussion of early mathematical concepts to their embodiment in some of the latest developments. In the first chapter, for example, she takes up the evolution of counting and the rise of systems of enumeration in the ancient world and immediately goes on to show the close connection of those early accomplishments with the new science of Cybernetics and the modern computing machine. Similarly, in the third chapter, *The Binding Energy of Thought*, she starts with a discussion of the meaning of algebra and the introduction of literal symbols by the French mathematician, Vieta. However, for illustrations of the use of algebraic symbols the author draws upon such topics, as: *Boolean Algebra*, Einstein's *Mass Energy Formula*, *Cosmic Rays*, etc. Finally, she shows how algebraic symbolism can be used in art by citing Professor Birkhoff's formula for aesthetic measure.

Among the best chapters of the book are: *Science and the Sweepstakes*, in which the author starts with the elementary illustration of tossing a coin and leads the reader through various applications of probability and statistics to the most advanced applications by Von Neuman and Morgenstern in Theory of Games; *From Alice to Einstein*, in which the modern concepts of the nature of mathematics are introduced by means of showing how geometry has changed under the influence of Hilbert and the non-Euclidean geometers; *The Paradise of Mathematicians*, which indicates the close connection between Zeno's paradoxes and the modern notion of the continuum as influenced by Cantor's transfinite numbers.

One may question, of course, how much of a real understanding the non-initiated—that mythical layman—for whom such books presumably are writ-

ten, can gain of such abstract concepts as: dense in itself, continuum, and so forth, by a concise definition, even if it is followed by an illustration. Every teacher knows how much even the advanced student has to struggle with these ideas when he meets with them for the first time. Such difficulties must arise in every attempt to make the abstractions of modern mathematics understandable to the non mathematician. However, the student of mathematics will read this book with profit and pleasure. He will delight in the warm personal touch with which the author introduces her short biographical sketches. The teacher of mathematics will find this book a most valuable addition to his reference material, to be assigned to his students for outside reading.

A. D. FLESHLER

Mathematics, Queen and Servant of Science. By E. T. Bell. McGraw-Hill Book Company, Inc., New York. 1951. 15+437 pages, $5.00.

This book is a thorough revision and an amplified integration of two popular volumes of mathematics, *The Queen of the Sciences*, 1931 and its sequel, *The Handmaiden of the Sciences*, 1937 by the author. Here is a fascinating account of selected topics from the huge accumulation available in the developments in pure and applied mathematics from the geometry of Euclid to the most recent findings in mathematical physics. Along with discussions in logic, rings, groups, transformations, topology, continuity, geometry, calculuses, infinity, probability, and others, the author introduces subjects such as the unified field theory (Einstein), Warings problem (Niven), Mersenne numbers (Lehmer), three-valued logic and quantum mechanics (Reichenbach and Post). Informally written and characterized by the author's provocative and stimulating style, mathematical ideas are presented and evaluated to suggest that mathematics is vigorously alive, is still growing, and is indispensable to an understanding of some sciences and technologies and for a deeper appreciation of the philosophy of science. This theme is the greatest virtue of the book.

Beyond this, the author's aims are modest. The sketch is not intended to be a substitute for a textbook in any of his chosen subjects. Some of these are presented in profuse detail. As an example, the chapter on "The Art Of Abstraction" is replete with postulates, terminology and notations from the logic of Tarski, through lattice theory to linear algebras including several other "structures" and abstractions. A splendid feature and a saving grace is the italic print of important terms. The reader may take what he likes and he must explore fuller accounts if he desires to know more. An appeal is made to the intuition of the reader in other chapters. This is inevitable because of the nature and scope of the topics, although (topology and integration for example) could be expanded or clarified. Again, although the sketch makes seemingly countless references to outstanding men and to hundreds of terms, it is not a history of mathematics. Only 202 of some 8000 mathematicians and physicists are actually mentioned.

This book should be valuable to many readers. The non-mathematician will enjoy the pervading spirit and vitality of living mathematics. The student will find new fields for fuller development by himself. The mathematical amateur will gain background and learn to appreciate techniques in research. The teacher will find stimulating material and methods for presentation. The philosopher and the specialist will find many topics of difference and disagreement in the authors overemphasis of some items and the lack of significant development of others.

F. G. GRAFF

Στοιχεια Θεωρητικης Γεωμετριας (*Elements of Theoretical Geometry*): 3 Vols. By N. Sakellariou. Athens, Pountza Book Company, 1950. Vol. I, 224 pages; Vol. II, 208 pages; Vol. III, 208 pages. 20,000 drachmas (about $1.35) per volume.

Elements of Theoretical Geometry is a comprehensive text-book in the field of Euclidean geometry, designed for general use in the upper grades of the Greek public gymnasia and for college preparation in the private schools.

Volume One is confined to the general topics contained in Books One and Two of Euclid's *Elements*. In addition to the ancient elementary concepts of Euclidean geometry, some advanced notions and modern extensions are introduced in the sections to which they apply. Examples of such material are the following: the orthocenter of a triangle; inscribed and circumscribed quadrilaterals; classical theorems concerning the Steiner-Miquel point, the Euler line, the nine-point circle, the Simson line, and the Brocard point.

Volume Two, continuing with the substance of Euclid's Books Three and Four, develops most completely the concepts of modern Euclidean geometry. It comprises such topics as the power of a point with respect to a circle, transversals, homothetic figures, poles and polars, inverses; the famous classical theorems of Clairaut, Stuart, Salmon, Compaignon, and Gergonne; the Lemoine point, antiparallels, transpolar triangles, the circle of Apollonius, the circles of Lemoine, Brocard, Tucker.

Volume Three, corresponding to Euclid's Books Five, Six and Seven, is a rather exhaustive work in the Euclidean geometry of three-space. A special distinction is the application to three-space of the concepts of the power of a point, homothetic figures, poles and polars, and so on.

The author maintains the same plan of presentation and technique throughout the three volumes. Definitions are systematically listed, and not carelessly scattered about. Proofs are given of all theorems except the simplest. Clear figures and illustrations are liberally employed. Exercises are select and abundant, but strong hints and even outlines of solution are given with each. Annexed to each chapter is a complete but concise summary.

A. G. FADELL

NEW BOOKS RECEIVED

Biometric Analysis, An Introduction. By A. E. Treloar. Minneapolis, Minn., Burgess Publishing Company, 1951. v+251 pp.

The New Physics. By Sir C. V. Raman. New York, The Philosophical Library, 1951. 144 pp. $3.75.

Secondary Mathematics, A Functional Approach for Teachers. By H. F. Fehr. Boston, D. C. Heath and Co., 1951. xi+431 pp. $4.25.

Differential Calculus. By B. S. Ray. Calcutta, Das Gupta and Company, Ltd., 1950. ii+245 pp. Rs. 6/8.

Dimensional Analysis and Theory of Models. By H. L. Langhaar. New York, 1951. xi+166 pp. $4.00.

Ontwakende Wetenschap. By B. L. Van der Waerden. Groningen, P. Noordhoff N. V., 1950. 332 pp. $3.58.

Infinite Matrices and Sequence Spaces. By R. G. Cooke. London, MacMillan and Company, 1950, 13+347 pp. $6.50.

Geometrie non Euclidienne, Par La Mathode Elementaire. By Gustave Verriest. Paris, Gauthier-Villars, 1951. 192 pp. $3.12.

Cours de Cinematique, Tome III, Geometrie et Cinematique Cayleyennes. By Rene Garner. Paris, Gauthier-Villars, 1951. 376 pp. $8.89.

Exercices de Mecanique, Tome I, Fascicule II. By H. Beghin and Julia G. Beghin. Paris, Gauthier-Villars, 1951. 240 pp.

Problemes Concrets D'Analyse Fonctionnelle. By Paul Levy. Paris, Gauthier-Villars, 1951. 484 pp. $11.75.

Intermediate Algebra. By P. K. Rees and F. W. Sparks. New York, McGraw-Hill, 1951. viii+329 pp. $3.25.

Hohere Algebra, 3rd Ed. By Helmut Hasse. I Lineare Gleichungen. 152 pp. Hohere Algebra II, Gleichungen Hoheren Grades. 158 pp. Berlin, Walter de Gruyter, 1951. No price given.

The Topology of Fibre Bundles (No. 14, Princeton Mathematical Series). By Norman Steenrod, New Jersey, Princeton University Press, 1951. viii+224 pp. $5.00.

Business Statistics. By J. R. Riggleman and I. N. Frisbee. New York, McGraw-Hill Book Company, 1951. xix+818 pp. $5.50.

An Introduction to Statistical Analysis. By W. J. Dixon and F. J. Massey. New York, McGraw-Hill, 1951. x+370 pp. $4.50.

National Income Behavior. By T. C. Schelling. New York, McGraw-Hill, 1951. x+291 pp. $4.50.

Trigonometry with Tables. By C. T. Holmes. New York, McGraw-Hill, 1951. ix+246 pp. $3.00.

Essentials of College Algebra. By J. B. Rosenbach and E. A. Whitman, Boston, Ginn and Company, 1951. x+322 pp+xxx pp. $3.00.

Johannes Kepler—Life and Letters. By Carola Baumgardt. New York, The Philosophical Library, 1951. 209 pp. $3.75.

CLUBS AND ALLIED ACTIVITIES

EDITED BY L. F. OLLMANN, Hofstra College

Send reports of all activities, such as club reports, special features, topics with references, student papers, and other material of interest to L. F. Ollmann, Hofstra College, Hempstead, New York.

Kappa Mu Epsilon, Eighth Biennial Convention

The Eighth Biennial Convention of *Kappa Mu Epsilon* was held in Springfield, Missouri on April 26–28, 1951 with *Missouri Alpha* chapter at Southwestern Missouri State College as host. A total of 188 delegates and members, representing thirty chapters and seventeen states, were registered. The president, Dr. H. Van Engen, presided at all sessions. His address as retiring president was called *You, Your Education and Kappa Mu Epsilon.*

The student papers presented to the convention were:

The configurations (9_3), by Hans Stetter of *Colorado Alpha*, at Colorado A. and M. College, Fort Collins

Some properties of the Simpson line, by Anne Robben, *Kansas Gamma* at Mount St. Scholastica College, Atchison

The gamma function, by Keith Stumpff, *Missouri Beta* at Central Missouri State College, Warrensburg

The odd perfect number, by James Barry, *Michigan Gamma* at Wayne University, Detroit

The squirrel cage slide rule, by Robert Green of *Kansas Alpha* at State Teachers College, Pittsburg

Angle trisection, by Wanda Ponder of *Iowa Alpha* at Iowa State Teachers College, Cedar Falls.

Dr. Leonard Blumenthal of the University of Missouri delivered the banquet address on *Relativity in mathematics.*

The officers elected for the coming biennium are: President, C. B. Tucker, State Teachers College, Emporia, Kansas; Vice-President, C. C. Richtmeyer, Central Michigan College of Education, Mount Pleasant, Michigan; Secretary, E. Marie Hove, Hofstra College, Hempstead, New York; Treasurer, L. F. Ollmann, Hofstra College, Hempstead, New York; Historian, Laura Greene, Washburn Municipal University, Topeka, Kansas.

Dr. H. D. Larsen, Albion College, Albion, Michigan was reappointed Editor of the *Pentagon*, the official publication of *Kappa Mu Epsilon.*

The delegates to the convention voted to incorporate *Kappa Mu Epsilon*, Honorary Mathematics Society under the laws of the State of New York.

The Ninth Biennial Convention will be held at St. Mary's Lake in Michigan on April 17–18, 1953 with the Michigan Chapters as hosts.

CLUB REPORTS

Mathematics Club, Kansas State College

During the year 1949–50, the *Mathematics Club* of Kansas State College had

a series of seven talks by people in the Department of Mathematics. The speakers and their topics were as follows:

Calendar problems, by Lois Carper

Fourier analysis by a method due to Stokes, by Prof. W. C. Janes

A method of testing hypotheses, by Robert Cell

Series, by Dr. S. T. Parker

Bessel functions, by Lansford Trapp

The theory of groups and their representation of matrices, by Violet Hachmeister

Pascal's triangle, by John Neff.

The officers for the year were: President, Wayne Cowell; Vice-President, Lansford Trapp; Secretary-Treasurer, Raymond Rinker.

Undergraduate Mathematics Society, Columbia College

The *Undergraduate Mathematics Society* of Columbia College met weekly during 1950–51 for the presentation of papers, the aim of which was to conclude, usually, with some significant result. The following lectures were given:

The transcendence of e, by M. Epstein

Peano's axioms, by C. Langley

Fixed point theorems, by S. Stein

The four-color problem, by I. Mann

Group theory, by R. Feldman

The calculus of finite differences, by C. Kaufman

Embedding theorems, by A. Fass

Transfinite cardinals, by Dr. M. Auslander

The concept of homotopy, by Prof. S. Eilenberg.

The officers for the year were: President, E. Auerbach; Vice-President, C. Kaufman.

Pi Mu Epsilon, Louisiana State University

The *Louisiana Alpha* chapter of *Pi Mu Epsilon* heard the following papers for 1950–51:

The number system, by Prof. F. A. Rickey

Plane continua, by Prof. N. E. Rutt

Development of quantum mechanics, by Prof. V. E. Parker

Some applications of mathematical concepts to chemical problems, by Prof. H. B. Williams

Groups in crystal structure and theory of equations, by Prof. Eugene Schenkman

Another way of doing it, by Prof. P. K. Rees

Calculus of variations, by Prof. B. B. Townsend

The exterior differential calculus of Cartan, by Prof. Eugenio Calabi.

Twenty-one members were initiated at the annual Banquet and Initiation.

Prof. C. C. MacDuffee, Director-General of *Pi Mu Epsilon*, gave the banquet address as part of the *Pi Mu Epsilon Lectures* for the year. The second address on *Curves in Minkowski space* was given the next day.

The following awards were presented at the banquet: Herbert W. Kelley received the Freshman Award, based on an Honors Examination; Horace C. Hearne and Roger W. Richardson, Jr. tied in the Senior Award, based on the amount of work taken in mathematics and the quality of work done.

Several new volumes were added to the Pi Mu Epsilon shelf during the year.

The officers serving for 1950–51 were: Director, James M. Turner; Vice-Director, Horace C. Hearne; Secretary, Delilah Stokes; Treasurer, Roger W. Richardson; Corresponding Secretary and Faculty Advisor, Prof. Houston T. Karnes.

Mathematics Club, Syracuse University

The plans for the Syracuse University *Mathematics Club* for 1950–51 include:

The publishing of a Club Paper to appear tri-semesterly. The paper will include a Problem Department.

Other plans call for the issuance of membership cards to the members.

The officers elected for 1950–51 are: President, Sherman Babcock; Vice-President, Harry Kagan; Secretary, Vera Rita; Treasurer, Frank Manley.

NEWS AND NOTICES

EDITED BY EDITH R. SCHNECKENBURGER, University of Buffalo

Readers are invited to contribute to the general interest of this department by sending news items to Edith R. Schneckenburger, University of Buffalo, Buffalo 14, New York. Items must be submitted at least two months before publication can take place.

ANNUAL CONVENTION OF THE CENTRAL ASSOCIATION OF SCIENCE AND MATHEMATICS TEACHERS

The annual Convention of the Central Association of Science and Mathematics Teachers will be held in Cleveland, Ohio, on November 22–24, 1951. The Hollenden Hotel has been selected for headquarters.

Interesting general meetings are scheduled for Friday morning and afternoon, November 23, and Saturday morning, November 24. Among the featured general speakers are the following: Dr. Keith Glennan, President of Case Institute of Technology and member of the Atomic Energy Commission; Dr. Elvin C. Stakman, Chief of the Division of Plant Pathology and Botany, University of Minnesota; and Dr. David Dietz, Science Editor, Scripps Howard Newspapers.

Dr. Dietz will speak also at the banquet Friday evening.

Section meetings in Elementary Mathematics, Elementary Science, General Science, Mathematics, Biology, Chemistry and Physics will be held Friday.

Group meetings in areas of Elementary Schools, Junior High Schools, Senior High Schools, and Conservation will be held on Saturday.

A NEW SCIENTIFIC JOURNAL

A new scientific journal, *Current Research*, will be published to furnish a medium for the prompt publication of brief papers recording the progress of science.

The journal is designed to serve the scientist in ways not done by any existing publication. Its unique features are: (1) Articles limited to 1,000 words; (2) Printed on one side of a single page; (3) Summary title printed along right-hand margin; (4) Inner edge of page perforated to facilitate easy removal and filing; (5) Authors assessed $10.00 for each article, enabling the journal to expand as necessary for prompt publication (15–30 days); (6) No editorial board; (7) Scope: the entire field of science; (8) International in character.

The first issue will be published on October 5, 1951. The subscription rate is $5.00. The editor is Ware Cattell, formerly editor of *The Scientific Monthly* and assistant editor of *Science.*

FELLOWSHIPS OF THE AMERICAN ASSOCIATION OF UNIVERSITY WOMEN

Twenty-five fellowships are offered by the American Association of University Women to American women for advanced study or research during the academic year 1952–53.

In general, the $1,500 fellowships are awarded to young women who have completed two years of residence work for the Ph.D. degree or who have already received the degree; the $2,000–$2,200 awards to more advanced students or those who may need to study abroad; the $3,000 awards to more mature scholars who need a year of uninterrupted work for writing and research. Unless otherwise specified, the fellowships are unrestricted as to subject and place of study.

Applications and supporting materials must reach the office in Washington by December 15, 1951. For detailed information concerning these fellowships and instructions for applying, address the Secretary, Committee on Fellowship Awards, American Association of University Women, 1634 Eye Street, N. W., Washington 6, D. C.

PERSONAL ITEMS

Professor E. M. Beesley, University of Nevada, was the representative of the Association at the Inauguration of President M. A. Love of the University of Nevada.

Professor Leonard Blumenthal of the University of Missouri represented the Association at the Inauguration of Dr. R. L. Woodward as President of

Central College on June 2, 1951.

Professor L. J. Green of Case Institute of Technology was appointed to represent the Association at the celebration of the One Hundred Twenty-Fifth Anniversary of Western Reserve University on June 11, 1951.

Professor F. E. Johnston, George Washington University, represented the Association at the thirty-fourth Annual Meeting of the American Council of Education which was held in Washington, D. C. on May 4–5, 1951.

Professor F. J. Murray of Columbia University was the delegate of the Association to the Academic Convocation at Manhattan College on April 30, 1951; this Convocation commemorated the Tercentenary of the birth of Saint John Baptist de la Salle.

Professor W. R. Ransom of Tufts College represented the Association at the Inauguration of President H. C. Case of Boston University on June 3, 1951.

Dr. Christine Williams Ayoub of Cornell University has been awarded the Sigma Delta Epsilon Post-Doctoral Fellowship for 1951–52.

The officers of the Mathematics Division of the American Society for Engineering Education for 1951–52 are: Professor H. M. Gehman, Chairman and Professor J. H. Zant, Secretary.

Professor H. P. Robertson, California Institute of Technology, and Professor G. T. Whyburn of the University of Virginia have been elected to membership in the National Academy of Sciences.

Cooper Union announces the following promotions in the School of Engineering: Assistant Professor J. N. Eastham to an associate professorship and Mr. C. H. Lehmann to an assistant professorship.

At Fresno State College: Professor F. R. Morris has been appointed Head of the Division of Physical Sciences; Associate Professor Alice K. Bell has been promoted to a professorship.

Kansas State College announces: Professor W. T. Stratton has retired after forty-one years service with the title of Professor Emeritus; Associate Professor Emma Hyde has retired after thirty-one years service with the title of Associate Professor Emeritus; Associate Professors P. M. Young and S. T. Parker have been promoted to professorships; Associate Professor Edison Greer has resigned to accept a position with Beech Aircraft, Wichita, Kansas; Mrs. Marilynn S. Dueker has resigned.

Kent State University announces the following promotions: Associate Professor Emma Olson to a professorship; Assistant Professor Paul Evans to an associate professorship; Miss Emalou Brumfield to an assistant professorship.

Massachusetts Institute of Technology reports the following: Professor W. T. Martin, chairman of the Department of Mathematics, will be at the Institute for Advanced Study during 1951–52; Assistant Professor G. W. Whitehead has been promoted to an associate professorship; Mr. A. H. Copeland, Jr., Mr. Felix Haas, Mr. E. B. Leach, Mr. P. P. Radkowski, and Mr. J. M. Stark have been promoted to instructorships.

University of Washington announces the following: Professor Z. W. Birn-

baum has been granted a leave of absence to accept a visiting professorship at Stanford University for 1951–52; Professor Edwin Hewitt has been granted a leave of absence for the academic year 1951–52 to accept an appointment as visiting lecturer at the University of Uppsala; Dr. D. B. Dekker has been promoted to an assistant professorship; Dr. M. G. Arsove, who has been studying at Grenoble on a Fulbright grant, has been appointed to an instructorship.

At Western Washington College of Education: Associate Professor S. A. Johnston, chairman of the Department of Mathematics, has been promoted to a professorship; during the summer Professor Johnston was a member of the staff of California Institute of Technology; Instructor H. M. Gelder has been promoted to an assistant professorship.

Assistant Professor Henry Antosiewicz of Montana State College has been promoted to an associate professorship.

Mr. J. S. Arend, formerly a part-time instructor at the University of Colorado, is now in the United States Air Force.

Dr. R. C. Briant, who has been at the Institute for Cooperative Research of Johns Hopkins University, has accepted the position of Project Director, Oak Ridge National Laboratory, Tennessee.

Mrs. Helen E. Brown, formerly an instructor at the University of Tennessee, is teaching at Fassifern School for Girls, Henderson, North Carolina.

Assistant Professor R. L. Calvert of the University of Wyoming has received an appointment in the Research Division of Sandia Corporation, Albuquerque, New Mexico.

Miss Mary Campbell of Lamar College retired on June 1, 1951.

Graduate Assistant JoAnn M. Cumming of Marquette University has a position as mathematician at the Aberdeen Proving Ground, Maryland.

Instructor F. W. Donaldson of the University of Texas has accepted a position on the technical staff of the Los Alamos Scientific Laboratory, New Mexico.

Dr. Nat Edmonson of Fairchild Engine and Airplane Corporation, Oak Ridge, Tennessee, has been appointed to the position of Principal Mathematician at the Oak Ridge National Laboratories.

Assistant Professor W. L. Fields of the University of Louisville has been appointed to an associate professorship at Hampton Institute.

Assistant Professor J. R. Foote, Iowa State College, is now at Wright-Patterson Air Force Base, Dayton, Ohio.

Professor L. R. Ford, chairman of the Department of Mathematics of Illinois Institute of Technology, was honored at a special meeting of the Men's Mathematics Club of Chicago on May 18, 1951.

Mr. J. H. Fountain, formerly a graduate student at the University of Buffalo, is now a research chemist with the Upson Company, Lockport, New York.

Mr. E. A. Franz of Culver-Stockton College has been recalled to active duty in the United States Army Signal Corps.

Mr. W. H. From, previously a graduate student at the University of Virginia, has accepted a position as Research Engineer at Aircraft Armaments

Incorporated, Baltimore, Maryland.

Professor W. H. Garrett of Baker University has retired with the title of Professor Emeritus of Mathematics and Astronomy; Professor Garrett was one of the founders of the Kansas Association of Teachers of Mathematics.

Mr. G. H. Gleissner, Columbia University, has a position as mathematician at the United States Naval Proving Ground, Dahlgren, Virginia.

Mr. R. B. Grekila, formerly a graduate assistant at the University of Maryland, has accepted a position as Assistant Chief Chemist at Ball Brothers Company, Incorporated, Muncie, Indiana.

Professor S. G. Hacker, State College of Washington, has been on leave of absence and has been serving as a mathematical consultant with Operations Analysis, Strategic Air Command, United States Air Force, Omaha, Nebraska.

Mr. P. C. Hanzel, formerly laboratory assistant at the University of California, has accepted a position as a research assistant in the Cryogenic Laboratory, Ohio State University.

Mr. Julius Honig of Raytheon Electronic Corporation, Waltham, Massachusetts, has a position as mathematician at the National Bureau of Standards, Washington, D. C.

Mrs. Verba W. Iturralde, previously an instructor at the College of William and Mary, is teaching in the El Paso Public Schools, Texas.

Dr. Hyman Kaufman of the Geophysical Department, Continental Oil Company, Ponca City, Oklahoma, has accepted a position in the Laboratory for Electronics, Incorporated, Boston, Massachusetts.

Mr. J. B. Kelly, University of Wisconsin, has been appointed a member of the Institute for Advanced Study.

Mr. R. S. Kingsbury, formerly a staff member of Operations Evaluation Group, Navy Department, Washington, D. C., is now a member of Operations Research Staff of Arthur D. Little, Incorporated, Cambridge, Massachusetts.

Visiting Assistant Professor Jacob Korevaar of Purdue University has been appointed to a professorship at the Institute of Technology, Delft, Netherlands.

Mrs. Mary M. Kruse of the Sandia Corporation, Albuquerque, New Mexico, has a position as mathematician at the Institute for Numerical Analysis, Los Angeles, California.

Assistant Professor Werner Leutert has been granted a leave of absence from the University of Maryland and has accepted a position as mathematician at the Ballistic Research Laboratories, Aberdeen Proving Ground, Maryland.

Mr. H. C. Mayer, Jr. of the University of Utah has a position as mathematician with the General Electric Corporation, Richland, Washington.

Dr. J. C. C. McKinsey, formerly a mathematician at the Rand Corporation, Santa Monica, California, has been appointed Visiting Professor of Philosophy at Stanford University.

Dr. C. E. Mullan of the Duquesne Light Company, Pittsburgh, Pennsylvania, and of the Carnegie Institute of Technology has been appointed Servomechanisms Engineer, Bell Aircraft Corporation, Buffalo, New York.

Professor P. R. Rider has a leave of absence from Washington University for the academic year 1951–52 to act as Mathematical and Statistical Adviser to the Chief of the Office of Air Research, Air Development Force, Air Research and Development Command, Wright-Patterson Air Force Base, Dayton, Ohio.

Assistant Professor Saul Rosen of Drexel Institute of Technology has joined the Research Division of the Burroughs Adding Machine Company, Philadelphia, Pennsylvania.

Assistant Professor W. C. Sangren, University of Miami, has a position as senior mathematician at the Oak Ridge National Laboratory, Tennessee.

Mr. R. J. Smurthwaite, formerly a graduate assistant at Lehigh University, has a position at Bell Aircraft Corporation, Buffalo, New York.

Dr. E. V. Somers of Westinghouse Electric Corporation, Pittsburgh, Pennsylvania, is now Senior Research Technologist, Magnolia Petroleum Company, Dallas, Texas.

Instructor R. D. Stalley of the University of Arizona has been appointed to a graduate assistantship at the University of Oregon.

Mr. W. J. Strauss has accepted a position as junior mathematician at the Air Weapons Research Center of the University of Chicago.

Dr. G. W. Tyler of the Navy Electronic Laboratory, San Diego, California, has a position as operations analyst, United States Air Force, Washington, D. C.

Dr. J. L. Ullman is on leave of absence from the University of Michigan and will be located at Stanford, California, during the year 1951–52.

Professor J. W. T. Youngs is on leave of absence from Indiana University and has a position as research consultant for the Atomic Energy Commission at Sandia Corporation, Albuquerque, New Mexico.

Professor G. A. Bliss, who was Martin A. Ryerson Distinguished Service Professor of Mathematics Emeritus at the University of Chicago, died on May 8, 1951. He was a charter member of the Association.

Reverend Hilary Doerfler of St. Gregory's College died on March 18, 1950.

Professor C. H. Gingrich, chairman of the Department of Mathematics, Carleton College, died on June 17, 1951. He had been a member of the Association for thirty-three years.

Professor T. W. Jackson, head of the Department of Mathematics, Jamestown College, North Dakota, died on June 10, 1950. He was a charter member of the Association.

Dr. R. B. Robbins of Charlottesville, Virginia, died on February 11, 1951.

Professor Emeritus W. D. A. Westfall of the University of Missouri died on April 28, 1951.

Professor Emeritus A. H. Wheeler of Clark University died on December 19, 1950. He was a charter member of the Association.

THE MATHEMATICAL ASSOCIATION OF AMERICA

Official Reports and Communications

NEW MEMBERS

Professor H. M. Gehman, Secretary-Treasurer, announces that the following ninety-two persons have been elected to membership by the Board of Governors on applications duly certified.

R. C. AILARA, B.S.E.E. (Illinois Tech.) Grad. Student, Illinois Institute of Technology, Chicago, Ill.

W. H. AMMANN, M.S. (St. Louis) Grad. Student, St. Louis University, Mo.

R. F. ATTRIDGE, Student, McMaster University, Hamilton, Ont.

FREDERICK BAGEMIHL, M.A. (Columbia) Asst. Professor, University of Rochester, N. Y.

J. J. BAILEY, M.A. (Michigan) Acting Head of Mathematics Department, Shattuck School, Faribault, Minn.

R. E. BAKER, M.A. (Texas) Asso. Professor, East Texas State Teachers College, Commerce, Tex.

R. S. BARTON, M.S. (Iowa) Instr., New Mexico School of Mines, Socorro, N. M.

T. F. BOBLAK, B.S. (Illinois Tech.) Grad. Student, Illinois Institute of Technology, Chicago, Ill.

J. V. BREGLIO, Student, George Pepperdine College, Los Angeles, Calif.

J. D. BRERETON, M.A. (Columbia) Vice-Principal, New Paltz High School, N. Y.

CHARLOTTE BROWN, M.S. (N.Y.U.) Part-time Instr., Brooklyn College, N. Y.

H. H. BROWN, A.B. (Penna. State) Research Engineer, Franklin Institute Laboratories, Philadelphia, Pa.

MARJORIE L. BROWNE, Ph.D. (Michigan) Instr., North Carolina College, Durham, N.C.

B. H. CHOVITZ, M.A. (Harvard) Mathematician, Army Map Service, Washington, D. C.

R. A. CHRISTENSEN, Student, Willamette University, Salem, Ore.

MRS. WILLIE E. CLARK, M.A. (Atlanta) Head of Mathematics Department, Agricultural Mechanical and Normal College, Pine Bluff, Ark.

R. C. CLELLAND, A.M. (Columbia) Instr., Hamilton College, Clinton, N. Y.

P. J. COCUZZA, Student, City College of the City of New York, N. Y.

M. L. COFFMAN, M.A. (Oklahoma) Instr., East Texas State Teachers College, Commerce, Tex.

ECKFORD COHEN, Ph.D. (Duke) Asst. Professor, Syracuse University, N. Y.

C. H. COOK, M.S. (Iowa) Asst. Professor, Western State College, Gunnison, Colo.

J. W. CORNWELL, 3RD, B.A. (Dartmouth) Senior Engineer, The Teleregister Corporation, New York, N. Y.

R. B. CROUCH, M.S. (Illinois) Instr., New Mexico College of Agricultural and Mechanic Arts, State College, N. M.

MRS. ELIZABETH H. CUTHILL, M.S. (Brown) Instr., Purdue University, Lafayette, Ind.

G. H. DUBAY, M.A. (Loyola, Chicago) Professor, University of St. Thomas, Houston, Tex.

D. L. ELLIOTT, Student, Pomona College, Claremont, Calif.

R. L. ELY, M.S. (C.I.T.) Research Engineer, Pittsburgh-Des Moines Company, Pittsburgh, Pa.

H. M. EMICH, Student, University of Rochester, N. Y.

D. O. ETTER, B.A. (Texas Christian) Grad. Student, Texas Christian University, Fort Worth, Tex.

BEN FITZPATRICK, JR., Student, Alabama Polytechnic Institute, Auburn, Ala.

D. D. FREDERICK, Student, George Pepperdine College, Los Angeles, Calif.

W. H. FUCHS, Ph.D. (Cambridge, England) Asso. Professor, Cornell University, Ithaca, N. Y.

F. W. GIBSON, Student, Los Angeles State College, Calif.

R. C. GILBERT, A.B. (Harvard) Engineering Technician, Lockheed Aircraft Corporation, Burbank, Calif.

F. J. GINIVAN, M.S. (Notre Dame) Teaching Fellow, University of Notre Dame, Ind.

M. L. GOLDWATER, M.A. (U.C.L.A.) Los Angeles, Calif.

E. P. GRANEY, B.S. (Rockhurst) Grad. Student, University of Notre Dame, Ind.

J. H. GRIESMER, Student, University of Notre Dame, Ind.

G. R. HAGEN, Student, University of Kentucky, Lexington, Ky.

CARL HAMMER, Ph.D. (Munich) Chairman, Division of Technical Education, Walter Hervey Junior College, New York, N. Y.

F. R. HARDING, Student, Marquette University, Milwaukee, Wis.

H. G. HARP, M.S. (Chicago) Asso. Professor, Ohio Northern University, Ada, Ohio

CLINTON HELTON, Student, Eastern Kentucky State College, Richmond, Ky.

J. E. HICKS, M.S. (DePaul) Teacher, Chicago Board of Education, Ill.

EVA JAGOE, B.A. (Alberta) Teacher, Calgary School Board, Alta.

W. J. JAMESON, Student, Montana State University, Missoula, Mont.

PHILIP JONAS, B.S. (Brooklyn C.) Instr., RCA Institutes, New York, N. Y.

F. A. KROS, M.S. (New Mexico A & M) Instr., University of Colorado, Boulder, Colo.

M. T. LANKALIS, B.S.E. (S.T.C., Bridgewater, Mass.) Instr., New Hampshire Technical Institute, Manchester, N. H.

E. H. LEE, Ph.D. (Stanford) Asso. Professor, Brown University, Providence, R. I.

G. R. LEHNER, Student, Loyola University, Chicago, Ill.

B. L. LERCHER, Student, University of Wisconsin, Madison, Wis.

MRS. NORMA L. LINDEMANN, A.M. (Indiana) Research Assistant, University of Notre Dame, Ind.

D. B. LLOYD, Ph.D. (Catholic) Instr., Wilson Teachers' College, Washington, D. C.

R. G. MCDERMOT, B.S. (Pittsburgh) Instr., Western Pennsylvania Institute of Technology, Pittsburgh, Pa.

J. A. MEIER, Student, Franklin and Marshall College, Lancaster, Pa.

R. B. MERKEL, A.B. (Sacramento S. C.) Instr., Sacramento Unified School District, Calif.

GEORGE MILLMAN, M.A. (Columbia) Mathematician, Army Map Service, Corps of Engineers, Washington, D. C.

J. T. MONTGOMERY, B.S. (Notre Dame) Teaching Fellow, University of Notre Dame, Ind.

T. D. OXLEY, Jr., Student, Texas Christian University, Fort Worth, Tex.

W. O. PORTMANN, Student, Kent State University, Ohio

CONRAD RENNEMANN, JR., B.S. (Nebraska) Grad. Assistant, University of Nebraska, Lincoln, Neb.

J. B. RYAN, B.S. (Seattle) Treasurer, Rice High School, New York, N. Y.

RAFAEL SANCHEZ-DIAZ, Ph.D. (California) Professor, New Mexico School of Mines, Socorro, N. M.

B. E. SEALANDER, B.S. (Seton Hall) Senior Research Engineer, Republic Aviation, Farmingdale, L. I., N. Y.

R. Q. SEALE, Ph.D. (Stanford) Professor, Southern Methodist University, Dallas, Tex.

SISTER MARY OLIVE NORMAN, M.S. (Marquette) Head of Mathematics Department, Cardinal Stritch College, Milwaukee, Wis.

SISTER MIRIAM PATRICK, B.S. (St. Mary's Indiana) Grad. Assistant, St. Mary's College, Notre Dame, Ind.

A. H. SMITH, Student, George Pepperdine College, Los Angeles, Calif.

J. D. SPEAS, Student, Butler University, Indianapolis, Ind.

H. E. SPEECE, M.A. (Texas Christian) Instr., North Carolina State College, Raleigh, N. C.

F. M. STEWART, Ph.D. (Harvard) Asst. Professor, Brown University, Providence, R. I.

S. SUBRAMU, M.Sc. (Mysore) Lecturer, Technical Institute, Tata Iron and Steel Company, Jamshedpur, Bihar, India

R. R. SUTTER, Student, Illinois Institute of Technology, Chicago, Ill.

W. W. TAYLOR, M.S. (East Texas S. T. C.) Instr., East Texas State Teachers College, Commerce, Tex.

G. C. THOMPSON, B.S. in E.E. (Union C., New York) Asso. Actuary, Security Mutual Life Insurance Company, Binghamton, N. Y.

F. J. WAGNER, M.S. (St. Louis) Grad. Fellow, University of Notre Dame, Ind.

J. G. WALL, M.A. (North Carolina) Grad.

Assistant, University of Georgia, Athens, Ga.
G. M. WEBSTER, M.A. (Oklahoma) Research Physicist, Carter Oil Company, Tulsa, Okla.
A. M. WENNER, Student, Gustavus Adolphus College, St. Peter, Minn.
R. A. WERDELIN, Student, Trinity College, Hartford, Conn.
H. W. WESSELY, Student, Illinois Institute of Technology, Chicago, Ill.
C. S. WEST, JR., B.S. (Southern Methodist) Assistant to the Actuary, Guardian International Life Insurance Company, Dallas, Tex.
H. E. WHEELER, Student, Northwestern University, Evanston, Ill.
R. S. WOLLAN, B.A. (Luther C., Iowa) Grad. Student, University of Georgia, Athens, Ga.
P. G. WOOD, B.Sc. (London) Mathematics Master, Eastleigh County High School, Hampshire, England
DALE WOODS, M.S. (Oklahoma A & M) Instr., North Dakota Agricultural College, Fargo, N. D.
T. M. WRIGHT, M.A. (Michigan) Asso. Professor, Berea College, Ky.
MELVIN YEDLIN, B.S. (Missouri) Automotive Design Engineer, Detroit Arsenal, Centerline, Mich.
R. L. YOUNG, B.S. (Davis and Elkins) Grad. Assistant, University of Denver, Colo.
E. J. ZIRKEL, Student, St. John's University, Brooklyn, N. Y.
M. N. ZUCKER, Student, New York University, N. Y.

NEW SECTIONAL GOVERNORS OF THE ASSOCIATION

The following have been elected Governors of the Association for a three-year term beginning July 1, 1951 by a mail vote of the membership of the Association in the Sections indicated:

Allegheny Mountain	J. B. Rosenbach, Carnegie Institute of Technology
Indiana	J. C. Polley, Wabash College
Kentucky	Aughtum S. Howard, Kentucky Wesleyan College
Metropolitan New York	T. F. Cope, Queens College
Nebraska	J. M. Earl, University of Omaha
Northern California	E. B. Roessler, University of California at Davis
Oklahoma	J. C. Brixey, University of Oklahoma
Rocky Mountain	C. F. Barr, University of Wyoming
Wisconsin	H. P. Pettit, Marquette University

Of the forty-one members of the Board of Governors, twenty-six Governors are elected as representatives of the Sections of the Association. In this way the Association attempts to make participation in its activities as widely spread as possible among its membership.

Each year the members in each of the Sections in which an election is being held show great interest in the conduct of the election. In the 1951 election, in six of the nine Sections more than half of those eligible to vote case their ballots in time to be counted. The highest percentage of votes cast was 74% in the Kentucky Section.

H. M. GEHMAN, *Secretary-Treasurer*

JOINT MEETING OF THE ASSOCIATION WITH A.S.E.E.

A meeting of the Mathematical Association of America was held at Michigan State College, East Lansing, Michigan, on Monday and Tuesday, June 25–26,

ARRANGEMENTS, ENTERTAINMENT AND RECREATION

The Committee on Arrangements for the meeting consisted of J. H. Bell, Chairman, H. M. Gehman, H. E. Stelson, B. M. Stewart, C. P. Wells, and J. H. Zant.

Registration headquarters were located in Shaw Hall of Michigan State College. Rooms in Shaw and Phillips dormitories were available for members of the Association and their families. Meals were served in the dormitory cafeterias.

A tea was given by the Mathematics Department of Michigan State College in the Conference Room of the Physics-Mathematics Building at the conclusion of the Tuesday afternoon session.

Other entertainment was provided under the auspices of the A.S.E.E. Visits to local automobile plants were arranged for each afternoon.

Sessions of the A.S.E.E. began on Monday and continued through Friday. In addition to the sessions mentioned above, the Mathematics Division of the A.S.E.E. met jointly with the Educational Methods Division on Wednesday afternoon.

At a brief business meeting of the Association, held on Tuesday evening, a motion was presented by Professor P. H. Daus expressing the appreciation of the members of the Association and of the Division for the hospitality of the authorities of Michigan State College and especially the local members of the Committee on Arrangements whose efforts had made possible this successful meeting.

H. M. GEHMAN, *Secretary-Treasurer*

MARCH MEETING OF THE SOUTHERN CALIFORNIA SECTION

The thirty-first regular meeting of the Southern California Section of the Mathematical Association of America was held at Whittier College, Whittier, California, on Saturday, March 10, 1951. Professor Herbert Busemann, Chairman of the Section, presided.

The attendance was eighty-four, including the following fifty-eight members of the Association: L. J. Adams, J. T. Ahlin, O. W. Albert, Florence R. Anderson, L. A. Aroian, Winifred A. Asprey, Lulu Bechtolsheim, May M. Beenken, C. M. Bell, Jr., Jonas Beraru, L. T. Black, Herbert Busemann, Frances L. Campbell, L. M. Coffin, W. V. Gamzon, W. H. Glenn, Jr., W. T. Guy, Jr., C. J. A. Halberg, H. J. Hamilton, V. C. Harris, A. F. Herbst, R. B. Herrera, M. R. Hestenes, P. G. Hoel, R. E. Horton, D. H. Hyers, C. G. Jaeger, P. B. Johnson, P. J. Kelly, Cornelius Lanczos, L. C. Lay, Margaret B. Lehman, G. F. McEwen, F. R. Morris, P. M. Niersbach, L. J. Paige, D. J. Peterson, W. T. Puckett, C. A. Pursel, H. R. Pyle, E. C. Rex, J. M. Robb, R. W. Shephard, Samuel Skolnik, J. C. Smith, Jr., R. H. Sorgenfrey, D. V. Steed, Robert Steinberg, E. B. Strutton, J. D. Swift, T. E. Sydnor, A. E. Taylor, S. E. Urner, F. A. Valentine, Morgan Ward, P. A. White, A. L. Whiteman, B. R. Wicker.

At the business meeting the following officers were elected for the academic

year 1951–52: Chairman, W. H. Glenn, Jr., Pasadena City Schools; Vice-Chairman, C. W. Trigg, Los Angeles City College; Program Committee, D. H. Hyers (Chairman) University of Southern California, P. B. Johnson, Occidental College, E. C. Rex, George Pepperdine College. The next meeting was scheduled for March 8, 1952, at Occidental College, Los Angeles, California.

The following program was presented:

1. *Unified freshman mathematics*, by Professor S. E. Urner, Los Angeles State College, Professor F. A. Valentine, University of California at Los Angeles, Professor H. R. Pyle, Whittier College.

2. *The nominalistic interpretation of mathematics*, by Professor L. A. Henkin, University of Southern California, introduced by Professor P. A. White.

The problem considered is: To provide a systematic method of reinterpreting the language of mathematics in such a way that no reference is made to abstract objects such as sets, functions, numbers, etc. It is required that logical terms such as "not," "if . . . then," "for every x," etc., retain their basic meanings, and that sentences which can be proved by ordinary methods of mathematical proof shall express true propositions under the new interpretations of the symbols. Using methods of mathematical logic, it is shown that there are a multiplicity of solutions to this problem.

3. *Mapping of the integers onto themselves preserving the partial ordering of division*, by Professor Morgan Ward, California Institute of Technology.

Mappings $\phi=\phi_n$ of the integers into themselves such that ϕ_n divides ϕ_m whenever n divides m are of frequent occurrence in the theory of numbers. A general theory of such mappings is here developed with emphasis on their lattice properties with respect to the partial ordering by division. Of particular interest are those mappings K with the two additional properties (i) $K_0=0$, $K_1=1$, $K_n\neq 0$, $n\neq 0$; (ii) For every integer m, the set of integers x such that $K_x\equiv 0 \pmod{m}$ is an ideal. (The Lucas function $(\alpha^n-\beta^n)/(\alpha-\beta)$, $\alpha+\beta$, $\alpha\beta$ coprime integers, is a K-mapping.) Such mappings have a number of interesting arithmetical properties. For example a ϕ-mapping with $\phi_n\neq 0$, $n\neq 0$, $\phi_0=0$, $\phi_1=1$ is a K-mapping if and only if $(\phi_n, \phi_m)=\phi((n, m))$. (Here the parentheses (. . .) indicate a greatest common divisor.)

4. *Experience of a Fulbright professor in England*, by Professor Elmer Tolsted, Pomona College, introduced by Professor C. G. Jaeger.

This speaker gave a description of the Fulbright Scholarship, and also discussed the general standard of living and teachers salaries in England, the B.A. requirements in mathematics at the University of London, and devices of the English to keep political influence out of their schools.

5. *Monte Carlo solution of linear equations*, by Dr. G. E. Forsythe, National Bureau of Standards, Los Angeles, California.

A method of solving certain classes of systems of linear equations (or inverting the coefficient matrices) by random sampling was devised by J. von Neumann and S. M. Ulam (unpublished). The method was expounded and extended by George E. Forsythe and Richard A. Leibler, *Mathematical Tables and Other Aids to Computation*, vol. 4 (1950), pp. 127–129, and vol. 5 (1951), p. 55. The present speaker explained the method and, with the assistance of Professor Paul G. Hoel, demonstrated it on a simple system with two unknowns.

6. *The Cobb-Douglas production function*, by Dr. R. W. Shephard, Rand Corporation.

The origin, economic meaning, and statistical analyses of the Cobb-Douglas production function are briefly reviewed. Following this, a mathematical development is given which furnishes a precise foundation for this well known empirical function and provides definitions for the aggregate quantities involved leading to the usual interpretations of the parameters. Thereby a rigorous foundation is provided for the function, enabling simplifications of the statistical analyses made heretofore. An exact correspondence between ratio of wage bill to total value added and the labor elasticity of output is found. As a by-product of the mathematical development, the cost function corresponding to the Cobb-Douglas production function is found to have equally simple expression.

W. T. Puckett, *Acting Secretary*

MARCH MEETING OF THE SOUTHWESTERN SECTION

The eleventh annual meeting of the Southwestern Section of the Mathematical Association of America was held at the University of New Mexico, Albuquerque, New Mexico, on March 23 and 24, 1951.

Dr. A. W. Boldyreff, Chairman of the Section, presided. Fifty-five persons attended the sessions, including the following thirty-eight members of the Association: J. W. Beach, A. W. Boldyreff, C. E. Buell, J. H. Butchart, C. R. Clark, G. A. Culpepper, Angelo Di Bella, D. G. Duncan, R J. Flanagan, R. S. Fouch, F. C. Gentry, R. F. Graesser, P. C. Hammer, W. P. Heinzman, M. S. Hendrickson, R. C. Hildner, B. W. Jones, Max Kramer, G. N. Landes, F. O. Lane, Rachael A. La Roe, L. E. Mahuron, R. W. Malley, H. V. Mathany, J. J. Miller, W. W. Mitchell, Leo Moser, J. W. Reed, N. W. Riebe, B. D. Roberts, H. P. Rogers, Henry Schutzberger, J. L. Slechticky, O. L. Wadkins, Earl Walden, D. L. Webb, R. L. Westhafer, Charles Wexler.

At the business meeting the following officers were elected for the year 1951–1952: Chairman, Charles Wexler, Arizona State College, Tempe, Arizona; Vice-Chairman, J. H. Butchart, Arizona State College, Flagstaff, Arizona; Lecturer, two-year term, D. L. Webb, University of Arizona. The section accepted the invitation of the University of Arizona to hold the spring meeting, April 11–12, 1952 in Tucson, Arizona.

Friday evening, March 23, Dr. B. W. Jones of the University of Colorado delivered an address by invitation entitled *Mathematics from the Cradle to the Grave*. Dr. Jones pointed out numerous examples of related problems in mathematics that can be introduced into standard mathematics courses at all age levels.

The following papers were read in the two-day session:

1. *Radial heat flow in a hypersphere*, by Professor R. L. Westhafer, New Mexico College of A. & M.A.

Upon obtaining the partial differential equation of radial heat flow in a hypersphere by a method analogous to that used in a three dimensional sphere, it was shown that the temperature function has the same behavior as the angular velocity of a viscous, incompressible fluid in an infinite right circular cylinder rotating about its axis.

2. *A necessary and sufficient condition for the existence of a unique least squares solution*, by Professor M. S. Hendrickson, University of New Mexico.

A necessary and sufficient condition that there be a unique solution to the problem of find-

ing by the method of least squares a function of the form $y=\sum_{j=1}^{m} a_j f_j(x)$ to fit the n points (x_i, y_i), $n>m$ is that the matrix $\|f_i(x_j)\|$ $i=l, \cdots, m$ column index, $j=1, \cdots, n$, row index, have rank m.

3. *Unsolved problems of convex bodies*, by Dr. P. C. Hammer, University of California, Los Alamos, New Mexico.

Let C be a closed convex body in a plane, and let x and y be any two points not on the boundary of C. The lengths of the chords lying on rays through x and y are determined. Is C the only convex body having the same chord length functions? This class of problems including generalization to higher dimensions and allowing x and y to go to infinity in two specified directions, the author proposed about three years ago. At present it is known that for two perpendicular directions in the plane the chord lengths do not uniquely determine a convex body. It is known that adding one more direction will uniquely determine the body or that some angle between two lines other than 90^0 will yield uniqueness.

The author defines an *outwardly simple* family of lines in the plane as a family of lines which simply cover the exterior of some circle including the points at infinity. Consider the convex hull H of the intersection points of such a family. It is conjectured that the total accumulated area swept out by the lines inside H is at most three times the area of H and that it is three times the area of H if and only if H is a triangle and the lines are its extended diameters.

4. *Numbers represented by certain quadratic forms*, by Professor B. W. Jones, University of Colorado.

If p is a prime congruent to 1 (mod 12), it is represented by one and only one of the forms $f=x^2+36y^2$, $g=9x^2+4y^2$. The background for this result was sketched as well as the basis for the fact that -3 is a biquadratic residue or non-residue of p according as $p=g$ or $p=f$ is solvable.

5. *On the distribution of digits of powers*, Professor Leo Moser, Texas Technological College.

The following theorem was proved and generalized in several directions. Every finite sequence of digits appears as the first digits of some power 2. Related solved and unsolved problems on the normality of certain sequences were discussed. The first part of this paper, written with co-author N. Macon, will be published in *Scripta Mathematica.*

6. *Vector methods in the geometry of the tetrahedron*, by Professor F. C. Gentry, Arizona State College, Tempe, Arizona.

Vectors were used to determine the configuration of points, lines, planes, and spheres associated with the bisecting planes of the internal and external dihedral angles of a tetrahedron. The six Apollonian spheres were shown to be co-axial.

7. *A class of infinite groups*, by Professor D. G. Duncan, University of Arizona.

The class of groups whose elements are vectors of the form $[c, f(x)]$, where $f(x)$ is a polynomial whose coefficients along with c belong to $GF(p)$ has been studied by L. Kaloujnine. The group operation X being defined here by $[c, f(x)]X[d, f(x)] = [c+d, f(x)+g(x+c)]$. The class of groups arising when the coefficients belong to certain infinite sets has been investigated by the speaker. The special cases where coefficients belong (i) to the integers, (ii) to the rational number field, were presented. In particular the structure of the lower central and derived series was determined as well as the generating elements of the Abelian factor groups and of the subgroups appearing in the central series.

8. *Some properties of generators of a three-valued Post algebra*, by Professor D. L. Webb, University of Arizona.

Limitations were developed on binary generators of a three-valued Post algebra. In the case of Boolean algebra these conditions were sufficient to determine the Sheffer stroke function.

9. *Demonstration of a machine for playing the game Nim*, by Mr. Frank Kros, Assistant P. Cioffi, Student Harold Connell, New Mexico College of A. & M. A., introduced by the Secretary.

Mr. Kros explained the principles of the game Nim and Mr. Cioffi and Mr. Connell demonstrated the machine built by Mr. Connell under the supervision of Professor Harold Brown of the A. & M. electrical engineering department. The machine was constructed following the specifications, with slight modifications, of the article by Raymond Redheffer in the June 1948 issue of this MONTHLY.

10. *Formation of a simple irrational equation*, by Professor J. W. Beach, University of New Mexico.

An expression, in terms of possible roots, was given for each of two forms of irrational equations which reduce to quadratics. The possible roots were chosen arbitrarily and the conditions were given which would make both roots satisfy, both be extraneous, or one satisfy while the other would be extraneous.

11. *Reducing the general conic*, by Professor J. H. Butchart, Arizona State College, Flagstaff, Arizona.

The speaker feels that texts on analytic geometry should call attention to the matrix of coefficients of the general conic as a means of recalling the equations satisfied by the coordinates of the center and the expression for the constant after translating the origin to the center. Then the final coefficients of x^2 and y^2 are easily found from the expression for their sum and their difference. For the parabola, the rotation is first handled in the same manner, after which translation may be effected by completing a square.

12. *On the logic and teaching of equality*, by Professor R. S. Fouch, Arizona State College, Tempe, Arizona.

Almost all high school and college texts take the equality as a concept already understood; most introductory texts state a number (from four to ten or more) of postulates of equality and make tacit use of several more. The possibility of making simple and meaningful proofs of these properties is generally neglected. Hilbert and others proved the complete properties of equality from the two postulates: $a=a$ and $a=b\rightarrow[P(a)\rightarrow P(b)]$. It was shown that the same thing can be accomplished from either of two definitions: (1) "$a=b$" $=df$ "$P(a)\leftrightarrow P(b)$" or (2) "$a=b$" $=df$ $(\exists|x)$ "$[aNx$ and $bNx]$", where "aNx" means that "a" is a name for x. It was argued that either or both of these definitions could be used in introductory courses and would produce pedagogic advantages such as greater clarity and a better foundation of basic concepts.

13. *Panel discussion: High school preparation for college mathematics*, by Earl Walden (Moderator), New Mexico College of A. & M. A., Charles Wexler, Arizona State College, Tempe, Max Kramer, New Mexico College of A. & M. A.

Two topics were discussed: *What should be the college entrance requirements in mathematics?* and *What can the colleges do to improve high school mathematics?* It was pointed out that by general cooperation colleges that have no entrance requirements in mathematics can and should insist on

two years of mathematics for entrance, provided that the selection be made from a broad base of mathematical subjects. The date for such change should be set sufficiently in advance to enable high schools to establish "two or three track" systems for their students.

Colleges can help high school mathematics programs by intensive training of prospective mathematics teachers in mathematics as well as education, deliberately stressing the relationship of advanced mathematics to the work of the secondary school, and by encouraging the most competent to teach. It was also argued that colleges provide intellectual stimulation for the gifted by visiting lecturers, awarding prizes, medals and scholarships, by sponsoring tournaments, clubs, and school publications. All students, however, should be provided with the guidance pamphlets recently issued by the M.A.A. and the N.C.T.M.

The panel of this discussion was appointed as a committee to meet with a similar committee of secondary school teachers to work on common problems and make recommendations to the Section at the 1952 meeting.

R. L. WESTHAFER, *Secretary*

CALENDAR OF FUTURE MEETINGS

Thirty-fifth Annual Meeting, Brown University, Providence, Rhode Island, December 29, 1951.

The following is a list of the Sections of the Association with dates of future meetings so far as they have been reported to the Associate Secretary.

ALLEGHENY MOUNTAIN, Waynesburg College, Waynesburg, Pennsylvania, May 10, 1952.

ILLINOIS, Western Illinois State College, Macomb, May 9–10, 1952.

INDIANA, Indiana University, Bloomington, Spring, 1952.

IOWA, Coe College, Cedar Rapids, April 18–19, 1952.

KANSAS

KENTUCKY, University of Kentucky, Lexington.

LOUISIANA-MISSISSIPPI, Northwestern State College, Natchitoches, Louisiana, February 15–16, 1952.

MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA, National Bureau of Standards, Washington, D. C., December 8, 1951.

METROPOLITAN NEW YORK, Spring, 1952.

MICHIGAN, University of Michigan, Ann Arbor, April 12, 1952.

MINNESOTA, North Dakota Agricultural College, Fargo, October 6, 1951.

MISSOURI, Lindenwood College, St. Charles, Spring, 1952.

NEBRASKA

NORTHERN CALIFORNIA, University of California, Berkeley, January 26, 1952.

OHIO, April 19, 1952.

OKLAHOMA, Oklahoma City University, October 12, 1951.

PACIFIC NORTHWEST, University of Oregon, Eugene, June 20, 1952.

PHILADELPHIA, University of Pennsylvania, Philadelphia, November 24, 1951.

ROCKY MOUNTAIN, Western State College, Gunnison, Colorado, May, 1952.

SOUTHEASTERN, Georgia Institute of Technology and Agnes Scott College, Atlanta, March 21–22, 1952.

SOUTHERN CALIFORNIA, Occidental College, Los Angeles, March 8, 1952.

SOUTHWESTERN, University of Arizona, Tucson, April 11–12, 1952.

TEXAS, East Texas State Teachers College, Commerce, April, 1952.

UPPER NEW YORK STATE, Colleges of the Seneca, Geneva, May, 1952.

WISCONSIN

GEORGE BANTA PUBLISHING COMPANY, MENASHA, WISCONSIN

THE AMERICAN
MATHEMATICAL MONTHLY

THE OFFICIAL JOURNAL OF

THE MATHEMATICAL ASSOCIATION OF AMERICA, INC.

VOLUME 58 NUMBER 8

CONTENTS

OCTOBER 1951

The AMERICAN MATHEMATICAL MONTHLY

(FOUNDED IN 1894 BY BENJAMIN F. FINKEL)

EDITORIAL CORRESPONDENCE should be addressed to the Editor, C. V. NEWSOM, State Education Building, Albany 1, N. Y. Papers submitted for publication should be sent to C. B. Allendoerfer, Department of Mathematics, University of Washington, Seattle 5, Washington.

ADVERTISING CORRESPONDENCE should be addressed to J. F. RANDOLPH, Morey Hall, University of Rochester, Rochester 3, N. Y.

NOTICE OF CHANGE OF ADDRESS by members of the Association as well as correspondence regarding subscriptions to the MONTHLY should be sent to the Secretary-Treasurer, H. M. GEHMAN, University of Buffalo, Buffalo 14, N. Y.

THIS IS THE OFFICIAL JOURNAL OF THE

MATHEMATICAL ASSOCIATION OF AMERICA, INC.

(Devoted to the Interests of Collegiate Mathematics)

Entered as second class matter at the post office at Menasha, Wis. Acceptance for mailing at special rate of postage provided for in the Act of February 28, 1925, embodied in Paragraph 4, Section 538, P. L. and R., authorized April 1, 1926.

SUBSCRIPTION PRICE: To Members, $4 a Year, To Others, $5 a Year.

PUBLISHED BY THE ASSOCIATION at Menasha, Wisconsin, and Albany, N. Y. during the months of January, February, March, April, May, June-July, August-September, October, November, December.

A HALF-CENTURY OF MATHEMATICS

HERMANN WEYL, Institute for Advanced Study

1. Introduction. Axiomatics. Mathematics, beside astronomy, is the oldest of all sciences. Without the concepts, methods and results found and developed by previous generations right down to Greek antiquity, one cannot understand either the aims or the achievements of mathematics in the last fifty years. Mathematics has been called the science of the infinite; indeed, the mathematician invents finite constructions by which questions are decided that by their very nature refer to the infinite. That is his glory. Kierkegaard once said religion deals with what concerns man unconditionally. In contrast (but with equal exaggeration) one may say that mathematics talks about the things which are of no concern at all to man. Mathematics has the inhuman quality of starlight, brilliant and sharp, but cold. But it seems an irony of creation that man's mind knows how to handle things the better the farther removed they are from the center of his existence. Thus we are cleverest where knowledge matters least: in mathematics, especially in number theory. There is nothing in any other science that, in subtlety and complexity, could compare even remotely with such mathematical theories as for instance that of algebraic class fields. Whereas physics in its development since the turn of the century resembles a mighty stream rushing on in one direction, mathematics is more like the Nile delta, its waters fanning out in all directions. In view of all this: dependence on a long past, other-worldliness, intricacy, and diversity, it seems an almost hopeless task to give a non-esoteric account of what mathematicians have done during the last fifty years. What I shall try to do here is, first to describe in somewhat vague terms general trends of development, and then in more precise language explain the most outstanding mathematical notions devised, and list some of the more important problems solved, in this period.

One very conspicuous aspect of twentieth century mathematics is the enormously increased role which the axiomatic approach plays. Whereas the axiomatic method was formerly used merely for the purpose of elucidating the foundations on which we build, it has now become a tool for concrete mathematical research. It is perhaps in algebra that it has scored its greatest successes. Take for instance the system of real numbers. It is like a Janus head facing in two directions: on the one side it is the field of the algebraic operations of addition and multiplication; on the other hand it is a continuous manifold, the parts of which are so connected as to defy exact isolation from each other. The one is the algebraic, the other the topological face of numbers. Modern axiomatics, simple-minded as it is (in contrast to modern politics), does not like such ambiguous mixtures of peace and war, and therefore cleanly separated both aspects from each other.

In order to understand a complex mathematical situation it is often convenient to separate in a natural manner the various sides of the subject in question, make each side accessible by a relatively narrow and easily surveyable

group of notions and of facts formulated in terms of these notions, and finally return to the whole by uniting the partial results in their proper specialization. The last synthetic act is purely mechanical. The art lies in the first, the analytic act of suitable separation and generalization. Our mathematics of the last decades has wallowed in generalizations and formalizations. But one misunderstands this tendency if one thinks that generality was sought merely for generality's sake. The real aim is simplicity: every natural generalization simplifies since it reduces the assumptions that have to be taken into account. It is not easy to say what constitutes a natural separation and generalization. For this there is ultimately no other criterion but fruitfulness: the success decides. In following this procedure the individual investigator is guided by more or less obvious analogies and by an instinctive discernment of the essential acquired through accumulated previous research experience. When systematized the procedure leads straight to axiomatics. Then the basic notions and facts of which we spoke are changed into undefined terms and into axioms involving them. The body of statements deduced from these hypothetical axioms is at our disposal now, not only for the instance from which the notions and axioms were abstracted, but wherever we come across an interpretation of the basic terms which turns the axioms into true statements. It is a common occurrence that there are several such interpretations with widely different subject matter.

The axiomatic approach has often revealed inner relations between, and has made for unification of methods within, domains that apparently lie far apart. This tendency of several branches of mathematics to coalesce is another conspicuous feature in the modern development of our science, and one that goes side by side with the apparently opposite tendency of axiomatization. It is as if you took a man out of a milieu in which he had lived not because it fitted him but from ingrained habits and prejudices, and then allowed him, after thus setting him free, to form associations in better accordance with his true inner nature.

In stressing the importance of the axiomatic method I do not wish to exaggerate. Without inventing new constructive processes no mathematician will get very far. It is perhaps proper to say that the strength of modern mathematics lies in the interaction between axiomatics and construction. Take algebra as a representative example. It is only in this century that algebra has come into its own by breaking away from the one universal system Ω of numbers which used to form the basis of all mathematical operations as well as all physical measurements. In its newly-acquired freedom algebra envisages an infinite variety of "number fields" each of which may serve as an operational basis; no attempt is made to embed them into the one system Ω. Axioms limit the possibilities for the number concept; constructive processes yield number fields that satisfy the axioms.

In this way algebra has made itself independent of its former master analysis and in some branches has even assumed the dominant role. This development in mathematics is paralleled in physics to a certain degree by the transition from

classical to quantum physics, inasmuch as the latter ascribes to each physical structure its own system of observables or quantities. These quantities are subject to the algebraic operations of addition and multiplication; but as their multiplication is non-commutative, they are certainly not reducible to ordinary numbers.

At the International Mathematical Congress in Paris in 1900 David Hilbert, convinced that problems are the life-blood of science, formulated twenty-three unsolved problems which he expected to play an important role in the development of mathematics during the next era. How much better he predicted the future of mathematics than any politician foresaw the gifts of war and terror that the new century was about to lavish upon mankind! We mathematicians have often measured our progress by checking which of Hilbert's questions had been settled in the meantime. It would be tempting to use his list as a guide for a survey like the one attempted here. I have not done so because it would necessitate explanation of too many details. I shall have to tax the reader's patience enough anyhow.

PART I. ALGEBRA. NUMBER THEORY. GROUPS.

2. Rings, Fields, Ideals. Indeed, at this point it seems impossible for me to go on without illustrating the axiomatic approach by some of the simplest algebraic notions. Some of them are as old as Methuselah. For what is older than the sequence of *natural numbers* 1, 2, 3, $\cdots$, by which we count? Two such numbers a, b may be added and multiplied ($a + b$ and $a \cdot b$). The next step in the genesis of numbers adds to these positive *integers* the negative ones and zero; in the wider system thus created the operation of addition permits of a unique inversion, subtraction. One does not stop here: the integers in their turn get absorbed into the still wider range of *rational numbers* (fractions). Thereby division, the operation inverse to multiplication, also becomes possible, with one notable exception however: division by zero. (Since $b \cdot 0 = 0$ for every rational number b, there is no inverse b of 0 such that $b \cdot 0 = 1$.) I now formulate the fundamental facts about the operations "plus" and "times" in the form of a table of axioms:

Table T

(1) The commutative and associative laws for addition,

$$a + b = b + a. \qquad a + (b + c) = (a + b) + c.$$

(2) The corresponding laws for multiplication.
(3) The distributive law connecting addition with multiplication

$$c \cdot (a + b) = (c \cdot a) + (c \cdot b).$$

(4) The axioms of subtraction: (4_1) There is an element o (0, "zero") such that $a + o = o + a = a$ for every a. (4_2) To every a there is a number $-a$ such that $a + (-a) = (-a) + a = o$.

(5) The axioms of division: (5_1) There is an element e (1, "unity") such that $a \cdot e = e \cdot a = a$ for every a. (5_2) To every $a \neq o$ there is an a^{-1} such that $a \cdot a^{-1} = a^{-1} \cdot a = e$.

By means of (4_2) and (5_2) one may introduce the difference $b - a$ and the quotient b/a as $b + (-a)$ and $b \cdot a^{-1}$, respectively.

When the Greeks discovered that the ratio ($\sqrt{2}$) between diagonal and side of a square is not measurable by a rational number, a further extension of the number concept was called for. However, all measurements of continuous quantities are possible only approximately, and always have a certain range of inaccuracy. Hence rational numbers, or even finite decimal fractions, can and do serve the ends of mensuration provided they are interpreted as approximations, and a calculus with approximate numbers seems the adequate numerical instrument for all measuring sciences. But mathematics ought to be prepared for any subsequent refinement of measurements. Hence dealing, say, with electric phenomena, one would be glad if one could consider the approximate values of the charge e of the electron which the experimentalist determines with ever greater accuracy as approximations of one definite *exact* value e. And thus, during more than two millenniums from Plato's time until the end of the nineteenth century, the mathematicians worked out an exact number concept, that of *real numbers*, that underlies all our theories in natural science. Not even to this day are the logical issues involved in that concept completely clarified and settled. The rational numbers are but a small part of the real numbers. The latter satisfy our axioms no less than the rational ones, but their system possesses a certain completeness not enjoyed by the rational numbers, and it is this, their "topological" feature, on which the operations with infinite sums and the like, as well as all continuity arguments, rest. We shall come back to this later.

Finally, during the Renaissance *complex numbers* were introduced. They are essentially pairs $z = (x, y)$ of real numbers x, y, pairs for which addition and multiplication are defined in such a way that all axioms hold. On the ground of these definitions $e = (1, 0)$ turns out to be the unity, while $i = (0, 1)$ satisfies the equation $i \cdot i = -e$. The two members x, y of the pair z are called its real and imaginary parts, and z is usually written in the form $xe + yi$, or simply $x + yi$. The usefulness of the complex numbers rests on the fact that every algebraic equation (with real or even complex coefficients) is solvable in the field of complex numbers. The analytic functions of a complex variable are the subject of a particularly rich and harmonious theory, which is the show-piece of classical nineteenth century analysis.

A set of elements for which the operations $a + b$ and $a \cdot b$ are so defined as to satisfy the axioms (1)–(4) is called a *ring*; it is called a *field* if also the axioms (5) hold. Thus the common integers form a ring I, the rational numbers form a field ω; so do the real numbers (field Ω) and the complex numbers (field Ω^*). But these are by no means the only rings or fields. The polynomials of all

possible degrees h,

$$f = f(x) = a_0 + a_1x + a_2x^2 + \cdots + a_hx^h, \tag{1}$$

with coefficients a_i taken from a given ring R (*e.g.* the ring I of integers, or the field ω), called "polynomials over R," form a ring $R[x]$. Here the variable or indeterminate x is to be looked upon as an empty symbol; the polynomial is really nothing but the sequence of its coefficients $a_0, a_1, a_2, \cdots$. But writing it in the customary form (1) suggests the rules for the addition and multiplication of polynomials which I will not repeat here. By substituting for the variable x a definite element ("number") γ of R, or of a ring P containing R as a subring, one projects the elements f of $R[x]$ into elements α of P, $f \rightarrow \alpha$: the polynomial $f = f(x)$ goes over into the number $\alpha = f(\gamma)$. This mapping $f \rightarrow \alpha$ is *homomorphic, i.e.*, it preserves addition and multiplication. Indeed, if the substitution of γ for x carries the polynomial f into α and the polynomial g into β then it carries $f + g$, $f \cdot g$ into $\alpha + \beta$, $\alpha \cdot \beta$, respectively.

If the product of two elements of a ring is never zero unless one of the factors is, one says that the ring is without null-divisor. This is the case for the rings discussed so far. A field is always a ring without null-divisor. The construction by which one rises from the integers to the fractions can be used to show that any ring R with unity and without null-divisor may be imbedded in a field k, the quotient field, such that every element of k is the quotient a/b of two elements a and b of R, the second of which (the denominator) is not zero.

Writing $1a, 2a, 3a, \cdots$, for $a, a + a, a + a + a$, *etc.*, we use the natural numbers $n=1, 2, 3, \cdots$, as multipliers for the elements a of a ring or a field. Suppose the ring contains the unity e. It may happen that a certain multiple ne of e equals zero; then one readily sees that $na = 0$ for every element a of the ring. If the ring is without null divisors, in particular if it is a field and p is the least natural number for which $pe = 0$, then p is necessarily a prime number like 2 or 3 or 5 or 7 or 11 $\cdots$. One thus distinguishes fields of prime characteristic p from those of characteristic 0 in which no multiple of e is zero.

Plot the integers $\cdots, -2, -1, 0, 1, 2, \cdots$ as equidistant marks on a line. Let n be a natural number ≥ 2 and roll this line upon a wheel of circumference n. Then any two marks a, a' coincide, the difference $a-a'$ of which is divisible by n. (The mathematicians write $a \equiv a' \pmod{n}$; they say: a congruent to a' modulo n.) By this identification the ring of integers I goes over into a ring I_n consisting of n elements only (the marks on the wheel), as which one may take the "residues" $0, 1, \cdots, n-1$. Indeed, congruent numbers give congruent results under both addition and multiplication: $a \equiv a'$, $b \equiv b' \pmod{n}$ imply $a + b \equiv a' + b'$, $a \cdot b \equiv a' \cdot b' \pmod{n}$. For instance, modulo 12 we have $7 + 8 = 3$, $5 \cdot 8 = 4$ because 15 leaves the residue 3 and 40 the residue 4 if divided by 12. The ring I_{12} is not without null divisors since $3 \cdot 4$ is divisible by 12, but neither 3 nor 4 is. However, if p is a natural prime number, then I_p has no null divisor and is even a field; for as the ancient Greeks proved by an ingenious procedure (Euclid's algorism), for every integer a not divisible by

p there is one, a', such that $a \cdot a' \equiv 1 \pmod{p}$. This Euclidean theorem is at the basis of the whole of number theory. The example shows that there are fields of any given prime characteristic p.

In any ring R one may introduce the notions of unit and prime element as follows. The ring element a is a unit if it has a reciprocal a' in the ring, such that $a' \cdot a = e$. The element a is composite if it may be decomposed into two factors $a_1 \cdot a_2$, neither of which is a unit. A prime number is one that is neither a unit nor composite. The units of I are the numbers $+1$ and -1. The units of the ring $k[x]$ of polynomials over a field k are the non-vanishing elements of k (polynomials of degree 0). According to the Greek discovery of the irrationality of $\sqrt{2}$ the polynomial $x^2 - 2$ is prime in the ring $\omega[x]$; but, of course, not in $\Omega[x]$, for there it splits into the two linear factors $(x - \sqrt{2})(x + \sqrt{2})$. Euclid's algorism is also applicable to polynomials $f(x)$ of one variable x over any field k. Hence they satisfy Euclid's theorem: Given a prime element $P = P(x)$ in this ring $k[x]$ and an element $f(x)$ of $k[x]$ not divisible by $P(x)$, there exists another polynomial $f'(x)$ over k such that $\{f(x) \cdot f'(x)\} - 1$ is divisible by $P(x)$. Identification of any elements f and g of $k[x]$, the difference of which is divisible by P, therefore changes the ring $k[x]$ into a field, the "residue field κ of $k[x]$ modulo P." Example: $\omega[x]$ mod $x^2 - 2$. (Incidentally the complex numbers may be described as the elements of the residue field of $\Omega[x]$ mod $x^2 + 1$.) Strangely enough, the fundamental Euclidean theorem does not hold for polynomials of two variables x, y. For instance, $P(x, y) = x - y$ is a prime element of $\omega[x, y]$, and $f(x, y) = x$ an element not divisible by $P(x, y)$. But a congruence

$$x \cdot f'(x, y) \equiv 1 \pmod{x - y}$$

is impossible. Indeed, it would imply $-1 + x \cdot f'(x, x) = 0$, contrary to the fact that the polynomial of one indeterminate x,

$$-1 + x \cdot f'(x, x) = -1 + c_1 x + c_2 x^2 + \cdots,$$

is not zero. Thus the ring $\omega[x, y]$ does not obey the simple laws prevailing in I and in $\omega[x]$.

Consider κ, the residue field of $\omega[x]$ mod $x^2 - 2$. Since for any two polynomials $f(x)$, $f'(x)$ which are congruent mod $x^2 - 2$ the numbers $f(\sqrt{2})$, $f'(\sqrt{2})$ coincide, the transition $f(x) \to f(\sqrt{2})$ maps κ into a sub-field $\omega[\sqrt{2}]$ of Ω consisting of the numbers $a + b\sqrt{2}$ with rational a, b. Another such projection would be $f(x) \to f(-\sqrt{2})$. In former times one looked upon κ as the part $\omega[\sqrt{2}]$ of the continuum Ω or Ω^* of all real or all complex numbers; one wished to embed everything into this universe Ω or Ω^* in which analysis and physics operate. But as we have introduced it here, κ is an algebraic entity the elements of which are not numbers in the ordinary sense. It requires for its construction no other numbers but the rational ones. It has nothing to do with Ω, and ought not to be confused with the one or the other of its two projections into Ω. More generally, if $P = P(x)$ is any prime element in $\omega[x]$ we can form the

residue field κ_P of $\omega[x]$ modulo P. To be sure, if δ is any of the real or complex roots of the equation $P(x)=0$ in Ω^* then $f(x) \rightarrow f(\delta)$ defines a homomorphic projection of κ_P into Ω^*. But the projection is not κ_P itself.

Let us return to the ordinary integers $\cdots, -2, -1, 0, 1, 2, \cdots$, which form the ring I. The multiples of 5, *i.e.*, the integers divisible by 5, clearly form a ring. It is a ring without unity, but it has another important peculiarity: not only does the product of any two of its elements lie in it, but all the integral multiples of an element do. The queer term *ideal* has been introduced for such a set: Given a ring R, an R-ideal $(\mathfrak{a})$ is a set of elements of R such that (1) sum and difference of any two elements of $(\mathfrak{a})$ are in $(\mathfrak{a})$, (2) the product of an element in $(\mathfrak{a})$ by any element of R is in $(\mathfrak{a})$. We may try to describe a divisor $\mathfrak{a}$ by the set of all elements divisible by $\mathfrak{a}$. One would certainly expect this set to be an ideal $(\mathfrak{a})$ in the sense just defined. Given an ideal $(\mathfrak{a})$, there may not exist an actual element a of R such that $(\mathfrak{a})$ consists of all multiples $j = m \cdot a$ of a (m any element in R). But then we would say that $(\mathfrak{a})$ stands for an "*ideal* divisor" $\mathfrak{a}$: the words "the element j of R is divisible by $\mathfrak{a}$" would simply mean: "j belongs to $(\mathfrak{a})$." In the ring I of common integers all divisors are actual.

But this is not so in every ring. An algebraic surface in the three-dimensional Euclidean space with the Cartesian coordinates x, y, z is defined by an equation $F(x, y, z) = 0$ where F is an element of ${}^3\Omega = \Omega[x, y, z]$, *i.e.*, a polynomial of the variables x, y, z with real coefficients. F is zero in all the points of the surface; but the same is true for every multiple $L \cdot F$ of F (L being any element of ${}^3\Omega$), in other words, for every polynomial of the ideal (F) in ${}^3\Omega$. Two simultaneous polynomial equations

$$F_1(x, y, z) = 0, \qquad F_2(x, y, z) = 0$$

will in general define a curve, the intersection of the surface $F_1 = 0$ and the surface $F_2 = 0$. The polynomials $(L_1 \cdot F_1) + (L_2 \cdot F_2)$ formed by arbitrary elements L_1, L_2 of ${}^3\Omega$ form an ideal (F_1, F_2), and all these polynomials vanish on the curve. This ideal will in general not correspond to an actual divisor F, for a curve is not a surface. Examples like this should convince the reader that the study of algebraic manifolds (curves, surfaces, *etc.*, in 2, 3, or any number of dimensions) amounts essentially to a study of polynomial ideals. The field of coefficients is not necessarily Ω or Ω^*, but may be a field of a more general nature.

3. Some achievements of algebra and number theory. I have finally reached a point where I can hint, I hope, with something less than complete obscurity, at some of the accomplishments of algebra and number theory in our century. The most important is probably the freedom with which we have learned to manage these abstract axiomatic concepts, like field, ring, ideal, *etc.* The atmosphere in a book like van der Waerden's *Moderne Algebra*, published about 1930, is completely different from that prevailing, *e.g.*, in the articles on algebra written for the *Mathematical Encyclopaedia* around 1900. More specif-

ically, a general theory of ideals, and in particular of polynomial ideals, was developed. (However, it should be said that the great pioneer of abstract algebra, Richard Dedekind, who first introduced the ideals into number theory, still belonged to the nineteenth century.) Algebraic geometry, before and around 1900 flourishing chiefly in Italy, was at that time a discipline of a type uncommon in the sisterhood of mathematical disciplines: it had powerful methods, plenty of general results, but they were of somewhat doubtful validity. By the abstract algebraic methods of the twentieth century all this was put on a safe basis, and the whole subject received a new impetus. Admission of fields other than Ω^*, as the field of coefficients, opened up a new horizon.

A new technique, the "primadic numbers," was introduced into algebra and number theory by K. Hensel shortly after the turn of the century, and since then has become of ever increasing importance. Hensel shaped this instrument in analogy to the power series which played such an important part in Riemann's and Weierstrass's theory of algebraic functions of one variable and their integrals (Abelian integrals). In this theory, one of the most impressive accomplishments of the previous century, the coefficients were supposed to vary over the field Ω^* of all complex numbers. Without pursuing the analogy, I may illustrate the idea of p-adic numbers by one typical example, that of quadratic norms. Let p be a prime number, and let us first agree that a congruence $a \equiv b$ modulo a power p^h of p for rational numbers a, b has this meaning that $(a - b)/p^h$ equals a fraction whose denominator is not divisible by p;

$$e.g., \frac{39}{4} - \frac{12}{5} \equiv 0 \pmod{7^2} \text{ because } \frac{39}{4} - \frac{12}{5} = 7^2 \cdot \frac{3}{20}.$$

Let now a, b be rational numbers, $a \neq 0$, and b not the square of a rational number. In the quadratic field $\omega[\sqrt{b}]$ the number a is a *norm* if there are rational numbers x, y such that

$$a = (x + y\sqrt{b})(x - y\sqrt{b}), \qquad \text{or} \qquad a = x^2 - by^2.$$

Necessary for the solvability of this equation is (1) that for every prime p and every power p^h of p the congruence $a \equiv x^2 - by^2 \pmod{p^h}$ has a solution. This is what we mean by saying the equation has a p-adic solution. Moreover there must exist rational numbers x and y such that $x^2 - by^2$ differs as little as one wants from a. This is what we mean by saying that the equation has an ∞-adic solution. The latter condition is clearly satisfied for every a provided b is positive; however, if b is negative it is satisfied only for positive a. In the first case every a is ∞-adic norm, in the second case only half of the a's are, namely, the positive ones. A similar situation prevails with respect to p-adic norms. One proves that these necessary conditions are also sufficient: if a is a norm locally everywhere, *i.e.*, if $a = x^2 - by^2$ has a p-adic solution for every "finite prime spot p" and also for the "infinite prime spot ∞," then it has a "global" solution, namely an exact solution in rational numbers x, y.

This example, the simplest I could think of, is closely connected with the theory of genera of quadratic forms, a subject that goes back to Gauss' *Disquisitiones arithmeticae*, but in which the twentieth century has made some decisive progress by means of the p-adic technique, and it is also typical for that most fascinating branch of mathematics mentioned in the introduction: class field theory. Around 1900 David Hilbert had formulated a number of interlaced theorems concerning class fields, proved some of them at least in special cases, and left the rest to his twentieth century successors, among whom I name Takagi, Artin and Chevalley. His norm residue symbol paved the way for Artin's general reciprocity law. Hilbert had used the analogy with the Riemann-Weierstrass theory of algebraic functions over Ω^* for his orientation, but the ingenious, partly transcendental methods which he applied had nothing to do with the much simpler ones that had proved effective for the functions. By the primadic technique a rapprochement of methods has occurred, although there is still a considerable gap separating the theory of algebraic functions and the much subtler algebraic numbers.

Hensel and his successors have expressed the p-adic technique in terms of the non-algebraic "topological" notion of ("valuation" or) *convergence*. An infinite sequence of rational numbers $a_1, a_2, \cdots$ is convergent if the difference $a_i - a_j$ tends to zero, $a_i - a_j \to 0$, provided i and j independently of each other tend to infinity; more explicitly, if for every positive rational number ϵ there exists a positive integral N such that $-\epsilon < a_i - a_j < \epsilon$ for all i and $j > N$. The completeness of the real number system is expressed by Cauchy's convergence theorem: To every convergent sequence $a_1, a_2, \cdots$ of rational numbers there exists a *real* number α to which it converges: $a_i - \alpha \to 0$ for $i \to \infty$. With the ∞-adic concept of convergence we have now confronted the p-adic one induced by a prime number p. Here the sequence is considered convergent if for every exponent $h = 1, 2, 3, \cdots$, there is a positive integer N such that $a_i - a_j$ is divisible by p^h as soon as i and $j > N$. By introduction of p-adic numbers one can make the system of rational numbers complete in the p-adic sense as the introduction of real numbers makes them complete in the ∞-adic sense. The rational numbers are embedded in the continuum of all real numbers, but they may be embedded as well in that of all p-adic numbers. Each of these embedments corresponding to a finite or the infinite prime spot p is equally interesting from the arithmetical viewpoint. Now it is more evident than ever how wrong it was to identify an algebraic number field with one of its homomorphic projections into the field Ω of real numbers; along with the (real) infinite prime spots one must pay attention to the finite prime spots which correspond to the various prime ideals of the field. This is a golden rule abstracted from earlier, and then made fruitful for later, arithmetical research; and here is one bridge (others will be pointed out later) joining the two most fascinating branches of modern mathematics: abstract algebra and topology.

Besides the introduction of the primadic treatment and the progress made in the theory of class fields, the most important advances of number theory

during the last fifty years seem to lie in those regions where the powerful tool of analytic functions can be brought to bear upon its problems. I mention two such fields of investigation: I. distribution of primes and the zeta function, II. additive number theory.

I. The notion of prime number is of course as old and as primitive as that of the multiplication of natural numbers. Hence it is most surprising to find the distribution of primes among all natural numbers is of such a highly irregular and almost mysterious character. While on the whole the prime numbers thin out the further one gets in the sequence of numbers, wide gaps are always followed again by clusters. An old conjecture of Goldbach's maintains that there even come along again and again pairs of primes of the smallest possible difference 2, like 57 and 59. However, the distribution of primes obeys at least a fairly simple *asymptotic* law: the number $\pi(n)$ of primes among all numbers from 1 to n is asymptotically equal to $n/\log n$. [Here $\log n$ is not the Briggs logarithm which our logarithmic tables give, but the natural logarithm as defined by the integral $\int_1^n dx/x$.] By asymptotic is meant that the quotient between $\pi(n)$ and the approximating function $n/\log n$ tends to 1 as n tends to infinity. In antiquity Eratosthenes had devised a method to sift out the prime numbers. By this sieve method the Russian mathematician Tchebycheff had obtained, during the nineteenth century, the first non-trivial results about the distribution of primes. Riemann used a different approach: his tool is the so-called zeta-function defined by the infinite series

$$\zeta(s) = 1^{-s} + 2^{-s} + 3^{-s} + \cdots. \tag{2}$$

Here s is a complex variable, and the series converges for all values of s, the real part of which is greater than 1, $\Re s > 1$. Already in the eighteenth century the fact that every positive integer can be uniquely factorized into primes had been translated by Euler into the equation

$$1/\zeta(s) = (1 - 2^{-s})(1 - 3^{-s})(1 - 5^{-s}) \cdots$$

where the (infinite) product extends over all primes $2, 3, 5, \cdots$. Riemann showed that the zeta-function has a unique "analytic continuation" to all values of s and that it satisfies a certain functional equation connecting its values for s and $1 - s$. Decisive for the prime number problem are the zeros of the zeta-function, *i.e.*, the values s for which $\zeta(s)=0$. Riemann's equation showed that, except for the "trivial" zeros at $s = -2, -4, -6, \cdots$, all zeros have real parts between 0 and 1. Riemann conjectured that their real parts actually equal $\frac{1}{2}$. His conjecture has remained a challenge to mathematics now for almost a hundred years. However, enough had been learned about these zeros at the close of the nineteenth century to enable mathematicians, by means of some profound and newly-discovered theorems concerning analytic functions, to prove the above-mentioned asymptotic law. This was generally considered a great triumph of mathematics. Since the turn of the century Rie-

mann's functional equation with the attending consequences has been carried over from the "classical" zeta-function (ii) of the field of rational numbers to that of an arbitrary algebraic number field (E. Hecke). For certain fields of prime characteristic one succeeded in confirming Riemann's conjecture, but this provides hardly a clue for the classical case. About the classical zeta-function we know now that it has infinitely many zeros on the critical line $\Re s = \frac{1}{2}$, and even that at least a fixed percentage, say 10 per cent, of them lie on it. (What this means is the following: Some percentage of those zeros whose imaginary part lies between arbitrary fixed limits $-T$ and $+T$ will have a real part equal to $\frac{1}{2}$, and this percentage will not sink below a certain positive limit, like 10 per cent, when T tends to infinity.) Finally about two years ago Atle Selberg succeeded, to the astonishment of the mathematical world, in giving an "elementary" proof of the prime number law by an ingenious refinement of old Eratosthenes' sieve method.

II. It has been known for a long time that every natural number n may be written as the sum of at most four square numbers, *e.g.*,

$$7 = 2^2 + 1^2 + 1^2 + 1^2, \qquad 87 = 9^2 + 2^2 + 1^2 + 1^2 = 7^2 + 5^2 + 3^2 + 2^2.$$

The same question arises for cubes, and generally for any k^{th} powers ($k=2, 3, 4, 5, \cdots$). In the eighteenth century Waring had conjectured that every non-negative integer n may be expressed as the sum of a limited number M of k^{th} powers,

$$n = n_1^k + n_2^k + \cdots + n_M^k, \tag{3}$$

where the n_i are also non-negative integers and M is independent of n. The first decade of the twentieth century brought two events: first one found that every n is expressible as the sum of at most 9 cubes (and that, excepting a few comparatively small n, even 8 cubes will do); and shortly afterwards Hilbert proved Waring's general theorem. His method was soon replaced by a different approach, the Hardy-Littlewood circle method, which rests on the use of a certain analytic function of a complex variable and yields asymptotic formulas for the number of different representations of n in the form (3). With some precautions demanded by the nature of the problem, and by overcoming some quite serious obstacles, the result was later carried over to arbitrary algebraic number fields; and by a further refinement of the circle method in a different direction Vinogradoff proved that every sufficiently large n is the sum of at most 3 primes. Is it even true that every even n is the sum of 2 primes? To show this seems to transcend our present mathematical powers as much as Goldbach's conjecture. The prime numbers remain very elusive fellows.

III. Finally, a word ought to be said about investigations concerning the arithmetical nature of numbers originating in analysis. One of the most elementary such constants is π, the area of the circle of radius 1. By proving that π is a transcendental number (not satisfying an algebraic equation with

rational coefficients) the age-old problem of "squaring the circle" was settled in 1882 in the negative sense; that is, one cannot square the circle by constructions with ruler and compass. In general it is much harder to establish the transcendency of numbers than of functions. Whereas it is easy to see that the exponential function

$$e^x = 1 + \frac{x}{1} + \frac{x^2}{1\cdot 2} + \frac{x^3}{1\cdot 2\cdot 3} + \cdots$$

is not algebraic, it is quite difficult to prove that its basis e is a transcendental number. C. L. Siegel was the first who succeeded, around 1930, in developing a sort of general method for testing the transcendency of numbers. But the results in this field remain sporadic.

4. Groups, vector spaces and algebras. This ends our report on number theory, but not on algebra. For now we have to introduce the *group* concept, which, since the young genius Evariste Galois blazed the trail in 1830, has penetrated the entire body of mathematics. Without it an understanding of modern mathematics is impossible. Groups first occurred as *groups of transformations*. Transformations may operate in any set of elements, whether it is finite like the integers from 1 to 10, or infinite like the points in space. *Set* is a premathematical concept: whenever we deal with a realm of objects, a set is defined by giving a criterion which decides for any object of the realm whether it belongs to the set or not. Thus we speak of the set of prime numbers, or of the set of all points on a circle, or of all points with rational coordinates in a given coordinate system, or of all people living at this moment in the State of New Jersey. Two sets are considered equal if every element of the one belongs to the other and vice versa. A *mapping* S of a set σ into a set σ' is defined if with every element a of σ there is associated an element a' of σ', $a \to a'$. Here a rule is required which allows one to find the "image" a' for any given element a of σ. This general notion of mapping we may also call of a premathematical nature. Examples: a real-valued function of a real variable is a mapping of the continuum Ω into itself. Perpendicular projection of the space points upon a given plane is a mapping of the space into the plane. Representing every space-point by its three coordinates x, y, z with respect to a given coordinate system is a mapping of space into the continuum of real number triples (x, y, z). If a mapping S, $a \to a'$ of σ into σ', is followed by a mapping S', $a' \to a''$ of σ' into a third set σ'', the result is a mapping SS': $a \to a''$ of σ into σ''. A *one-to-one mapping* between two sets σ, σ' is a pair of mappings, S: $a \to a'$ of σ into σ', and S': $a' \to a$ of σ' into σ, which are inverse to each other. This means that the mapping SS' of σ into σ is the identical mapping E of σ which sends every element a of σ into itself, and that $S'S$ is the identical mapping of σ'. In particular, one is interested in one-to-one mappings of a set σ into itself. For them we shall use the word *transformation*. Permutations are nothing but transformations of a finite set.

The inverse S' of a transformation S, $a \to a'$ of a given set σ, is again a transformation and is usually denoted by S^{-1}. The result ST of any two transformations S and T of σ is again a transformation, and its inverse is $T^{-1}S^{-1}$ (according to the rule of dressing and undressing: if in dressing one begins with the shirt and ends with the jacket, one must in undressing begin with the jacket and end with the shirt. The order of the two "factors" S, T is essential.) A *group of transformations* is a set of transformations of a given manifold which (1) contains the identity E, (2) contains with every transformation S its inverse S^{-1}, and (3) with any two transformations S, T their "product" ST. Example: One could define congruent configurations in space as point sets of which one goes into the other by a congruent transformation of space. The congruent transformations, or "motions," of space form a group; a statement which, according to the above definition of group, is equivalent to the threefold statement that (1) every figure is congruent to itself, (2) if a figure F is congruent to F', then F' is congruent to F, and (3) if F is congruent to F' and F' congruent to F'', then F is congruent to F''. This example at once illuminates the inner significance of the group concept. *Symmetry* of a configuration F in space is described by the group of motions that carry F into itself.

Often manifolds have a structure. For instance, the elements of a field are connected by the two operations of plus and times; or in Euclidean space we have the relationship of congruence between figures. Hence we have the idea of structure-preserving mappings; they are called *homomorphisms*. Thus a homomorphic mapping of a field k into a field k' is a mapping $a \to a'$ of the "numbers" a of k into the numbers a' of k' such that $(a + b)' = a' + b'$ and $(a \cdot b)' = a' \cdot b'$. A homomorphic mapping of space into itself would be one that carries any two congruent figures into two mutually congruent figures. The following terminology (suggestive to him who knows a little Greek) has been agreed upon: homomorphisms which are one-to-one mappings are called isomorphisms; when a homomorphism maps a manifold σ into itself, it is called an endomorphism, and an automorphism when it is both: a one-to-one mapping of σ into itself. Isomorphic systems, *i.e.*, any two systems mapped isomorphically upon each other, have the same structure; indeed nothing can be said about the structure of the one system that is not equally true for the other.

The *automorphisms* of a manifold with a well-defined structure form a *group*. Two sub-sets of the manifold that go over into each other by an automorphism deserve the name of *equivalent*. This is the precise idea at which Leibniz hints when he says that two such sub-sets are "indiscernible when each is considered in itself"; he recognized this general idea as lying behind the specific geometric notion of similitude. The general problem of relativity consists in nothing else but to find the group of automorphisms. Here then is an important lesson the mathematicians learned in the twentieth century: whenever you are concerned with a structured manifold, study its group of automorphisms. Also the inverse problem, which Felix Klein stressed in his famous Erlangen program (1872), deserves attention: Given a group of transformations of a manifold σ,

determine such relations or operations as are invariant with respect to the group.

If in studying a group of transformations we ignore the fact that it consists of transformations and look merely at the way in which any two of its transformations S, T give rise to a composite ST, we obtain the abstract composition schema of the group. Hence an *abstract group* is a set of elements (of unknown or irrelevant nature) for which an operation of composition is defined generating an element st from any two elements s, t such that the following axioms hold:

(1) There is a unit element e such that $es=se=s$ for every s.

(2) Every element s has an inverse s^{-1} such that $ss^{-1} = s^{-1}s = e$.

(3) The associative law $(st)u = s(tu)$ holds.

It is perhaps the most astonishing experience of modern mathematics how rich in consequences these three simple axioms are. A realization of an abstract group by transformations of a given manifold σ is obtained by associating with every element s of the group a transformation S of σ, $s \rightarrow S$, such that $s \rightarrow S$, $t \rightarrow T$ imply $st \rightarrow ST$. In general, the commutative law $st = ts$ will not hold. If it does, the group is called commutative or Abelian (after the Norwegian mathematician Niels Henrik Abel). Because composition of group elements in general does not satisfy the commutative law, it has proved convenient to use the term "ring" in the wider sense in which it does not imply the commutative law for multiplication. (However, in speaking of a field one usually assumes this law.)

The simplest mappings are the linear ones. They operate in a vector space. The vectors in our ordinary three-dimensional space are directed segments AB leading from a point A to a point B. The vector AB is considered equal to $A'B'$ if a parallel displacement (translation) carries AB into $A'B'$. In consequence of this convention one can add vectors and one can also multiply a vector by a number (integral, rational or even real). Addition satisfies the same axioms as enumerated for numbers in the table **T**, and it is also easy to formulate the axioms for the second operation. These axioms constitute the general axiomatic notion of vector space, which is therefore an algebraic and not a geometric concept. The numbers which serve as multipliers of the vectors may be the elements of any ring; this generality is actually required in the application of the axiomatic vector concept to topology. However, here we shall assume that they form a field. Then one sees at once that one can ascribe to the vector space a natural number n as its dimensionality in this sense: there exist n vectors $e_1, \cdots, e_n$ such that every vector may be expressed in one and only one way as a linear combination $x_1e_1 + \cdots + x_ne_n$, where the "coordinates" x_i are definite numbers of the field. In our three-dimensional space n equals 3, but mechanics and physics give ample occasion to use the general algebraic notion of an n-dimensional vector space for higher n.

The endomorphisms of a vector space are called its *linear mappings*; as such they allow composition ST (perform first the mapping S, then T), but they also allow addition and multiplication by numbers γ: if S sends the

arbitrary vector x into xS, T into xT, then $S + T$, γS are those linear mappings which send x into $(xS) + (xT)$ and $\gamma \cdot xS$, respectively. We must forego to describe how in terms of a vector basis $e_1, \cdots, e_n$ a linear mapping is represented by a square matrix of numbers.

Often rings occur—they are then called *algebras*—which are at the same time vector spaces, *i.e.*, for which three operations, addition of two elements, multiplication of two elements and multiplication of an element by a number, are defined in such manner as to satisfy the characteristic axioms. The linear mappings of an n-dimensional vector space themselves form such an algebra, called the complete matric algebra (in n dimensions). According to quantum mechanics the observables of a physical system form an algebra of special type with a non-commutative multiplication. In the hands of the physicists abstract algebra has thus become a key that unlocked to them the secrets of the atom. A realization of an abstract group by linear transformations of a vector space is called *representation*. One may also speak of representations of a ring or an algebra: in each case the representation can be described as a homomorphic mapping of the group or ring or algebra into the complete matric algebra (which indeed is a group and a ring and an algebra, all in one).

5. Finale. After spending so much time on the explanation of the notions I can be brief in my enumeration of some of the essential achievements for which they provided the tools. If g is a subgroup of the group G, one may identify elements s, t of G that are congruent mod g, *i.e.*, for which st^{-1} is in g; g is a "self-conjugate" subgroup if this process of identification carries G again into a group, the "factor group" G/g. The group-theoretic core of Galois' theory is a theorem due to C. Jordan and O. Hölder which deals with the several ways in which one may break down a given finite group G in steps $G = G_0, G_1, G_2, \cdots$, each G_i being a self-conjugate subgroup of the preceding group G_{i-1}. Under the assumption that this is done in as small steps as possible, the theorem states, the steps (factor groups) G_{i-1}/G_i ($i = 1, 2, \cdots$) in one such "composition series" are isomorphic to the steps, suitably rearranged, in a second such series. The theorem is very remarkable in itself, but perhaps the more so as its proof rests on the same argument by which one proves what I consider the most fundamental proposition in all mathematics, namely the fact that if you count a finite set of elements in two ways, you end up with the same number n both times. The Jordan-Hölder theorem in recent times received a much more natural and general formulation by (1) abandoning the assumption that the breaking down is done in the smallest possible steps, and (2) by admitting only such subgroups as are invariant with respect to a given set of endomorphic mappings of G. It thus has become applicable to infinite as well as finite groups, and provided a common denominator for quite a number of important algebraic facts.

The theory of representations of finite groups, the most systematic and substantial part of group theory developed shortly before the turn of the century

by G. Frobenius, taught us that there are only a few irreducible representations, of which all others are composed. This theory was greatly simplified after 1900 and later carried over, first to continuous groups that have the topological property of compactness, but then also to all infinite groups, with a restrictive imposition (called almost-periodicity) on the representations. With these generalizations one trespasses the limits of algebra, and a few more words will have to be said about it under the title analysis. New phenomena occur if representations of finite groups in fields of prime characteristic are taken into account, and from their investigation profound number-theoretic consequences have been derived. It is easy to embed a finite group into an algebra, and hence facts about representations of a group are best deduced from those of the embedding algebra. At the beginning of the century algebras seemed to be ferocious beasts of unpredictable behavior, but after fifty years of investigation they, or at least the variety called semi-simple, have become remarkably tame; indeed the wild things do not happen in these superstructures, but in the underlying commutative "number" fields. In the nineteenth century geometry seemed to have been reduced to a study of invariants of groups; Felix Klein formulated this standpoint explicitly in his Erlangen program. But the full linear group was practically the only group whose invariants were studied. We have now outgrown this limitation and no longer ignore all the other continuous groups one encounters in algebra, analysis, geometry and physics. Above all we have come to realize that the theory of invariants has to be subsumed under that of representations. Certain infinite discontinuous groups, like the unimodular and the modular groups, which are of special importance to number theory, witness Gauss' class theory of quadratic forms, have been studied with remarkable success and profound results. The macroscopic and microscopic symmetries of crystals are described by discontinuous groups of motions, and it has been proved for n dimensions, what had long been known for 3 dimensions, that in a certain sense there is but a finite number of possibilities for these crystallographic groups. In the nineteenth century Sophus Lie had reduced a continuous group to its "germ" of infinitesimal elements. These elements form a sort of algebra in which the associative law is replaced by a different type of law. A Lie algebra is a purely algebraic structure, especially if the numbers which act as multipliers are taken from an algebraically defined field rather than from the continuum of real numbers Ω. These Lie groups have provided a new playground for our algebraists.

The constructions of the mathematical mind are at the same time free and necessary. The individual mathematician feels free to define his notions and to set up his axioms as he pleases. But the question is, will he get his fellow-mathematicians interested in the constructs of his imagination. We can not help feeling that certain mathematical structures which have evolved through the combined efforts of the mathematical community bear the stamp of a necessity not affected by the accidents of their historical birth. Everybody who looks

at the spectacle of modern algebra will be struck by this complementarity of freedom and necessity.

PART II. ANALYSIS. TOPOLOGY. GEOMETRY. FOUNDATIONS.

6. Linear operators and their spectral decomposition. Hilbert space. A mechanical system of n degrees of freedom in stable equilibrium is capable of oscillations deviating "infinitely little" from the state of equilibrium. It is a fact of fundamental significance not only for physics but also for music that all these oscillations are superpositions of n "harmonic" oscillations with definite frequencies. Mathematically the problem of determining the harmonic oscillations amounts to constructing the principal axes of an ellipsoid in an n-dimensional Euclidean space. Representing the vectors x in this space by their coordinates $(x_1, x_2, \cdots, x_n)$ one has to solve an equation

$$x - \lambda \cdot Kx = 0,$$

where K denotes a given linear operator (= linear mapping); λ is the square of the unknown frequency ν of the harmonic oscillation, whereas the "eigenvector" x characterizes its amplitude. Define the scalar product (x, y) of two vectors x and y by the sum $x_1y_1 + \cdots + x_ny_n$. Our "affine" vector space is made into a metric one by assigning to any vector x the length $\|x\|$ given by $\|x\|^2 = (x, x)$, and this metric is the Euclidean one so familiar to us from the 3-dimensional case and epitomized by the "Pythagoras." The linear operator K is symmetric in the sense that $(x, Ky) = (Kx, y)$. The field of numbers in which we operate here is, of course, the continuum of all real numbers. Determination of the n frequencies ν or rather of the corresponding eigen-values $\lambda = \nu^2$ requires the solution of an algebraic equation of degree n (often known as the secular equation, because it first appeared in the theory of the secular perturbations of the planetary system).

More important in physics than the oscillations of a mechanical system of a finite number of degrees of freedom are the oscillations of continuous media, as the mechanical-acoustical oscillations of a string, a membrane or a 3-dimensional elastic body, and the electromagnetic-optical oscillations of the "ether." Here the vectors with which one has to operate are continuous functions $x(s)$ of a point s with one or several coordinates that vary over a given domain, and consequently K is a linear *integral* operator. Take for instance a straight string of length 1, the points of which are distinguished by a parameter s varying from 0 to 1. Here (x, x) is the integral $\int_0^1 x^2(s) \cdot ds$, and the problem of harmonic oscillations (which first suggested to the early Greeks the idea of a universe ruled by harmonious mathematical laws) takes the form of the integral equation

$$[1] \qquad x(s) - \lambda \int_0^1 K(s, t)x(t)dt = 0, \qquad (0 \leqq s \leqq 1),$$

where

$$[1'] \qquad K(s, t) = \left(\frac{a}{\pi}\right)^2 \cdot \begin{cases} s(1-t) & \text{for } s \leqq t \\ (1-s)t & \text{for } s \geqq t \end{cases},$$

and a is a constant determined by the physical conditions of the string. The solutions are

$$\lambda = (na)^2, \qquad x(s) = \sin n\pi s,$$

where n is capable of all positive integral values $1, 2, 3, \cdots$. This fact that the frequencies of a string are integral multiples na of a ground frequency a is the basic law of musical harmony. If one prefers an optical to an acoustic language one speaks of the *spectrum* of eigen-values λ.

After Fredholm at the very close of the 19th century had developed the theory of linear integral equations it was Hilbert who in the next decade established the general *spectral theory of symmetric linear operators* K. Only twenty years earlier it had required the greatest mathematical efforts to prove the existence of the ground frequency for a membrane, and now constructive proofs for the existence of the whole series of harmonic oscillations and their characteristic frequencies were given under very general assumptions concerning the oscillating medium. This was an event of great consequence both in mathematics and theoretical physics. Soon afterwards Hilbert's approach made it possible to establish those asymptotic laws for the distribution of eigen-values the physicists had postulated in their statistical treatment of the thermodynamics of radiation and elastic bodies.

Hilbert observed that an arbitrary continuous function $x(s)$ defined in the interval $0 \leqq s \leqq 1$ may be replaced by the sequence

$$x_n = \sqrt{2} \int_0^1 x(s) \cdot \sin n\pi s \cdot ds, \qquad n = 1, 2, 3, \cdots,$$

of its Fourier coefficients. Thus there is no inner difference between a vector space whose elements are functions $x(s)$ of a continuous variable and one whose elements are infinite sequences of numbers $(x_1, x_2, x_3, \cdots)$. The square of the "length," $\int_0^1 x^2(s) \cdot ds$ equals $x_1^2 + x_2^2 + x_3^2 + \cdots$. Between the two forms in which one may pass from a finite sum to a limit, the infinite sum $a_1 + a_2 + a_3 + \cdots$ and the integral $\int_0^1 a(s) \cdot ds$, there is therefore here no essential difference. Thus an axiomatic formulation is called for. To the axioms for an (affine) vector space one adds the postulate of the existence of a scalar product (x, y) of any two vectors (x, y) with the properties characteristic for Euclidean metric: (x, y) is a number depending linearly on either of the two argument vectors x and y; it is symmetric, $(x, y) = (y, x)$; and $(x, x) = \|x\|^2$ is positive except for $x = 0$. The axiom of finite dimensionality is replaced by a denumerability axiom of more general character. All operations in such a space are greatly facilitated if it is assumed to be complete in the same sense that the system of real numbers is complete; *i.e.*, if the following is true: Given a

"convergent" sequence $x', x'', \cdots$ of vectors, namely, one for which $\|x^{(m)} - x^{(n)}\|$ tends to zero with m and n tending to infinity, there exists a vector a toward which this sequence converges, $\|x^{(n)} - a\| \to 0$ for $n \to \infty$. A non-complete vector space can be made complete by the same construction by which the system of rational numbers is completed to form that of real numbers. Later authors have coined the name "Hilbert space" for a vector space satisfying these axioms.

Hilbert himself first tackled only integral operators in the strict sense as exemplified by [1]. But soon he extended his spectral theory to a far wider class, that of bounded (symmetric) linear operators in Hilbert space. Boundedness of the linear operator requires the existence of a constant M such that $\|Kx\|^2 \leqq M \cdot \|x\|^2$ for all vectors x of finite length $\|x\|$. Indeed the restriction to integral operators would be unnatural since the simplest operator, the identity $x \to x$, is not of this type. And now one of those events happened, unforeseeable by the wildest imagination, the like of which could tempt one to believe in a pre-established harmony between physical nature and mathematical mind: Twenty years after Hilbert's investigations *quantum mechanics* found that the observables of a physical system are represented by the linear symmetric operators in a Hilbert space and that the eigen-values and eigen-vectors of that operator which represents *energy* are the energy levels and corresponding stationary quantum states of the system. Of course this quantum-physical interpretation added greatly to the interest in the theory and led to a more scrupulous investigation of it, resulting in various simplifications and extensions.

Oscillations of continua, the boundary value problems of classical physics and the problem of energy levels in quantum physics, are not the only titles for applications of the theory of integral equations and their spectra. One other somewhat isolated application is the solution of *Riemann's monodromy problem* concerning analytic functions of a complex variable z. It concerns the determination of n analytic functions of z which remain regular under analytic continuation along arbitrary paths in the z-plane provided these avoid a finite number of singular points, whereas the functions undergo a given linear transformation with constant coefficients when the path circles one of these points.

Another surprising application is to the establishment of the fundamental facts, in particular of the completeness relation, in the theory of *representations of continuous compact groups*. The simplest such group consists of the rotations of a circle, and in that case the theory of representations is nothing but the theory of the so-called Fourier series, which expresses an arbitrary periodic function $f(s)$ of period 2π in terms of the harmonic oscillations

$$\cos ns, \qquad \sin ns, \qquad\qquad n = 0, 1, 2, \cdots.$$

In Nature functions often occur with hidden non-commensurable periodicities. The mathematician Harald Bohr, the brother of the physicist Niels Bohr, prompted by certain of his investigations concerning the Riemann zeta function, developed the general mathematical theory of such *almost periodic func-*

tions. One may describe his theory as that of almost periodic representations of the simplest continuous group one can imagine, namely, the group of all translations of a straight line. His main results could be carried over to arbitrary groups. No restriction is imposed on the group, but the representations one studies are supposed to be almost-periodic. For a function $x(s)$, the argument s of which runs over the group elements, while its values are real or complex numbers, almost-periodicity amounts to the requirement that the group be compact in a certain topology induced by the function. This relative compactness instead of absolute compactness is sufficient. Even so the restriction is severe. Indeed the most important representations of the classical continuous groups are not almost-periodic. Hence the theory is in need of further extension, which has busied a number of American and Russian mathematicians during the last decade.

7. Lebesgue's integral. Measure theory. Ergodic hypothesis. Before turning to other applications of operators in Hilbert space I must mention the, in all probability final, form given to the idea of integration by Lebesgue at the beginning of our century. Instead of speaking of the area of a piece of the 2-dimensional plane referred to coordinates x, y, or the volume of a piece of the 3-dimensional Euclidean space, we use the neutral term *measure* for all dimensions. The notions of measure and *integral* are interconnected. Any piece of space, any set of space points can be described by its characteristic function $\chi(P)$, which equals 1 or 0 according to whether the point P belongs or does not belong to the set. The measure of the point set is the integral of this characteristic function. Before Lebesgue one first defined the integral for continuous functions; the notion of measure was secondary; it required transition from continuous to such discontinuous functions as $\chi(P)$. Lebesgue goes the opposite and perhaps more natural way: for him measure comes first and the integral second. The one-dimensional space is sufficient for an illustration. Consider a real-valued function $y = f(x)$ of a real variable x which maps the interval $0 \leqq x \leqq 1$ into a finite interval $a \leqq y \leqq b$. Instead of subdividing the interval of the argument x Lebesgue subdivides the interval (a, b) of the dependent variable y into a finite number of small subintervals $a_i \leqq y < a_{i+1}$, say of lengths $< \epsilon$, and then determines the measure m_i of the set S_i on the x-axis, the points of which satisfy the inequality $a_i \leqq f(x) < a_{i+1}$. The integral lies between the two sums $\sum_i a_i m_i$ and $\sum_i a_{i+1} m_i$ which differ by less than ϵ, and thus can be computed with any degree of accuracy. In determining the measure of a point set—and this is the more essential modification—Lebesgue covers the set with infinite sequences, rather than finite ensembles, of intervals. Thus, to the set of rational x in the interval $0 \leqq x \leqq 1$ no measure could be ascribed before Lebesgue. But these rational numbers can be arranged in a denumerable sequence $a_1, a_2, a_3, \cdots$, and, after choosing a positive number ϵ as small as one likes, one can surround the point a_n by an interval of length $\epsilon/2^n$ with the center a_n. Thus the whole set of rational points is enclosed in a

sequence of intervals of total length

$$\epsilon(1/2 + 1/2^2 + 1/2^3 + \cdots) = \epsilon;$$

and according to Lebesgue's definition its measure is therefore less than (the arbitrary positive) ϵ and hence zero. The notion of *probability* is tied to that of measure, and for this reason mathematical statisticians are deeply interested in measure theory. Lebesgue's idea has been generalized in several directions. The two fundamental operations one can perform with sets are: forming the intersection and the union of given sets, and thus sets may be considered as elements of a "*Boolean algebra*" with these two operations, the properties of which may be laid down in a number of axioms resembling the arithmetical axioms for addition and multiplication. Hence one of the questions which has occupied the more axiomatically minded among the mathematicians and statisticians is concerned with the introduction of measure in abstract Boolean algebras.

Lebesgue's integral is important in our present context, because those real-valued functions $f(x)$ of a real variable x ranging over the interval $0 \leqq x \leqq 1$, the squares of which are Lebesgue-integrable, form a complete Hilbert space —provided two functions $f(x)$, $g(x)$ are considered equal if those values x for which $f(x) \neq g(x)$ form a set of measure zero (Riesz-Fischer theorem).

The mechanical equations for a system of n degrees of freedom in Hamilton's form uniquely determine the state tP at the moment t if the state P at the moment $t = 0$ is given. Such is the precise formulation of the law of causality in mechanics. The possible states P form the points of a $(2n)$-dimensional phase space, and for a fixed t and an arbitrary P the transition $P \to tP$ is a measure-preserving mapping (t). These transformations form a group: $(t_1)(t_2) = (t_1 + t_2)$. For a given P and a variable t the point tP describes the consecutive states which this system assumes if at the moment $t = 0$ it is in the state P. Considering P as a particle of a $(2n)$-dimensional fluid which fills the phase-space and ascribing to the particle P the position tP at the time t, one obtains the picture of an incompressible fluid in stationary flow. The statistical derivation of the laws of thermodynamics makes use of the so-called *ergodic hypothesis* according to which the path of an arbitrary individual particle P (excepting initial states P which form a set of measure zero) covers the phase-space (or at least that $(2n - 1)$-dimensional sub-space of it where the energy has a given value) everywhere dense, so that in the course of its history the probability of finding it in this or that part of the space is the same for any parts of equal measure. Nineteenth century mathematics seemed to be a long way off from proving this hypothesis with any degree of generality. Strangely enough it was proved shortly after the transition from classical to quantum mechanics had rendered the hypothesis almost valueless, and it was proved by making use of the mathematical apparatus of quantum physics. Under the influence of the mapping (t), $P \to tP$, any function $f(P)$ in phase-space is transformed into the function $f' = U_t f$, defined by the equation $f'(tP) = f(P)$. The U_t form a group of

operators in the Hilbert space of arbitrary functions $f(P)$, $U_{t_1}U_{t_2} = U_{t_1+t_2}$, and application of spectral decomposition to this group enabled J. von Neumann to deduce the ergodic hypothesis with two provisos: (1) Convergence of a sequence of functions $f_n(P)$ toward a function $f(P)$, $f_n \rightarrow f$, is understood (as it would in quantum mechanics, namely) as convergence in Hilbert space where it means that the total integral of $(f_n - f)^2$ tends to zero with $n \rightarrow \infty$; (2) one assumes that there are no subspaces of the phase-space which are invariant under the group of transformations (t) except those spaces that are in Lebesgue's sense equal either to the empty or the total space. Shortly afterwards proofs were also given for other interpretations of the notion of convergence.

The laws of nature can either be formulated as differential equations or as "principles of variation" according to which certain quantities assume extremal values under given conditions. For instance, in an optically homogeneous or non-homogeneous medium the light travels along that road from a given point A to a given point B for which the time of travel assumes minimal value. In potential theory the quantity which assumes a minimum is the so-called Dirichlet integral. Attempts to establish directly the existence of a minimum had been discouraged by Weierstrass' criticism in the 19th century. Our century, however, restored the direct methods of the Calculus of Variation to a position of honor after Hilbert in 1900 gave a direct proof of the Dirichlet principle and later showed how it can be applied not only in establishing the fundamental facts about functions and integrals ("algebraic" functions and "abelian" integrals) on a compact Riemann surface (as Riemann had suggested 50 years earlier) but also for deriving the basic propositions of the *theory of uniformization*. That theory occupies a central position in the theory of functions of one complex variable, and the first decade of the 20th century witnessed the first proofs by P. Koebe and H. Poincaré of these propositions conjectured about 25 years before by Poincaré himself and by Felix Klein. As in an Euclidean vector space of finite dimensionality, so in the Hilbert space of infinitely many dimensions, this fact is true: Given a linear (complete) subspace E, any vector may be split in a uniquely determined manner into a component lying in E (orthogonal projection) and one perpendicular to E. Dirichlet's principle is nothing but a special case of this fact. But since the function-theoretic applications of orthogonal projection in Hilbert space which we alluded to are closely tied up with topology we had better turn first to a discussion of this important branch of modern mathematics: topology.

8. Topology and harmonic integrals. Essential features of the modern approach to *topology* can be brought to light in its connection with the, only recently developed, theory of *harmonic integrals*. Consider a stationary magnetic field h in a domain G which is free from electric currents. At every point of G it satisfies two differential conditions which in the usual notations of vector analysis are written in the form div $h = 0$, rot $h = 0$. A field of this type is called harmonic. The second of these conditions states that the line integral of

h along a closed curve (cycle) C, $\int_C h$, vanishes provided C lies in a sufficiently small neighborhood of an arbitrary point of G. This implies $\int_C h = 0$ for any cycle C in G that is the boundary of a surface in G. However, for an arbitrary cycle C in G the integral is equal to the electric current surrounded by C.

Let the phrase "C homologous to zero," $C \sim 0$, indicate that the cycle C in G bounds a surface in G. One can travel over a cycle C in the opposite sense, thus obtaining $-C$, or travel over it $2, 3, \cdots$ times, thus obtaining $2C, 3C, \cdots$; and cycles may be added and subtracted from each other (if one does not insist that cycles are of one piece). Two cycles C, C' are called homologous, $C \sim C'$, if $C - C' \sim 0$. Note that $C \sim 0$, $C' \sim 0$ imply $-C \sim 0$, $C + C' \sim 0$. Hence the cycles form a commutative group under addition, the "Betti group," if homologous cycles are considered as one and the same group element. These notions of *cycles and their homologies* may be carried over from a three-dimensional domain in Euclidean space to any n-dimensional manifold, in particular to closed (compact) manifolds like the two-dimensional surfaces of the sphere or the torus; and on an n-dimensional manifold we can speak not only of 1-dimensional, but also of 2-, 3-, $\cdots$, n-dimensional cycles. The notion of a harmonic vector field permits a similar generalization, harmonic tensor field (harmonic form) of rank r ($r = 1, 2, \cdots, n$), provided the manifold bears a Riemannian metric, an assumption the meaning of which will be discussed later in the section on geometry. Any tensor field (linear differential form) of rank r may be integrated over an r-dimensional cycle.

The fundamental problem of *homology theory* consists in determining the structure of the Betti group, not only for 1-, but also for 2-, $\cdots$, n-dimensional cycles, in particular in determining the number of linearly independent cycles (Betti number). [ν cycles $C_1, \cdots, C_\nu$ are linearly independent if there exists no homology $k_1C_1 + \cdots + k_\nu C_\nu \sim 0$ with integral coefficients k except $k_1 = \cdots = k_\nu = 0$.] The fundamental theorem for harmonic forms on compact manifolds states that, given ν linearly independent cycles $C_1, \cdots, C_\nu$, there exists a harmonic form h with pre-assigned periods

$$\int_{C_1} h = \pi_1, \cdots, \qquad \int_{C_\nu} h = \pi_\nu.$$

H. Poincaré developed the algebraic apparatus necessary to formulate exactly the notions of cycle and homology. In the course of the twentieth century it turned out that in most problems co-homologies are easier to handle than homologies. I illustrate this for 1-dimensional cycles. A line C_1 leading from a point p_1 to p_2, when followed by a line C_2 leading from p_2 to a third point p_3, gives rise to a line $C_1 + C_2$ leading from p_1 to p_3. The line integral $\int_C h$ of a given vector field h along an arbitrary (closed or open) line C is an additive function $\phi(C)$ of C, $\phi(C_1 + C_2) = \phi(C_1) + \phi(C_2)$. If moreover rot h vanishes everywhere, then $\phi(C) = 0$ for any closed line C that lies in a sufficiently small neighborhood of a point, whatever this point may be. Any real-

valued function ϕ satisfying these two conditions may be called an abstract integral. The co-homology $\phi \sim 0$ means that $\phi(C) = 0$ for any closed line C, and thus it is clear what the co-homology $k_1\phi_1 + \cdots + k_\nu\phi_\nu \sim 0$ with arbitrary real coefficients $k_1, \cdots, k_\nu$ means. The homology $C \sim 0$ could now be defined, not by the condition that the cycle C bounds, but by the requirement that $\phi(C) = 0$ for every abstract integral ϕ. With the convention that any two abstract integrals ϕ, ϕ' are identified if $\phi - \phi' \sim 0$, these integrals form a vector space, and the dimensionality of this vector space is now introduced as the Betti number. And the fundamental theorem for harmonic integrals on a compact manifold now asserts that for any given abstract integral ϕ there exists one and only one harmonic vector field h whose integral is cohomologous to ϕ, $\int_C h = \phi(C)$, for every cycle C (realization of the abstract integral in concreto by a harmonic integral).

J. W. Alexander discovered an important result connecting the Betti numbers of a manifold M that is embedded in the n-dimensional Euclidean space R_n with the Betti numbers of the complement $R_n - M$ (*Alexander's duality theorem*).

The difficulties of topology spring from the double aspect under which one can consider continuous manifolds. Euclid looked upon a figure as an assemblage of a finite number of geometric elements, like points, straight lines, circles, planes, spheres. But after replacing each line or surface by the set of points lying on it one may also adopt the set-theoretic view that there is only one sort of elements, points, and that any (in general infinite) set of points can serve as a figure. This modern standpoint obviously gives geometry far greater generality and freedom. In topology, however, it is not necessary to descend to the points as the ultimate atoms, but one can construct the manifold like a building from "blocks" or cells, and a finite number of such cells serving as units will do, provided the manifold is compact. Thus it is possible here to revert to a treatment in Euclid's "finitistic" style (combinatorial topology).

On the first standpoint, manifold as a point set, the task is to formulate that *continuity* by which a point p approaching a given point p_0 becomes gradually indistinguishable from p_0. This is done by associating with p_0 the *neighborhoods* of p_0, an infinite shrinking sequence of sub-sets $U_1 \supset U_2 \supset U_3 \supset \cdots$, all containing p_0. [$U \supset V$ means: the set U contains V.] For example, in a plane referred to Cartesian coordinates x, y we may choose as the nth neighborhood U_n of a point $p_0 = (x_0, y_0)$ the interior of the circle of radius $1/2^n$ around p_0. The notion of convergence, basic for all continuity considerations, is defined in terms of the sequence of neighborhoods as follows: A sequence of points $p_1, p_2, \cdots$ converges to p_0 if for every natural number n there is an N so that all points p_ν with $\nu > N$ lie in the nth neighborhood U_n of p_0. Of course, the choice of the neighborhoods U_n is arbitrary to a certain extent. For instance, one could also have chosen as the nth neighborhood V_n of (x_0, y_0) the square of side $2/n$ around (x_0, y_0), to which a point (x, y) belongs, if

$$-1/n < x - x_0 < 1/n, \qquad -1/n < y - y_0 < 1/n.$$

However the sequence V_n is equivalent to the sequence U_n in the sense that for every n there is an n' such that $U_{n'} \subset V_n$ (and thus $U_\nu \subset V_n$ for $\nu \geqq n'$), and also for every m an m' such that $V_{m'} \subset U_m$; and consequently the notion of convergence for points is the same, whether based on the one or the other sequence of neighborhoods. It is clear how to define continuity of a mapping of one manifold into another. A one-to-one mapping of two manifolds upon each other is called topological if continuous in both directions, and two manifolds that can be mapped topologically upon each other are topologically equivalent. Topology investigates such properties of manifolds as are invariant with respect to topological mappings (in particular with respect to continuous deformations).

A continuous function $y=f(x)$ may be approximated by piecewise linear functions. The corresponding device in higher dimensions, the method of *simplicial approximations* of a given continuous mapping of one manifold into another, is of great importance in set-theoretic topology. It has served to develop a general *theory of dimensions*, to prove the topological invariance of the Betti groups, to define the decisive notion of the degree of mapping ("Abbildungsgrad," L. E. J. Brouwer) and to prove a number of interesting fixed point theorems. For instance, a continuous mapping of a square into itself has necessarily a fixed point, *i.e.*, a point carried by the mapping into itself. Given two continuous mappings of a (compact) manifold M into another M', one can ask more generally for which points p on M both images on M' coincide. A famous formula by S. Lefschetz relates the "total index" of such points with the homology theory of cycles on M and M'.

Application of fixed point theorems to functional spaces of infinitely many dimensions has proved a powerful method to establish the existence of solutions for non-linear differential equations. This is particularly valuable, because the hydrodynamical and aerodynamical problems are almost all of this type.

Poincaré found that a satisfactory formulation of the homology theory of cycles was possible only from the second standpoint where the n-dimensional manifold is considered as a conglomerate of n-dimensional cells. The boundary of an n-dimensional cell (n-cell) consists of a finite number of $(n-1)$-cells, the boundary of an $(n-1)$-cell consists of a finite number of $(n-2)$-cells, *etc.* The *combinatorial skeleton* of the manifold is obtained by assigning symbols to these cells and then stating in terms of their symbols which $(i-1)$-cells belong to the boundary of any of the occurring i-cells $(i = 1, 2, \cdots, n)$. From the cells one descends to the points of the manifold by a repeated process of sub-division which catches the points in an ever finer net. Since this subdivision proceeds according to a fixed combinatorial scheme, the manifold is in topological regard completely fixed by its combinatorial skeleton. And at once the question arises under what circumstances two given combinatorial skele-

tons represent the same manifold, *i.e.*, lead by iterated sub-division to topologically equivalent manifolds. We are far from being able to solve this fundamental problem. Algebraic topology, which operates with the combinatorial skeletons, is in itself a rich and beautiful theory, linked in various ways with the basic notions and theorems of algebra and group theory.

The connection between algebraic and set-theoretic topology is fraught with serious difficulties which are not yet overcome in a quite satisfactory manner. So much, however, seems clear that one had better start, not with a division into cells, but with a covering by patches which are allowed to overlap. From such a pattern the fundamental topologically invariant concepts are to be developed. The above notion of an abstract integral, which relates homology and co-homology, is an indication; it can indeed be used for a direct proof of the invariance of the first Betti number without the tool of simplicial approximation.

9. Conformal mapping, meromorphic functions, Calculus of Variation in the large. Homology theory, in combination with the Dirichlet principle or the method of orthogonal projection in Hilbert space, leads to the theory of harmonic integrals, in particular for the lowest dimension $n=2$ to the theory of abelian integrals on Riemann surfaces. But for Riemann surfaces the Dirichlet principle also yields the fundamental facts concerning uniformization of analytic functions of one variable if one combines it with the *homotopy* (not homology) *theory* of closed curves. Whereas a cycle is homologous to zero if it bounds, it is homotopic to zero if it can be contracted into a point by continuous deformation. The homotopy theory of 1- and more-dimensional cycles has recently come to the fore as an important branch of topology, and the group-theoretic aspect of homotopy has led to some surprising discoveries in abstract group theory. Homotopy of 1-dimensional cycles is closely related with the idea of the *universal covering manifold* of a given manifold. Given a continuous mapping $p \rightarrow p'$ of one manifold M into another M', the point p' may be considered as the trace or projection in M' of the arbitrary point p on M, and thus M becomes a manifold covering M'. There may be no point or several points p on M which lie over a given point p' of M' (which are mapped into p'). The mapping is without ramifications if for any point p_0 of M it is one-to-one (and continuous both ways) in a sufficiently small neighborhood of p_0. Let p_0 be a point on M, p_0' its trace on M', and C' a curve on M' beginning at p_0'. If M covers M' without ramifications we can follow this curve on M by starting at p_0, at least up to a certain point where we would run against a "boundary of M relative to M'." Of chief interest are those covering manifolds M over a given M' for which this never happens and which therefore cover M' without ramifications and relative boundaries. The best way of defining the central topological notion "simply connected" is by describing a simply connected manifold as one having no other unramified unbounded covering but itself. There is a strongest of all unramified unbounded covering manifolds, the universal

covering manifold, which can be described by the statement that on it a curve C is closed only if its trace C' is (closed and) homotopic to zero. The proof of the fundamental theorem on uniformization consists of two parts: (1) constructing the universal covering manifold of the given Riemann surface, (2) constructing by means of the Dirichlet principle a one-to-one conformal mapping of the covering manifold upon the interior of a circle of finite or infinite radius.

All we have discussed so far in our account of analysis, is in some way tied up with operators and projections in Hilbert space, the analogue in infinitely many dimensions of Euclidean space. In H. Minkowski's *Geometry of Numbers* distances $|AB|$, which are different from the Euclidean distance but satisfy the axioms that $|BA| = |AB|$ and that in a triangle ABC the inequality $|AC| \leq |AB| + |BC|$ holds, were used to great advantage for obtaining numerous results concerning the solvability of inequalities by integers. We do not find time here to report on the progress of this attractive branch of number theory during the last fifty years. In infinitely many dimensions spaces endowed with a metric of this sort, of a more general nature than the Euclid-Hilbert metric, have been introduced by Banach, not however for number-theoretic but for purely analytic purposes. Whether the importance of the subject justifies the large number of papers written on *Banach spaces* is perhaps questionable.

The Dirichlet principle is but the simplest example of the direct methods of the Calculus of Variation as they came into use with the turn of the century. It was by these methods that the theory of *minimal surfaces*, so closely related to that of analytic functions, was put on a new footing. What we know about non-linear differential equations has been obtained either by the topological fixed point method (see above) or by the so-called continuity method or by constructing their solutions as extremals of a suitable functional.

A continuous function on an n-dimensional compact manifold assumes somewhere a minimum and somewhere else a maximum value. Interpret the function as altitude. Besides summit (local maximum) and bottom (local minimum) one has the further possibility of a saddle point (pass) as a point of "stationary" altitude. In n dimensions the several possibilities are indicated by an inertial index k which is capable of the values $k = 0, 1, 2, \cdots, n$, the value $k = 0$ a minimum and $k = n$ characterizing a maximum. Marston Morse discovered the inequality $M_k \geqq B_k$ between the number M_k of stationary points of index k and the Betti number B_k of linearly independent homology classes of k-dimensional cycles. In their generalization to functional spaces these relations have opened a line of study adequately described as *Calculus of Variation in the large.*

Development of the theory of uniformization for analytic functions led to a closer investigation of *conformal mapping* of 2-dimensional manifolds in the large, which resulted in a number of theorems of surprising simplicity and beauty. In the same field there is to register an enormous extension of our

knowledge of the behavior of *meromorphic functions*, *i.e.*, single-valued analytic functions of the complex variable z which are regular everywhere with the exception of isolated "poles" (points of infinity). Towards the end of the previous century Riemann's zeta function had provided the stimulus for a deeper study of "entire functions" (functions without poles). The greatest stride forward, both in methods and results, was marked by a paper on meromorphic functions published in 1925 by the Finnish mathematician Rolf Nevanlinna. Besides meromorphic functions in the z-plane one can study such functions on a given Riemann surface; and in the way in which the theory of algebraic functions (equal to meromorphic functions on a compact Riemann surface) as a theory of algebraic curves in two complex dimensions may be generalized to any number of dimensions, so one can pass from meromorphic functions to meromorphic curves.

The theory of *analytic functions of several complex variables*, in spite of a number of deep results, is still in its infancy.

10. Geometry. After having dealt at some length with the problems of analysis and topology I must be brief about geometry. Of subjects mentioned before, minimal surfaces, conformal mapping, algebraic manifolds and the whole of topology could be subsumed under the title of geometry. In the domain of *elementary axiomatic geometry* one strange discovery, that of von Neumann's pointless "continuous geometries" stands out, because it is intimately interrelated with quantum mechanics, logic and the general algebraic theory of "lattices." The 1-, 2-, $\cdots$, n-dimensional linear manifolds of an n-dimensional vector space form the 0-, 1-, $\cdots$, $(n-1)$-dimensional linear manifolds in an $(n-1)$-dimensional projective point space. The usual axiomatic foundation of projective geometry uses the points as the primitive elements or atoms of which the higher-than-zero-dimensional manifolds are composed. However, there is possible a treatment where the linear manifolds of all dimensions figure as elements, and the axioms deal with the relation "B contains A" ($A \subset B$) between these elements and the operation of intersection, $A \cap B$, and of union, $A \cup B$, performed on them; the union $A \cup B$ consists of all sums $x+y$ of a vector x in A and a vector y in B. In quantum logic this relation and these operations correspond to the relation of implication ("The statement A implies B") and the operations 'and,' 'or' in classical logic. But whereas in classical logic the distributive law

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$$

holds, this is not so in quantum logic; it must be replaced by the weaker axiom:

$$\text{If} \quad C \subset A \quad \text{then} \quad A \cap (B \cup C) = (A \cap B) \cup C.$$

On formulating the axioms without the implication of finite dimensionality one will come across several possibilities; one leads to the Hilbert space in which quantum mechanics operates, another to von Neumann's continuous geometry

with its continuous scale of dimensions, in which elements of arbitrarily low dimensions exist but none of dimension zero.

The most important development of geometry in the twentieth century took place in differential geometry and was stimulated by general relativity, which showed that the world is a 4-dimensional manifold endowed with a Riemannian metric. A piece of an n-dimensional manifold can be mapped in one-to-one continuous fashion upon a piece of the n-dimensional "arithmetical space" which consists of all n-uples $(x_1, x_2, \cdots, x_n)$ of real numbers x_i. A *Riemann metric* assigns to a line element which leads from the point $P = (x_1, \cdots, x_n)$ to the infinitely near point $P' = (x_1 + dx_1, \cdots, x_n + dx_n)$ a distance ds the square of which is a quadratic form of the relative coordinates dx_i,

$$ds^2 = \sum g_{ij}dx_idx_j, \qquad (i, j = 1, \cdots, n)$$

with coefficients g_{ij} depending on the point P but not on the line element. This means that, in the infinitely small, Pythagoras' theorem and hence Euclidian geometry are valid, but in general not in a region of finite extension. The line elements at a point may be considered as the infinitesimal vectors of an n-dimensional vector space in P, the tangent space or the compass at P; indeed an arbitrary (differentiable) transformation of the coordinates x_i induces a linear transformation of the components dx_i of any line element at a given point P. As Levi-Civita found in 1915 the development of Riemannian geometry hinges on the fact that a Riemannian metric uniquely determines an infinitesimal parallel displacement of the vector compass at P to any infinitely near point P'. From this a general scheme for differential geometry arose in which each point P of the manifold is associated with a homogeneous space Σ_P described by a definite group of "authomorphisms," this space now taking over the role of the tangent space (whose group of authomorphisms consists of all non-singular linear transformations). One assumes that one knows how this associated space Σ_P is transferred by infinitesimal displacement to the space $\Sigma_{P'}$ associated with any infinitely near point P'. The most fundamental notion of Riemannian geometry, that of curvature, which figures so prominently in Einstein's equations of the gravitational field, can be carried over to this general scheme. Thus one has erected general differential affine, projective, conformal, geometries, *etc*. One has also tried by their structures to account for the other physical fields existing in nature beside the gravitational one, namely the electromagnetic field, the electronic wave-field and further fields corresponding to the several kinds of elementary particles. But it seems to the author that so far all such speculative attempts of building up a unified field theory have failed. There are very good reasons for interpreting gravitation in terms of the basic concepts of differential geometry. But it is probably unsound to try to "geometrize" all physical entities.

Differential geometry in the large is an interesting field of investigation which relates the differential properties of a manifold with its topological structure. The schema of differential geometry explained above with its associated spaces

Σ_P and their displacements has a purely topological kernel which has recently developed under the name of *fibre spaces* into an important topological technique.

Our account of progress made during the last fifty years in analysis, geometry and topology had to touch on many special subjects. It would have failed completely had it not imparted to the reader some feeling of the close relationship connecting all these mathematical endeavors. As the last example of fibre spaces (beside many others) shows, this unity in diversity even makes a clear-cut division into analysis, geometry, topology (and algebra) practically impossible.

11. Foundations. Finally a few words about the *foundations of mathematics.* The nineteenth century had witnessed the critical analysis of all mathematical notions including that of natural numbers to the point where they got reduced to pure logic and the ideas "set" and "mapping." At the end of the century it became clear that the unrestricted formation of sets, sub-sets of sets, sets of sets *etc.*, together with an unimpeded application to them as to the original elements of the logical quantifiers "there exists" and "all" [cf. the sentences: the (natural) number n is even if there exists a number x such that $n = 2x$; it is odd if n is different from $2x$ for all x] inexorably leads to antinomies. The three most characteristic contributions of the twentieth century to the solution of this Gordian knot are connected with the names of L. E. J. Brouwer, David Hilbert and Kurt Gödel. Brouwer's critique of "mathematical existentialism" not only dissolved the antinomies completely but also destroyed a good part of classical mathematics that had heretofore been universally accepted.

If only the historical event that somebody has succeeded in constructing a (natural) number n with the given property P can give a right to the assertion that "there exists a number with that property" then the alternative that there either exists such a number or that all numbers have the opposite property non-P is without foundation. The principle of excluded middle for such sentences may be valid for God who surveys the infinite sequence of all natural numbers, as it were, with one glance, but not for human logic. Since the quantifiers "there is" and "all" are piled upon each other in the most manifold way in the formation of mathematical propositions, Brouwer's critique makes almost all of them meaningless, and therefore Brouwer set out to build up a new mathematics which makes no use of that logical principle. I think that everybody has to accept Brouwer's critique who wants to hold on to the belief that mathematical propositions tell the sheer truth, truth based on evidence. At least Brouwers' opponent, Hilbert, accepted it tacitly. He tried to save classical mathematics by converting it from a system of meaningful propositions into a game of meaningless formulas, and by showing that this game never leads to two formulas, F and non-F, which are inconsistent. Consistency, not truth, is his aim. His attempts at proving consistency revealed the astonishingly complex logical structure of mathematics. The first steps were promising indeed. But

then Gödel's discovery cast a deep shadow over Hilbert's enterprise. Consistency itself may be expressed by a formula. What Gödel showed was this: If the game of mathematics is actually consistent then the formula of consistency cannot be proved within this game. How can we then hope to prove it at all?

This is where we stand now. It is pretty clear that our theory of the physical world is not a description of the phenomena as we perceive them, but is a bold symbolic construction. However, one may be surprised to learn that even mathematics shares this character. The success of the anti-phenomenological constructive method is undeniable. And yet the ultimate foundations on which it rests remain a mystery, even in mathematics.

MATHEMATICAL NOTES

EDITED BY F. A. FICKEN, *New York University*

Material for this department should be sent to F. A. Ficken, Institute for Mathematics and Mechanics, New York University, 45 Fourth Ave., New York 3, New York.

ON EVEN NUMBERS m DIVIDING 2^m-2

N. G. W. H. BEEGER, Amsterdam, Netherlands

In a recent paper P. Erdös [1] notes that D. H. Lehmer has recently detected the first case of an even number $m=2.73.1103=161038$ which satisfies the congruence

$$2^m - 2 \equiv 0 \pmod{m}. \tag{1}$$

It is the purpose of this paper to give three more such numbers and to prove that there exists an infinity.

In the first place it is clear from (1) that m is not divisible by 4 so that setting $m=2n$ (n odd), congruence (1) is equivalent to

$$2^{2n-1} - 1 \equiv 0 \pmod{n}. \tag{2}$$

In what follows we denote by e_p the exponent of 2 (mod p), p a prime, that is, the least positive x for which $2^x \equiv 1$ (mod p). It is well known that e_p divides $p-1$ as well as any other x for which $2^x \equiv 1$ (mod p). In particular if p is any prime factor of n in (2) then clearly e_p is odd since it divides $2n-1$.

THEOREM 1. *If* (2) *holds then n has at least two prime factors.*

Proof. If possible let $n=p^k$, then

$$2p^k - 1 = 2(p^k - 1) + 1$$

is divisible by e_p. But e_p divides $p-1$ and hence p^k-1. Therefore $e_p=1$, which is impossible.

THEOREM 2. *If* (2) *holds and if n is divisible by the square of a prime p, then either* $p=3511$, or $p>25000$.

Proof. Let $n=kp^2$ and let

$$2^{e_p} \equiv 1 + ph \pmod{p^2}.$$

Then by (2) e_p divides $2n-1$ so that $2n-1=ye_p$. Hence

$$1 \equiv 2^{2n-1} \equiv (2^{e_p})^y \equiv (1 + hp)^y \equiv 1 + hyp \pmod{p^2}.$$

Thus p divides hy. But p does not divide $ye_p=2n-1=2kp^2-1$ so that p does not divide y. Hence p divides h or in other words

$$2^{e_p} \equiv 1 \pmod{p^2}.$$

The only primes less than 25000 having this property are 1093 and 3511 with exponents 364 and 1755 respectively [2]. Since e_p must be odd the theorem follows.

THEOREM 3. If $n=p_1p_2 \cdots p_k$, *then for* (2) *to hold it is necessary and sufficient that* e_{p_i} *divide* $2np_i^{-1}-1$ *for* $i=1, 2, \cdots, k$.

Proof. The congruence (2) is equivalent to the condition that $2n-1$ is divisible by each e_{p_i}. But

$$2n - 1 = 2np_i^{-1}(p_i - 1) + 2np_i^{-1} - 1.$$

Three new solutions of (1) may be obtained by applying Theorem 3 to the case $n=23.31.p$. In this case

$$e_{23} = 11, \qquad e_{31} = 15,$$

so that by Theorem 3, p must satisfy

$$\begin{aligned} 62p &\equiv 1 \pmod{11} \\ 46p &\equiv 1 \pmod{15} \\ 1426p &\equiv 1 \pmod{e_p}. \end{aligned}$$

The first two conditions imply that

$$p \equiv 151 \pmod{330} \tag{3}$$

while the last implies that e_p is some divisor k of 1425, whose divisors are

$$k = 1, 3, 5, 15, 19, 25, 57, 75, 95, 285, 475, 1425.$$

Tables of the factors of 2^k-1 give the primes having specified exponents, the primitive prime factors of 2^k-1 being the only primes having exponent k. Because the factorization of 2^k-1 is incomplete for $n=95,285,475$ and 1425 not all primes p for which e_p divides 1425 are known. Most of those which are known fail to satisfy (3). In fact only three values $p=151$, 1801, 100801 satisfy (3). [3] This gives us the three values of m satisfying (1)

$$215326, \qquad 2568226, \qquad 143742226.$$

Finally we have

THEOREM 4. *Congruence* (1) *has an infinity of solutions.*

Proof. The proof consists in showing that given any solution m of (1) there exists a prime p depending on m such that mp is also a solution.

In fact let $m=2n$ be any solution of (1) so that (2) holds and let p be any primitive prime factor [4] of $2^{2n-1}-1$ so that $e_p=2n-1$ and

$$2^{2n-1} \equiv 1 \pmod{p}.$$

Since e_p divides $p-1$

$$p = (2n-1)x + 1 > n.$$

Hence p and n are coprime. To show that mp is a solution of (1) it therefore suffices to establish that

$$2^{2pn-1} \equiv 1 \pmod{m}$$
$$2^{2pn-1} \equiv 1 \pmod{p}$$

or to show that $2pn-1$ is divisible by $2n-1$ and e_p respectively. But $e_p=2n-1$ and

$$2pn - 1 = 2n - 1 + 2n(p-1) = (2n-1)(1+2nx).$$

This completes the proof of Theorem 4.

References

1. This MONTHLY v. 57, p. 405 (1950).

2. Messenger of Math. v. 51, pp. 149–152 (1922); v. 55, pp. 17–24 (1925). Nieuw Arch. v. Wskde. v. 20, pp. 51–54 (1939). Unpublished result of D. H. Lehmer.

3. *Factorization of* $y^n \pm 1$, A. J. C. Cunningham and H. J. Woodall, London, 1925.

4. That p exists follows from a theorem of Bang, Tidskrift of Mat. s. 5, v. 4, 1886, 130–137.

AN ELEMENTARY METHOD FOR EVALUATING AN INFINITE INTEGRAL

M. R. SPIEGEL, Rensselaer Polytechnic Institute

1. Introduction. In the March, 1951, issue of this MONTHLY W. Kozakiewicz proved the relation

$$\int_0^\infty \frac{\sin x}{x}\,dx = \frac{\pi}{2} \tag{1}$$

by appealing to the Riemann theorem; namely, that if $f(x)$ is Riemann integrable in $a \leqq x \leqq b$ then

$$\lim_{k\to\infty} \int_a^b f(x) \begin{Bmatrix} \sin kx \\ \cos kx \end{Bmatrix} dx = 0. \tag{2}$$

The purpose of this article is two-fold. The first is to point out how one may avoid the Riemann theorem in Kozakiewicz' paper by use of a more elementary theorem. The second is to arrive at the relation (1) by evaluating directly an equivalent integral, namely,

$$\int_0^\infty \frac{\sin^2 x}{x^2}\,dx, \tag{3}$$

by making use of our theorem.

2. Theorem. The more elementary theorem may be stated as follows:

THEOREM. *If $f(x)$ is Riemann integrable and bounded in $a \leqq x \leqq b$, then*

$$\lim_{k\to\infty} \int_a^b f(x)\left\{\frac{1-\cos kx}{k}\right\}dx = 0. \tag{4}$$

Proof.

$$\left|\int_a^b f(x)\left\{\frac{1-\cos kx}{k}\right\}dx\right| \leqq \frac{2(b-a)M}{k} \tag{5}$$

where M is an upper bound of $|f(x)|$ in $a \leqq x \leqq b$. Letting $k\to\infty$ in (5) we have the result (4).

3. Remark Concerning Kozakiewicz' Paper. It is possible to avoid the Riemann theorem in Kozakiewicz' paper, since use is made of the Riemann theorem there only to prove that

$$\lim_{k\to\infty} \int_0^\pi \left[\frac{1}{x} - \frac{1}{2\sin\frac{x}{2}}\right]\sin kx dx = 0 \tag{6}$$

where $k = n+1/2$. This may be proved directly by use of our theorem. For we may show that

$$f(x) = \frac{d}{dx}\left[\frac{1}{x} - \frac{1}{2\sin\frac{x}{2}}\right]$$

satisfies the conditions of our theorem by reasoning analogous to that used immediately following (10). Thus

$$\lim_{k\to\infty} \int_0^\pi \frac{d}{dx}\left[\frac{1}{x} - \frac{1}{2\sin\frac{x}{2}}\right]\left[\frac{1-\cos kx}{k}\right]dx = 0.$$

An integration by parts easily yields the result (6).

4. Evaluation of the Integral. We shall now prove (1) by evaluating directly the equivalent integral (3), (the equivalence may be proved easily by an integration by parts). We make use of certain easily proved trigonometric formulas; namely, if

$$\sigma_n(x) = \tfrac{1}{2} + \cos x + \cos 2x + \cdots + \cos nx \tag{7}$$

then

$$S_k(x) = \frac{1}{k}\left[\sigma_0(x) + \sigma_1(x) + \cdots + \sigma_{k-1}(x)\right] = \frac{1}{2k}\left[\frac{\sin \dfrac{kx}{2}}{\sin \dfrac{x}{2}}\right]^2 \tag{8}$$

and

$$\int_0^\pi S_k(x)dx = \frac{\pi}{2}\,. \tag{9}$$

From (9) we have

$$\int_0^\pi \frac{1}{k}\left\{\frac{1 - \cos kx}{4 \sin^2 \dfrac{x}{2}}\right\} dx = \frac{\pi}{2}\,. \tag{10}$$

Calling

$$f(x) = \frac{1}{x^2} - \frac{1}{4 \sin^2 \dfrac{x}{2}}$$

we may show that

$$\lim_{x\to 0} f(x) = -\frac{1}{12}\,.$$

Hence, if we define $f(0) = -1/12$, then $f(x)$ is continuous in $0 \leqq x \leqq \pi$.

Hence, $f(x)$ satisfies the conditions of our theorem and, therefore,

$$\lim_{k\to\infty} \int_0^\pi \left[\frac{1}{x^2} - \frac{1}{4 \sin^2 \dfrac{x}{2}}\right]\left[\frac{1 - \cos kx}{k}\right] dx = 0. \tag{11}$$

Thus, by (10)

$$\lim_{k\to\infty} \int_0^\pi \frac{1}{x^2}\left[\frac{1 - \cos kx}{k}\right] dx = \frac{\pi}{2}\,. \tag{12}$$

or

$$\lim_{k\to\infty}\int_0^{\pi}\frac{\sin^2\frac{kx}{2}}{kx^2}\,dx=\frac{\pi}{4}\cdot$$

This becomes, when $kx/2$ is replaced by x,

$$\lim_{k\to\infty}\int_0^{k\pi/2}\frac{\sin^2 x}{x^2}\,dx=\int_0^{\infty}\frac{\sin^2 x}{x^2}\,dx=\frac{\pi}{2}\cdot$$

Since

$$\int_0^{\infty}\frac{\sin^2 x}{x^2}\,dx=\int_0^{\infty}\frac{\sin x}{x}\,dx,$$

the relation (1) follows.

A NOTE ON INEQUALITIES FOR THE NORM OF A MATRIX

P. STEIN, University of Natal, Durban, South Africa

Let $A=\|a_{ij}\|$ be a square matrix of order n with real elements and $B=\frac{1}{2}(A+A')$ the associated symmetric matrix with elements $b_{ij}=\frac{1}{2}(a_{ij}+a_{ji})$. It is the purpose of this note to relate the minimum and maximum roots of B to the trace and norm of A^{-1}.

Let x be a matrix of one column and x' its transpose (a row matrix). Then $x'Ax=x'A'x$ is a scalar and

$$x'Bx = x'Ax. \tag{1}$$

If m and M are the (necessarily real) minimum and maximum characteristic roots of B and the components x_i are real, a well known theorem* gives

$$m(x'x) \leqq x'Bx \leqq M(x'x). \tag{2}$$

Applying this result successively to each of the column vectors of a square matrix X and summing the resulting inequalities yields

$$m(\text{trace } X'X) \leqq \text{trace } X'BX \leqq M(\text{trace } X'X). \tag{3}$$

In particular, for $X=(\text{adj } A)'$, $X'BX=\frac{1}{2}\Delta[(\text{adj } A)+(\text{adj } A)']$, so that

$$m\cdot N^2(\text{adj } A) \leqq \Delta\cdot\text{trace }(\text{adj } A) \leqq M\cdot N^2(\text{adj } A), \tag{4}$$

where $\Delta=$determinant of A and $N^2(A)=\text{trace } A'A=\sum_{i,j}(a_{ij})^2$ is the square of the norm of A. Using $X=A^{-1}$ if $\Delta\neq 0$,

$$m\cdot N^2(A^{-1}) \leqq \text{trace } A^{-1} \leqq M\cdot N^2(A^{-1}). \tag{5}$$

In case A is symmetric, $B=A$ and m and M are the minimum and maximum

* R. Courant and D. Hilbert, Methoden der Mathematischen Physik I, Berlin 1931.

roots of A itself. An orthogonal T exists such that $TAT^{-1}=TAT'=A^*$ is diagonal and of course has the same roots, say λ_i $(i=1, \cdots, n)$, that A does. Also $N^2((A^*)^{-1})=N^2(A^{-1})$ and trace $(A^*)^{-1}=$ trace A^{-1}. Hence, (5) reduces to the obvious inequality

$$(\min \lambda_i) \sum_j (\lambda_j)^{-2} \leqq \sum_j (\lambda_j)^{-1} \leqq (\max \lambda_i) \sum_j (\lambda_j)^{-2}.$$

A further case one might mention is the case $A=I+S$, with $S'=-S$. A is then known to be non-singular. Since $B=I$, we have $m=M=1$, and the inequality (5) reduces to the equality

$$N^2(A^{-1}) = \text{trace } (A^{-1}).$$

Further, since for any square matrix M, $N^2(M)=$ trace MM' we have the result that, if $S'=-S$,

$$\text{trace } (I - S^2)^{-1} = \text{trace } (I - S)^{-1}.$$

If A contains complex elements, inequalities similar to (4) and (5) may be obtained by taking $B=\frac{1}{2}(A+\overline{A}')$. B is then Hermitian. Corresponding to (3) we have the inequalities

$$m\cdot \text{trace } (\overline{X}'X) \leqq \text{trace } (\overline{X}'BX) \leqq M\cdot \text{trace } (\overline{X}'X) \tag{6}$$

and corresponding to (4), the inequality

$$m\cdot N^2(\text{adj } A) \leqq \tfrac{1}{2}\Delta \text{ trace } (\text{adj } \overline{A}) + \tfrac{1}{2}\overline{\Delta} \text{ trace } (\text{adj } A). \tag{7}$$

If $\Delta\neq 0$, corresponding to (5) we then have the inequality

$$m\cdot N^2(A^{-1}) \leqq \text{Real part } \{\text{trace } (A^{-1})\} \leqq M\cdot N^2(A^{-1}).$$

Finally, I wish to acknowledge with thanks the assistance of the referee in putting this note into a form suitable for publication.

CLASSROOM NOTES

Edited by C. B. Allendoerfer, University of Washington

All material for this department should be sent to G. B. Thomas, Department of Mathematics, Massachusetts Institute of Technology, Cambridge, 39, Mass.

TAYLOR'S THEOREM IN A FIRST COURSE

C. P. Nicholas, U. S. Military Academy

A sophomore's first encounter with Taylor's Theorem is likely to be frustrating. If his text is one of several in common use throughout the United States, it will prove Taylor's Theorem by appealing to Rolle's Theorem, introducing for

that purpose an elaborate auxiliary function seemingly drawn out of thin air. This is a bewildering approach for beginners: it shatters their faith in the directness of mathematics.

A well-known alternative is to derive the theorem by means of n successive integrations, starting arbitrarily with the integral of $f^{(n)}(x)dx$. This also is artificial diet for an undergraduate. What he needs instead is a discussion starting along paths his own logic might suggest.

A derivation designed to appeal intuitively to sophomores is offered below. It presupposes that the student is prepared to evaluate the successive integral

$$\int_a^x \int_a^x \cdots \int_a^x (dx)^n = \frac{(x-a)^n}{n!}$$

when he meets it in due course. If not, he will need a brief preliminary explanation, but otherwise no special preparation is required.

We assume that the student has been led into the subject carefully. Thus, he understands that his problem is to find an approximate value for $f(x)$ at a point x, based on known values of $f(x)$ and its first n derivatives at a neighboring point $x=a$. Also, he knows that $f(x)$ and its first n derivatives are continuous throughout the closed interval (a, x). The unknown quantity whose approximation he seeks is represented by the ordinate at x in Figure 1.

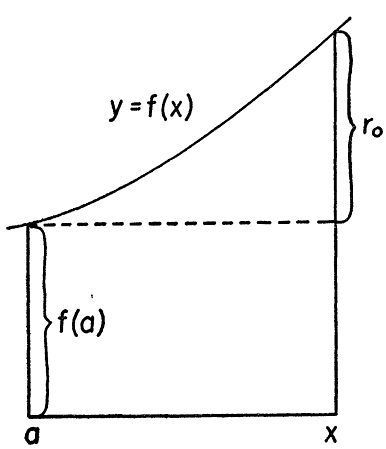


FIG. 1

Since $f(a)$ is known, clearly the heart of the problem is to find an approximation for the increment r_0. Therefore, we introduce a tool designed especially to express the value of such an increment. It is based on an elementary relation between successive derivative curves.

For each curve considered, we shall let r denote the increment in the function as x changes from a to x. A subscript to r will denote that the increment pertains to a derivative curve of order corresponding to the subscript. Thus, r_n

will denote a change in $f^{(n)}(x)$ as x changes from a to x.

Consider now the relationship between any function $G(x)$ and its first derivative, as shown between Figures 2 and 3.

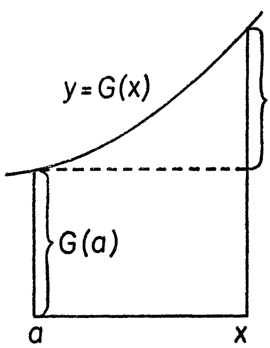


FIG. 2

$$r_0 = G(x) - G(a) \quad \text{(Fig. 2)}$$
$$= \int_a^x G'(x)dx$$
$$= \text{area under } G'(x)$$
$$= \text{area } \underline{A} + \text{area } \underline{B} \quad \text{(Fig. 3)}.$$

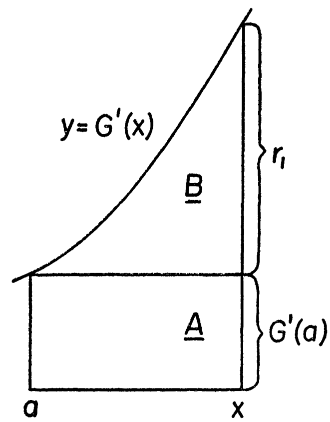


FIG. 3

Therefore:

$$r_0 = G'(a)\int_a^x dx + \int_a^x r_1 dx.$$

The foregoing result is one case of a general relationship between *any* two consecutive derivative curves. We may express it by substituting $f^{(n-1)}(x)$ for $G(x)$, thus:

$$\text{(I)} \qquad r_{n-1} = f^{(n)}(a)\int_a^x dx + \int_a^x r_n dx.$$

At one point in the ensuing discussion, we shall have occasion to use a different (and more obvious) form of the above relationship. Since $f^{(n)}(a)+r_n=f^{(n)}(x)$, we may write:

$$\text{(II)} \qquad r_{n-1} = \int_a^x f^{(n)}(x)dx.$$

We shall now use (I) and (II) to expand a function, $f(x)$, in the vicinity of $x=a$. By our previous definition of r we may state:

$$f(x) = f(a) + r_0.$$

Applying (I) to r_0, we have:

$$f(x) = f(a) + f'(a)\int_a^x dx + \int_a^x r_1 dx.$$

Again, applying (I) to r_1 we have:

$$f(x) = f(a) + f'(a)\int_a^x dx + \int_a^x \left[f''(a)\int_a^x dx + \int_a^x r_2 dx \right] dx$$
$$= f(a) + f'(a)\int_a^x dx + f''(a)\int_a^x \int_a^x (dx)^2 + \int_a^x \int_a^x r_2(dx)^2.$$

Continuing, we apply (I) successively to r_2, r_3, r_4, and so on, until we have expanded $f(x)$ to any desired number of terms. We then stop the expansion by applying II to the final term. Thus:

$$f(x) = f(a) + f'(a)\int_a^x dx + f''(a)\int_a^x \int_a^x (dx)^2 + f'''(a)\int_a^x \int_a^x \int_a^x (dx)^3$$
$$+ \cdots + f^{n-1}(a)\int_a^x \int_a^x \cdots \int_a^x (dx)^{n-1}$$
$$+ \int_a^x \int_a^x \int_a^x \cdots \int_a^x f^{(n)}(x)(dx)^n.$$

Evaluating each integral except the last, we obtain Taylor's Formula:

$$f(x) = f(a) + f'(a)\cdot(x-a) + f''(a)\frac{(x-a)^2}{2!} + f'''(a)\frac{(x-a)^3}{3!} + \cdots$$
$$+ f^{(n-1)}(a)\frac{(x-a)^{n-1}}{(n-1)!} + R_n$$

where R_n, the remainder after n terms, is given by

$$R_n = \int_a^x \int_a^x \cdots \int_a^x f^{(n)}(x)(dx)^n.$$

Since we cannot evaluate the above integral conveniently, this form of R_n is not useful. Therefore we shall replace it by an equivalent expression more easily evaluated. We do this by substituting for the variable $f^{(n)}(x)$ a constant selected in such a way that the value of R_n will not be altered by the substitution. It can be shown* that $f^n(\xi)$ is such a constant, where ξ is some fixed but unknown value between a and x. Making this substitution, we have:

$$R_n = f^{(n)}(\xi)\int_a^x \int_a^x \cdots \int_a^x (dx)^n, \qquad a < \xi < x.$$

Integrating, we obtain the Lagrange form of the remainder:

$$R_n = f^{(n)}(\xi)\frac{(x-a)^n}{n!}, \qquad a < \xi < x.$$

* The proof is substantially the same as for the Mean Value Theorem for Integrals.

AN APPLICATION OF A FAMOUS INEQUALITY

N. S. MENDELSOHN, University of Manitoba

The proofs usually given in elementary textbooks of the fact that the sequence $S_n = (1+1/n)^n$ approaches a limit as $n\to\infty$ are either incorrect or quite messy. The correct proofs are usually based on the following facts:

(1) The sequence $S_n = (1+1/n)^n$ is an increasing function of n for positive integral n.

(2) The sequence $T_n = (1+1/n)^{n+1}$ is a decreasing function of n for positive integral n.

(3) $T_n > S_n$ for every positive integer n.

The proof of (3) is quite trivial. In this note we address ourselves to the problem of obtaining interesting proofs of (1) and (2). The proofs are based on the famous inequality which states that the arithmetic mean of a set of k positive numbers is greater than their geometric mean.

Proof of (1). Consider the set of $(n+1)$ numbers

$$1,\ 1+\frac{1}{n},\ 1+\frac{1}{n},\ 1+\frac{1}{n},\ \cdots,\ 1+\frac{1}{n}\cdot$$

These have an arithmetic mean of $1+1/(n+1)$ and a geometric mean of $(1+1/n)^{n/(n+1)}$. Hence

$$1+\frac{1}{n+1} > \left(1+\frac{1}{n}\right)^{n/(n+1)} \quad\text{or}\quad \left(1+\frac{1}{n+1}\right)^{n+1} > \left(1+\frac{1}{n}\right)^{n}.$$

Hence S_n increases with n.

Proof of (2). Consider the set of $(n+2)$ numbers.

$$1,\ \frac{n}{n+1},\ \frac{n}{n+1},\ \frac{n}{n+1},\ \cdots,\ \frac{n}{n+1}\cdot$$

These have an arithmetic mean of $(n+1)/(n+2)$ and a geometric mean of $(n/n+1)^{(n+1)/(n+2)}$. Hence

$$\frac{n+1}{n+2} > \left(\frac{n}{n+1}\right)^{(n+1)/(n+2)}.$$

On taking reciprocals this becomes

$$1+\frac{1}{n+1} < \left(1+\frac{1}{n}\right)^{(n+1)/(n+2)} \quad\text{or}\quad \left(1+\frac{1}{n+1}\right)^{n+2} < \left(1+\frac{1}{n}\right)^{n+1}.$$

Hence T_n is a decreasing function of n.

The proof of (1) was first given by a first year student at the University of Toronto about 10 years ago. The proof of (2) rests with the author of this note.

ELEMENTARY PROBLEMS AND SOLUTIONS

EDITED BY HOWARD EVES, Oregon State College

Send all communications concerning Elementary Problems and Solutions to Howard Eves, Mathematics Department, Champlain College, Plattsburg, New York. This department welcomes problems believed to be new, and demanding no tools beyond those ordinarily furnished in the first two years of college mathematics. To facilitate their consideration, solutions should be submitted on separate, signed sheets, within three months after publication of problems.

PROBLEMS FOR SOLUTION

E 980. *Proposed by Victor Thébault, Tennie, Sarthe, France*

If a sphere through the vertices A, B, C of a tetrahedron $ABCD$ cuts the edges DA, DB, DC in A', B', C', then the distances of its center from those of the spheres $ABCD$ and $A'B'C'D$ are equal, respectively, to the radii of the spheres $A'B'C'D$ and $ABCD$.

E 981. *Proposed by David Mandelbaum, Hillside, N. J.*

Derive a formula for the sum of the first n terms of a progression in which the first term is a, each even placed term is obtained from its preceding term by multiplying by the constant u, and each odd placed term (after the first) is obtained from its preceding term by multiplying by the constant v.

E 982. *Proposed by C. W. Trigg, Los Angeles City College*

Show that there are two different sets of three congruent right cylindrical surfaces which may be folded and assembled into a cube with no open edges.

E 983. *Proposed by A. W. Goodman, University of Kentucky*

Let M and N be two points one unit apart. With M and N as centers and with unit radii draw arcs ANB and AMB. Let Q be any point on arc AMB and P_1 and P_2 any points on arc ANB such that N is the midpoint of arc P_1P_2. Show that

$$QP_1 + QP_2 \leqq 2 \leqq (QP_1)^2 + (QP_2)^2.$$

E 984. *Proposed by Joseph Rosenbaum, Hartford, Conn.*

(a) Find $f(x)$ when $f[f(x)] = x^2 - 2$.

(b) More generally, find $f_1(x)$ when $f_n(x) = x^2 - 2$, where $f_n(x)$ is defined by the relation $f_{r+1}(x) = f_1[f_r(x)]$.

E 985. *Proposed by Leo Moser, Texas Technological College*

What is the smallest integer N such that if the integers $1, 2, 3, \cdots, N$ are distributed into three classes in any manner whatsoever at least one of the classes will contain a solution of $a+b=c$.

SOLUTIONS

Inequality of K. O. Friedrichs

E 949 [1951, 37]. *Proposed by Robert Oeder, Oregon State College*

Show that if D_n is a determinant whose elements are a_{ij}, $i, j=1, 2, \cdots, n$, and, for all i,

$$a_{ii} \geqq (1/2)\sum_{k=1}^{n} | a_{ik} |,$$

then

$$D_n \geqq a_{11}\prod_{i=2}^{n}\left(a_{ii} - \sum_{k=1}^{i-1} | a_{ik} |\right).$$

Solution by the Proposer. Olga Taussky, in *A recurring theorem on determinants*, this MONTHLY, 1949, pp. 672–675, has shown that $D_n \geqq 0$. Now

$$D_n = \begin{vmatrix} a_{11} & a_{12} & a_{13} \cdots a_{1n} \\ 0 & a_{22} - | a_{21} | & a_{23} \cdots a_{2n} \\ a_{31} & a_{32} & a_{33} \cdots a_{3n} \\ \cdot & \cdot & \cdot \cdots \cdot \\ a_{n1} & a_{n2} & a_{n3} \cdots a_{nn} \end{vmatrix} + \begin{vmatrix} a_{11} & a_{12} & a_{13} \cdots a_{1n} \\ a_{21} & | a_{21} | & 0 \cdots 0 \\ a_{31} & a_{32} & a_{33} \cdots a_{3n} \\ \cdot & \cdot & \cdot \cdots \cdot \\ a_{n1} & a_{n2} & a_{n3} \cdots a_{nn} \end{vmatrix}.$$

By Taussky's result, the second determinant in the sum is non-negative, and D_n is therefore not less than the first determinant in the sum. By applying this process to the remaining $n-2$ rows one obtains

$$D_n \geqq \begin{vmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ 0 & a_{22} - | a_{21} | & a_{23} & \cdots & a_{2n} \\ 0 & 0 & a_{33} - | a_{31} | - | a_{32} | & \cdots & a_{3n} \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ 0 & 0 & 0 & \cdots & a_{nn} - \sum_{k=1}^{n-1} | a_{nk} | \end{vmatrix}$$

$$= a_{11}\prod_{i=2}^{n}\left(a_{ii} - \sum_{k=1}^{i-1} | a_{ik} |\right).$$

Also solved by E. V. Haynsworth and O. E. Stanaitis.

Attention should be called to G. B. Price, *Bounds for determinants with dominant principal diagonal*, Proc. Am. Math. Soc., 1951, pp. 497–502. The result of the problem is a special case of Theorem 1 of this paper.

Powers of a Binomial Surd

E 950 [1951, 108]. *Proposed by W. R. Ransom, Tufts College*

Show that every positive integral power of $\sqrt{2}-1$ is of the form $\sqrt{m}-\sqrt{m-1}$.

Solution by S. T. Thompson, Tacoma, Washington. We shall in (1), obtain a considerably more general result. Let a and b, $a \geqq b$, be two non-negative real numbers, and n a positive integer. Set

$$p = [(a+b)^n + (a-b)^n]/2.$$

Then

$$p^2 - (a^2 - b^2)^n = [(a+b)^n - (a-b)^n]^2/4.$$

It follows that

$$(a-b)^n = \sqrt{p^2} - \sqrt{p^2 - (a^2-b^2)^n}. \tag{1}$$

The following special cases, of which (C) is the required result, are worthy of note:

(A) Take $a=\sqrt{s}$, $b=\sqrt{s-k}$, s and k positive integers, $s \geqq k$. Then p^2 is a positive integer, and

$$(\sqrt{s} - \sqrt{s-k})^n = \sqrt{p^2} - \sqrt{p^2 - k^n}.$$

(B) In (A) take $k=1$. Then

$$(\sqrt{s} - \sqrt{s-1})^n = \sqrt{p^2} - \sqrt{p^2-1}.$$

(C) In (B) take $s=2$. Then

$$(\sqrt{2} - 1)^n = \sqrt{p^2} - \sqrt{p^2-1}.$$

Here $p = [(\sqrt{2}+1)^n + (\sqrt{2}-1)^n]/2$.

Also solved by A. N. Aheart, H. L. Alder, F. Bagemihl, E. W. Banhagel, A. W. Boldyreff, D. H. Browne, Aaron Buchman, I. A. Dodes, Arnold Dresden, D. G. Duncan, Ragnar Dybvik, R. E. Edwards, R. E. Ekstrom, A. L. Epstein, Leopold Flatto, Walter Fleming, Calvin Foreman, A. E. Foster, A. E. Franz, Albert Furman, Bernard Greenspan, N. G. Gunderson, Frank Harary, Vern Hoggatt, L. N. Howard, R. Huck, Ray Jurgensen, C. W. Karns, Abraham Karrass, R. S. Kingsbury, G. J. Kleinhesselink, W. J. Klimczak, Sam Kravitz, Sidney Kravitz, H. R. Leifer, A. J. Lohwater, Fred Marer, D. R. Morrison, Leo Moser, Prasert Na Nagara, C. S. Ogilvy, J. H. Oppenheim, F. D. Parker, R. G. Paxman, C. G. Phipps, C. F. Pinzka, Daniel Resch, L. A. Ringenberg, Cal Rogers, Alex Rosenberg, C. M. Sandwick, Sr., J. W. Sawyer, W. Seidel, O. D. Smith, Robert Spira, O. E. Stanaitis, Elijah Swift, G. W. Walker, Alan Wayne, J. E. Weidlich, Albert Wilansky, L. M. Winer, E. M. Zaustinsky, and the proposer.

Concerning Symmedians

E 951 [1951, 108]. *Proposed by C. S. Venkataraman, Trichur, South India*

Two circles are drawn touching the sides AB, AC of a triangle ABC at the ends of the base BC and also passing through the midpoint D of BC. If E is the other point of intersection of the circles show that AE is a symmedian of triangle ABC.

Solution by Miriam Huggins, Woman's College of the University of North Carolina. Since $\measuredangle ACD=\measuredangle DEC$ and $\measuredangle ABD=\measuredangle BED$ it follows that $\measuredangle BEC$ and $\measuredangle BAC$ are supplementary and therefore that $ABEC$ is cyclic. Hence $\measuredangle BCE=\measuredangle BAE$. But also $\measuredangle ABE=\measuredangle EDC$, since each is equal to $\measuredangle BED+\measuredangle DBE$. Therefore triangles ABE and CDE are similar, whence $BE/DE=AB/DC=AB/BD$. It now follows that triangles BED and ABD are similar. Then $\measuredangle BAD=\measuredangle EBD=\measuredangle EAC$, or AE is a symmedian.

Also solved by N. Balasubramanian, Vern Hoggatt, Prasert Na Nagara, A. Sisk, O. D. Smith, and Roscoe Woods.

P-slopes

E 952 [1951, 108]. *Proposed by Albert Wilansky, Lehigh University*

Let $P:(r, \theta)$ be given in plane polar coordinates. Define the *p-slope* of the segment P_1P_2 to be $(r_2-r_1)/$(distance from P_1 to P_2). If A, B, C are three collinear points, with B between A and C, show that p-slope $CB\leqq p$-slope $CA\leqq p$-slope BA. (*Note*: The inequality signs in the original statement of the problem should be reversed, as above.)

Solution by the Proposer. Let Q be the foot of the perpendicular from the origin O on the line of collinearity of A, B, C, and let P be any point on the line. Designate the lengths of OQ, OP, QP by c, r, ρ, where c and r are taken as positive but ρ is a signed length. Then $r=(\rho^2+c^2)^{1/2}$ is a convex function of ρ. Designate the (r, ρ) coordinates of A, B, C by (r_A, ρ_A), (r_B, ρ_B), (r_C, ρ_C). By convexity

$$\frac{r_B-r_A}{|\rho_B-\rho_A|}\leqq\frac{r_C-r_A}{|\rho_C-\rho_A|}\leqq\frac{r_C-r_B}{|\rho_C-\rho_B|},$$

and the theorem is proved.

Also solved by Vern Hoggatt, F. D. Parker, and C. M. Sandwick, Sr.

Tetrahedron and Four Planes

E 953 [1951, 108]. *Proposed by Joseph Langr, Prague, Czechoslovakia*

Four planes, drawn through a point, parallel to the faces of a tetrahedron, cut the edges of the tetrahedron in 12 points lying on a quadric surface.

Solution by S. T. Thompson, Tacoma, Washington. The four planes cut the three edges of each face in six points which, by the converse of Carnot's theorem,

are easily shown to lie on a conic. These four conics intersect in pairs in two points. Since, in general, nine points determine a quadric surface, it now follows that the four conics lie on a common quadric surface.

The analogous theorem in the plane is well known.

The theorem may be slightly generalized to the following: If a plane cuts the four faces of a tetrahedron in four lines, then any four concurrent planes passing through these four lines intersect the edges of the tetrahedron in 12 points lying on a quadric surface.

Triangular Squares

E 954 [1951, 108]. *Proposed by G. W. Walker, Buffalo, N. Y.*

Let T_c be the cth positive integer which is both a triangular number and a square number. Find T_c as a function of c.

Solution by W. J. Klimczak, Iola Sanatorium, Rochester, N. Y. The kth triangular number and the nth square are equal when $k(k+1)/2=n^2$. Solving this as a quadratic in k we find $k=[-1\pm\sqrt{1+8n^2}]/2$. Since k and n are positive integers we must have $1+8n^2=x^2$, say, where x is a positive integer. From the solution $n=1$, $x=3$ of this equation we obtain as the general solution for n,

$$n = [(3+\sqrt{8})^c - (3-\sqrt{8})^c]/2\sqrt{8}.$$

Then

$$T_c = n^2 = [(17+12\sqrt{2})^c + (17-12\sqrt{2})^c - 2]/32.$$

The first five values of T_c are 1, 36, 1225, 41616, and 1413721.

Also solved by J. H. Braun, D. H. Browne, Sam Kravitz, Leo Moser, Prasert Na Nagara, P. A. Piza, L. A. Ringenberg, J. E. Sanders, C. M. Sandwick, Sr., E. P. Starke, Elijah Swift, and the proposer.

Evaluation of a Determinant

E 955 [1951, 108]. *Proposed by C. L. Dunsmore, University of California, Los Angeles*

Evaluate the nth order determinant $|a_{ij}|$, where $a_{ij}=1/(i+j-1)!$.

Solution by J. T. Morse, Case Institute of Technology. We shall solve a slight generalization of the given problem. Let $I_{n,k}$ be the nth order determinant $|a_{ij}|$, where $a_{ij}=1/(i+j+2k-1)!$ for $k=0, 1, 2, \cdots$. Multiply the ith row by $i+2k$ and subtract the $(i-1)$th row for $i=n, n-1, \cdots, 2$. Factor $1-j$ from the jth column for $j=2, 3, \cdots, n$. Then

$$\begin{aligned} I_{n,k} &= [(-1)^{n-1}(n-1)!/(n+2k)!]I_{n-1,k+1} \\ &= (-1)^{n(n-1)/2}\prod_{r=1}^{n}[(r-1)!/(n+2k+r-1)!]. \end{aligned}$$

Also solved by F. Bagemihl, N. J. Fine, A. E. Livingston, F. D. Parker, F. A. E. Pirani, O. E. Stanaitis, L. M. Winer, the proposer, and an anonymous solver.

Bagemihl remarked that the proposed determinant was evaluated, incorrectly, by H. W. Segar, *The deduction of certain determinants from others of indeterminate form*, The Messenger of Mathematics, XXII (1892–1893), p. 67. Pirani showed that the problem is equivalent to finding the number of distinct plays in the game of linearized Chinese Chequers, invented by Professor K. F. Vomberg.

ADVANCED PROBLEMS AND SOLUTIONS

EDITED BY E. P. STARKE, Rutgers University

Send all communications concerning Advanced Problems and Solutions to E. P. Starke, Rutgers University, New Brunswick, New Jersey. All manuscript should be typewritten, with double spacing and margins at least one inch wide. Problems containing results believed to be new or extensions of old results are especially sought. Proposers of problems should also enclose any solutions or information that will assist the editor. In general, problems in well known textbooks or results in readily accessible sources should not be proposed for this department.

PROBLEMS FOR SOLUTION

4453. *Proposed by H. F. Sandham, Trinity College, Ireland*

Prove that

$$\frac{1^5}{e^{\pi}+1}+\frac{3^5}{e^{3\pi}+1}+\frac{5^5}{e^{5\pi}+1}+\cdots=\frac{31}{504}.$$

4454. *Proposed by R. C. Lyndon, Princeton University*

The "Burnside problem" for semi-groups without cancellation can be given the following form. Let $S(n, m)$ be the set of all words, or finite sequences, formed from an alphabet of m letters, which have the property that no (non-empty) part is repeated as many as n consecutive times. Show that, for $m, n>1$, and with the single exception $m=n=2$, the set $S(n, m)$ is infinite.

4455. *Proposed by M. S. Klamkin, Polytechnic Institute of Brooklyn*

If

$$\left[\frac{f(x)}{g(x)}\right]'=\frac{f'(x)}{g'(x)},$$

solve for $f(x)$ in terms of $g(x)$, and for $g(x)$ in terms of $f(x)$. (This is an extension of one of the problems proposed for the 1951 William Lowell Putnam Prize

Competition.)

4456. *Proposed by Victor Thébault, Tennie, Sarthe, France*

If through the vertices A, B, C, D of a tetrahedron, parallel planes are drawn cutting a given line L in points A_2, B_2, C_2, D_2, and if A_1, B_1, C_1, D_1 are the points in which the lines AA_2, BB_2, CC_2, DD_2 cut the planes BCD, CDA, DAB, ABC, then

$$\frac{AA_2}{AA_1}+\frac{BB_2}{BB_1}+\frac{CC_2}{CC_1}+\frac{DD_2}{DD_1}=2.$$

4457. *Proposed by Israel Halperin, Queens University, Kingston, Ontario*

If

$$\sum_{m=1}^{\infty}|b_m|<\infty \quad \text{and} \quad \lim_{t\to 0}\sum_{m=1}^{\infty} b_m \cos\frac{1}{mt}=0,$$

show that every $b_m=0$.

SOLUTIONS

Number of Integers in One Random Set Greater than the Maximum of Another

4377 [1950, 41]. *Proposed by Paul Brock, Reeves Instrument Company, New York City*

Consider two non-decreasing sequences of n positive integers, $(n<M)$, the integers being chosen at random. What is the most probable number of integers in the sequence containing the maximum integer, each of which is larger than the maximum of the second sequence?

Solution by the Proposer. Given two sets,* each of n positive integers chosen at random. Assume an upper bound M on the values chosen, and a number $k\leqq M$. The probability that the first set has at least one element equal to k, and no element greater than k is

$$\left(\frac{k}{M}\right)^n-\left(\frac{k-1}{M}\right)^n.$$

The probability that the second set has i $(i>0)$ elements greater than k and $n-i$ elements not greater than k is

$${}^nC_i\left(\frac{k}{M}\right)^{n-i}\left(1-\frac{k}{M}\right)^i.$$

The probability of both events simultaneously is

* There seems to be no significance to the requirement that these sets be non-decreasing sequences.

$$2{}^nC_i\left(\frac{k}{M}\right)^{n-i}\left(1-\frac{k}{M}\right)^i\left\{\left(\frac{k}{M}\right)^n-\left(\frac{k-1}{M}\right)^n\right\},$$

where the possibility of interchanging the sets has been taken into account. Now the total probability of i integers in one set greater than the maximum of the other is the sum of this last probability taken over all k:

$$P_M(i) = 2{}^nC_i\sum_{k=1}^{M-1}\left(\frac{k}{M}\right)^{n-i}\left(1-\frac{k}{M}\right)^i\left\{\left(\frac{k}{M}\right)^n-\left(\frac{k-1}{M}\right)^n\right\}.$$

Let $k/M=x$, $\Delta k/M=\Delta x$, where $\Delta k=1$, and we have

$$P_M(i) = 2{}^nC_i\sum_{x=1/M}^{1-1/M} x^{n-i}(1-x)^i\{x^n-(x-\Delta x)^n\}.$$

We now let $M\to\infty$, eliminate higher order differentials, and replace summation by integration:

$$P(i) = 2n{}^nC_i\int_0^1 x^{2n-i-1}(1-x)^i dx.$$

Upon replacing the definite integral by $\beta(2n-i, i+1)$ we have finally, since n and i are integers,

$$P(i) = {}^nC_i/{}^{2n-1}C_i.$$

We note in passing that the probability of the two sets having the same maximum is

$$\sum_k\left\{\left(\frac{k}{M}\right)^n-\left(\frac{k-1}{M}\right)^n\right\}^2,$$

which approaches 0 as $M\to\infty$. As a check it is easy to show that $\sum_1^n P(i)=1$.

Since $P(1)=n/(2n-1)>\frac{1}{2}$, the value of i most likely to occur is 1. If "most probable value" is interpreted to mean the expected value, the answer is given by $\sum_1^n iP(i)$ which turns out to be $2n/(n+1)$.

Also solved by Roger Lessard, and J. G. Millar.

Laws of Attraction

4381 [1950, 119]. *Proposed by R. J. Walker, Cornell University*

The inverse square law has the property that the attraction, at an external point, due to a sphere of uniform density is the same as if the sphere were concentrated at its center. Are there any other laws of attraction which have this property?

Solution by J. L. Ericksen, University of Indiana. By choosing coördinates properly we may assume that the sphere is of radius a with center at the origin and that the attracted particle has coördinates $(0, 0, z)$ where $z>a\geqq 0$. Then,

with proper choice of units, we look for a continuous function $f(r)$, where r denotes distance, such that

$$a^2\int_0^\pi\int_0^{2\pi}\{(a\cos\theta - z)/r\}f(r)d\phi\sin\theta d\theta = -4\pi a^2 f(z) \tag{1}$$

where $r^2=a^2+z^2-2az\cos\theta$. This is easily reduced to the equivalent condition

$$I=\int_{z-a}^{z+a}(a^2-z^2-r^2)f(r)dr = -4az^2f(z).$$

We have then

$$\begin{aligned}\partial I/\partial a &= f(z+a)(-2az-2z^2)+f(z-a)(2az-2z^2)\\ &\quad + 2a\int_{z-a}^{z+a}f(r)dr = -4z^2f(z),\end{aligned} \tag{2}$$

$$\begin{aligned}\partial I/\partial z &= f(z+a)(-2az-2z^2)-f(z-a)(2az-2z^2)\\ &\quad - 2z\int_{z-a}^{z+a}f(r)dr = -8azf(z)-4az^2f'(z).\end{aligned} \tag{3}$$

Thus $f'(z)$ and, similarly, $f''(z)$ exist and are continuous. Eliminating the integral between (2) and (3), we have

$$2a^2zf'(z) = f(z+a)(a+z)^2+f(z-a)(z-a)^2-f(z)(4a^2+2z^2). \tag{4}$$

If this is differentiated twice and then a is allowed to approach zero there results

$$z^2f''(z)+4zf'(z)-2f(z)=0$$

which has the solution $f(z)=Az+Bz^{-2}$, which satisfies the required condition (1). Thus the most general solution is $f(r)=Ar+Br^{-2}$.

Also solved by F. G. Fender, J. W. Green, Frank Herlihy, M. S. Klamkin, Roger Lessard, E. H. Sondheimer, A. E. Taylor, and the proposer.

Editorial Note. I. N. Sneddon and K. C. Thornhill, in the *Proceedings of the Cambridge Philosophical Society*, 1949, p. 318, treat a more general problem which reduces to the present discussion for the special case $\lambda=0$. However, the limit as $\lambda\to 0$ was incorrectly taken with the result that the direct first power law was overlooked and the Newtonian potential was the only result found.

Klamkin notes that only the inverse square law provides that the force exerted on a particle inside a hollow sphere is zero. Herlihy notes that under the direct first power law an arbitrary body attracts as if its mass were concentrated at its center of gravity.

Modified Harmonic Series

4384 [1950, 120]. *Proposed by H. F. Sandham, Trinity College, Ireland*

In the following modification of the harmonic series there is a change of sign after the reciprocal of each square. Prove that

$$\frac{1}{1}-\frac{1}{2}-\frac{1}{3}-\frac{1}{4}+\frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\frac{1}{8}+\frac{1}{9}-\frac{1}{10}-\frac{1}{11}-\frac{1}{12}-\cdots$$
$$=\sum_{n=1}^{\infty}\frac{\pi}{\sqrt{n}\sinh\pi\sqrt{n}}\cdot$$

Solution by the Proposer. In his solution of E 824 [1949, 264] Herzog gives necessary and sufficient conditions for the convergence of a general series of the first type. Let us write

$$f(x)=\frac{1}{1+x}-\frac{1}{2+x}-\frac{1}{3+x}-\frac{1}{4+x}+\frac{1}{5+x}+\frac{1}{6+x}+\cdots,\ x\geqq 0.$$

An application of Herzog's argument shows that

$$\frac{1}{1+x},\quad \frac{1}{2+x}+\frac{1}{3+x}+\frac{1}{4+x},\quad \frac{1}{5+x}+\frac{1}{6+x}+\cdots,\quad\cdots$$

is a decreasing sequence. It follows easily that the series for $f(x)$ is convergent. We note, too, that $0<f(x)<1/(1+x)$, and that therefore as $x\to\infty$, $\lim f(x)=0$.

Now*

$$f(x-1)-f(x)=\frac{1}{x}-\frac{2}{x+1}+\frac{2}{x+4}-\frac{2}{x+9}+\cdots=\frac{\pi}{\sqrt{x}\sinh\pi\sqrt{x}}$$

for $x\geqq 1$. Letting x have integral values and summing from 1 to ∞ gives the result required.

Also solved by N. J. Fine and O. E. Stanaitis.

Cycles of n-digit Binary Integers

4385 [1950, 188]. *Proposed by P. Ungar, University College, London, England*

The three-digit sections of the sequence 1110001011 represent all three-digit numbers in the binary system exactly once each. For a given positive integer n an analogous sequence is obtained in the following manner: write down n 1's to begin with, and in each subsequent place write 0 unless the n-digit section thus completed occurs previously, in which case put 1. Show that the resulting sequence of 2^n+n-1 digits has the same property as the case $n=3$ cited at the outset.

Solution by Roger Lessard, École Polytechnique, Montreal, Canada. Let a_t stand for the n-digit binary number whose last digit occupies the tth place in

* Whittaker-Watson, *Modern Analysis*, 7.4, ex. 5.

the proposed sequence, and let $a_{\bar{t}}=a_t-1$. Note that, according to the construction rule, the occurrence of an odd $a_{\bar{t}}$ implies the previous occurrence of a_t. If, now, duplicate numbers occur in the sequence, let a_i be the first; that is

$$a_i = a_j, \qquad i < j.$$

As a_i is odd (a_j cannot be even by the construction rule) there exists a k such that

$$a_k = a_{\bar{i}} \qquad k < i$$

unless a_i is the first number of the sequence, $a_i=a_n=2^n-1$. Then a_k, a_i, a_j differ only in their last digit and therefore the numbers a_{k-1}, a_{i-1}, a_{j-1} have the same $n-1$ last digits; whence two of them are the same, contrary to the hypothesis.

It follows that the first number to repeat itself* is $a_n=a_j=2^n-1$. This last result implies

$$a_{j-t} \equiv 1 \pmod 2, \qquad 0 \leqq t < n.$$

As a_{j-1} is odd $a_{\overline{j-1}}$ occurs before it. In other words:

Corresponding to a_{j-1} there exist before a_j two numbers which have 0 as first digit followed by $n-2$ 1's, viz. $a_{j-1}=2^{n-1}-1$, $a_{\overline{j-1}}=2^{n-1}-2$.

The two numbers a_{j-2} and $a_{\overline{j-1}-1}$ have the same $n-1$ last digits so that they differ in the first digit only. As both are odd (same parity, a_{j-2} is odd) $a_{\overline{j-2}}$ and $a_{\overline{\overline{j-1}-1}}$ occur before them, so that:

Corresponding to a_{j-2} we have before a_j 2^2 distinct numbers having 0 as second digit followed by $n-3$ 1's. Two of them are $2^{n-2}-1$ and $2^{n-2}-2$.

As these numbers can be grouped in pairs having the same $n-1$ first digits the same reasoning can be repeated, giving:

Corresponding to $a_{j-k}(k=3, 4, \cdots, n-1)$ we have before $a_j 2^k$ distinct numbers having 0 as the kth digit followed by $n-k-1$ 1's. Two of them are $2^{n-k}-1$ and $2^{n-k}-2$.

We have then $2+4+\cdots+2^k+\cdots+2^{n-1}=2^n-2$ numbers before a_j. As these numbers include all numbers having 0 as a digit at any place except the last, the two remaining numbers of the 2^n numbers are 2^n-1 and 2^n-2, that is a_n and a_{n+1}. It follows that all 2^n numbers occur in the sequence before the first number a_n repeats, so that if the sequence is terminated at a_{j-1} we have each of the 2^n numbers exactly once.

Also solved by the proposer.

Editorial Note. The following statement of the problem has some advantages:

Let $a_1=2^n-1$, and form the sequence of n-digit binary integers, $a_1a_2a_3, \cdots$, where

$$a_{j+1} \equiv 2a_j \pmod{2^n}$$

* We do not at this stage consider the sequence terminated at 2^n+n-1 digits.

unless for some $i<j$, $a_i \equiv 2a_j$. In that case put

$$a_{j+1} \equiv 2a_j + 1 \pmod{2^n},$$

Then every n-digit integer will appear exactly once among the first 2^n members of the sequence.

A moment's reflection shows that, with the above rule of construction, if the particular integer 2^n-1 is not at the beginning it must be preceded by $2^{n-1}-1$ and 2^n-2 must occur earlier. But the immediate predecessor of 2^n-2 must be either 2^n-1 or $2^{n-1}-1$. Hence a necessary condition that the prescribed construction shall produce all n-digit integers is that the initial number be 2^n-1 or 2^n-2. The sequence for $a_1=2^n-2$ is the same as that for $a_1=2^n-1$ except that the number 2^n-1 is transposed from the left end to the right. The condition is therefore also sufficient.

D. D. Wall contributes the following notes. In the scale of notation r, a sequence of r^n digits which when interpreted cyclically contains each of the r^n different sequences of n digits as a consecutive subsequence—and hence each one once and only once—is called a cycle. N. G. de Bruijn (*Nederlandsche Akademie van Wetenschappen, Proceedings*, 1946, pp. 758–764) has shown that in base 2 the number of cycles is

$$2^{(2^{n-1}-n)}.$$

I. J. Good and D. Rees (*Journal of the London Mathematical Society*, 1946, pp. 167–172) have given existence proofs that such cycles exist for every scale of notation. The problem is also of interest in relation to normal numbers as defined in Hardy and Wright, *Theory of Numbers.*

RECENT PUBLICATIONS

EDITED BY E. P. VANCE, Oberlin College

All books for review should be sent directly to the editor of this department, American Mathematical Monthly, 80 Waterman Street, Providence 6, Rhode Island, and not to any of the other editors or officers of the Association.

Introduction to Algebraic Geometry. By W. G. Welchman. New York, Cambridge University Press, 1950. 10+349 pp. $4.50.

In writing this text the author had in mind a university course which would lead "as rapidly as possible to the study of configurations, loci and transformations in space of three, four and five dimensions." In order to facilitate progress, it seemed "desirable to apply to the more elementary problems the types of reasoning that are used in advanced work, instead of employing methods that are not capable of extension." Hence, we have a treatment of the

theory of conics which is "as much concerned with the development of technique as with the proving of theorems, and for this reason a theorem will often be proved by several different methods."

The particular algebraic geometry, to which this volume is an introduction, may be more exactly entitled "The Algebraic Geometry of the Field of Complex Numbers" in which the elements in a space of any number of dimensions are defined by ordered sets of complex numbers, but in which the only curves, surfaces and other constructs to be discussed are those which may be defined by algebraic equations whose coefficients are complex numbers and whose variables are complex variables. A theorem may be proved by non-algebraic methods, but the theorem itself must be a statement about algebraic equations. If we think of projective geometry as that part of algebraic geometry which may be deduced from the study of linear transformations, then nearly all of the geometry in this book is projective.

The foundations necessary for the later developments are established in the first three chapters (98 pages) and some of the topics discussed are the following: linear dependence, linear systems, ratio sets, polynomial equations, freedom of algebraic systems, geometric constructs in the plane, algebraic correspondence and rational systems of freedom one. The definition of a space of n dimensions is followed by quite detailed discussions of spaces of one, two and three dimensions. Duality and cross-ratio are introduced early and throughout the entire volume they play a very important role. Naturally enough many of these basic topics are treated abstractly and often briefly. Lest the beginner become discouraged, the author suggests that he read these chapters lightly at first and proceed with the theory of conics which follows. The frequent cross references given will gradually strengthen the reader's understanding of the foundations.

Chapter IV (The conic) deals with Joachimsthal's equation (name not given), parametric representation, cross-ratio on a conic, involution on a conic, polar theory and pencils of conics. Proofs are given for the conic locus and for its dual, the conic scroll. Chapter V (Configurations) considers the Desargues figure, the quadrangle and quadrilateral and the corresponding point and line pencils of conics. Chapter VI (Metrical geometry) introduces projective meanings for metrical terms and shows that in many cases the theorems of algebraic geometry reduce to theorems of Euclidean geometry. In Chapter VII (Homographic ranges on a conic) many of the theorems considered are essentially theorems about values of a parameter and, therefore, they may be stated as theorems about homographies in the elements of an arbitrary rational system of freedom one.

In Chapter VIII (Two conics. Reciprocation of one conic into another. Particular cases) two proper conics may have four distinct common points, simple, three-point, four-point or proper double contact. In each case standard forms for the equations are obtained and the reciprocation of a conic into a conic is investigated. In Chapter IX (Two conics. Apolarity) two proper conics determine certain conic loci and scrolls whose properties lead to the

consideration of the system of triangles inscribed in one conic and circumscribed to another. Linear systems of conic loci and conic scrolls are introduced. Chapter X (Two-two correspondences) discusses symmetric (2, 2) correspondences in order to derive properties of two conics leading to a proof of Poncelet's Theorem.

In Chapter XI (Application of matrix algebra) elementary matrix notation is introduced as an aid in the discussion of the configuration determined by two proper conics. Finally, Chapter XII (Invariants and covariants) deals with certain projective invariants and covariants which tend to unify the subject matter of the entire volume.

Of the dozen titles to which the author has referred, the most important is Robson's Introduction to Analytical Geometry. It is evident that great care and effort have gone into the writing and printing of this volume. However, the reviewer does feel that an index would increase the value of the book for reference purposes and that some well chosen problems would clarify and stimulate the reader's thinking.

C. H. YEATON

Analytic Geometry and Calculus. H. M. Gay. New York, McGraw-Hill, 1950. vii+485 pages+Tables, Answers, and Index. $5.00.

Analytic Geometry and Calculus. L. M. Kells. New York, Prentice-Hall, 1950. iii+552 pages+Appendix, Answers and Index. $4.75.

Elements of Mathematical Analysis, S. E. Urner-W. B. Orange. Boston, Ginn and Company, 1950. xi+506 pages+Appendix, Answers, and Index. $4.00.

These three books are written to provide college students an early introduction to the calculus, developing analytic geometry along side the calculus and introducing integral calculus before considering the derivatives of the logarithmic, exponential, or trigonometric functions. The Gay and the Kells assume that all necessary preparation has been made in algebra and trigonometry; the Urner-Orange though written for students with a background of intermediate algebra and trigonometry assumes a "considerable workout in the algebraic techniques will still be needed" and in trigonometry the authors "decided not to take anything for granted but to supply all basic subject matter."

The Urner-Orange is one of the "unified" mathematics texts and the authors state that the departure from "compartment" presentation has increased the general serviceability of the subject. The book starts with three chapters on functions in which are discussed graphs, formulas, and the rates of change of such functions as ax^2, the quadratic function, ax^3, and ax^4. Here the h notation is employed rather than the Δx and limits are found intuitively. In the next chapter, the ideas of limits and derivatives are presented with the customary theorems on the derivatives of products, quotients, *etc.* The appendix gives a further discussion of limits and continuity. Next comes a chapter on applica-

velocities, accelerations, and rates.

Next comes integration, in three chapters, showing such applications as areas, volumes by slices and of revolution, length of arc, moments, fluid pressures, work, the trapezoidal and Simpson's rules. This book has a chapter on Vectors, "which requires their differentiation, interpretations of them as velocities, accelerations." Also included are linear differential equations with constant coefficients and applications.

The Kells does not follow the convention of having the exercises set in a smaller size type than the body of the text which, to this reviewer, makes more difficult a quick review and the making of assignments for classes. The other books are provided with answers to the odd numbered problems with a few exceptions in the case of the Gay. The Kells apparently gives most of the answers except to certain drill problems but no statement as to the system for giving answers appeared in the usual places. All three books are testimonials to the excellent work of compositors and publishers.

E. A. WHITMAN

NEW BOOKS RECEIVED

A New Theory of Gravitation. By Jakob Mandelker. New York, The Philosophical Library, 1951. 25 pp. $4.75.

Abstract Algebra, Vol. I. By Nathan Jacobson. D. Van Nostrand Company, New York. $5.00. 1951.

Brief Course in Analytics. M. A. Hill, Jr. and J. B. Linker. New York, Henry Holt, 1951. xi+224 pp.

College Algebra. By H. L. Rietz, A. R. Crathorne and J. W. Peters. New York, Henry Holt, 1951. xv+387 pp.

Fourier Transforms. By I. N. Sneddon. New York, McGraw-Hill, 1951. xii +542 pp. $10.00.

Analytic Geometry, Second Edition. By J. W. Cell. New York, John Wiley and Sons, 1951. xii+326 pp. $3.75.

Algebra for Commerce and Liberal Arts. By Bettinger and Dwyer. New York, Pitman Publishing Corporation, 1951. xi+225 pp. $3.00.

Quantum Mechanics of Particles and Wave Fields. By Arthur March. New York, John Wiley and Sons, Inc., 1951. 10+292 pp. $5.50.

Tables Relating to Mathieu Functions. National Bureau of Standards, The Computation Laboratory. New York, Columbia University Press, 1951. xviii +278 pp.

Using Your Mind Effectively. By J. L. Mursell. New York, McGraw-Hill, ix+264 pp. $3.00.

Teaching the New Arithmetic, Second Edition. G. M. Wilson. New York, McGraw-Hill, 1951. xiv+483 pp. $4.50.

CLUBS AND ALLIED ACTIVITIES

EDITED BY L. F. OLLMANN, Hofstra College

All material for this department should be sent to Professor H. D. Larsen, Albion College, Albion, Michigan.

CLUB REPORTS, 1950–51

Mu Alpha Theta, Yeshiva College

The Yeshiva College *Mathematics Club, Mu Alpha Theta,* devoted its program to lectures, delivered by the students, on various mathematical topics, including the following:

Abstract algebra; *Construction of the number system*; *Mathematical logic*, a series of four lectures by Mr. Rosenfeld

Vector algebra, by Mr. Wenger

Mathematical inversion, by Mr. Solomon

Probability, two lectures by Mr. Lieberman and Mr. Hertzberg

Elementary topology, by Mr. Hellerstein

Stieltjes integration, by Mr. Haber

Cryptography, by Mr. Levy

Infinite series, two lectures by Mr. Hellerstein

Special relativity, by Mr. Landau

The prime pair theorem, by Mr. Klein.

Other activities included a number of parties, and several informal discussions on miscellaneous problems in recreational and educational mathematics.

The present officers of the club are: President, Irwin Wenger; Vice-President, Simon Hellerstein.

The Mathematics Society, The Cooper Union

The following talks were presented to the *Mathematics Society* of The Cooper Union during 1950–51:

Solution of algebraic equations, by Peter Redmond

Elementary number theory, by Alan Berndt

Theory and use of the slide rule, by Jerry Davis

Calculus of finite differences, by Walter Kahn

Coaxial circles, by Peter Redmond

Inversion, by Peter Redmond

Introduction to matrices, by Irving Lowe

Nomographs, by Walter Kahn

Iteration, by Walter Kahn

Higher plane curves, by Harry Schwartzlander

Laplace transform, by Peter Redmond

Symbolic logic, by Arthur Robinson.

The officers for 1951–52 are: President, Alan Berndt; Vice-President and Treasurer, Harry Schwartzlander; Secretary, Walter Pecota; Faculty Adviser, Prof. J. N. Eastham.

Kappa Mu Epsilon, Hofstra College

The *New York Alpha* chapter of *Kappa Mu Epsilon*, during the academic year 1950–51, held monthly meetings with talks by faculty members, students, and alumni members. The topics and speakers were:

Report on the International Mathematics Congress, by Dr. E. R. Stabler
Probabilities, by Peter Hinrichs
The wineglass problem, by Mr. Frank Hawthorne
Continued fractions, by Peter Marshall
The normal law of error, by Dr. Mildred Dean
The trisection problem, by Richard Jaeger
Laplace transforms, by Sam Reynolds
Paradox, by Mr. W. Lysle Marshall
Selective sequence electronic calculator, by Ruth Mayer
Mathematics in the field of teaching, by Gertrude Decker
Mathematics in musical scales, by Robert Blasch
Probability—geometrical methods, by Kenneth Feldmann
The ladder problem, by Alexander Basil and Lee Dunbar.

Mr. Preston R. Bassett, President of Sperry Gyroscope Co. spoke on *The inventor discovers mathematics* at the Tenth Annual Initiation Banquet. Leaders of Long Island industries were invited as guests for the occasion. Mr. Bassett was initiated as an honorary member of the *New York Alpha* chapter.

Each year the *New York Alpha* chapter presents an award to the best student of freshman mathematics. The winner, Sue Rae Waldman, was selected on the basis of her grades in freshman mathematics and a two-hour competitive examination.

Miss Jane Brandt received a Fulbright scholarship for study in Biological Statistics in England for the academic year 1951–52.

The chapter has sponsored student help sections where *Kappa Mu Epsilon* members gave their time to help the weaker students in mathematics.

Eight of the members attended the national convention held in Springfield, Missouri and reported to the group at the last regular meeting of the year. Miss E. Marie Hove and Dr. L. F. Ollmann were reelected National Secretary and National Treasurer, respectively, at the convention.

Other activities included a square dance and Christmas party, and a picnic.

The newly elected officers to serve for 1951–52 are: President, Robert Blasch; Vice-President, Richard Lamm; Secretary, Marjorie Liers; Treasurer, Louis Bronzo; Historian, Mary Pawelko; Corresponding Secretary, Dr. Mildred Dean; Sponsor, E. Marie Hove.

Mathematics Club, Wellesley College

The Wellesley College *Mathematics Club* had for its 1950–51 program a series of lectures pertaining to various phases of mathematics. The lectures and speakers were:

Job possibilities for mathematics majors, by Mrs. R. L. Bishop of the Wellesley

Placement Office

Mathematics of philosophy, by Miss Onderdonk of the Wellesley Philosophy Department

The measurement theory, by Walter Rudin, Instructor at Massachusetts Institute of Technology.

The officers for 1950–51 were: President, Elizabeth Robinson; Vice-President, Ursula Loengard; Secretary, Barbara Kuehn; Treasurer, Eleanore Torstenson; Junior Executive, Marjorie Matsis; Sponsor, Helen G. Russell.

Mathematics Club, Indiana State Teachers College

The *Mathematics Club* of Indiana State Teachers College, Indiana, Pennsylvania had a very successful year, the high spot being a mathematical Christmas Party. The meetings, which proved to be very interesting consisted of the following talks:

Non-Euclidean geometry, by John Toman

Use of the slide rule, by Donald Leffler

Recreational mathematics, by Barbara Johnson, Leonora Murray, and Shirley Smith

Problems of a student teacher in mathematics, by Nancy Wallace, H. C. McKowen, and Robert Whistner

Applications of mathematics in road construction, by C. L. Playfoot, Pennsylvania Department of Highways.

Officers for the year 1950–51 were: President, James Norman Cornell; Secretary, Shirley Reynolds; Sponsors, Dr. Joy Mahachek and Dr. I. L. Stright.

Pi Mu Epsilon, University of Nebraska

Four meetings were held by the *Nebraska Alpha* chapter of *Pi Mu Epsilon* during 1950–51, in addition to social meetings. The program consisted of the following talks:

An application of the solution of a special recursive series, by Dr. H. B. Ribeiro

A method of quality production control, by J. W. Adams

Biology and mathematics, by Prof. B. H. Burma

Movement of pressure systems, by Prof. L. K. Jackson.

The new officers for 1951–52 are: Director, George Cobel; Vice-Director, Bruce Emmons; Secretary, Donna Grueber; Treasurer, Winfred Zacharias.

Mathematics Club, Kansas State Teachers College, Pittsburg

The activities of the *Mathematics Club* of the Kansas State Teachers College at Pittsburg included two picnics and the following programs:

History of calendar forms, by Joe Butler

Mathematical recreations, by Tom Clark

Magic squares, by Prof. I. G. Wilson

The language of mathematics, by James Pike

Spherical trigonometry, by William Brumbaugh

Celestial navigation, by Prof. R. W. Hart

The squirrel cage slide rule, by Robert Green, a paper also given at the National Convention of *Kappa Mu Epsilon* in Springfield, Missouri.

Morse Mathematics and Physics Club, University Heights Branch of New York University

The *Morse Mathematics and Physics Club* of University Heights held bi-weekly meetings during the academic year 1950–51. Lectures presented were:

Discharges in gases, by Prof. L. H. Fisher

Approximate homomorphisms, by Prof. H. N. Shapiro

Matric methods in the solution of systems of linear differential equations, by Prof. Avron Douglis

Nuclear emulsions, by Arthur Beiser

Cosmic rays, by Prof. Serge Korff

Special methods in the evaluation of definite integrals, by Prof. P. L. Thorne

Random noise, by Prof. Yardley Beers

Coding of problems for electronic computers, by Herbert Goertzel

Matrices as applied to mechanics, by Mr. L. Leitner

Farey series, by Dr. Sholom Arzt

The calculus of variations, by Charles Solky.

Bi-weekly seminars were also held, at which Harold Bohr's work, *Almost Periodic Functions*, was discussed.

The officers for the year 1950–51 were: President, Charles Solky; Vice-President, Burton Wendroff; Secretary, Herbert Kranzer.

Pi Mu Epsilon, St. Louis University

The *Missouri Gamma* chapter of *Pi Mu Epsilon*, during the academic year 1950–51 held three regular meetings, including business meetings and elections, besides the annual initiation banquet. The programs consisted of lectures on the following topics:

Foundations of mathematics, by Bernard Derwort

Partial differential equations, by John Hoffschwelle

Line geometry, by Dr. Paolo Lanzano

Chance, by Dr. Paul Rider, Professor of Mathematics, Washington University.

The fifth Annual Prize Essay Contest open to undergraduate students only, was conducted by Mr. Alois Lorenz. The senior division was won by Miss Helen Fagan, a senior of Maryville College. The title of her paper was *The mind of Newton as reflected in the Principia*. Her award was D. E. Smith's *Source Book of Mathematics*. Miss Maureen Burke, a sophomore of Fontbonne College, won the junior award for her essay *Isaac Newton: his life and works*. She was awarded E. T. Bell's *Men of Mathematics*.

James Krebs received the Chemical Rubber Publishing Co.'s *Book of Tables*

for being the freshman with the highest scholastic standing in mathematics. Roger Ahrens was given the Garneau Award of twenty-five dollars for being the highest ranking senior majoring in mathematics.

Seventy-five new members were inducted into the chapter at the spring meeting, bringing the total number of active members to two hundred and ninety-one and the total membership since the charter was granted to five hundred and fifty-three.

The Missouri Gamma News, the chapter bulletin, will be published during the summer.

The officers for 1950–51 were: Director, Joseph Santner; Vice-Director, Richard Kern; Secretary-Treasurer, Virginia Herre; Faculty Advisor and Permanent Secretary-Treasurer, Dr. Francis Regan.

NEWS AND NOTICES

EDITED BY EDITH R. SCHNECKENBURGER, University of Buffalo

Readers are invited to contribute to the general interest of this department by sending news items to Edith R. Schneckenburger, University of Buffalo, Buffalo 14, New York. Items must be submitted at least two months before publication can take place.

EDUCATIONAL TESTING SERVICE PSYCHOMETRIC FELLOWSHIPS

The Educational Testing Service is offering for 1952–53 its fifth series of research fellowships in psychometrics leading to the Ph.D. degree at Princeton University. Open to men who are acceptable to the Graduate School of the University, the two fellowships each carry a stipend of $2,375 a year and are normally renewable.

Fellows will be engaged in part-time research in the general area of psychological measurement at the offices of the Educational Testing Service and will, in addition, carry a normal program of studies in the Graduate School. Competence in mathematics and psychology is a pre-requisite for obtaining these fellowships. The closing date for completing applications is January 18, 1952. Information and application blanks will be available about November 1 and may be obtained from: Director of Psychometric Fellowship Program, Educational Testing Service, 20 Nassau Street, Princeton, New Jersey.

THE TENSOR SOCIETY

A new mathematical organization, the Tensor Society, has been founded recently at Hokkaido University, Sapporo, Japan. Membership is open to all interested persons. The dues are $2.00 per year. The Society's journal, *Tensor, New Series*, will appear in two or three issues annually and will contain forty to eighty pages to the issue. *Tensor* will be devoted primarily to short, high grade

articles in the general field of vector and tensor analysis and certain domains of application such as mathmatical physics (including statics and dynamics), engineering, astronomy, geodesy, physiology, and philosophy. Application blanks for membership may be obtained from Professor A. Kawaguchi, Faculty of Science, Hokkaido University, Sapporo, Japan, or from Professor H. V. Craig, Department of Applied Mathematics, University of Texas, Austin, Texas.

MATHEMATICS PRIZE COMPETITION AT LOS ANGELES CITY COLLEGE

Los Angeles City College held its first annual William B. Orange Mathematics Prize Competition for high school students of the Los Angeles City High Schools on May 25, 1951. The competition was inaugurated as a memorial to the late William B. Orange, formerly chairman of the Department of Mathematics of Los Angeles City College.

The competition consisted of a two hour written examination. Schools were permitted to send two teams of three students each. There were one hundred sixty-one participants from thirty-three high schools. The team prize was a bronze trophy which was won by Eagle Rock High School. Also, several individual prizes were awarded.

A second competition will be held next spring. The bronze trophy will be presented annually to the winning school.

PERSONAL ITEMS

Dr. J. A. Clarkson, head of the Department of Mathematics of Tufts College, has succeeded Professor J. R. Kline in the position of Executive Secretary of the Division of Mathematics, which was recently established by the National Research Council.

Professor J. J. Nassau, director of the Observatory, Case Institute of Technology, has been elected Treasurer of the American Astronomical Society.

Illinois Institute of Technology announces the following: Professor L. R. Ford will be absent from the campus on special assignments during the year 1951–52; Professor Haim Reingold is Executive Officer of the Department of Mathematics for the year; Assistant Professor S. S. Shu has been promoted to an associate professorship.

At New York State College for Teachers, Albany: Professor Harry Birchenough, chairman of the Department of Mathematics, has retired with the title of Professor Emeritus; Professor R. A. Beaver has been promoted to the chairmanship of the department.

University of New Hampshire reports: Associate Professor M. R. Solt has been promoted to a professorship; Mr. R. O. Kimball has been promoted to an assistant professorship; Dr. R. B. Davis and Dr. H. G. Rice have been appointed to assistant professorships; Mr. Frederic Cunningham has been appointed to an instructorship.

Mr. R. C. Ailara, who has been a graduate student at Illinois Institute of Technology, has accepted a position as Junior Systems Engineer at Chance

Vought Aircraft, Dallas, Texas.

Mr. P. M. Anselone, formerly a graduate student at the College of Puget Sound, has a position as mathematician at General Electric—Hanford Works, Richland, Washington.

Assistant Professor Winifred A. Asprey of Vassar College has been promoted to an associate professorship.

Mr. J. H. Braun, formerly a student at Illinois Institute of Technology, has a position as mathematical physicist at the Naval Gun Factory, Washington, D. C.

Mr. R. L. Brooks, previously a graduate student at Catholic University, has accepted a position as mathematician at the National Bureau of Standards, Washington, D. C.

Mr. L. P. Burton, who has been a graduate student at the University of North Carolina, is now at the University of California at Davis.

Assistant Professor R. C. Campbell of the United States Naval Postgraduate School has been promoted to an associate professorship.

Assistant Professor F. M. Carpenter, Colorado School of Mines, has been promoted to an associate professorship.

Mr. T. E. Cheatham, Jr., has been appointed to a research assistantship at Purdue University.

Mr. Carl Cohen of Cambridge Junior College has been appointed Lecturer at Wheaton College, Norton, Massachusetts.

Mr. K. J. Cohen, formerly a student at Reed College, is now a Rhodes Scholar at Queen's College, Oxford University, Oxford, England.

Dr. E. P. Coleman has accepted a position as mathematical statistician with the Hughes Aircraft Company, Culver City, California.

Mr. D. F. Coulter, Jr., formerly a teaching assistant at State College of Washington, has a position as mathematician at the United States Navy Mine Countermeasures Station, Panama City, Florida.

Mr. E. L. Eagle of the University of Tennessee has accepted a position as electromechanical engineer at Glen L. Martin Aircraft Company, Baltimore, Maryland.

Mr. James Eastcott, formerly a teaching fellow at the University of Michigan, is now a research scientist with the Defense Research Board, Ottawa, Canada.

Mr. R. P. Eisinger, previously a student at Swarthmore College, is enrolled in the College of Physicians and Surgeons, Columbia University.

Dr. H. Margaret Elliott of Washington University has been promoted to an assistant professorship.

Graduate Assistant P. G. Engstrom of Kansas State College is now at Augustana Theological Seminary.

Professor C. H. Forsyth of Dartmouth College has retired.

Assistant Professor Stanley Frankel of the Department of Applied Mathematics, California Institute of Technology, was at Victoria University of

Manchester, England, during the summer and was engaged in research for the British government.

Dr. L. E. Fuller of the University of Wisconsin has a position as senior development engineer with the Goodyear Aircraft Corporation, Akron, Ohio.

Assistant Professor R. W. Gardner of Olivet Nazarene College has been promoted to the position of Professor and Head of the Department of Mathematics.

Mr. R. N. Goss of the University of Tulsa has a position in the United States Navy Electronics Laboratory, San Diego, California.

Mr. H. W. Gould, formerly a student at the University of Virginia, is now in the United States Army.

Mr. E. P. Graney of the University of Notre Dame has been appointed Research Assistant at Willow Run Research Center, University of Michigan.

Associate Professor J. W. Green of the University of California at Los Angeles is at the Institute for Advanced Study.

Dr. W. T. Guy, Jr., previously a graduate assistant at California Institute of Technology, has been appointed Assistant Professor of Applied Mathematics and Astronomy at the University of Texas.

Assistant Professor J. J. Hayes of the University of Utah has retired.

Associate Professor P. W. Healy of Southwestern College has been appointed Assistant Professor of Mathematics and Astronomy at the University of New Mexico.

Dr. L. Aileen Hostinsky of Temple University has been promoted to an assistant professorship.

Associate Professor S. T. Hu of Tulane University delivered a series of lectures on Cohomology Theory of Topological Groups at the University of Michigan during the Summer Session of 1951.

Professor C. M. Huber, chairman of the Division of Mathematics and Business Education, James Ormond Wilson Teachers College, has been appointed Dean of Instruction, Montclair State Teachers College, New Jersey.

Assistant Professor W. J. Klimczak of the University of Rochester has been appointed to an assistant professorship at Trinity College, Hartford, Connecticut.

Mr. W. G. Koellner, formerly a student at New Jersey State Teachers College, is teaching at Hillside High School, Hillside, New Jersey.

Mr. Charles Kurland, previously a graduate student at the University of Buffalo, has accepted a position as research engineer, Locomotive Development Company, Alco Plant, Dunkirk, New York.

Associate Professor J. P. LaSalle of the University of Notre Dame is now in the United States Navy.

Dr. D. B. Lloyd has been appointed to an instructorship at James Ormond Wilson Teachers College.

Associate Professor N. H. Mewaldt of Northern State Teachers College, Aberdeen, South Dakota, has been promoted to a professorship.

Professor W. L. Miser, who has retired from the position of Head of the Applied Mathematics Department, Vanderbilt University, has accepted a position at McKendree College.

Dr. Deane Montgomery of the Institute for Advanced Study has been promoted to the position of Professor of Mathematics.

Associate Professor R. H. Moorman has been promoted to the position of Professor and Head of the Department of Mathematics of Tennessee Polytechnic Institute.

Mr. W. L. Murdock, formerly of Cornell University, has a position as consulting and research mathematician at General Electric Electronics Laboratory, Syracuse, New York.

Graduate Assistant J. D. Neff of Kansas State College has been appointed to a graduate assistantship at Purdue University.

Associate Professor Zeev Nehari of Washington University has been promoted to a professorship.

Mr. George Ogawa, previously a student at the University of Chicago, is now in the United States Army.

Mr. S. V. Parter, recently a student at Illinois Institute of Technology, has been appointed to a research assistantship at the Los Alamos Scientific Laboratory of the University of California.

Mr. M. F. Pollack of Stockton College is teaching now at the Union Linden High School, Linden, California.

Assistant Professor M. H. Protter of Syracuse University is a member of the Institute for Advanced Study during 1951–52.

Sister M. Virgilia, who has been a student at Notre Dame University, is now at Madonna College, Plymouth, Michigan.

Associate Professor T. H. Southard of Wayne University is at the Institute for Numerical Analysis.

Mr. A. L. Stamper, recently a student at Eastern Kentucky State College, is in military service.

Associate Professor J. E. Stewart of the University of Maine has been appointed Dean of Men.

Lecturer N. Y. Tang of the University of Washington has accepted a position with the Actuarial Department, Manhattan Life Insurance Company, New York City.

Associate Professor Alexander Tartler of Lafayette College has been promoted to a professorship.

Mr. R. E. Walters, formerly a graduate student at Kent State University, has a position as mathematician at Ballistics Research Laboratory, Aberdeen Proving Ground, Maryland.

Mr. M. E. White of Hunter College is in the United States Army.

Mr. W. F. Winn is Military Instructor at Camp Wood, New Jersey.

Mr. E. G. Wohlford, who has been teaching at the Iolani School, Honolulu, Hawaii, is now in the United States Army.

Associate Professor H. E. Wolfe of Indiana University has been promoted to a professorship.

Assistant Professor A. D. Ziebur of Oklahoma Agricultural and Mechanical College has been appointed to an instructorship at Ohio State University.

Mr. R. L. Adey, who was a student at Oregon State College, was killed in an automobile accident on May 12, 1951. The annual mathematics award of Oregon Beta Chapter of Pi Mu Epsilon has been named after him as a commemoration.

Professor Emeritus Abraham Cohen of Johns Hopkins University died on April 25, 1951. He was a charter member of the Association.

Miss Florence L. Munroe, who had retired from her position of Head of the Department of Mathematics, Northampton High School, Northampton, Massachusetts, died in September, 1949. She was a charter member of the Association.

Reverend Joseph Wilczewski, S.J., of the Mathematics Department of Marquette University, died on March 6, 1951. He had been a member of the Association for twenty-eight years.

THE MATHEMATICAL ASSOCIATION OF AMERICA

Official Reports and Communications

NEW MEMBERS

Professor H. M. Gehman, Secretary-Treasurer, announces that the following seventy-six persons have been elected to membership by the Board of Governors on applications duly certified:

F. H. Applebaum, Student, University of Pennsylvania, Philadelphia, Pa.

J. T. Archibald, Student, St. John's College, Booklyn, N. Y.

W. L. Baily, Jr., Student, Massachusetts Institute of Technology, Cambridge, Mass.

Mrs. Barbara B. Blair, M.S. (Iowa) Part-time Instr., State University of Iowa, Iowa City, Iowa

I. L. Bossler, M.S. (Purdue) Grad. Assistant, Purdue University, Lafayette, Ind.

H. A. Bott, Student, Illinois Institute of Technology, Chicago, Ill.

O. C. Braune, Student, St. Mary's University, San Antonio, Tex.

Anita Bredt, Student, University of Pennsylvania, Philadelphia, Pa.

Dorothy C. Breynaert, Student, University of New Hampshire, Durham, N. H.

Eva Burkhalter, M.S. (Oregon) Teacher, Klamath Union High School, Klamath Falls, Ore.

H. E. Chrestenson, M.A. (Washington S.C.) Teaching Assistant, State College of Washington, Pullman, Wash.

Helen E. Core, M.S. (Michigan S.C.) Research Assistant, University of Michigan, Ann Arbor, Mich.

D. F. Coulter, Jr., M.A. (Washington S.C.) Mathematican, U. S. Navy, Panama City, Fla.

D. O. DAVIDSON, B.S. (Oregon S. C.) 1833 5th Street, Manhattan Beach, Calif.

T. E. DEPKOVICH, B.S. (Indiana S.T.C.) Asst. Professor, College of Steubenville, Ohio.

JOAN DRAKE, Student, Seton Hill College, Greensburg, Pa.

D. G. DUNCAN, Ph.D. (Michigan) Asst. Professor, University of Arizona, Tucson, Ariz.

R. P. Eisinger, Student, Swarthmore College, Pa.

MILTON FISHMAN, B.S.E. (Michigan) Automobile Design Engineer, Detroit Arsenal, Centerline, Mich.

D. J. R. FOULIS, Student, University of Miami, Coral Gables, Fla.

MRS. HERTA T. FREITAG, M.A. (Columbia) Asst. Professor, Hollins College, Va.

H. N. GARBER, Student, University of Pennsylvania, Philadelphia, Pa.

S. H. GREENE, M.A. (Pennsylvania) Lieutenant, U. S. Army, Fort Bliss, Tex.

R. C. GUNNING, Student, University of Colorado, Boulder, Colo.

H. M. GURK, Student, University of Pennsylvania, Philadelphia, Pa.

SIMON HELLERSTEIN, Student, Yeshiva University, New York, N. Y.

J. A. HUMMEL, B.S. (C.I.T.) U. S. Army, Red Bank, N. J.

ELEANOR L. JENKINS, M.A. (North Carolina) Asso. Professor, Queens College, Charlotte, N. C.

C. H. JONES, 1178 King Street, Charleston, S. C.

M. W. KALLETT, A.B. (Buffalo) Teacher, North Park Junior High School, Lockport, N. Y.

MRS. ROSELLA KANARIK, Ph.D. (Pittsburgh) Lecturer, University of Southern California, Los Angeles, Calif.

ROSEMARY KANE, Student, Seton Hill College, Greensburg, Pa.

GENE KERNS, A.B. (Chico S.C.) P.O. Box 535, Susanville, Calif.

SIDNEY KISSIN, B.S. (Brooklyn C.) 2111 67 Street, Brooklyn, N. Y.

PATIENCE B. KLOPP, Student, Seton Hill College, Greensburg, Pa.

W. G. KOELLNER, Student, New Jersey State Teachers College, Upper Montclair, N. J.

T. J. LEE, 89–12 205 Street, Hollis, N. Y.

MILTON LEES, Student, University of California, Berkeley, Calif.

JULIUS LIEBLEIN, M.A. (Brooklyn C.) Mathematician, National Bureau of Standards, Washington, D. C.

J. L. LOCKER, Student, Alabama Polytechnic Institute, Auburn, Ala.

P. H. LORD, A.B. (Princeton) Lieutenant, U. S. Army, Fort Sill, Okla.

A. R. MATTHEWS, Student, Alabama Polytechnic Institute, Auburn, Ala.

R. B. MCMURDO, B.S.E.E. (U.S. Naval Academy) Instr., Montana State College, Bozeman, Mont.

E. A. MEANS, Student, Friends University, Wichita, Kans.

A. E. MILLER, Student, University of Pennsylvania, Philadelphia, Pa.

F. D. MILLER, A.M. (Michigan) Mathematician, Bell Aircraft Corporation, Niagara Falls, N. Y.

J. S. MORISON, S.B. (Chicago) 1445-C Stanford Street, Santa Monica, Calif.

D. E. MORRILL, B.A. (Mississippi) Grad. Fellow, University of Mississippi, University, Miss.

H. T. MUHLY, Ph.D. (Johns Hopkins) Asso. Professor, State University of Iowa, Iowa City, Iowa.

ZEEV NEHARI, Ph.D. (Hebrew U., Jerusalem) Professor, Washington University, St. Louis, Mo.

S. I. NEUWIRTH, M.S. (N.Y.U.) Bio-statistician, Schering Corporation, Bloomfield, N. J.

LEONARD NICHOLS, Student, New Jersey State Teachers College, Upper Montclair, N. J.

W. A. NICHOLSON, JR., Transmission Man, American Telephone and Telegraph Company, Charlotte, N. C.

J. L. O'DAY, Student, Gonzaga University, Spokane, Wash.

MARTIN ORR, M.A. (Columbia) Mathematician, Signal Corps Center, Ft. Monmouth, N. J.

F. D. PARKER, Ph.D. (Case) Asst. Professor, St. Lawrence University, Canton, N. Y.

MRS. LOIS K. PARRISH, Student, Eastern Kentucky State Teachers College, Richmond, Ky.

M. F. POLLACK, Master (Berlin) Teacher, Stockton College, Calif.

P. J. REDMOND, Student, Cooper Union,

New York, N. Y.

H. E. REINHARDT, M.A. (Washington S.C.) Teaching Assistant, State College of Washington, Pullman, Wash.

E. C. RICE, M.A. (George Peabody) Grad. Student, George Peabody College for Teachers, Nashville, Tenn.

JUDITH A. RICHMAN, Student, University of Pennsylvania, Philadelphia, Pa.

A. E. RICHMOND, Certificate (Multnomah) Instr., Multnomah College, Portland, Ore.

M. B. RITTERMAN, M.S. (N.Y.U.) Instr., Long Island University, Brooklyn, N. Y.

G. K. ROSENBLUM, Student, University of Pennsylvania, Philadelphia, Pa.

W. T. SANDLIN, A.B. (Marshall) Manager and Advertising Director, C. E. Sandlin & Son, Richmond, Ky.

J. J. SCHODERBEK, M.S. (Carnegie) Electronics Engineer, Glen L. Martin Company, Baltimore, Md.

M. J. SENDROW, Student, University of Michigan, Ann Arbor, Mich.

SISTER ST. THOMAS OF CANTERBURY, M.A. (Fordham) Teacher, Villa Maria Academy, Bronx, N. Y.

O. E. STANAITIS, Ph.D. (Würzburg) Asst. Professor, St. Olaf College, Northfield, Minn.

MARIE A. STEIN, Student, Adelphi College, Garden City, N. Y.

F. D. STROW, M.A. (Bowling Green) Head of Mathematics Department, Perrysburg High School, Ohio.

C. A. SWANSON, B.A. (British Columbia) Grad. Student, University of British Columbia, Vancouver, B. C.

HERMAN VON SCHELLING, Ph.D. (Berlin) Biomathematician, Naval Medical Research Laboratory, U. S. Submarine Base, New London, Conn.

F. J. WALL, B. S. (Sul Ross S.C.) Mathematical Analyst, University of California, Los Alamos, N. M.

J. F. WILLIAMS, M.A. (Tennessee) Asst. Professor, Memphis State College, Tenn.

JANUARY MEETING OF THE NORTHERN CALIFORNIA SECTION

The thirteenth annual meeting of the Northern California Section of the Mathematical Association of America was held at the University of San Francisco, San Francisco, California, on Saturday, January 27, 1951. Mr. S. A. Francis, Chairman of the Section, presided at both the morning and afternoon sessions.

The attendance was seventy-six including the following forty-five members of the Association: H. L. Alder, H. M. Bacon, G. A. Baker, Alice K. Bell, M. T. Bird, A. C. Burdette, M. A. Dernham, Brother Dominic, Roy Dubisch, Hazel E. Eggett, E. B. Eilertsen, G. C. Evans, Ruth A. Fish, S. A. Francis, N. S. Free, C. M. Fulton, L. C. Graue, J. G. Herriot, Marjorie L. Hoffman, D. W. Hullinghorst, V. F. Ivanoff, Helene G. Kusick, D. H. Lehmer, Sophia L. McDonald, W. H. McKenzie, A. B. Mewborn, E. D. Miller, H. S. Moredock, F. R. Morris, W. H. Myers, Andrewa R. Noble, C. D. Olds, George Pólya, Raphael M. Robinson, Kathryn B. Rolfe, Sister Madeleine Rose, Mary V. Sunseri, Irving Sussman, Gabor Szegö, M. J. Vitousek, K. J. Waider, L. A. Walker, Elizabeth Weinstock, Anna P. Wheeler, A. R. Williams.

The following officers were elected for the coming year: Chairman, Professor D. H. Lehmer, University of California; Vice-Chairman, Professor C. D. Olds, San Jose State College; Secretary-Treasurer, Mrs. Marjorie L. Hoffman, San Mateo Junior College.

By invitation of the Section, Professor J. G. van der Corput of the University of Amsterdam and visiting Professor at Stanford University delivered an address during the morning session.

The following papers were presented:

1. *Integral solutions of $x^2+y^2=z^2$ in quadratic fields*, by Professor C. D. Olds, San Joseph State College.

Every Diophantine equation whose history is given in Dickson's *History of the Theory of Numbers* admits of a generalized treatment in algebraic fields of the second, third, and higher degrees. The treatment of many such equations is well within the reach of first year graduate students. Using the above equation as an example, the author indicates how by considering special cases the student can be encouraged to attempt the general solution. In this case general formulas for x, y, and z can be obtained. It is also possible to prove that in every quadratic field there is an infinity of solutions through algebraic integers x, y, z belonging to the same field $R(\sqrt{t})$.

2. *On plausible reasoning*, by Professor George Pólya, Stanford University.

The speaker discussed historical examples of plausible reasoning in mathematical matters (by Euler, Descartes, Lord Rayleigh) and showed how these examples lead naturally to certain patterns of plausible inference which we seem to follow also in everyday matters, in court proceedings, in experimental science, etc. The simplest pattern of this kind is discussed in the speaker's booklet "How to Solve It," 5th printing, pp. 220–224. The title of the talk was the same, and the content almost the same, as that of an invited address given by the speaker at the International Congress of Mathematicians in Cambridge, on September 2, 1950.

3. *The second pearl of the theory of numbers*, by Professor J. G. van der Corput, University of Amsterdam, introduced by the Secretary.

If the number of positive elements $\leqq h$ of a sequence A, formed by integers $\geqq 0$, is $\geqq \alpha h$ for, $h=1, 2, \cdots, g-1$, where g denotes an integer >1, α being the largest number with that property, then α is called the lower density of A below g. The purpose of the lecture is to discuss the $\alpha+\beta$ theorem which can be stated as follows: If two sequences A and B, formed by integers, both contain 0, and if α and β are their respective lower densities below g, then the lower density of the sum sequence $A+B$ (formed by the numbers which can be written in at least one way as $a+b$, where a belongs to A, b to B) is $\geqq\alpha+\beta$, unless $\alpha+\beta\geqq 1$, when it is 1 and therefore each positive integer $<g$ can be written in at least one way as $a+b$.

Mann has proved this theorem in a more general form: Let A and B be two sequences formed by integers $\geqq 0$, both containing 0, such that

$$(\text{number of elements } \leqq h \text{ of } A)+(\text{number of elements } \leqq h \text{ of } B)\geqq \gamma h$$

for $h=1, 2, \cdots, g-1$, where $\gamma\leqq 1$. Then the lower density of $A+B$ below g is $\geqq\gamma$. Denoting by $A(h)$ the number of elements $<h$ of A, Mann's theorem can be formulated also as follows: If the sequences A and B, formed by integers $\geqq 0$, both contain 0, and if $A(h)>B(h) \geqq\phi(h)+1$, then $(A+B)(h)\geqq\phi(h)$ for $0<h\leqq g$, where $\phi(h)=\gamma(h-1)+1$, $\gamma\leqq 1$. A simple proof of this theorem can be given, showing a more general result, namely that the elements of A and B need not be integers, need not contain 0, and $\phi(h)$ can be any monotonic nondecreasing function satisfying $\phi(h+h')\leqq\phi(h)+\phi(h')$ for $h>0$, $h'>0$, $h+h'<g$.

4. *General mathematics in the college*, by Professor Arthur J. Hall, San Francisco State College, introduced by the Secretary.

The speaker presented an overview of the two-unit general mathematics course now required at San Francisco State College of nearly all candidates for the bachelor's degree. Now in the second experimental year, the course resulted from the re-study of the lower division requirements of the college. The course is related to other phases of the required lower division program. Its purpose is to assist the student to utilize mathematics in personal affairs and in solving social problems, to maintain or improve computational skills, to understand mathematics as a way of thinking, communicating, and solving quantitative problems. The content includes elementary statistics,

budgeting, installment buying, and probability. The text used is *Mathematics for the Consumer* with supplementary materials. The activities provided center around two projects. These projects involve campus problems such as "How do State Students Provide for Their Lunch?", "The Student Body Card Situation and Reasons for Not Buying," and installment buying problems such as "Furnishing a Home" or "Buying a Home." Weekly assignments of exercises from the text are made as the student needs practice on concepts to be used in the project. Students work in groups of 3 or 4 on joint projects. Evaluation reveals that the majority of students feel the projects are well worth their time, the text inappropriate, that the course should be required, and that both students and faculty feel course materials are useful. Standardized tests show statistically significant gains (at 1% level) in computational and reasoning ability in arithmetic.

5. *Calculation of moments for a Cantor-Vitali function*, by Professor G. C. Evans, University of California.

The speaker evaluated the moments $M^n = \int_0^1 x^n f(x)dx$ for the Cantor-Vitali continuous function $f(x)$, with $f(0)=0, f(1)=1$, which is obtained by displacing the open middle third of the interval (0, 1) to the level $y=1/2$, the open middle thirds of the remaining intervals to the levels $y=1/4$ and $y=3/4$, respectively, etc., and by making the natural definitions of $y=f(x)$ on the remaining set of content zero. He started with the obvious value $M^0=1/2$ and found $M'=5/16$, $M^2=11/48$, $M^3=233/1280$, by deriving the recurrence formula $2M^n=1/(n+1)+[(2+M)^n-M^n]/(3^{n+1}-1)$, in which $(2+M)^n$ stands for its binomial expansion beginning 2^nM^0. The formula is obtained by elementary means in terms of the relations $f(3x)=2f(x), f(x+2/3)=1/2+f(x)$, valid for $0 \leqq x \leqq 1/3$.

6. *Representation of the integers by positive integers*, by Professor Roy Dubisch, Fresno State College.

There is a 1–1 correspondence between the integers and the positive integers given by $n \leftrightarrow 2n+1$, $(n=0, 1, 2, \cdots)$, and $-n \leftrightarrow 2n$, $(n=1, 2, 3, \cdots)$. In this paper it is observed that this correspondence can be extended to an isomorphism by defining addition and multiplication of positive integers suitably.

E. B. ROESSLER, *Secretary*

APRIL MEETING OF THE MISSOURI SECTION

The Missouri Section of the Mathematical Association of America met at Central College in Fayette on Friday, April 6, 1951. Professor F. F. Helton, Chairman of the Section, and Professor L. O. Jones, Vice-Chairman of the Section, presided at the morning session. Professor Helton presided at the afternoon session and Professor Margaret F. Willerding, Secretary-Treasurer of the Section, introduced the guest speaker.

Fifty-three persons were in attendance including the following thirty-three members of the association: L. W. Akers, J. J. Andrews, S. Louise Beasley, J. W. Blattner, L. M. Blumenthal, C. A. Bridger, P. B. Burcham, Mary Cummings, C. E. Denny, W. C. Doyle, R. E. Ekstrom, M. B. Evans, G. M. Ewing, C. V. Fronabarger, J. W. Gaddum, M. A. Golub, Nola A. Haynes, F. F. Helton, C. A. Johnson, L. O. Jones, R. E. Lee, Jessie McLean, Ella Marth, R. J. Michel, Marie A. Moore, Margaret Owchar, Gordon Pall, R. M. Rankin, Francis Regan, W. L. Stamey, W. R. Utz, Margaret F. Willerding, J. L. Zemmer.

At the business meeting, the following officers were elected for the coming year: Chairman, Francis Regan, St. Louis University; Vice-Chairman, P. B. Burcham, University of Missouri; Secretary-Treasurer, S. Louise Beasley, Lin-

denwood College.

The annual meeting of 1952 will be held at Lindenwood College, the date to be announced later.

Professor Gordon Pall of Illinois Institute of Technology was the guest speaker. Professor Pall presented a paper on *Generalized Quaternions* at the afternoon session.

The following papers were presented:

1. *The distance set for the Cantor discontinuum*, by Professor W. R. Utz, University of Missouri.

This paper will appear in this MONTHLY.

2. *Modern mathematics for college freshmen*, by Rev. W. C. Doyle, Rockhurst College.

Little of the mathematics developed in the last hundred years appears in standard lower division courses. The paper suggested topics and viewpoints which could be included in college algebra without much revision of content. In analytic geometry two and three dimensions can be treated as a unit by adopting direction numbers (vectors) in place of slope. These number sets easily generalize into matrices. Rotation of axes then becomes a simple application of theory on diameters and the reduction of quadratic forms by means of matrices.

3. *What's wrong with mathematics textbooks?*, by Professor C. A. Johnson, School of Mines and Metallurgy of the University of Missouri.

The quality of pedagogical practices in the teaching of mathematics is still being called into question by prominent educators. It is the function of groups such as the Mathematical Association of America to look into the validity of these criticisms as they apply in every phase of the educational process. The mathematics textbook in itself constitutes one vital phase of this process that appears to need closer examination. The writing of modern mathematics textbooks seems to be vulnerable to attack on the following bases: (1) Failure to follow out the scientific method in the presentation of new theory; (2) Failure to develop a psychologically sound approach to logical organization of the material; (3) Substitution of long lists of "exercises" for real problems; (4) Development of the subject matter in a generally expository and non-reflective manner.

To improve the writing of mathematics textbooks it is recommended that in the future they be the joint effort of the mathematician and the educational psychologist.

4. *An attempt to broaden the background of prospective teachers of mathematics*, by Professor Nola A. Haynes, University of Missouri.

The speaker discussed the subject matter and objectives of a course recently organized at the University of Missouri and designed for prospective teachers of mathematics in secondary schools. The purpose of the course is to provide a broader and more mature understanding of the fundamental concepts of elementary mathematics.

5. *On the maximum likelihood estimation of a single parameter*, by C. A. Bridger, Bureau of Vital Statistics.

Since the estimators obtained by the method of maximum likelihood usually meet Fischer criteria of consistency, efficiency, and sufficiency, they are widely used. The principle of likelihood arose in an attempt to overcome practical difficulties associated with inverse probability. This principle asserts that if A refers to a universe and C to a sample, the probability that C in fact came from A is proportional to the likelihood that C is representative of A. The maximizing of

the likelihood relative to the parameters characterizing A is done in most cases by standard mathematical procedures.

6. *Generalized quaternions*, by Professor Gordon Pall, Illinois Institute of Technology.

With every ternary quadratic form is associated a quaternion algebra, which is closely related to the automorphs of the form. A change of basis of the algebra is accompanied by a corresponding linear transformation of the form. A ring of integral elements with unity is associated in a simple way with a ternary form having integral coefficients. Classes of ideals within such rings are associated with classes of quaternary quadratic forms within the genus of the norm form. The quadratic congruence $\bar{x}ax \equiv b \pmod{p^r}$, where a and b are given pure quaternions, p is a rational prime, and x is a variable quaternion, has interesting properties. For example, if a and b are primitive and of norm zero mod p, and $p \nmid d$, $r=1$, and p is odd, then the congruence is solvable if and only if the trace $2\sum A_{\alpha\beta}a_\alpha b_\beta$ is a quadratic residue mod p, with a special interpretation if the trace is zero mod p. Thus there are two residue systems of such pure vectors mod p. In a similar situation for the prime 2, there are four residue systems. The results for the prime 2 and odd discriminant are given in detail. There is a reciprocity law similar to the Hilbert form of the quadratic reciprocity law. Here primes p behave differently according as the Hansel-Hasse invariant c_p of the ternary form is $+1$ or -1.

MARGARET F. WILLERDING, *Secretary*

MATHEMATICS CONTEST OF THE METROPOLITAN NEW YORK SECTION

The Committee on Contests and Awards of the Metropolitan New York Section of the Association has announced the results of its mathematics contest which was held on May 10, 1951. This contest was open to high school students in the area of the Section. More than 5,000 students participated. It seems evident that the contest is accomplishing its purpose of creating the greatest possible interest in mathematics.

The Association's Bronze Cup was awarded to Brooklyn Technical High School for a score of 402 points out of a possible 450 points. This trophy is given to the school that has the highest total score on the three best papers which it submits.

Certificates of Merit were awarded to the highest ranking student in each county. The following students received these awards: J. S. Lew, Mamaroneck Senior High School, Westchester County; P. H. Monsky, Brooklyn Technical High School, Kings County; Lewis Iscol, Bronx High School of Science, Bronx County; Alicia E. Larde, Marymount School, New York County; T. E. Douglass, Saint Bonaventure's High School, Passaic County; E. J. Stolz, St. Joseph's Normal Institute, Dutchess County; R. A. Howard, Lynbrook High School, Nassau County, G. L. Smith, Chatham High School, Morris County; J. A. Wolf, College High School, Essex County; R. J. Sibner, Forest Hills High School, Queens County; and J. R. Hastings, Rutherford High School, Bergen County.

In addition to the above awards, Honor Keys were awarded to the following seven students who ranked first in their respective schools for the contest of 1950 and also that of 1951: J. S. Lew, Mamaroneck Senior High School, Mamaroneck, New York; P. H. Monsky, Brooklyn Technical High School, Brooklyn,

New York; Stephen Weingram, DeWitt Clinton High School, Bronx, New York; G. L. Smith, Chatham High School, Chatham, New Jersey; Lois B. Mitchell, A. B. Davis High School, Mount Vernon, New York; P. G. Federbush, Weequahic High School, Newark, New Jersey; and Anna Chernovitz, Washington Irving High School, New York, New York. Also individual awards, in the form of a button or pin bearing the facsimile of the seal of the Mathematical Association of America, were given to each student who won first place in his school.

Copies of the contest questions and other information about the contest may be obtained from Professor W. H. Fagerstrom, the City College of the City of New York, New York 31, New York. Professor Fagerstrom has served as Chairman of the Committee on Contests and Awards and the other members of the committee are as follows: Brother Bernard Alfred, Manhattan College; Professor H. F. Fehr, Teachers College, Columbia University; Mr. Julius Freilich, Brooklyn Technical High School; Professor J. J. Kinsella, School of Education, New York University; Professor V. S. Mallory, State Teachers College, Montclair, New Jersey; Mr. K. B. Morgan, Mt. Kisco High School, Mt. Kisco, New York; Mr. Aaron Shapiro, Midwood High School, Brooklyn, New York.

CALENDAR OF FUTURE MEETINGS

Thirty-fifth Annual Meeting, Brown University, Providence, Rhode Island, December 29, 1951.

The following is a list of the Sections of the Association with dates of future meetings so far as they have been reported to the Associate Secretary.

ALLEGHENY MOUNTAIN, Waynesburg College Waynesburg, Pennsylvania, May 10, 1952.
ILLINOIS, Western Illinois State College, Macomb, May 9–10, 1952.
INDIANA, Indiana University, Bloomington, Spring, 1952.
IOWA, Coe College, Cedar Rapids, April 18–19, 1952.
KANSAS
KENTUCKY, University of Kentucky, Lexington.
LOUISIANA-MISSISSIPPI, Northwestern State College, Natchitoches, Louisiana, February 15–16, 1952.
MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA, National Bureau of Standards, Washington, D. C., December 8, 1951.
METROPOLITAN NEW YORK, Spring, 1952.
MICHIGAN, University of Michigan, Ann Arbor, April 12, 1952.
MINNESOTA
MISSOURI, Lindenwood College, St. Charles, Spring, 1952.
NEBRASKA
NORTHERN CALIFORNIA, University of California, Berkeley, January 26, 1952.
OHIO, April 19, 1952.
OKLAHOMA, Oklahoma City University, October 12, 1951.
PACIFIC NORTHWEST, University of Oregon, Eugene, June 20, 1952.
PHILADELPHIA, University of Pennsylvania, Philadelphia, November 24, 1951.
ROCKY MOUNTAIN, Western State College, Gunnison, Colorado, May, 1952.
SOUTHEASTERN, Georgia Institute of Technology and Agnes Scott College, Atlanta, March 21–22, 1952.
SOUTHERN CALIFORNIA, Occidental College, Los Angeles, March 8, 1952.
SOUTHWESTERN, University of Arizona, Tucson, April 11–12, 1952.
TEXAS, East Texas State Teachers College, Commerce, April, 1952.
UPPER NEW YORK STATE, Hobart and William Smith Colleges, Geneva, May, 1952.
WISCONSIN

GEORGE BANTA PUBLISHING COMPANY, MENASHA, WISCONSIN

THE AMERICAN
MATHEMATICAL MONTHLY

THE OFFICIAL JOURNAL OF

THE MATHEMATICAL ASSOCIATION OF AMERICA, INC.

VOLUME 58 NUMBER 9

CONTENTS

NOVEMBER 1951

The AMERICAN MATHEMATICAL MONTHLY

(FOUNDED IN 1894 BY BENJAMIN F. FINKEL)

EDITORIAL CORRESPONDENCE should be addressed to the Editor, C. V. NEWSOM, State Education Building, Albany 1, N. Y. Papers submitted for publication should be sent to C. B. Allendoerfer, Department of Mathematics, University of Washington, Seattle 5, Washington.

ADVERTISING CORRESPONDENCE should be addressed to J. F. RANDOLPH, Morey Hall, University of Rochester, Rochester 3, N. Y.

NOTICE OF CHANGE OF ADDRESS by members of the Association as well as correspondence regarding subscriptions to the MONTHLY should be sent to the Secretary-Treasurer, H. M. GEHMAN, University of Buffalo, Buffalo 14, N. Y.

THIS IS THE OFFICIAL JOURNAL OF THE

MATHEMATICAL ASSOCIATION OF AMERICA, INC.

(Devoted to the Interests of Collegiate Mathematics)

Entered as second class matter at the post office at Menasha, Wis. Acceptance for mailing at special rate of postage provided for in the Act of February 28, 1925, embodied in Paragraph 4, Section 538, P. L. and R., authorized April 1, 1926.

SUBSCRIPTION PRICE: To Members, $4 a Year, To Others, $5 a Year.

PUBLISHED BY THE ASSOCIATION at Menasha, Wisconsin, and Albany, N. Y. during the months of January, February, March, April, May, June-July, August-September, October, November, December.

BROOK TAYLOR AND THE MATHEMATICAL THEORY OF LINEAR PERSPECTIVE*

P. S. JONES, University of Michigan

One can distinguish four overlapping and interrelated periods in the development of the mathematical theory of linear perspective: (1) the "prehistory" period in which, for example, the Greeks are reported to have made some use of perspective drawing in their theater, (2) the 15th and 16th century period of the origin of the theory with the artists-architects-engineers of the Renaissance (Brunelleschi, Franceschi, Alberti, and da Vinici), (3) a period of geometrical expositions typified by the works of del Monte and Stevin in the 17th century, and, finally, (4) the period of a generalized, complete, and even abstract theory.

This last period falls largely in the 18th century and is typified by the work of William Jacob Gravesand in Holland, Humphrey Ditton and Brook Taylor in England, and of the Alsacian (he was born in Mülhausen in the period when it was allied with Switzerland) mathematician Johanne Heinrich Lambert.

Of these, the work of Brook Taylor was certainly the most widely translated and reproduced, although the later work of Lambert rivals it in interest and perhaps in its total effect [1].

Brook Taylor published only two books in his lifetime of 46 years. Both of these appeared in 1715 when he was 30, and both of them exerted wide influence. He is, of course, best known for his *Methodus Incrementorum Directa et Inversa* in which appears the well known expansion of $f(x+h)$ which bears his name.

The other book was *LINEAR / PERSPECTIVE OR, A / New METHOD / Of Representing justly all manner of / OBJECTS as they appear to the EYE / IN ALL / SITUATIONS. / A Work necessary for PAINTERS, / ARCHITECTS, &c. to Judge of, and / Regulate Designs by.* This work is today only rarely and sparingly referred to in histories of either mathematics or art. This alone is of interest in view of a study which shows that the original appeared in four editions (or five, if Ware's revision be counted), the latest as recent as 1811, that it appeared in three translations, one French and two Italian, and that Taylor's English disciples in perspective number nine and were responsible for twelve books and twenty-two editions from 1715 through 1888 [2]. By disciples I here mean men who used Taylor's name in the titles or body of their own works which works in turn followed more or less closely Taylor's sequence and method.

One reason for this lack of recognition of Taylor's *Perspective* is perhaps the same defect as that upon which John Bernoulli is said to have seized when, according to Taylor's grandson and biographer, he called the book "abstruse to all and unintelligible to artists for whom it was more especially written" [3]. I have not found these exact words but it is quite likely both that Bernoulli

* Presented on September 1, 1950 to Section VII of the International Congress of Mathematicians held at Cambridge, Massachusetts.

said them and that one must discount them a little because of the heated and sharp nature of the controversy carried on by these two men in the pages of *Acta Eruditorum* and the *Philosophical Transactions*, beginning with a letter by Bernoulli in 1716 [4]. This controversy was over priority and the proper recognition of sources used in both Taylor's *Methodus Incrementorum* and also in his publications on the vibrating string and on an isoperimetric problem. More than this, however, it was a part of the continuation by their partisans of the Newton-Leibniz Controversy which was not always conducted in a fair and rational vein.

Taylor himself, however, recognized the excessive conciseness and abstractness of his first book on perspective when he expanded it from 42 pages to the 70 found in the second, or 1719 edition, and when he added a few plates showing the application of his method to actual drawings of physical objects in addition to the purely geometric diagrams of the first edition.

A later evaluation by Monge and Lacroix is interesting. In An 9 (1801) they recommended to the Académie des Sciences of the Institut de France that it not sanction the publication of a French translation of the first edition by B. Lavite [5]. In the introduction to their report, however, they remarked that Taylor's work was "distinguished from a crowd of others dealing with perspective by its originality and the fruitfulness of the principle upon which it was based." They also termed it "elegant," "expeditious," and "not lacking in a sort of generality." They explained that they did not favor printing it in spite of this for two reasons; namely, that additional work or study of perspective was unnecessary for those who already knew "Stéréotomie," and that Taylor's work was too geometrical for most artists who were not versed in Stéréotomie. This seems a fair and rational evaluation when one recalls that it was made by the founder of descriptive geometry and one of his followers.

More recently Julian Coolidge has referred to Taylor's work as the "capstone of the whole edifice" of perspective [6]. In spite of this and the fact that Gino Loria also has paid some attention to Taylor's work [7], the tabulation of editions, translations and extensions which is noted above and detailed in the notes has not been made before, nor is there a discussion of the first or 1715 edition available since later writers on perspective used the second edition and the historians have used either it or versions still more remote from the original edition.

In this paper are presented only three of the items of especial interest which appeared in the 1715 edition but not in later editions. First, however, it will be helpful to note that Taylor found it necessary to, as he said, "Consider this subject entirely anew." To this end he gave new terms, four axioms (in the 1719 edition), and then developed his theory in a formal and rigorous fashion with theorems, corollaries, problems, and proofs. He defined the "vanishing line" of any "original plane" to be the intersection with the picture plane of a plane through the eye of the beholder parallel to the original plane. This means that his basic three dimensional diagram as shown in Figure 1 (Plate 1 of Taylor's

book) consisted of four planes parallel in pairs, the picture plane, the "directing plane" through the eye of the beholder and parallel to the picture, the original plane, and the plane through the eye parallel to it. The "vanishing point" of any "original line" is the intersection with the picture of a line through the eye

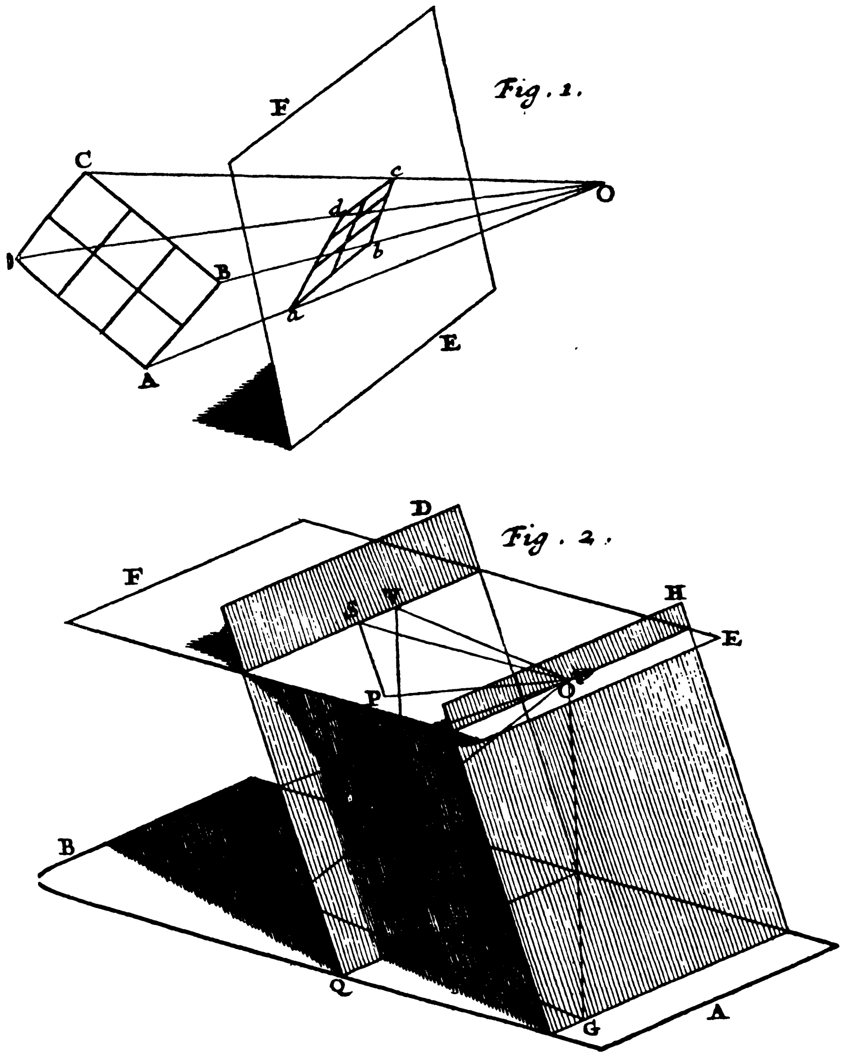


FIG. 1

parallel to the original line. Since the intersection of any original line with the picture is its own perspective, it follows as "PROPOSITION I, THEOREM 1" that "*The representation of a Line is Part of a Line passing thro' the Intersection and Vanishing Point of the Original Line.*"

The above discussion of Taylor's terminology and Theorem I indicates three

things about his work; namely, his formal, mathematical formulation, the generality of his concepts and procedures (he has, for example, no need to distinguish a special ground line and horizon line), and the completely new concise synthesis which he did achieve of procedures not all of which were original with him.

The first of the three specific items which will be discussed here is his construction for the perspective of a triangle ABC (see Figure 20 of Taylor's Plate 7 as reproduced in our Figure 2). Consider the plane of the drawing to represent the picture plane with two other planes rotated into coincidence with it. Below ED, the intersection of the original plane and the picture, is the original plane itself, containing ABC (to be thought of as "behind" the picture from the viewpoint of the observer) rotated about ED into the picture. The plane through the eye parallel to the ground plane has been rotated upward about FH. Above FH then is O, the eye point. Extend AB to meet ED in D, its "intersection." Draw a line through O parallel to AB to meet FH in F, the vanishing point of AB. FD is then the indefinite perspective of AB, *i.e.*, the perspective of AB produced. Join O to A and B. The intersections of these lines with FD determine perspective points a and b. A similar determination of c (c could also be located as the intersection of EH and IG) would give a diagram in which the corresponding sides of the two triangles meet on ED and the lines joining corresponding vertices concur in O. Although the Desargues triangle theorem is neither mentioned nor stated, note how completely it is implicit in this construction and the accompanying diagram [8]. Both the problem and the diagram were modified in the second edition and the relationship, though still implicit, became less obvious.

Also in the 1715 edition but omitted in the second edition is the problem of finding the perspective of the shadow of a triangle on a plane. Not only does this associate with the three dimensional case of the Desargues theorem, but of particular interest is Taylor's second solution of the problem which is, as he terms it, by putting the rules of perspective in perspective. In this same vein he elsewhere gives constructions for such things as the vanishing point of lines perpendicular to a given plane for the specific purpose of making it possible to draw directly in perspective without first having an orthogonal projection. In this Taylor anticipated Lambert who took this as one of the major objectives of his *Freye Perspective* (1759). Taylor's work with such problems led him to make repeated use in the 1715 edition of the idea of associating infinitely distant intersections with parallel lines.

A second construction which is both unique to the 1715 edition and which has for its purpose the construction of drawings directly in perspective is Taylor's solution of the problem of completing the construction of the perspective of a circle, given the perspective of its center and of one of its points. The diagram for this is to be found in "Fig. 21" of Taylor's Plate 7 which is our Figure 2. C is the perspective of the center of a circle, A the perspective of a point on

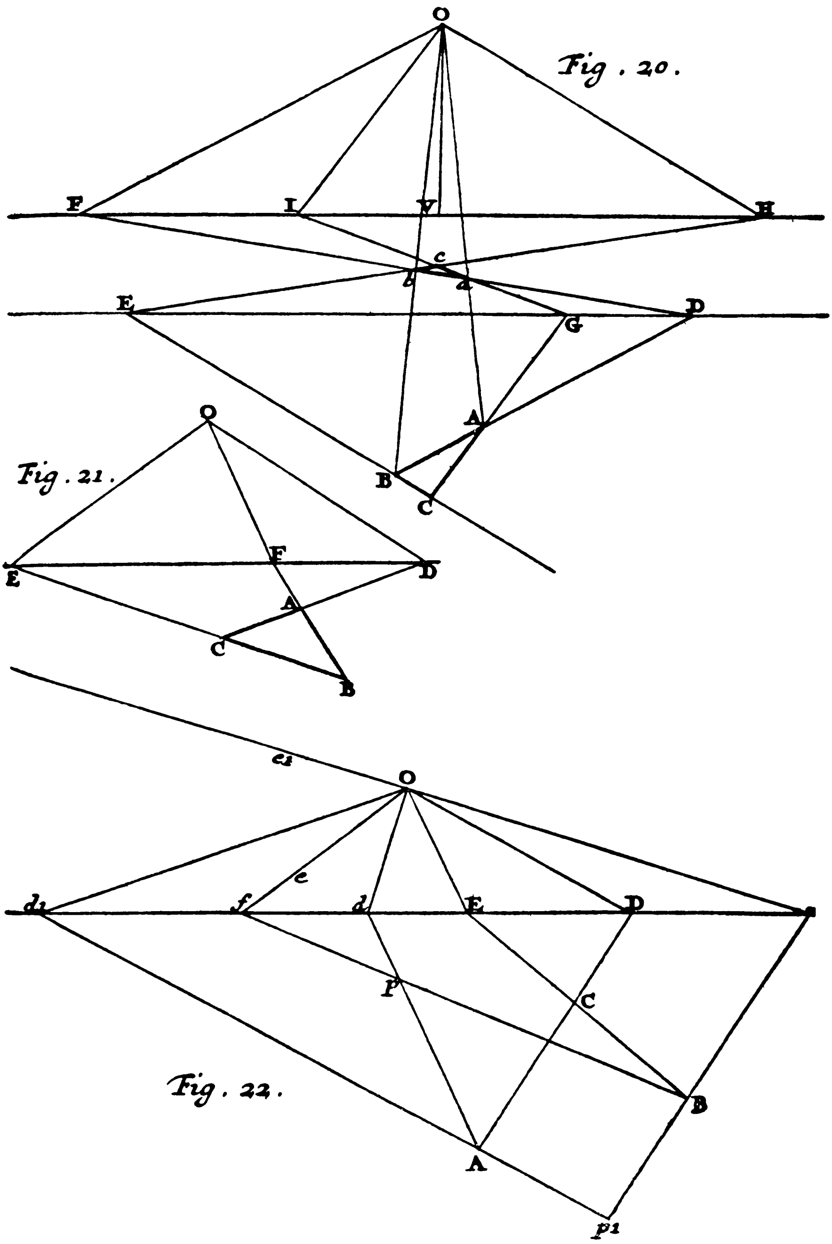


Fig. 20.
O
F
I
V
H
c
b
a
E
G
D
A
B
C
Fig. 21.
O
F
E
D
A
C
B
e1
O
e
d1
f
d
E
D
P
C
B
A
p1
Fig. 22.

FIG. 2

the circle, ED the vanishing line of its plane, and O the eye rotated into the picture. CA then represents a radius. Draw any line through C to meet the vanishing line in E and extend CA to intersect it in D. Bisect angle EOD to locate point F on ED. The join of F and A meets EC in B, another point of the perspective circle. Taylor's reasoning was based on the fact that since the angles at A and B are perspectives of equal angles then CA and CB are perspectives of the sides of an isosceles triangle and hence are the perspectives of equal lines. CB must then represent a radius, and B is the perspective of a point on the circle. This is another example of Taylor's thinking and drawing directly in perspective. It is also interesting to note that if the construction were extended

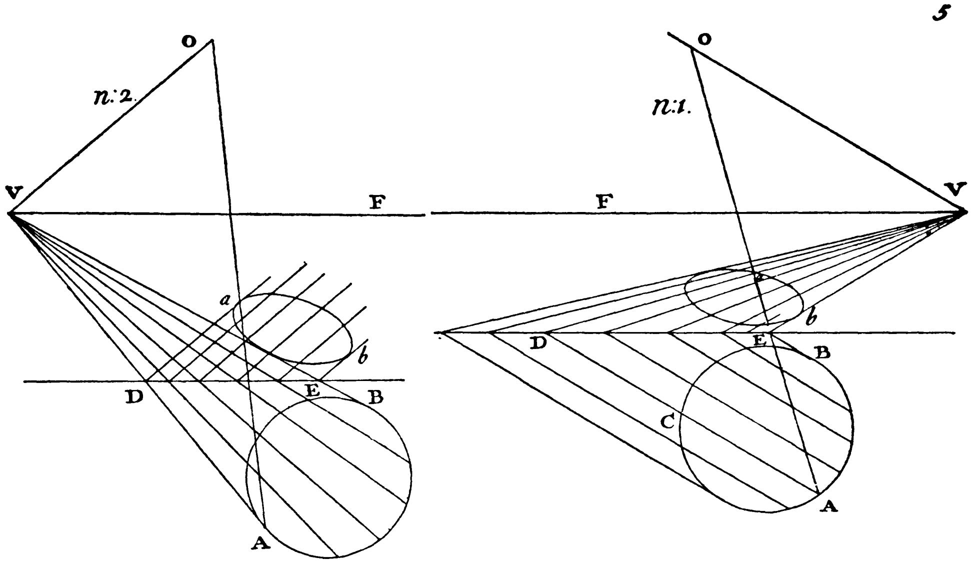


FIG. 3

to determine the second point on each radius by bisecting the supplement of EOD we would have an harmonic set of points on ED, and further that C and ED are pole and polar with respect to the conic which represents the circle.

The first explicit use which the author has found of the terms pole and polar in perspective is in the work of Cousinery in 1828. However, John Hamilton, one of Taylor's followers, had also read LaHire's *Sectiones Conicae* (1685) in which much use is made of harmonic sets. Book III of Hamilton's *Stereography or a Compleat Body of Perspective* (1738) makes extensive use of harmonic sets and some use of theorems on poles and polars although without using the latter terms.

Our Figure 3 shows, for contrast with the above, the two constructions for the perspective of a circle which were given as Figure 13 in the 1719 edition of Taylor's book. They are more conventional, use the orthogonal projection of the original circle, and are described in much more detail in the text.

Taylor gives no proof or explanation of the third unique construction which is here presented from the 1715 edition. The construction we refer to is "n:2." in Figure 32 of Taylor's Plate 12 which is shown here as Figure 4. Both "n:1." and "n:2." are constructions for a line through a given point and the inaccessible intersection of two other given lines. Today, "n:2." would be regarded as an application of harmonic sets related to complete quadrilaterals. Knowing that he did use both the idea that lines meeting on a vanishing line are parallel and its converse, we can guess that Taylor might have proven it quickly and

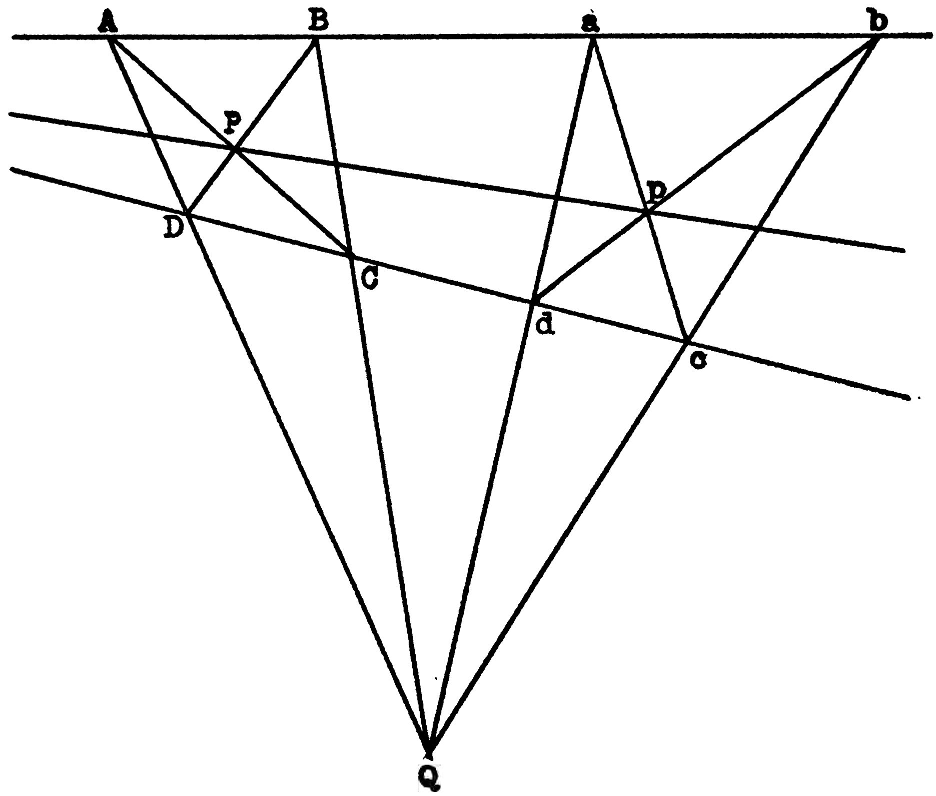


FIG. 4

easily by thinking in a "perspective" geometry where *ABCD* and *abcd* would be parallelograms rather than in a "Euclidean" geometry. In any case, Taylor was the first writer on perspective to treat this problem.

The only original work on perspective printed in England prior to Taylor's was Humphrey Ditton's *A Treatise of Perspective* of 1712, which deserves more note than it has had in the past but which is not comparable to Taylor's *Linear Perspective* in generality or originality. Following Taylor in England only Hamilton showed much originality while on the continent Lambert's work was outstanding in this century. Another feature which, though first met in Guido

Ubaldo Monte's *Perspectivae Librix Sex* (Pisa, 1600), was developed by Taylor and then carried much farther by Lambert was the solution of the inverse problem of perspective. This problem; namely, given data about a perspective drawing to draw inferences about the original, is basic in the modern science of photogrammetry.

This discussion represents only a portion of a complete study which the author has made of the development of the mathematical theory of perspective. It shows that Brook Taylor contributed a mathematically clear, concise, and logical, but abstract, formulation of extraordinary generality, including some treatment of the inverse problem. The editions, translations, and sequels to his work noted here extended his influence beyond both his homeland and his chronological period.

In conclusion, we note for those who might wonder at the interest of Taylor in this subject that not only is this interest consistent with the mathematical and cultural interests of the time (Desargues, Stevin, Ozanam, were earlier mathematicians who wrote on this topic), but also that Taylor grew up in a home where music and art were popular diversions. According to his grandson, Taylor himself in his painting "favored landscapes and water colors.—They have a force of color, a freedom of touch, a varied disposition of planes of distance, and a learned use of aerial as well as linear perspective which all professional men who have seen these paintings have admired" [9].

Notes

1. Max Steck, Johann Heinrich Lambert Schriften zur Perspektive. (Berlin: 1943), p. 48 lists Jacquier's French translation of Taylor's work, as among the books in Lambert's library and adds parenthetically that it was "von Lambert im II Teil des Hauptwerkes benützt."

2. Since the writer found no other at all complete enumeration of these works, it seems appropriate to preserve this data in detail for future reference. The books referred to are:

Editions:

Brook Taylor, Linear Perspective. London: 1715.

———, New Principles of Linear Perspective. London: 1719.

———, New Principles of Linear Perspective—third edition corrected by J. Colson. London: 1749.

———, Method of Perspective. 1766. The Dictionary of National Biography (London: 1899 LIX, p. 359) lists this under Isaac Ware who, it says, prepared the edition.

———, New Principles of Linear Perspective:—The fourth edition, revised. London: 1811.

Translations:

Francois Jacquier, Elementi di perspectiva secondo li principii di Brook Taylor, con varie aggiente. Roma: 1753.

Antoine Rivoire, Nouveaux principes de la perspective lineaire, traduction de deux ouvrages, l'un Anglois, due Docteur Brook Taylor, l'autre Latin, de M. Patrice Murdoch. Amsterdam: 1759.

Jacopo Stellini, Opere varie. Padova: 1781. Contains in volume II Taylor's "Nuovi principij della prospettiva lineare" according to Pietro Riccardi in his Biblioteca Matematica Italiana.

Disciples:

John Hamilton, Stereography or a compleat body of perspective. London: 1738, 1740, 1748.

John Joshua Kirby, Dr. Brook Taylor's method of perspective made easy both in theory and practice. Ipswich: 1754, 1755; London: 1765, 1768.

John Joshua Kirby, The perspective of architecture—deduced from the principles of Dr. Brook Taylor. London: 1761.

———, Dr. Brook Taylor's method of perspective, compared with examples lately published on this subject, as Sirigatti's by Isaac Ware. London: 1767.

Daniel Fournier, A treatise of the theory and practice of perspective. Wherein the principles—laid down by Dr. Brook Taylor are explained by moveable schemes. London: 1761, 1762, 1763, 1764.

Joseph Highmore, Practice of perspective on the principles of Dr. Brook Taylor. London: 1763.

Thomas Malton, A compleat treatise on perspective in theory and practice on the true principles of Dr. Brook Taylor. London: 1775, 1776, 1779.

———, An appendix or second part to the compleat treatise on perspective containing a brief history of perspective. London: 1783.

James Malton, The young painter's maulstick; being a practical treatise on perspective;—with the theoretic principles of—B. Taylor. London: 1800.

Edward Edwards, A practical treatise of perspective on the principles of Dr. Brook Taylor. London: 1803.

Joseph Jopling, Taylor's principles of linear perspective, new edition with additions by Joseph Jopling. London: 1835.

George Blacker, John Heywood's second grade perspective—adapted from Dr. Brook Taylor. Manchester: 1885–88.

3. Contemplatio Philosophica: A Posthumous Work of the Late Brook Taylor, L.L.D., F.R.S. some time Secretary of the Royal Society. To which is prefixed a life of the author by his grandson, Sir William Young, Bart. F.R.S., A.S.S. (London: Printed by W. Bulmer and Co., 1793), p. 29. The title page of this book bears the printed note *Not Published*. The book also includes some letters to and from Taylor to which we will refer later.

4. "Epistola Pro Eminente Mathematico Dn. Johanne Bernoullio, contra quendam ex Anglia antagonistam scripta." Acta Eruditorum. (July, 1716), pp. 296–315. The article preceding this one in Acta was "Methodus Incrementorum Directa & Inversa; Autore Broock (sic!) Taylor, L.L.D. & Regiae Societatis Secretario," a summary of the book with comments, references to Leibniz and his procedures and to Collins' Commercium Epistolicum. This "review" was probably written by Leibniz himself according to Heinrich Auchter, Brook Taylor der Mathematiker und Philosoph. (Wurzburg: Konrad Triltsch, 1937), p. 79.

The Taylor-Bernoulli dispute as it appeared in Acta and the Philosophical Transactions is somewhat expanded in details and clarified by the letters printed by Young in the Contemplatio Philosophica and in Auchter, *op. cit.*

Taylor wrote on February 5, 1719 to Count Raymond de Montmort in reply to a letter from Bernoulli which Montmort had forwarded, "For if the book be so very obscure, as he says it is, that the best artists, those already acquainted with the subjects, cannot well understand it—." This may be the source for Young's quotation. Taylor, however, seems to have been referring to his Methodus rather than his work on perspective.

5. Institut de France, Académie des Sciences, Procès-Verbaux des Séances de l'Académie. Tome II, An VIII–XI (1800–1804), 1912, p. 360 ff.

6. Julian L. Coolige, A History of Geometrical Methods. (Oxford, 1940), p. 108.

7. Gino Loria, Storia della Geometria Descrittiva. (Milano, 1921), pp. 43–51.

8. The copy of Taylor's book used originally in this study is in the Rare Book Room of the University of Michigan. The author, happening recently to have purchased a copy for himself, was startled to find lines *OaA* and *ObB* in his copy to have been drawn in with ordinary pen and

ink after printing. Further comparison of the two copies showed that a number of corrections to the plates were made during the printing process, appearing inked in in the author's copy and printed in the library's copy. It should be remarked that practically all of the original works cited are to be found in the University of Michigan's collection built up by Professor L. C. Karpinski whose suggestions and advice aided significantly in the study of which this paper is a partial report.

9. Sir W. Young, *op. cit.*, pp. 28–29.

THE PROBLEM OF n LIARS AND MARKOV CHAINS

WILLIAM FELLER, Princeton University

1. Introduction. In the Advanced Problems section of this MONTHLY [4288, Vol. 57 (1950) pp. 43–45] we find a solution of the following problem, first treated by A. S. Eddington: If A, B, C, D each speak the truth once in three times (independently), and *A affirms that B denies that C declares that D is a liar, what is the probability that D was telling the truth?* There are only eight different statements which A can make (or deny) and their enumeration shows that the required probability is 13/41.

John Riordan's remark that this problem applies also to switching circuits led me to look for the appropriate probability model. It turns out that the problem of n liars leads directly to the simplest Markov chain, and that natural variations of the same problem correspond to more general chains. The example may be of some didactic use since the theory of Markov chains grows in importance, but simple illustrative examples are hard to get.

2. Model for the simplest Markov chain. In the original formulation, persons C, B, and A issue successively three statements which may be true or false and which may contradict each other. However, taken at face value, each statement would imply either that D is telling the truth or that he lies. Thus, if B denies that C declares that D is a liar, he implies that D tells the truth. If A (rightly or wrongly) denies that B denied *etc.*, then he implies that D is a liar. In a continued process, an even number of denials cancel, and the implication of a statement like "A_1 asserts that A_2 denies that $A_3 \cdots$" depends on the evenness of the total number of denials.

To use a neutral terminology, we shall speak of a chance process with two possible states; at any time the observed state is 1 or 2 according as the last statement implies that D is honest or a liar. We imagine that the statements are issued at times $1, 2, 3, \cdots$. Initially, or at time 0, the "observed" state is 1 or 2 according as D tells the truth or lies. (In the original problem actually only D and C know the initial state.) Every possible sample sequence of the process is represented by a succession of the digits 1 and 2, and *vice versa.*

The fundamental assumption (to which the Markov character of the process is due) is that each person knows only the statement of the last speaker, but

not the past history of the system. Moreover, at time n the observed state changes or remains unchanged according as the nth speaker tells the truth or lies. Thus we arrive at the following model which represents the simplest Markov chain:

We have a process with two possible states 1, 2. *Initially* (*or at time* 0) *the probabilities of the two states are* α *and* β, *respectively* $(\alpha+\beta=1)$. *Whatever the development up to time* n, *there is probability* p *that at time* n *the observed state does not undergo a change and probability* $q=1-p$ *that it does. We seek the conditional probabilities* x_n *and* y_n *that the process actually started from state* 1, *given that at time* n *the observed state is* 1 *or* 2, *respectively.*

In the original formulation $\alpha=p=\frac{1}{3}$, $n=3$, and only x_n is required.

3. Each individual step consists in one of the four possible transitions $1\to1$, $1\to2$, $2\to1$, $2\to2$, and the corresponding transition probabilities are, by assumption,

$$p_{11} = p_{22} = p, \qquad p_{12} = p_{21} = q. \tag{1}$$

Suppose now that at a certain time the system is in state j and let $p_{jk}^{(n)}$ be the probability (on this hypothesis) that n steps later the observed state is k. The $p_{jk}^{(n)}$ are called the n-step transition probabilities. Clearly

$$p_{jk}^{(1)} = p_{jk},$$

and

$$p_{jk}^{(2)} = p_{j1}p_{1k} + p_{j2}p_{2k}. \tag{2}$$

More generally, we can calculate $p_{jk}^{(n)}$ from the obvious recursion formulas

$$p_{jk}^{(n+1)} = p_{j1}^{(n)}p_{1k} + p_{j2}^{(n)}p_{2k}. \tag{3}$$

Now these are just the formulas for matrix multiplication: if the matrix (p_{jk}) is denoted by P, then the $p_{jk}^{(n)}$ are elements of P^n.

Since the initial probabilities of the states 1 and 2 are α and β, the probability $a_k^{(n)}$ of observing at time n the state k is obviously

$$a_k^{(n)} = \alpha p_{1k}^{(n)} + \beta p_{2k}^{(n)}. \tag{4}$$

We find therefore for the two required probabilities

$$x_n = \frac{\alpha p_{11}^{(n)}}{a_1^{(n)}} \quad \text{and} \quad y_n = \frac{\alpha p_{12}^{(n)}}{a_2^{(n)}}. \tag{5}$$

For explicit formulas we must, of course, calculate $p_{jk}^{(n)}$. The formula

$$P^n = \frac{1}{2}\begin{pmatrix}1 & 1\\ 1 & 1\end{pmatrix} + \frac{1}{2}(p-q)^n\begin{pmatrix}1 & -1\\ -1 & 1\end{pmatrix} \tag{6}$$

follows directly from the canonical decomposition of matrices, and can easily be verified. Substituting from (6) and (4) into (5) we find

$$
(7)\qquad
\begin{aligned}
x_n &= \frac{\alpha\{1 + (p - q)^n\}}{1 + (\alpha - \beta)(p - q)^n},\\
y_n &= \frac{\alpha\{1 - (p - q)^n\}}{1 - (\alpha - \beta)(p - q)^n}.
\end{aligned}
$$

For $n=3$, $\alpha=p=\frac{1}{3}=13/41$ in agreement with the solutions quoted. Moreover, $y_n=7/20$. Clearly $x_n\to\alpha$ as $n\to\infty$. It is interesting that x_n decreases monotonically if $p>q$, while x_n is alternatingly larger and smaller than α if $p<q$.

4. Preferential lying. Up to now we have assumed that the chances of a person telling the truth are in no way dependent on the statement he is supposed to relay. Suppose now that each person has a preference to claim that D is honest. Then a transition $1\to 1$ is more probable than $2\to 2$, while $1\to 2$ is less probable than $2\to 1$.

We can treat this general case in the same way as before, provided we replace (1) by the general matrix of transition probabilities

$$
(8)\qquad p_{11} = p, \qquad p_{12} = q, \qquad p_{21} = q', \qquad p_{22} = p',
$$

where, of course, $p+q=p'+q'=1$. All formulas of the preceding section apply, except that (6) must be replaced by

$$
(9)\qquad P^n = \frac{1}{q+q}\begin{pmatrix} q' & q \\ q' & q \end{pmatrix} + \frac{(p'-q)^n}{q+q'}\begin{pmatrix} q & -q \\ -q' & q' \end{pmatrix}.
$$

The final result is now

$$
(10)\qquad x_n = \frac{\alpha\{q' + q(p' - q)^n\}}{q' + (\alpha q - \beta q')(p' - q)^n}.
$$

5. Generalizations. In the preceding examples we had a message capable of two forms transmitted in successive steps. It is easy to generalize this scheme to the case where the message can assume N different forms $1, 2, \cdots, N$. (For example, the message may be a digit which is repeatedly copied and is subject to copying errors.) The preceding theory applies, except that the matrix $P=(p_{jk})$ of transition probabilities is now of order N, and that more questions can be asked. We are, in this way, led to the general Markov chain with N possible states and constant (or stationary) transition probabilities. Finally, if we admit that the transition probabilities vary from step to step (variable proneness to lie) then we are led to the most general Markov chain with finitely many states.*

* For the general theory cf. Chapters 15 and 16 of An Introduction to Mathematical Probability and Its Applications by W. Feller (Wiley, New York, 1950).

A SIMPLE ITERATIVE SOLUTION FOR CERTAIN SIMULTANEOUS QUADRATIC EQUATIONS

W. S. LOUD, University of Minnesota

This short paper describes an iterative solution for a class of simultaneous quadratic equations. It shows how the non-linearity of the equations places a severe restriction on the possibility of the process. It also illustrated a case for which, once the possibility of the iterative solution is known, a starting point can easily be found from which the process always converges.

An iterative solution of the system,

$$x = f(x, y), \qquad y = g(x, y),$$

involves the selection of an initial point (x_1, y_1), and the determination of a succession of points (x_n, y_n) by the formulas

$$x_{n+1} = f(x_n, y_n), \qquad y_{n+1} = g(x_{n+1}, y_n) = h(x_n, y_n).$$

It is known that if (x_0, y_0) is a solution of the system, the process will converge to (x_0, y_0) if (x_1, y_1) is chosen sufficiently close to (x_0, y_0) and if both of the characteristic values of the matrix of the Jacobian

$$\frac{\partial(f, h)}{\partial(x, y)}$$

are less than unity in magnitude in a neighborhood of (x_0, y_0).

We shall consider the case where f is a quadratic function of y alone and g is a quadratic function of x alone. Without loss of generality we may consider the system,

$$y = x^2 + A, \qquad x = y^2 + B, \tag{1}$$

where A and B are constants.

For the system (1) the convergence condition is $|4x_0y_0| < 1$ where (x_0, y_0) is a real solution of the system.

To determine the conditions on A and B which will imply this condition, it is helpful to consider an A, B-plane and to analyze the system (1) in terms of regions in this plane. There are five possibilities for roots of (1): four real and distinct, four real with two coincident, two real and distinct and two complex, two real and coincident and two complex, and four complex. If there are two coincident roots, the pair always lies on the hyperbola $4xy = 1$. To find the points in the A, B-plane which give rise to this, we eliminate x and y from the three equations $4xy = 1$ and the system (1). The result is:

$$F_1(A, B) = 256A^2B^2 + 256A^3 + 256B^3 + 288AB - 27 = 0.$$

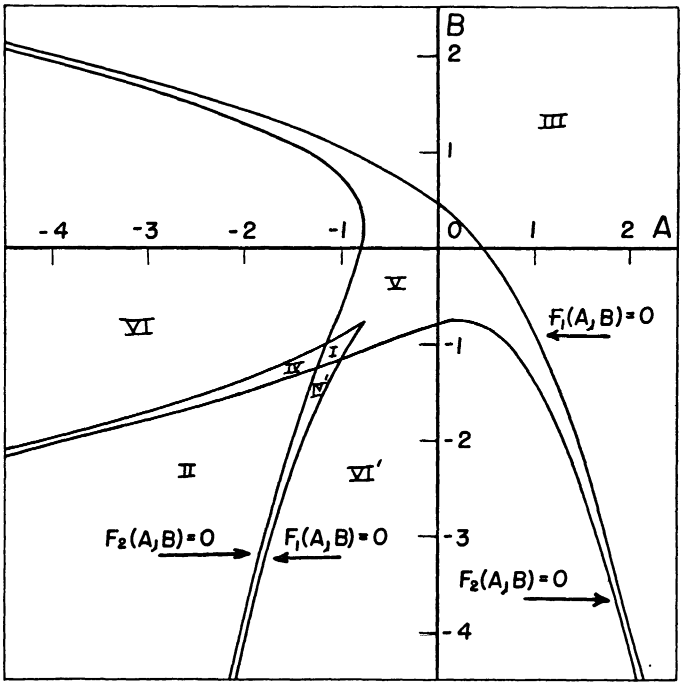
If $F_1(A, B)$ is not zero, we will have one of the three cases with no coincident roots.

The locus of $F_1 = 0$ in the A, B-plane will also be a boundary of the region

in that plane of values of A and B for which the convergence condition will be satisfied. The other boundary will be the locus of those values of A and B for which (1) has a root on the hyperbola $4xy=-1$. By the same process as above we find:

$$F_2(A, B) = 256A^2B^2 + 256A^3 + 256B^3 + 160AB + 125 = 0.$$

In the accompanying figure, the loci of $F_1=0$ and $F_2=0$ are plotted in the A, B-plane. There are formed eight regions, and the system (1) has a root for which the process converges if (A, B) lies in one of a certain four of these regions.



Region	Sign of F_1	Sign of F_2	Number of Real Roots	Number of Roots for which Iterative Process Converges
I	+	+	4	2
II	+	+	4	0
III	+	+	0	0
IV–IV′	+	−	4	1
V	−	+	2	1
VI–VI′	−	−	2	0

It will be necessary to know the regions of the x, y-plane in which the roots of (1) will fall for the various cases. It is interesting that the function F_1 intro-

duced earlier also plays a role here. The two branches of the curve $F_1(-x/2, -y/2)=0$ and the two branches of the hyperbola $4xy=1$ divide the x, y-plane into six regions. (The lower left branches are tangent at $(-1/2, -1/2)$.) If the system (1) has two distinct real roots and two complex, there will be one real root in each of the two regions between the branches of $F_1(-x/2, -y/2)=0$. If there are four distinct real roots, one will lie in each of the four remaining regions.

We now consider the problem of determining a starting point for the iterative process. The principal result is: If (x_0, y_0) is a root of the system (1) and $|4x_0y_0|<1$, then at least one of the points $(0, A)$ and $(B, 0)$, the vertices of the graphs of the two equations, may be used as the starting point of an iterative process as described, and this process will converge to (x_0, y_0).

It should be remarked that if the system has two "convergent" roots, the iterative processes starting at the two vertices both converge, one to each of the roots.

To prove the above, two cases must be considered, that where the product x_0y_0 is non-negative, and that where it is negative. If the root lies on one of the axes, the process terminates immediately. If the root lies in the first or third quadrant of the x, y-plane, we may suppose without loss of generality that $|x_0| \leqq |y_0|$. If the process is started at $(0, A)$ it is not difficult to show that the values $0, x_1, x_2, \cdots$ form a monotonic, bounded sequence whose only limit can be x_0. This fact is immediately suggested if a diagram of the process is drawn. The values $y_1, y_2, \cdots$ must also converge to y_0. If $|x_0| \geqq |y_0|$, identical reasoning, with starting-point $(B, 0)$, establishes convergence in that case.

If the point (x_0, y_0) lies in the second or fourth quadrant, we do not have monotonic sequences. This fact can be seen from a diagram in which the points (x_1, y_1), (x_2, y_1), (x_2, y_2), $\cdots$ are joined by straight lines. A rectilinear "spiral" is formed, which will converge to (x_0, y_0) if (x_1, y_1) is sufficiently close to (x_0, y_0). We must show that one of the vertices $(0, A)$ and $(B, 0)$ is indeed sufficiently close. Let P_1 be a point in the second quadrant from which the process converges to a root in that quadrant. If the spiral through P_1 is followed outward, one of two things can occur. First the spiral may cross one of the axes. If this is the case, it is clear from a diagram that the spiral starting from the vertex on that axis will wind in between the turns of the spiral through P_1 and so will converge to the root (x_0, y_0). The other possibility is that the spiral through P_1, if followed outward, will converge to a limiting rectangle lying entirely within the second quadrant. If this were the case, a spiral starting from a vertex could never enter this rectangle, and could not converge to (x_0, y_0). It is clear that such a rectangle must have all four vertices on the two parabolas (1). It is not difficult to establish that the center of such a rectangle must lie on the hyperbola $4xy=-1$. It then follows, after some calculation, that the root (x_0, y_0) within the rectangle (and the only root in the second quadrant) must have $4x_0y_0<-1$, so it is not a root to which the iterative process would converge. Thus, if (x_0, y_0) is a convergent root, the iterative process, when initiated from a

vertex, will converge to it.

The convergence criterion cited earlier, though well known, does not seem to be readily available in present-day literature. It is used, for example, by Levinson in the theory of non-linear differential equations.* The theory of iterative processes is old, dating back to 1674,** so it is very likely that the criterion involving the characteristic roots of the Jacobian appears in older literature. The convergence condition given in Scarborough's *Numerical Mathematical Analysis* for the iterative process

$$x_{n+1} = f(x_n, y_n), \qquad y_{n+1} = h(x_n, y_n), \tag{2}$$

is that, in the neighborhood of a solution (x_0, y_0),

$$|f_x| + |h_x| < 1 \quad \text{and} \quad |f_y| + |h_y| < 1.$$

The quantities involved need not be real. This condition is more stringent than necessary. This same condition is given in Cassini's *Calcoli Numerici Grafici E Meccanici.*

Although the Jacobian condition does not imply the above, it is always possible to reduce the general case, in which the Jacobian condition is satisfied, to a special case, in which the above condition is satisfied by means of a linear transformation of variables.

Without loss of generality, we may take the solution to be found by iteration as (0, 0). Let the characteristic values of the Jacobian matrix of f and h both be less than 1 in magnitude at (0, 0) and have continuity in a neighborhood. Then by a suitable (not necessarily real) transformation,

$$x = au + bv, \qquad y = cu + dv,$$

the system (2) will become

$$u_{n+1} = F(u_n, v_n), \qquad v_{n+1} = H(u_n, v_n), \tag{3}$$

where the Jacobian matrix of F and H will have at (0, 0) the same characteristic values, but will be in a *canonical form,* either

$$\begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} \lambda & 0 \\ \epsilon & \lambda \end{pmatrix}.$$

The λ's are the characteristic values in question. The second form may occur if they are equal, but the quantity ϵ may be chosen arbitrarily small, so that $|\lambda| + |\epsilon| < 1$. With the hypothesis that the λ's are less than 1 in magnitude it is clear that (3) satisfies the convergence condition of Scarborough. Thus the u's and v's converge, so the x's and y's do also.

---

* Levinson: Transformation Theory of Non-Linear Differential Equations of the Second Order, Annals of Math. Vol. 45 (1944). See page 729.

** Whittaker and Robinson: The Calculus of Observations, London (1924), pp. 79 ff.

References

Scarborough: Numerical Mathematical Analysis, Johns Hopkins Press, 1930, Chapter IX.
Cassini: Calcoli Numerici Grafici E Meccanici, Pisa, 1928.

MATHEMATICAL NOTES

EDITED BY F. A. FICKEN, *New York University*

Material for this department should be sent to F. A. Ficken, Institute for Mathematics and Mechanics, New York University, 45 Fourth Ave., New York, 3, New York

A NOTE ON CROSS RATIO*

A. J. HOFFMAN, Institute for Advanced Study

Let F be an arbitrary field and m the projective line over F. Assuming further that F is not of characteristic 2, one sees readily that if τ is any 1-1 mapping of m onto itself that fixes 0, 1 and ∞ and preserves cross ratio -1 (*i.e.*, preserves harmonic sets), then τ is an automorphism of F. It is natural to inquire (*i*) whether it is possible to substitute some other number for -1 in the hypothesis so that the conclusion remains valid, and (*ii*) whether some more general statement can be made which will include the case of characteristic 2.

The purpose of this note is to answer these questions by the following theorem, valid for F of arbitrary characteristic. We denote τx by x'.

THEOREM. *If τ is any $1-1$ mapping of m orto itself leaving 0, 1 and ∞ fixed such that for some $k \in F$, $k \neq 0$, 1, τ preserves cross ratio k, then τ is an automorphism of F.*

We first prove a simple lemma that will be useful later.

LEMMA. *If $a \in F$ is left fixed by every τ satisfying the hypothesis of the theorem, then τ preserves cross ratio a.*

Proof. Let $\mathrm{R}(x_1, x_2, x_3, x_4) = a$ and let ρ be the projective transformation that takes x_1 into ∞, x_2 into 0, and x_3 into 1. Then $\rho x_4 = a$. Let σ be the projective transformation such that $\sigma: x_1' \to \infty$, $x_2' \to 0$, $x_3' \to 1$. We are required to show that $\sigma x_4' = q$. Now $\sigma\tau\rho^{-1}$ fixes ∞, 0 and 1, it is 1–1, and it preserves cross ratio k, so by hypothesis $\sigma\tau\rho^{-1}a = a$. But $\sigma\tau\rho^{-1}a = \sigma x_4'$.

Now to prove the theorem. Since $\mathrm{R}(\infty, 0, -x/k, -x) = k$, it follows that

$$(1) \qquad (-x/k)' = (-x)'/k.$$

Since $\mathrm{R}(x_1, x_2, x_3, x_4) = k$ if and only if $\mathrm{R}(x_1, x_3, x_2, x_4) = 1 - k$, it follows that τ

---

* Written while the author was working under a grant from ONR.

preserves cross ratio $1-k$. It follows, in view of (1) and the equation $\mathrm{R}(0, -x/k, x/(1-k), \infty)=k$, that

$$(-x)' = -x'. \tag{2}$$

To show that $(x+y)'=x'+y'$: If x or y is 0, this is immediate. If $y=-x/k$, it is a consequence of (1), (2), and $\mathrm{R}(\infty, -x/k, 0, x-(x/k))=k$. If none of these relations between x and y hold, it follows from $\mathrm{R}(y, -x/k, (x+y)/(1-k), \infty)=k$.

By (1), induction, repeated use of the lemma and the fact that τ is an automorphism of the additive group of F, it follows that τ preserves any cross ratio in K, where K is the subfield of F generated by k. It is worth remarking that since $-1\in K$ the theorem now follows, in case the characteristic of F is not 2, from the fact mentioned in the introduction. But a proof for the general case is not much longer.

To prove that τ is an automorphism of the multiplicative group of F, we first show that $(1/x)'=1/x'$. This is obtained from $\mathrm{R}(x-k+1, 0, 1, k+(k-k^2)/x)=k$. $(x^2)'=(x')^2$ follows from $\mathrm{R}(x^2-2kx+k^2, 1, x-k, 1+kx-2k^2+(k^3-k)/x)=k$. In each of these cases, two elements inside the cross ratio symbol can coincide only if $x\in K$, where the assertions to be proven are already known to hold.

Next, $((xy)')^2=(x'y')^2$ follows from $\mathrm{R}((1/x^2)+y-ky, 0, y, ky+k(1-k)\cdot(xy)^2)=k$, provided all entries are distinct. But each of the cases in which the entries are not distinct can easily be handled separately. Hence, $(xy)'=\pm x'y'$. If F is of characteristic 2, we are finished. Otherwise, if for some $xy\neq 0$, we have $(xy)'=-x'y'$, then $\pm(x'y'+x')=(x(y+1))'=(xy+x)'=-x'y'+x'$, which is impossible.

COROLLARY. *If τ is any* 1-1 *mapping onto itself such that for some $k\in F$, $k\neq 0$,* 1, *τ preserves cross ratio k, then τ is the composition of a projective transformation of m and an automorphism of F.*

THE ROW-SUMS OF THE INVERSE MATRIX

ALBERT WILANSKY, Lehigh University

This note closes with an unsolved problem, and begins with the following surprising, yet almost trivial, fact about a matrix with an inverse.

If the sum of the elements in each row of a square matrix is k, then the sum of the elements in each row of the inverse matrix is $1/k$.

The proof is as follows. Let A be $m\times m$, non-singular, with the stated property. Let B be its inverse. Then for $n\leqq m$,

$$1 = \sum_{r=1}^{m}\delta_{nr} = \sum_{r=1}^{m}\sum_{s=1}^{m} b_{ns}a_{sr} = \sum_{s=1}^{m}\sum_{r=1}^{m} b_{ns}a_{sr} = k\sum_{s=1}^{m} b_{ns}.$$

This completes the proof. (A is singular if $k=0$.)

The result has an obvious application to magic squares.

Since the crucial point is an interchange of summations we can conclude:

The same is true for an infinite matrix A such that the inverse matrix has finite rows (*i.e.* almost all the elements in each row are zero). *Furthermore, the inverse matrix need be only a left inverse.*

A triangular matrix, *i.e.* one such that the nth row has no non-zero entries beyond the nth place, can be shown to have a two-sided inverse if and only if it is normal, *i.e.* has no zeros on its main diagonal. Such a matrix is easily shown to have a normal inverse. Thus we conclude:

The result holds for normal matrices.

We observe that the row-sums of the inverse need not exist. For example, let A have the rows $(1, 1, 0, 0, 0, \cdots)$, $(0, 1, 1, 0, 0, 0, \cdots)$, $(0, 0, 1, 1, 0, 0, 0, \cdots)$, $\cdots$, then the two-sided inverse has the rows $(1, -1, 1, -1, \cdots)$, $(0, 1, -1, 1, -1, \cdots) \cdots$. We notice, however, the suggestive fact that the rows of the inverse matrix are summable by many of the classical methods to the expected value $\frac{1}{2}$. That this is not generally true is shown by replacing, in the nth row of A as defined above, the pair $(1, 1)$ by (a, b) or by (b, a) respectively according as n is odd or even, where a, b are any numbers satisfying $ab \neq 0$.

A further observation is that it is not sufficient to assume only that the inverse is a right inverse. For let A be formed by adjoining a column of zeros on the left side of the identity matrix. It has, as right inverse, the matrix obtained by adjoining a row of zeros to the top of the identity matrix. The row-sums of A are one, not so for the right inverse.

I conjecture that "no" is the answer to the Problem: *Will the result hold for a matrix A and its inverse* (or left inverse) *if it is merely assumed that each row of the inverse matrix has a sum?*

REPRESENTATION OF THE INTEGERS BY POSITIVE INTEGERS

ROY DUBISCH, Fresno State College

The integers are commonly constructed from the positive integers by the use of pairs of positive integers. That is, one considers the integers as pairs* (a, b) for which

$$\begin{aligned} (a, b) &= (c, d) \quad \text{if} \quad a + d = b + c, \\ (a, b) + (c, d) &= (a + c, b + d), \\ (a, b) \times (c, d) &= (ac + bd, ad + bc), \end{aligned} \tag{1}$$

for positive integers a, b, c, and d. From (1) and the properties of positive integers it is then possible to establish the commutative, associative, distributive laws, *etc.* for integers.

We propose here to consider the integers as single positive integers. To do this we consider the set of positive integers as divided up into two classes, the even numbers (the negative integers) and the odd numbers (the non-negative integers). Addition ($\oplus$) is then defined by

* More exactly, as classes of equivalent pairs.

$$
(2)\qquad
\begin{aligned}
&m \oplus n = m + n \quad \text{if} \quad m \text{ and } n \text{ are both even},\\
&m \oplus n = m + n - 1 \quad \text{if} \quad m \text{ and } n \text{ are both odd},\\
&m \oplus n = n \oplus m = m - n + 1 \quad \text{if} \quad m \text{ is even, } n \text{ odd, and } m > n,\\
&m \oplus n = n \oplus m = m - n \quad \text{if} \quad m \text{ is odd, } n \text{ even, and } m > n.
\end{aligned}
$$

Here, of course, the addition on the right is the ordinary addition of positive integers while $m-n$ is defined by $m-n=c$ (m, n, c positive integers) if $m=n+c$. Likewise $m>n$ means there exists a positive integer k such that $m=n+k$.

Similarly, multiplication ($\otimes$) is defined by

$$
(3)\qquad
\begin{aligned}
&m \otimes n = (mn + 2)/2 \quad \text{if} \quad m \text{ and } n \text{ are both even},\\
&m \otimes n = (mn - m - n + 3)/2 \quad \text{if} \quad m \text{ and } n \text{ are both odd},\\
&m \otimes n = n \otimes m = (nm - m)/2 \quad \text{if} \quad m \text{ is even, } n \text{ odd, and } n > 1,\\
&1 \otimes n = n \otimes 1 = 1 \quad \text{for all positive integers } n.
\end{aligned}
$$

Here the multiplication on the right is the ordinary multiplication of positive integers.

We now claim that the set of positive integers under $\oplus$ and $\otimes$ is isomorphic to the set of pairs of positive integers (*i.e.*, the integers) under the correspondence

$$
(4)\qquad (a, b) \leftrightarrow (2a + 1) \oplus 2b.
$$

By virtue of $(2)_4$ and $(2)_3$ respectively, (4) can be re-written as

$$
(5)\qquad
\begin{aligned}
&(a, b) \leftrightarrow (2a + 1) - 2b = 2(a - b) + 1^* \quad \text{if} \quad a \geqq b,\\
&(a, b) \leftrightarrow 2b - (2a + 1) + 1 = 2(b - a) \quad \text{if} \quad a < b.
\end{aligned}
$$

For example, $(4, 1)\leftrightarrow 2(4-1)+1=7$ and $(2, 6)\leftrightarrow 2(6-2)=8$. Then $(4, 1)+(2, 6)=(4+2, 1+6)=(6, 7)$ by $(1)_2$. But $7\oplus 8=8-7+1=2$ by $(2)_3$ and $(6, 7)\leftrightarrow 2(7-6)=2$. Similarly, $(4, 1)\times(2, 6)=(4\cdot 2+1\cdot 6, 4\cdot 6+1\cdot 2)=(14, 26)$ by $(1)_3$. But $7\otimes 8=(7\cdot 8-8)/2=24$ by $(3)_3$ and $(14, 26)\leftrightarrow 2(26-14)=24$.

The proof of this isomorphism is straightforward but somewhat tedious due to the complexity of the definitions of $\oplus$ and $\otimes$ and will be left to the interested reader.

A PRIME-REPRESENTING FUNCTION

E. M. WRIGHT, University of Aberdeen, Scotland

Recently Mills [1] proved that there is a fixed number A such that, for all positive integral values of x, the number

$$
[A^{3^x}]
$$

is a prime. Here $[y]$ denotes as usual the greatest integer $\leqq y$. The proof de-

* If $a=b$, $a-b$ is not a positive integer and we should consider $(2b+1)-2b=1$. We write $2(a-b)+1$ for brevity and understand that $2(b-b)+1=1$ without defining $b-b$.

pends on the result that

$$p_{n+1} - p_n < Kp_n^{5/8},$$

where K is a fixed positive number and p_n is the nth prime. This result (due to Ingham [2]) is fairly deep.

Here I prove

THEOREM *There is a number α such that, if*

$$g_0 = \alpha, \qquad g_{n+1} = 2^{g_n}, \qquad (n \geqq 0),$$

then

$$[g_n] = [2^{2^{\cdot^{\cdot^{\cdot^{\alpha}}}}}], \qquad (n \geqq 1),$$

is always a prime.

I require only the very elementary result that, for every $N \geqq 2$, there is a prime between N and $2N$ (see [2]). By this, we can choose a sequence of primes $\{P_n\}$ such that $P_1 = 2$ or 3 and

$$2^{P_n} < P_{n+1} < P_{n+1} + 1 < 2^{P_n+1}.$$

We take all logarithms to base 2, and write,

$$u_n = \log^{(n)} P_n, \qquad v_n = \log^{(n)} (P_n + 1),$$

where $\log^{(n)}$ denotes the nth iterate of the logarithm. We have

$$P_n < \log P_{n+1} < \log (P_{n+1} + 1) < P_n + 1$$

and so

$$u_n < u_{n+1} < v_{n+1} < v_n.$$

Hence $u_n \to \alpha$ (say) as $n \to \infty$ and

$$u_n < \alpha < v_n, \qquad P_n < g_n < P_n + 1$$

and so

$$P_n = [g_n].$$

There are, of course, any number of possible values of α. For example, if

$$\alpha = 1.9287800 \cdots,$$

we have

$$P_1 = 3, \qquad P_2 = 13, \qquad P_3 = 16381$$

and P_4 has some 5000 digits.

If we use only the almost trivial result that there is a number B such that, for every $N \geqq 2$, there is a prime between N and BN (see [3]), we can prove that,

for some β, $[h_n]$ is always a prime, where

$$h_0 = \beta, \qquad h_{n+1} = B^{h_n} \qquad (n \geqq 0).$$

References

1. W. H. Mills, "A prime-representing function," *Bull. Amer. Math. Soc.* 53 (1947), 604.
2. E. Landau, *Handbuch der Lehre von der Verteilung der Primzahlen* I, 92 (Leipzig 1909).
3. G. H. Hardy and E. M. Wright, *Introduction to the Theory of Numbers*, second edition, 346 (Oxford, 1945).

ON THE DENSITY OF SETS OF GAUSSIAN INTEGERS

LUTHER CHEO, University of Oregon

1. Introduction. We present two theorems and a remark on the density of sets of Gaussian integers in the first quadrant of the complex plane. A Gaussian integer is a complex number of the form $a+a'i$ where a and a' are integers. Attention is confined exclusively to the set Q of those Gaussian integers $a+a'i$ with $a \geqq 0$ and $a' \geqq 0$ but not both $a=0$ and $a'=0$.

If S is a set of positive integers, Schnirelman [1] lets $S(x)$ denote the number of elements s of S with $s \geqq x$ and defines the density of S to be the greatest lower bound of the fraction $S(x)/x$ for $x=1, 2, \cdots$. By analogy, we adopt the following.

Definition: Let A be a set of Gaussian integers (subset of Q). For each Gaussian integer $x+yi$, let $A(x+yi)$ denote the number of Gaussian integers $a+a'i$ in A with $a \leqq x$ and $a' \leqq y$. The density of A is defined to be

$$\text{g.l.b.} \frac{A(x+yi)}{xy+x+y}$$

taken over all $x+yi$ in Q.

2. Two theorems. If A and B are subsets of Q, the sum $A+B$ is defined to be the set C consisting of all elements $a+a'i$, $b+b'i$ and all $a+b+(a'+b')i$ with $a+a'i$ in A and $b+b'i$ in B. Let the densities of A, B, and C be α, β and γ respectively.

THEOREM 1. *If* $\alpha+\beta \geqq 1$, *then* $\gamma = 1$.

Proof. Suppose $\gamma < 1$, then there is an element $u+vi$ not in C. Whenever $a+a'i$ is in A, $u-a+(v-a')i$ is not in B. Therefore $A(u+vi)+B(u+vi) \leqq uv+u+v-1$. Thus

$$\frac{A(u+vi)+B(u+vi)}{uv+u+v} < 1.$$

That is, $\alpha+\beta < 1$. This contradicts the hypothesis.

THEOREM 2. *If B contains all numbers ji for $j=1, 2, \cdots$,* then $\gamma \geqq \alpha_0+\beta-\alpha_0\beta$, where

$$\alpha_0 = \underset{x}{\text{g.l.b.}} \frac{A(x+0i)}{x}\cdot$$

Proof. Take any number $x+yi$ in Q. Define B_j to be the set of all elements $b+ji$ whenever it is in B, and $B_j(x)$ denotes the number of elements $b+ji$ in B_j with $b\leqq x$. Similarly define A_j and C_j. Let $k_{j1}, k_{j2}, \cdots, k_{jn_j}$ be all the numbers less than x such that $k_{jt}+ji$ is in B_j but $k_{jt}+1+ji$ is not in B_j, $t=1, 2, \cdots, n_j$. Let $m_{j1}, m_{j2}, \cdots, m_{jn_j}$ be the lengths of corresponding gaps such that no element $u+ji$ in the range $k_{jt}<u<k_{jt}+m_{jt}$ belongs to B_j, but $k_{jt}+m_{jt}+1+ji$ belongs to B_j, and define $m_{jn_j}=x-k_{jn_j}$ if $x+ji$ is not in B_j. When

$$j = 0, \qquad B_0(x) = x - \sum_{t=1}^{n_0} m_{0t},$$

and

$$\begin{aligned} C_0(x) &\geqq B_0(x) + \sum_{t=1}^{n_0} A_0(m_{0t}) \geqq B_0(x) + \alpha_0\sum_{t=1}^{n_0} m_{0t} \\ &= B_0(x) + \alpha_0(x - B_0(x)) = (1-\alpha_0)B_0(x) + \alpha_0 x. \end{aligned}$$

When $j>0$, we have

$$B_j(x) = x + 1 - \sum_{t=1}^{n_j} m_{jt},$$

and then

$$\begin{aligned} C_j(x) &\geqq B_j(x) + \sum_{t=1}^{n_j} A_0(m_{jt}) \geqq B_j(x) + \alpha_0\sum_{t=1}^{n_j} m_{jt} \\ &= B_j(x) + \alpha_0(x + 1 - B_j(x)) \\ &= (1-\alpha_0)B_j(x) + \alpha_0(x+1). \end{aligned}$$

Hence

$$\begin{aligned} C(x+yi) &= \sum_{j=0}^{y} C_j(x) \geqq (1-\alpha_0)\sum_{j=0}^{y} B_j(x) + \alpha_0 x + \alpha_0 y(x+1) \\ &= (1-\alpha_0)B(x+yi) + \alpha_0(xy+x+y) \\ &\geqq (1-\alpha_0)\beta(xy+x+y) + \alpha_0(xy+x+y). \end{aligned}$$

Therefore

$$\frac{C(x+yi)}{xy+x+y} \geqq (1-\alpha_0)\beta + \alpha_0 = \alpha_0 + \beta - \alpha_0\beta.$$

3. Remark. Let A_0, B_0 be sets of positive integers with densities α_0 and β_0 respectively, and let $A_0+B_0=C_0$ have density γ_0. Mann [2] proved that $\gamma_0 \geqq \alpha_0+\beta_0$ or 1. By our definition and the two theorems proved in section 2 one might conjecture that Mann's theorem also holds in the present case of two dimensions. But, however, this is not true as shown by the following example: let A be the set consisting of numbers 1, 2, 5, 6, 9, 10, 13, 14, 17, 18, i, $1+i$, $4+i$, $5+i$, $8+i$, $9+i$, $12+i$, $13+i$, $16+i$, $17+i$, $20+i$, and all numbers $a+a'i$ with $a>20$, or $a'>1$; and let B be the set consisting of the numbers 1, 4, 5, 8, 9, 13, 17, i, $3+i$, $4+i$, $8+i$, $12+i$, $16+i$, $20+i$, and all $b+b'i$ with $b>20$ or $b'>1$. C then is the set consisting of all numbers in A, all numbers in B and numbers 3, 7, 11, 15, 19, $2+i$, $6+i$, $10+i$, $14+i$, and $18+i$. It is readily calculated that $\alpha=1/2$, $\beta=1/3$ and $\gamma=34/41$ which is less than $\alpha+\beta$.

References

1. Schnirelmann, L., Über additive Eigenschaften von Zahlen, Math. Annalen, vol. 107 (1933), pp. 649–690.

2. Mann, H. B., A Proof of the Fundamental Theorem on the Density of Sums of Sets of Positive Integers. Ann. of Math., vol. 43 (1942) pp. 523–527.

3. Erdös, P., and Niven, I., The $\alpha+\beta$ Hypothesis and Related Problems. This MONTHLY, vol. 53 (1946) pp. 314–317.

ON FEUERBACH'S THEOREM*

VICTOR THÉBAULT, Tennie, Sarthe, France

1. A general theorem. *If the distances O_2O_3, O_3O_1, O_1O_2 between the centers of three circles (O_1), (O_2), (O_3), having radii R_1, R_2, R_3, satisfy a relation of the form*

$$(1) \qquad (O_2O_3)R_1 \pm (O_3O_1)R_2 \pm (O_1O_2)R_3 = 0,$$

then the circumcircle (O) of triangle $O_1O_2O_3$ is tangent to the radical circle (O') of the circles (O_1), (O_2), (O_3), and the radical center O' is at the center of one of the tritangent circles of the triangle determined by the radical axes of circle (O) in association with each of the circles (O_1), (O_2), (O_3).

The locus of points the ratio of whose distances from O_2 and O_3 is equal to R_2/R_3 is a circle Γ_{23} having for diameter the segment joining the centers of homothety of circles (O_2) and (O_3), and Γ_{23} belongs to the pencil of circles defined by (O_2) and (O_3). Circle Γ_{23} and the two analogous circles Γ_{31} and Γ_{12} have, in general, two common points P and P'. Since circle Γ_{23}, for example, is orthogonal to all circles passing through O_2 and O_3, it follows that the three circles Γ_{23}, Γ_{31}, Γ_{12} are orthogonal to circle (O), of radius R. Also, since circle Γ_{23} belongs to the pencil of circles determined by circles (O_2) and (O_3), it cuts orthogonally all circles orthogonal to both (O_2) and (O_3). Thus the three circles Γ_{23}, Γ_{31}, Γ_{12} are also orthogonal to the radical circle (O') of the three circles (O_1), (O_2), (O_3). It follows that the points P and P' are inverses of each other in

* Translated from the French by Howard Eves.

both circle (O) and circle (O'). But the distances of P and P' from the vertices of triangle $O_1O_2O_3$ are proportional to R_1, R_2, R_3. Therefore, by (1),

$$(O_1P)(O_2O_3) \pm (O_2P)(O_3O_1) \pm (O_3P)(O_1O_2) = 0,$$

and P (and similarly P') lies on circle (O). Points P and P' are then coincident, and circle (O) is tangent to circle (O').

On the other hand, if we denote by d_1, d_2, d_3 the algebraic distances of O' from the radical axes of the pairs of circles (O) and (O_1), (O) and (O_2), (O) and (O_3), we find that $d_1=d_2=d_3$, for

$$\begin{aligned}(O'O)^2 - R^2 - (O'O_1)^2 + R_1^2 &= (O'O)^2 - R^2 - (O'O_2)^2 + R_2^2 \\ &= (O'O)^2 - R^2 - (O'O_3)^2 + R_3^2 \\ &= 2(O'O)d_1 = 2(O'O)d_2 = 2(O'O)d_3.\end{aligned}$$

This completes the proof of the theorem.

2. Applications. Let (O) be the circumcircle of a triangle $T\equiv ABC;$ A_1, B_1, C_1 and A', B', C' the midpoints of the sides BC, CA, AB, and the feet of the corresponding altitudes; (N) the nine point circle; and D, E, F the points of contact of one of the tritangent circles, (I), with sides BC, CA, AB.

(A) *Feuerbach's Theorem. The radical circle of the three circles* (A_1, A_1D), (B_1, B_1E), (C_1, C_1F) *clearly coincides with circle* (I), *and since*

$$(B_1C_1)(A_1D) \pm (C_1A_1)(B_1E) \pm (A_1B_1)(C_1F) = 0,$$

circles (I) *and* (N) *are tangent to each other.*

(B) *In a triangle T, the radical circle (O') of the three circles having for centers the midpoints of the sides of T and passing through the feet of the corresponding altitudes is one of the tritangent circles of the anticomplementary triangle $T''=A''B''C''$ of the orthic triangle $T'=A'B'C'$ of T.*

The evident relation

$$(B_1C_1)(A_1A') \pm (C_1A_1)(B_1B') \pm (A_1B_1)(C_1C') = 0 \tag{2}$$

shows immediately that the radical circle (O') of the three circles $(O_1)\equiv(A_1, A_1A')$, $(O_2)\equiv(B_1, B_1B')$, $(O_3)\equiv(C_1, C_1C')$ is concentric with one of the tritangent circles of the triangle determined by the radical axes of the circles (N) and (O_1), (N) and (O_2), (N) and (O_3). Now these axes pass through A', B', C' and are perpendicular to NA_1, NB_1, NC_1, and, consequently, to the lines OA, OB, OC, which proves that the triangle which they determine is the triangle T''. Since triangles T and T'' have the same nine point circle (N), it follows, from (1) and (2), that (N) is tangent to (O').

If T is acute angled, (O') is interior to (N), and if T is obtuse angled, (O') is exterior to (N). In either case, by applying Feuerbach's theorem to triangle T'', we see that (O') is one of the tritangent circles of T''. If T is right angled at

A, then circle (O') reduces to the point A'.

3. Additional remarks. If T is acute angled, the radius of circle (O') is given by

$$\rho = 4R \cos A \cos B \cos C,$$

and if T is obtuse angled, at A say,

$$\rho = (2S \cot A)/R,$$

where S denotes the area of the fundamental triangle. The point of contact of circles (N) and (O') is one of the Feuerbach points of triangle T''. We have given other properties of this point. See this MONTHLY, [1947, p. 448].

CLASSROOM NOTES

EDITED BY C. B. ALLENDOERFER, University of Washington

All material for this department should be sent to G. B. Thomas, Department of Mathematics, Massachusetts Institute of Technology, Cambridge, 39, Mass.

COMPLEX NUMBERS AND VECTOR ALGEBRA

D. E. RICHMOND, Williams College

1. Introduction. Vectors in the plane are frequently represented by complex numbers and treated by the corresponding rules. However, in many applications they are treated by vector algebra and multiplied to form vector and scalar products. The student is sometimes puzzled about the relation between these two schemes of representation. The literature appears to contain no explicit discussion of this point. The present paper supplies this lack and also presents complex numbers in a somewhat new light.

Let us imagine that we are faced with the task of inventing an algebra of directed quantities, that is, quantities representable by arrows or vectors in the plane of the paper. By parallel displacement, any such vector may be drawn from the origin of a rectangular coordinate system to some point P in the plane (see Figure 1). It is natural to use letters, say α, β, γ, $\cdots$, to represent such vectors and to try to give meanings to sums, differences and products like $\alpha+\beta$, $\alpha-\beta$, $\alpha\beta$. Keeping in mind simple physical applications like the composition of forces or velocities, it is easy to arrive at the usual geometric interpretations of the addition and subtraction of two vectors and of the product of a vector by a scalar. The corresponding algebra is also simple.

Let $\mathbf{H}$ denote a unit vector along the horizontal axis and $\mathbf{V}$ a unit vector along the vertical axis (see Figure 1). In terms of vector addition and of mul-

tiplication by a scalar, we may represent an arbitrary vector in the form

$$\alpha = a_1\mathbf{H} + a_2\mathbf{V}$$

with real numbers a_1 and a_2. We may also define the magnitude of α, written $|\alpha|$, as $\sqrt{a_1^2+a_2^2}$.

The difficulty is to find an appropriate definition for the product of two vectors α and β. It is natural to attempt to define the product in such a way that magnitudes shall multiply, that is, so that $|\alpha\beta| = |\alpha|\,|\beta|$. This stipulation leads to an essentially unique result if we add the requirement that $\mathbf{H}$ shall act

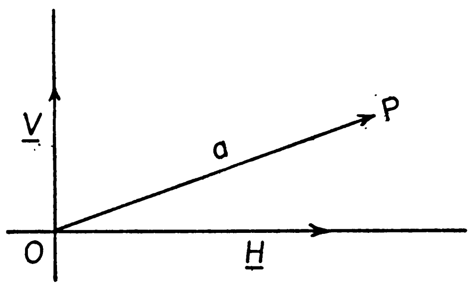


FIG. 1

like unity, so that specifically.

$$\mathbf{HV} = \mathbf{VH}(=\mathbf{V}) \quad \text{and} \quad \mathbf{HH} = \mathbf{H} \text{ (i.e., } \mathbf{H}^2 = \mathbf{H}).$$

We proceed to derive this unique result.

To begin with, it is necessary to evaluate $\mathbf{V}^2$. Let

$$\mathbf{V}^2 = a\mathbf{H} + b\mathbf{V}$$

where a and b are to be determined.

Since $|\mathbf{V}| = 1$, we have, equating the squares of the magnitudes,

$$a^2 + b^2 = 1.$$

Now consider

$$(\mathbf{H} + \mathbf{V})(\mathbf{H} - \mathbf{V}) = \mathbf{H}^2 + \mathbf{VH} - \mathbf{HV} - \mathbf{V}^2 = \mathbf{H}^2 - \mathbf{V}^2$$
$$= \mathbf{H} - a\mathbf{H} - b\mathbf{V} = (1 - a)\mathbf{H} - b\mathbf{V}.$$

Equating squares of magnitudes,

$$4 = (1 - a)^2 + b^2.$$

Combining with

$$1 = a^2 + b^2$$

we have

$$a = -1, \qquad b = 0.$$

Hence

$$\mathbf{V}^2 = -\mathbf{H}.$$

If

$$\alpha = a_1\mathbf{H} + a_2\mathbf{V}$$

and

$$\beta = b_1\mathbf{H} + b_2\mathbf{V}$$

are two vectors, their product is easily found to be

$$\alpha\beta = (a_1b_1 + a_2b_2)\mathbf{H} + (a_1b_2 + a_2b_1)\mathbf{V}. \tag{1}$$

Then

$$\begin{aligned} |\alpha\beta|^2 &= (a_1b_1 - a_2b_2)^2 + (a_1b_2 + a_2b_1)^2 \\ &= (a_1^2 + a_2^2)(b_1^2 + b_2^2) = |\alpha|^2 |\beta|^2 \end{aligned}$$

for arbitrary a_1, a_2, b_1, b_2. Hence the stipulation $|\alpha\beta| = |\alpha|\,|\beta|$ can be carried through in general.

The scheme just discussed is, except for notation, that of complex numbers. In fact, if we write $\alpha = a_1 + ib_1$ and $\beta = a_2 + ib_2$ and multiply as usual, replacing i^2 by -1, the result corresponds to (1). The scheme of complex numbers is therefore essentially the only one for which the magnitude of the product equals the product of the magnitudes.

2. Vector algebra. Vector algebra uses a somewhat different scheme of representation of vectors in the plane. The vector α of section 1 would be written

$$\alpha = a_1\mathbf{i} + a_2\mathbf{j}$$

replacing $\mathbf{H}$ by $\mathbf{i}$ and $\mathbf{V}$ by $\mathbf{j}$.

So long as one deals only with addition, subtraction and multiplication by scalars, no difference occurs other than this trivial notational one. But as soon as one speaks of multiplying vectors α and β, an essential difference comes in.

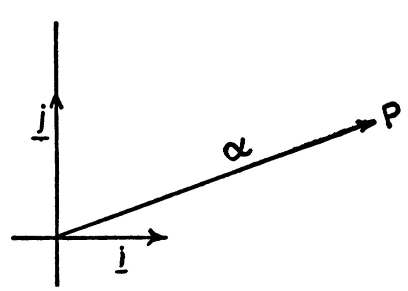


FIG. 2

As is well known, there are two types of product used in vector algebra, the scalar product and the vector product. If

$$\alpha = a_1\mathbf{i} + a_2\mathbf{j}$$
$$\beta = b_1\mathbf{i} + b_2\mathbf{j}$$

these products are

$$(2) \qquad \alpha\cdot\beta = a_1b_1 + a_2b_2 \qquad \text{(scalar product)}$$

$$(3) \qquad \alpha \times \beta = (a_1b_2 - a_2b_1)\mathbf{k} \qquad \text{(vector product)}$$

where $\mathbf{k}$ is a unit vector perpendicular to the plane of α and β.

The choice of these expressions to represent products is dictated by physical applications (work done by a force, moment of a force). Here we are concerned only with the formal relations among the definitions (1), (2) and (3).

The relations are the following.

$$\alpha\cdot\beta = \operatorname{Re}(\bar{\alpha}\beta)$$
$$\alpha \times \beta = \operatorname{Im}(\bar{\alpha}\beta)\mathbf{k}$$

where Re stands for "real part of" and Im for "imaginary part of" and where $\bar{\alpha}$ is the conjugate of α, that is, $a_1 - a_2i$. In fact,

$$\begin{aligned}\bar{\alpha}\beta &= (a_1b_1 + a_2b_2) + (a_1b_2 - b_1a_2)i,\\ \operatorname{Re}(\bar{\alpha}\beta) &= a_1b_1 + a_2b_2 = \alpha\cdot\beta,\\ \operatorname{Im}(\bar{\alpha}\beta) &= a_1b_2 - b_1a_2, \text{ the coefficient of } \mathbf{k} \text{ in } \alpha \times \beta.\end{aligned}$$

To illustrate this relationship, consider two examples from geometry.

EXAMPLE 1. To prove that an angle inscribed in a semicircle is a right angle.

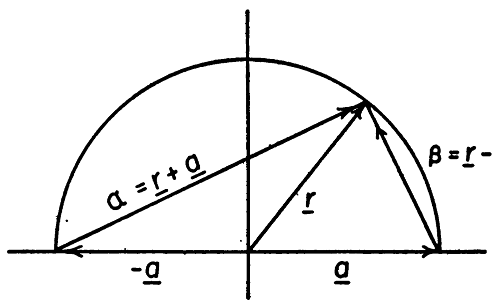


FIG. 3

Vector algebra proof:

$$\begin{aligned}\alpha\cdot\beta &= (\mathbf{r} + \mathbf{a})\cdot(\mathbf{r} - \mathbf{a})\\ &= \mathbf{r}^2 - \mathbf{a}^2\\ &= 0, \quad \text{since} \quad |\mathbf{r}| = |\mathbf{a}|.\end{aligned}$$

Hence $\alpha \perp \beta$.

Complex number proof:

$$\begin{aligned}\bar{\alpha}\beta &= [(x + a) - iy][(x - a) + iy] \\ &= x^2 - a^2 + y^2 + 2aiy \\ &= 2aiy, \quad \text{since} \quad |x + iy| = a.\end{aligned}$$

$$\operatorname{Re}(\bar{\alpha}\beta) = 0.$$

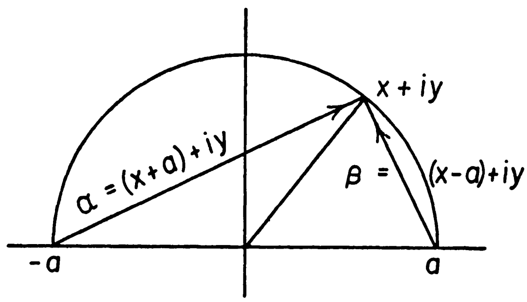


FIG. 4

Hence $\alpha \perp \beta$.

EXAMPLE 2. To derive the difference formulas of trigonometry.

Vector algebra proof:

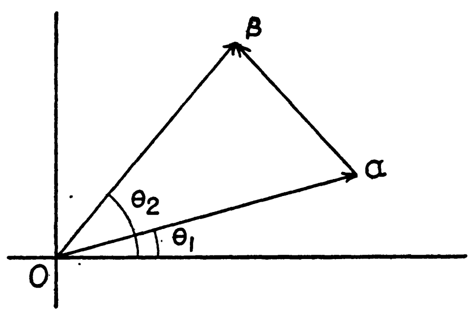


FIG. 5

$$\begin{aligned}\alpha &= \cos\theta_1\mathbf{i} + \sin\theta_1\mathbf{j} \\ \beta &= \cos\theta_2\mathbf{i} + \sin\theta_2\mathbf{j} \\ \alpha\cdot\beta &= \cos\theta_1\cos\theta_2 + \sin\theta_1\sin\theta_2.\end{aligned}$$

But

$$\alpha\cdot\beta = |\alpha||\beta|\cos(\theta_2 - \theta_1) = \cos(\theta_2 - \theta_1).$$

Hence

$$\begin{aligned}\cos(\theta_2 - \theta_1) &= \cos\theta_1\cos\theta_2 + \sin\theta_1\sin\theta_2 \\ \alpha \times \beta &= (\sin\theta_2\cos\theta_1 - \cos\theta_2\sin\theta_1)\mathbf{k} \\ &= |\alpha||\beta|\sin(\theta_2 - \theta_1)\mathbf{k} = \sin(\theta_2 - \theta_1)\mathbf{k}.\end{aligned}$$

Hence

$$\sin(\theta_2 - \theta_1) = \sin\theta_2 \cos\theta_1 - \cos\theta_2 \sin\theta_1.$$

Note:

Since the formulas $\alpha\cdot\beta = |\alpha||\beta| \cos(\theta_2-\theta_1)$ and $\alpha\times\beta = |\alpha||\beta| \sin(\theta_2-\theta_1)\mathbf{k}$, used in this proof, are often established by way of the difference formulas, the proof may give the appearance of circularity. However, there is no difficulty in avoiding circularity. In fact,

$$(\beta - \alpha)^2 = \beta^2 + \alpha^2 - 2\alpha\cdot\beta.$$

By the law of cosines

$$(\beta - \alpha)^2 = |\beta - \alpha|^2 = \beta^2 + \alpha^2 - 2|\alpha||\beta| \cos(\theta_2 - \theta_1).$$

Equating these two results gives the first formula. The second follows by equating two expressions for the area of the triangle $O\alpha\beta$.

Complex number proof:

$$\alpha = \cos\theta_1 + \sin\theta_1 i$$

$$\beta = \cos\theta_2 + \sin\theta_2 i$$

$$\begin{aligned}\bar{\alpha}\beta &= (\cos\theta_2\cos\theta_1 + \sin\theta_2\sin\theta_1) + (\sin\theta_2\cos\theta_1 - \cos\theta_2\sin\theta_1)i\\ &= \cos(\theta_2-\theta_1) + \sin(\theta_2-\theta_1)i.\end{aligned}$$

Conclusions as above.

Note that

$$\alpha\cdot\beta = \mathrm{Re}\,(\bar{\alpha}\beta), \qquad \alpha\times\beta = \mathrm{Im}\,(\bar{\alpha}\beta)\mathbf{k}.$$

3. Dimensions higher than two. It is of interest to show that there exists no method of multiplication which yields $|\alpha\beta| = |\alpha||\beta|$ for arbitrary vectors α and β in *three* dimensions, if we make the additional assumption that the associative law holds, $(\alpha\beta)\gamma = \alpha(\beta\gamma)$.

Let

$$\alpha = a_1\mathbf{u}_1 + a_2\mathbf{u}_2 + a_3\mathbf{u}_3$$

where $\mathbf{u}_1$, $\mathbf{u}_2$, $\mathbf{u}_3$ are unit vectors along the axes of a rectangular coordinate system. Assume that $\mathbf{u}_1$ acts like unity so that

$$\mathbf{u}_1^2 = \mathbf{u}_1, \qquad \mathbf{u}_1\mathbf{u}_2 = \mathbf{u}_2\mathbf{u}_1(=\mathbf{u}_2), \qquad \mathbf{u}_1\mathbf{u}_3 = \mathbf{u}_3\mathbf{u}_1(=\mathbf{u}_3).$$

If

$$\mathbf{u}_2^2 = a\mathbf{u}_1 + b\mathbf{u}_2 + c\mathbf{u}_3$$

with undetermined a, b and c,

$$(\mathbf{u}_1 - \mathbf{u}_2)(\mathbf{u}_1 + \mathbf{u}_2) = \mathbf{u}_1^2 - \mathbf{u}_2^2 = \mathbf{u}_1 - (a\mathbf{u}_1 + b\mathbf{u}_2 + c\mathbf{u}_3)$$
$$= (1 - a)\mathbf{u}_1 - b\mathbf{u}_2 - c\mathbf{u}_3.$$

Equating squares of magnitudes

$$4 = (1 - a)^2 + b^2 + c^2.$$

Since $a^2+b^2+c^2=1$, we have

$$a = -1, \qquad b = c = 0.$$

Thus

$$\mathbf{u}_2^2 = -\mathbf{u}_1.$$

Similarly,

$$\mathbf{u}_3^2 = -\mathbf{u}_1.$$

Now let

$$\mathbf{u}_2\mathbf{u}_3 = r\mathbf{u}_1 + s\mathbf{u}_2 + t\mathbf{u}_3.$$

Then

$$\mathbf{u}_2(\mathbf{u}_2 + \mathbf{u}_3) = \mathbf{u}_2^2 + \mathbf{u}_2\mathbf{u}_3$$
$$= (r - 1)\mathbf{u}_1 + s\mathbf{u}_2 + t\mathbf{u}_3.$$

Equating squares of magnitudes

$$2 = (r - 1)^2 + s^2 + t^2.$$

Since $r^2+s^2+t^2=1$, $r=0$ and

$$\mathbf{u}_2\mathbf{u}_3 = s\mathbf{u}_2 + t\mathbf{u}_3.$$

Multiply on the left by $\mathbf{u}_2$. Since

$$\mathbf{u}_2(\mathbf{u}_2\mathbf{u}_3) = (\mathbf{u}_2^2)\mathbf{u}_3 = -\mathbf{u}_1\mathbf{u}_3 = -\mathbf{u}_3,$$
$$-\mathbf{u}_3 = -s\mathbf{u}_1 + t(s\mathbf{u}_2 + t\mathbf{u}_3).$$

Hence $s=0$, $t^2=-1$. This result is inconsistent with the fact that t must be real. It also conflicts with the necessary condition $t^2+s^2=1$.

For arbitrary vectors in 4 dimensions, it is possible to define the product so that magnitudes *do* multiply. In fact the methods of this section lead directly to quaternions which have this property. The same methods show that if multiplication is associative, the condition $|\alpha\beta| = |\alpha|\,|\beta|$ is impossible to satisfy for arbitrary vectors in a space of more than 4 dimensions. The proofs of both statements are well within the ability of any good student and should prove illuminating.

PARAMETRIC EQUATIONS AND PROPER INTERPRETATION OF MATHEMATICAL SYMBOLS

A. ERDÉLYI, California Institute of Technology

The point raised recently (this MONTHLY, vol. 58, pp. 106–107, 1951) by A. D. Fleshler is a pertinent one and deserves further discussion. The examples given by Fleshler show very clearly that mathematical symbols have no existence of their own: they are given a lease of life by an exact definition, and cannot function except in conjunction with that definition.

The chain of operations

$$\int_0^{4\pi} \sqrt{1 - \cos x}\, dx = \sqrt{2} \int_0^{4\pi} \sin \frac{x}{2}\, dx = 0 \tag{1}$$

is reprehensible *not* because the result is wrong, but rather because the point of departure is not well defined. As a matter of fact, it is possible to advance a definition of $\sqrt{1-\cos x}$ which will make the above operation justifiable, and its result correct. However, when we write down

$$\int_0^{4\pi} \sqrt{1 - \cos x}\, dx$$

we tacitly assume that everybody knows that by $\sqrt{1-\cos x}$ we mean the non-negative square root, and with *that* definition (1) is incorrect.

In a parametric representation $x=\phi(t)$, $y=\psi(t)$ it is necessary to define unambiguously not only the functions $\phi(t)$ and $\psi(t)$, but also the region over which t varies. For example, if t is assumed to run through *all real values*, the equations

$$x = e^t, \qquad y = e^t \tag{2}$$

represent the portion of the line $x=y$ in the first quadrant. If t varies over *positive real values* only, then our equations represent only that portion of the line which lies to the right of the point (1, 1); and the equations (2) represent the entire line $x=y$ (with the exception of the origin) if t is allowed to assume complex values, and its region of variability in the complex t plane consists of two lines, the real axis and a line parallel to it at the distance π.

Often the tacit assumption will be that t runs over all real values, but this convention should be made clear to the students, and one should also explain that it is not always appropriate. Take another one of Mr. Fleshler's examples,

$$x = \sin t, \qquad y = \sin t. \tag{3}$$

These equations can be made to represent the entire line $x=y$ by letting t vary in the complex plane over a broken line which joins the points $(-\pi/2)-i\infty$, $(-\pi/2)$, $(\pi/2)$, $(\pi/2)+i\infty$ in this order. If one lets t vary over all real values, one obtains *not* so much *the segment* joining $(-1, -1)$ and $(1, 1)$, *but* rather *a curve* which covers this segment infinitely often (and has an infinite

length!). If this is pointed out to a student familiar with the periodic property of trigonometric functions, he will see it, but on being challenged to specify a reasonable region of variation for t, his first guess is likely to be $0 \leqq t \leqq 2\pi$, which leads again not to the segment but to a closed curve covering this segment twice. Actually, one possible interval of definition is $-\pi/2 \leqq t \leqq \pi/2$. It is not the only one to lead to the desired result, but some such specification is necessary. In some case one will have to consider equations (3) in the interval $0 \leqq t \leqq \pi/2$, when they represent the segment joining the origin and the point (1, 1), and so on.

By a discussion of (3) in various (real) ranges of t the student is introduced at an early stage to the necessity of a precise definition, and the ground is prepared for the demand of the more advanced theory that a correct definition of a function should always include (a) the region of variability of the independent variable, and (b) an unambiguous rule for finding the value of the function for each admissible value of the independent variable.

AN ALTERNATE DERIVATION OF A WELL-KNOWN INTEGRATION FORMULA

A. WOLINSKY, New York University

The derivation of the formula for the indefinite integral of the trigonometric tangent function commonly given in classrooms and in textbooks is the following:

$$\int \tan x dx = \int \frac{\sin x}{\cos x} dx = \int \frac{-d(\cos x)}{\cos x} = -\int \frac{d(\cos x)}{\cos x} \tag{1}$$
$$= -\ln \cos x + C = \ln \sec x + C.$$

This derivation, including the integration, is based on four formulas, from trigonometry and calculus, namely,

$$\tan x = \frac{\sin x}{\cos x}, \tag{2}$$

$$d(\cos x) = -\sin x dx, \tag{3}$$

$$\int a dv = a \int dv, \tag{4}$$

$$\int \frac{dv}{v} = \ln v + C. \tag{5}$$

The following derivation of the formula is perhaps more appealing because it is shorter and more direct:

$$\int \tan x dx = \int \frac{d(\sec x)}{\sec x} = \ln \sec x + C = -\ln \cos x + C. \tag{6}$$

This derivation, including the integration, is based on only two formulas, from

calculus, namely,

$$d(\sec x) = \sec x \tan x dx \tag{7}$$

and (5).

The alternate method of derivation applies, of course, also to the trigonometric cotangent function, and to the hyperbolic tangent and cotangent functions.

Generally, the differential equation

$$df(x) = f(x)g(x)dx, \tag{8}$$

of which (7) is a special case, and where $g(x)$ is an arbitrary function of x, has the solution

$$f(x) = C'e^{\int g(x)dx}, \tag{9}$$

so that for any integrable function $g(x)$ there is a function $f(x)$, given by (9), which satisfies (8) and, therefore,

$$\int g(x)dx = \ln f(x) + C. \tag{10}$$

ELEMENTARY PROBLEMS AND SOLUTIONS

EDITED BY HOWARD EVES, Champlain College

Send all communications concerning Elementary Problems and Solutions to Howard Eves, Mathematics Department, Champlain College, Plattsburg, New York. This department welcomes problems believed to be new, and demanding no tools beyond those ordinarily furnished in the first two years of college mathematics. To facilitate their consideration, solutions should be submitted on separate, signed sheets, within three months after publication of problems.

PROBLEMS FOR SOLUTIONS

E 986. *Proposed by W. R. Ransom, Tufts College*

Show that there are fifty right triangles whose sides are integers less than 100.

E 987. *Proposed by W. L. Stamey and J. L. Zemmer, Jr., University of Missouri*

Prove that the interior of the strip, on a cartesian grid, bounded by two arbitrarily close parallel straight lines, contains either no points with integer coordinates or infinitely many such points.

E 988. *Proposed by Leo Moser, Texas Technological College*

Let a, b, x_0 be positive integers and let $x_1=ax_0+b, \cdots, x_n=ax_{n-1}+b$. Prove that x_n cannot be prime for all n.

E 989. *Proposed by C. W. Bruce, Tennessee Polytechnic Institute*

If the angle between the blades of scissors is less than the angle of friction of the metal on the cloth being cut, there will be a little tendency for the cloth to move forward as the scissors are being closed. How can a pair of scissors be constructed so that one blade is straight and there is a constant angle between the cutting edges?

E 990. *Proposed by Victor Thébault, Tennie, Sarthe, France*

If a sphere with center at the orthocenter of an orthocentric tetrahedron intersects the lines joining the midpoints of the edges of the tetrahedron, then the 24 distances of the vertices from the intersections on those lines joining the midpoints of the edges issuing from these vertices are all equal. (The analogous property for the triangle was announced by Steiner and proved by Droz-Farny. See Johnson, *Modern Geometry*, p. 256.)

SOLUTIONS

Arithmetic-Harmonic Mean

E 956 [1951, 189]. *Proposed by H. F. Sandham, Trinity College, Ireland*

Let a_0 and b_0 be positive numbers and define a_{n+1} and b_{n+1} as the arithmetic and harmonic means, respectively, of a_n and b_n. Show that

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n = (a_0b_0)^{1/2}.$$

Solution by Chandler Davis, University of Michigan. Since the case $a_0=b_0$ is trivial we assume $a_0>b_0$. Then $a_0>a_1> \cdots >b_1>b_0$, and the sequences $\{a_n\}$ and $\{b_n\}$ both converge. Call the limits a and b. From the expressions for the arithmetic and harmonic means we find

$$a_{n+1} - b_{n+1} = (a_n - b_n)^2/2(a_n + b_n) < (a_n - b_n)/2,$$

so that $a_n-b_n\to 0$, or $a=b$. Similarly we find that $a_{n+1}b_{n+1}=a_nb_n$, or

$$a^2 = \lim a_nb_n = a_0b_0,$$

and the proof is complete.

Also solved by F. Bagemihl and W. Seidel (jointly), L. F. Boron, A. L. Epstein, E. Franz, V. C. Harris, R. E. Horton, R. Huck, H. A. James, P. G. Kirmser, M. S. Klamkin, H. C. Kranzer, A. E. Livingston, Jerome Manheim, Burnett Meyer, Prasert Na Nagara, S. T. Parker, L. B. Rall, L. A. Ringenberg, Azriel Rosenfeld, E. C. Smith, M. R. Spiegel, O. E. Stanaitis, Elijah Swift, P. M. Treuenfels, F. Underwood, J. H. Wahab, Albert Wilansky, R. E. Wild, F. H. Young, and the proposer.

Palindromes

E 957 [1951, 189]. *Proposed by C. O. Oakley, Haverford College*

A *palindrome* is an integer that reads the same forwards as backwards as, for example, 2332. By definitions the ten digits 0, 1, 2, $\cdots$, 9 will be considered as palindromes. How many palindromes are there of at most n digits?

Solution by H. C. Kranzer, New York University. A palindrome of an even number, $2k$, or an odd number, $2k-1$, of digits is uniquely determined by its first k digits, which are arbitrary except that the first digit may not be zero (with the single trivial exception of 0 itself). Thus, if we let $f(p)$ be the number of non-zero palindromes of exactly p digits, we have, for $k \geqq 1$,

$$f(2k-1) = f(2k) = 9\cdot 10^{k-1}.$$

We wish to find

$$F(n) = 1 + \sum_{p=1}^{n} f(p).$$

Let n be even. Then

$$F(n) = 1 + 2\sum_{k=1}^{n/2} f(2k) = 1 + 18\sum_{k=1}^{n/2} 10^{k-1} = 2\cdot 10^{n/2} - 1.$$

Let n be odd. Then

$$F(n) = F(n-1) + f(n) = 11\cdot 10^{(n-1)/2} - 1.$$

Also solved by R. V. Andree, D. R. Clutterham, Carl Cohen, R. E. Horton, H. A. James, M. S. Klamkin, G. R. Mach, Jerome Manheim, Prasert Na Nagara, J. D. Neff, C. S. Ogilvy, Bart Park, S. T. Parker, Azriel Rosenfeld, J. J. Rowland, E. P. Starke, C. W. Trigg, and the proposer.

An Interesting Determinant

E 958 [1951, 189]. *Proposed by Herbert Scarf, Temple University*

If $a_n = 1/n!$ show that

$$\begin{vmatrix} a_1 & a_0 & 0 & 0 \cdots 0 \\ a_2 & a_1 & a_0 & 0 \cdots 0 \\ a_3 & a_2 & a_1 & a_0 \cdots 0 \\ \cdot & \cdot & \cdot & \cdot \cdots \cdot \\ a_n & \cdot & \cdot & \cdot \cdots a_1 \end{vmatrix} = a_n.$$

I. *Solution by J. H. Wahab, University of North Carolina.* The result is obviously true for determinants of orders 1 and 2. If the result is assumed for all orders less than n, then expansion by cofactors of the elements of the first rows yields for the determinant D_n of order n:

$$D_n = a_{n-1} - a_2a_{n-2} + a_3a_{n-3} - \cdots (-1)^{n+1}a_n$$

$$= (1/n!)\left[\binom{n}{1} - \binom{n}{2} + \binom{n}{3} - \cdots (-1)^{n+1}\right] = 1/n! = a_n.$$

II. *Solution by F. H. Young, University of Oregon.* Subtract the first column from the second; in the resulting determinant subtract twice the second column from the third; in the resulting determinant subtract three times the third column from the fourth; and continue. We may show by induction that when this process is completed the resulting determinant is triangular with the jth element along the principal diagonal equal to $1/j$, from which the result follows.

III. *Solution by R. G. Buschman, University of Colorado.* Let $f(x) = 1/p(x)$, where $p(x) = \sum_{k=0}^{\infty} a_k x^k$, $a_0 \neq 0$. Then the power series expansion for $f(x)$ is $\sum_{k=0}^{\infty} b_k x^k$, where

$$b_n = (-1)^n D_n / b_0^{n+1},$$

D_n being the determinant of the problem. Therefore

$$D_n = (-1)^n b_n b_0^{n+1}.$$

Now consider the special case $a_n = 1/n!$. Then $p(x) = e^x$, $f(x) = e^{-x}$, and $b_n = (-1)^n/n!$. Hence $D_n = 1/n! = a_n$.

Also solved by A. C. Aitken, P. M. Anselone, F. Bagemihl, H. D. Block, Ralph Calvert, Chandler Davis, J. C. Eaves, C. V. Fronabarger, W. W. Funkenbusch, E. I. Gale, Louisa Grinstein, E. Ikenberry and W. A. Rutledge (jointly), H. A. James, R. S. Kingsbury, M. S. Klamkin, Violet Larney, S. T. Parker, F. A. E. Pirani, Azriel Rosenfeld, M. R. Spiegel, O. E. Stanaitis, Elijah Swift, F. Underwood, Louise Wolf, and the proposer.

A Special Kasner Oval

E 959 [1951, 189]. *Proposed by C. S. Ogilvy, Columbia University*

Let C_0 be a square, C_{n+1} the convex polygon obtained from C_n by measuring off one fourth of the length of every side of C_n and cutting off the corners, and $C = \lim_{n\to\infty} C_n$. Prove that C consists of four parabolic arcs.

Solution by Chandler Davis, University of Michigan. Locate C_0 with vertices at $(1, 0)$, $(0, 1)$, $(-1, 0)$, $(0, -1)$. We will show that C consists of four arcs of parabolas with axes parallel to the coordinate axes. By symmetry we need consider only the half of the first quadrant lying above $y = x$.

Now it is clear that the midpoints of the sides of C_n are midpoints of sides of C_{n+1}, C_{n+2}, $\cdots$, and therefore are on C. Also they are dense on C. On C_1, the points $(1/2, 1/2)$ and $(0, 3/4)$ are such midpoints. At those points C_1 is tangent to the parabola $y = 3/4 - x^2$. To finish the solution it is enough to note the following known property of a parabola: If the tangents to a parabola at any

two points A and B on the parabola meet in T, and if P is the midpoint of AT and Q the midpoint of BT, then PQ is tangent to the parabola at the midpoint of PQ.

Also solved by H. A. James, H. C. Kranzer, J. H. Wahab, and the proposer.

The proposer pointed out the following definition of a *Kasner oval*. Let P_0 be a plane convex polygon, and let P_{n+1} be the convex polygon obtained from P_n by measuring off $(1/r)$th of the length of every side of P_n from each vertex, and cutting off the corners. Then $\lim_{n\to\infty} P_n$ is called a *Kasner oval*. Of course, that the successive polygons remain convex and Jordan, places a restriction on the range of r.

A Chain of Circles in a Circle

E 960 [1951, 189]. *Proposed by Victor Thébault, Tennie, Sarthe, France*

Let AB be a diameter of a circle (O) and let C and D be the midpoints of the radii OA, OB. Let the circles $A(O)$, $B(O)$ cut (O) in A_1, B_1 on the same side of AB. On the other side of AB draw the semicircles $C(O)$ and $D(O)$. Consider the sequence of circles (O_1), (O_2), (O_3), $\cdots$, (O_n), $\cdots$, tangent to the arcs OA_1, OB_1 of $A(O)$ and $B(O)$ and, in turn, to (O), (O_1), (O_2), $\cdots$, (O_{n-1}), $\cdots$. Also consider the sequence of circles (O_1'), (O_2'), (O_3'), $\cdots$, (O_n'), $\cdots$ tangent to the semicircles $C(O)$ and $D(O)$ and, in turn, to (O), (O_1'), (O_2'), $\cdots$, (O_{n-1}'), $\cdots$. Show that the sum of the lengths of the circumferences of all the circles (O_n) and (O_n') is equal to the length of the circumference of (O).

Solution by C. S. Ogilvy, Columbia University. From the symmetry it is obvious that all the circles (O_n) and (O_n') form a row of externally tangent circles, with centers on a diameter of (O), and whose end members are tangent to (O). Thus if C_n is the circumference and D_n the diameter of circle (O_n), and similarly for the primed sequence, we have

$$\sum_{n=1}^{\infty} C_n + \sum_{n=1}^{\infty} C_n' = \pi\left(\sum_{n=1}^{\infty} D_n + \sum_{n=1}^{\infty} D_n' \right) = \pi(AB).$$

Of course any row whatever of circles ranged in this way along a diameter has the same property.

Also solved by H. A. James, M. S. Klamkin, Joseph Langr, and the proposer.

If r_n, r_n' are the radii of circles (O_n), (O_n'), (O), then it is easily shown that

$$r_n = r/2n(n+1), \qquad r_n' = r/(4n^2 - 1),$$

and that $\Sigma r_n = \Sigma r_n' = r/2$.

ADVANCED PROBLEMS AND SOLUTIONS

EDITED BY E. P. STARKE, Rutgers University

Send all communications concerning Advanced Problems and Solutions to E. P. Starke, Rutgers University, New Brunswick, New Jersey. All manuscripts should be typewritten, with double spacing and with margins at least one inch wide. Problems containing results believed to be new or extensions of old results are especially sought. Proposers of problems should also enclose any solutions or information that will assist the editor. In general problems in well known text books or results found in readily accessible sources should not be proposed for this department.

PROBLEMS FOR SOLUTION

4458. *Proposed by Z. A. Melzak, McGill University, Montreal*

Let $f(x)$ be a polynomial of degree n. Then

$$f(x+y) = \frac{y(y+1)(y+2)\cdots(y+n)}{n!}\sum_{k=0}^{n}(-1)^k\binom{n}{k}\frac{f(x-k)}{y+k}\,.$$

4459. *Proposed by D. J. Newman, New York University*

Find an asymptototic expression for the number of integers, not exceeding x, each of which has the property that each of its prime divisors divides it to the second power at least.

4460. *Proposed by Paul Erdös, University of Aberdeen, Scotland*

Let $f(1)=1$,

$$f(n) = (1+c_1)f(n-1) - c_2 f(n-2) - \cdots - c_{n-1}f(1),$$

$c_i \geqq 0$, $i=1, 2, \cdots$, $c_1 = c_2 + c_3 + \cdots$. What is the necessary and sufficient condition for the convergence of $f(n)$?

4461. *Proposed by George Grossman, De Witt Clinton High School, New York City*

A die is made of homogenous material in the shape of a rectangular parallelepiped and is tossed onto a level surface from which it will not bounce. What is the probability that a particular face will come up on top?

4462. *Proposed by M. R. Spiegel, Rensselaer Polytechnic Institute, Troy, New York*

If

$$S_{2m} = \frac{1}{2^{2m}} + \frac{1}{4^{2m}} + \frac{1}{6^{2m}} + \cdots.$$

Find the value of

$$\frac{S_2}{2\cdot 3} + \frac{S_4}{4\cdot 5} + \frac{S_6}{6\cdot 7} + \cdots.$$

SOLUTIONS

A Determinant Related to the Permutation Matrix

4383 [1950, 120]. *Proposed by Robert Steinberg, University of California, Los Angeles*

Let n symbols be ordered in two different ways, and let a_{ij} denote the number of symbols common to the first i in the first ordering and the first j in the second ordering. Prove that the determinant $|a_{ij}|$ is 1 if the transition from the first ordering to the second is effected by an even permutation and -1 if it is effected by an odd permutation.

Solution by the Proposer. Let the symbols in the first ordering be numbered $1, 2, \cdots, n$, and in the second ordering $p_1, p_2, \cdots, p_n$. Then, after defining $a_{0j}=a_{i0}=0$ for convenience, one can readily verify the following equalities which are true for $i, j=1, 2, \cdots, n$:

If $i>p_j$, then $a_{i,j}-a_{i,j-1}=a_{i-1,j}-a_{i-1,j-1}=1$.

If $i<p_j$, then $a_{i,j}-a_{i,j-1}=a_{i-1,j}-a_{i-1,j-1}=0$.

If $i=p_j$, then $a_{i,j}-a_{i,j-1}=1$ and $a_{i-1,j}-a_{i-1,j-1}=0$.

Hence the expression $a_{i,j}-a_{i,j-1}-a_{i-1,j}+a_{i-1,j-1}$ is 1 if $i=p_j$ and 0 otherwise. Thus, if we subtract the $(n-1)$st row of the matrix (a_{ij}) from the nth, the $(n-2)$nd from the $(n-1)$st, $\cdots$, the 1st from the 2nd, and do the same to the columns, we obtain the matrix (δ_{ip_j}). This latter matrix is just the permutation matrix corresponding to the permutation $i\rightarrow p_i$, and has the determinant 1 or -1 according as the permutation is even or odd.

A compact expression for the relationship between (a_{ij}) and (δ_{ip_j}) can be obtained as follows. The operations performed on the rows of (a_{ij}) are equivalent to left multiplication by the matrix which has 1's in the main diagonal, -1's in the sub main diagonal, and 0's elsewhere. This matrix has as inverse the matrix M_{sub} which has 1's on and below the main diagonal and 0's elsewhere. A corresponding result holds for the columns of (a_{ij}): one need only replace the words left, sub, below by right, super, above. The final result is

$$(a_{ij}) = M_{\text{sub}}(\delta_{ip_j})M_{\text{sup}}.$$

Orthoptic Locus of Conic Inscribed in a Triangle

4386 [1950, 188]. *Proposed by Victor Thébault, Tennie Sarthe, France*

In any triangle the Monge circle (orthoptic locus) of the inscribed conic concentric with the circumcircle is orthogonal to the polar circle.

Solution by L. M. Kelly, Michigan State College. The hypothesis that the center of the conic be the circumcenter is superfluous. We will prove the theorem for any inscribed conic and its associated orthoptic locus, appealing to the following well established properties.

1. The tangents from a point P to a conic inscribed in a complete quadrilateral are corresponding lines in an involution. The lines joining the point P to the opposite ends of the diagonals are also pairs of corresponding

elements in this involution. (Desargues)

2. Circles on the diagonals of a complete quadrilateral (as diameters) are coaxial. (Gauss-Bodenmiller)
3. A circle orthogonal to two given circles is orthogonal to every circle coaxial with them.

Consider any conic inscribed in a triangle and draw a fourth tangent to the conic. The conic is now inscribed in a complete quadrilateral. The circles on the diagonals intersect in two points P and P', by (2). The tangents from P to the conic and the pairs of lines joining P to the opposite vertices of the quadrilateral are elements of an involutory pencil, by (1). Furthermore the pairs of rays joining P to the opposite ends of a diagonal are at right angles. Thus the involution is circular and the tangents from P to the conic are perpendicular. Similarly for P'. Thus the orthoptic locus passes through P and P' and being in general a circle (in the case of a parabola, a line) is coaxial with the three circles on the diagonals. Each of these circles passes through a vertex and corresponding altitude foot of the original triangle. Thus each is orthogonal to the polar circle (Johnson, *Modern Geometry*, p. 177). By (3) the polar circle is orthogonal to the orthoptic locus as required.

Also solved by A. E. Landry and the proposer.

Sums of rth Powers of Divisors

4387 [1950, 188]. *Proposed by Paul Bateman, University of Illinois*

If $\sigma_r(n)$ denotes the sum of the rth powers of the divisors of the positive integer n, prove that

$$\sigma_r(n)\sigma_r(m) = \sum_{d\mid(n,m)} d^r\sigma_r(nm/d^2),$$

where d runs through all the common divisors of n and m.

Solution by the Proposer. Since for $(n, m)=1$ it is clear that $\sigma_r(nm)=\sigma_r(n)\sigma_r(m)$, it suffices to prove the formula in the case where n and m are both powers of the same prime p. Suppose $n=p^a$, $m=p^b$, $0<a\leqq b$. Then

$$\begin{aligned}\sigma_r(p^a)\sigma_r(p^b) &= (1+p^r+p^{2r}+\cdots+p^{ar})(1+p^r+\cdots+p^{br})\\ &= 1+2p^r+3p^{2r}+\cdots+(a+1)p^{ar}+(a+1)p^{(a+1)r}+\cdots\\ &\quad+(a+1)p^{br}+ap^{(b+1)r}+\cdots\\ &\quad+3p^{(a+b-2)r}+2p^{(a+b-1)r}+p^{(a+b)r}.\end{aligned}$$

On the other hand,

$$\begin{aligned}\sum_{d\mid(p^a,p^b)} d^r\sigma_r\left(\frac{p^ap^b}{d^2}\right) &= \sum_{\lambda=0}^{a} p^{\lambda r}\sigma_r(p^{a+b-2\lambda})\\ &= \sum_{\lambda=0}^{a} p^{\lambda r}(1+p^r+\cdots+p^{(a+b-2\lambda)r}).\end{aligned}$$

It is easily seen that these two expressions are the same.

Also solved by T. M. Apostol, Emilie V. Haynsworth, Roger Lessard, and N. T. Seeley, Jr.

Editorial Note. Apostol proves the problem as a special case of the following theorem: *Let $f(n)$ be a multiplicative function, $f(1)=1$, and let $g(n)$ be a completely multiplicative function, $g(1)=1$. Then the equation*

$$f(m)f(n) = \sum_{d|(m,n)} g(d)f(mn/d^2) \tag{1}$$

is equivalent to the equation

$$f(p)f(p^a) = g(p)f(p^{a-1}) + f(p^{a+1}), \qquad p = \text{prime}, a \geqq 1, \tag{2}$$

and also to the equation

$$\sum_{n=1}^{\infty} f(n)n^{-s} = \prod_p \frac{1}{1 - f(p)p^{-s} + g(p)p^{-2s}} \tag{3}$$

whenever the Dirichlet series is absolutely convergent. The product is extended over all primes.

Since equation (2) is easily verified for $f(n)=\sigma_r(n)$ with $g(n)=n^r$, this theorem solves the proposed problem. Equation (3) also solves the problem since, for the same choice of $f(n)$ and $g(n)$, both sides are the same as $\zeta(s)\zeta(s-r)$, where $\zeta(s)$ is the Riemann zeta-function.

Components of the Set $|(z-z_1)\cdots(z-z_n)| \leqq 1$

4388 [1950, 188]. *Proposed by Paul Erdös, University of Aberdeen, and W. H. Fuchs, Cornell University*

Let

$$f(z) = \prod_{i=1}^{n} (z - z_i), \qquad |z_i| \leqq 1.$$

Consider the set $|f(z)| \leqq 1$. Prove that it consists of at most $n-1$ components.

Solution by G. Szegö, Stanford University. A confluence of the components of the lemniscate $|f(z)| =$constant takes place whenever $f'(\zeta)=0$. Let $\zeta_1, \zeta_2, \cdots, \zeta_{n-1}$ be the roots of this equation; we have to show that min $|f(\zeta_i)| \leqq 1$. But

$$\begin{aligned} \min |f(\zeta_i)|^{n-1} &\leqq |f(\zeta_1)||f(\zeta_2)| \cdots |f(\zeta_{n-1})| \\ &= n^{-n}|f'(z_1)| \cdots |f'(z_n)| = n^{-n}\prod_{i\neq k}|z_i - z_k|. \end{aligned}$$

The maximum of this product is attained for $f(z)=z^n-1$ (see I. Schur, *Mathematische Zeitschrift*, 1918, p. 385) in which case

$$|f'(z_1)| = \cdots = |f'(z_n)| = n.$$

Hence min $|f(\zeta_i)| \leqq 1$.

Also solved by the proposers.

Simultaneous Linear Equations

4389 [1950, 188]. *Proposed by F. J. Dyson, Institute for Advanced Study*

Given N numbers a_m satisfying the N equations

$$\sum_{m=1}^{N} \frac{a_m}{m+n} = \frac{4}{2n+1}, \qquad n = 1, 2, \cdots, N,$$

prove that

$$\sum_{m=1}^{N} \frac{a_m}{2m+1} = 1 - \frac{1}{(2N+1)^2}.$$

This problem arose in an investigation of the diffraction of light (H. Levine and J. Schwinger, *Physical Review*, 1948, p. 970).

I. *Solution by Ernest Trost, Technikum Winterthur, Switzerland.* We can prove the following more general theorem: *If the N numbers a_m satisfy the N equations*

$$\sum_{m=1}^{N} \frac{a_m}{A_m + B_n} = \frac{-a_{N+1}}{A_{N+1} + B_n}, \qquad (n = 1, 2, \cdots, N),\ A_m + B_n \neq 0,$$

we have

$$\sum_{m=1}^{N} \frac{a_m}{A_m + x} = \frac{a_{N+1}}{A_{N+1} + x}\left\{\prod_{k=1}^{N} \frac{(x - B_k)(A_{N+1} - A_k)}{(x + A_k)(A_{N+1} + B_k)} - 1\right\}.$$

Consider the polynomial of the Nth degree in x

$$f(x) = \prod_{k=1}^{N+1} (x + A_k) \sum_{m=1}^{N+1} \frac{a_m}{x + A_m}. \tag{1}$$

$f(x)$ vanishes for $x = B_n$ $(n = 1, 2, \cdots, N)$. Hence we have

$$f(x) = C \cdot \prod_{i=1}^{N} (x - B_i), \tag{2}$$

where C denotes a constant. Putting $x = -A_{N+1}$ we obtain

$$a_{N+1} \cdot \prod_{k=1}^{N} (A_k - A_{N+1}) = (-1)^N \cdot C \cdot \prod_{k=1}^{N} (A_{N+1} + B_k),$$

and thus

$$C = a_{N+1} \prod_{k=1}^{N} \frac{(A_{N+1} - A_k)}{(A_{N+1} + B_k)}.$$

From (1) and (2) we deduce

$$\sum_{m=1}^{N+1} \frac{a_m}{x + A_m} = \frac{a_{N+1}}{x + A_{N+1}} \prod_{k=1}^{N} \frac{(x - B_k)(A_{N+1} - A_k)}{(x + A_k)(A_{N+1} + B_k)},$$

which proves the theorem.

Putting $A_m = m$, $B_n = n$ $(m, n = 1, 2, \cdots, N)$, $A_{N+1} = x = \frac{1}{2}$, $a_{N+1} = -2$, we have the special case of the problem.

II. *Solution by G. Szegö, Stanford University.* For every polynomial $\phi(x)$ of degree $N-1$:

$$\phi(x) = \sum_{m=1}^{N} \frac{\phi(-m)}{f'(-m)} \frac{f(x)}{x + m}$$

identically. Take, in particular,

$$\phi(x) = \frac{2}{f(\frac{1}{2})} \frac{f(\frac{1}{2})f(x) - f(-\frac{1}{2})f(-x)}{x + \frac{1}{2}}$$

so that

$$\phi(n) = \frac{2}{n + \frac{1}{2}} f(n), \qquad n = 1, 2, \cdots, N,$$

$$\frac{2}{n + \frac{1}{2}} = \sum_{m=1}^{N} \frac{\phi(-m)}{f'(-m)} \frac{1}{m + n}.$$

Consequently

$$a_m = \frac{\phi(-m)}{f'(-m)} = \frac{4(-1)^{m-1}(N + m)!}{(2m - 1)(2N + 1)(N - m)!m!(m - 1)!}.$$

Now

$$\sum_{m=1}^{N} \frac{a_m}{2m + 1} = \sum_{m=1}^{N} \frac{\phi(-m)}{f'(-m)} \frac{1}{2m + 1} = \frac{1}{2} \frac{\phi(\frac{1}{2})}{f(\frac{1}{2})}$$

$$= \frac{(f(\frac{1}{2}))^2 - (f(-\frac{1}{2}))^2}{(f(\frac{1}{2}))^2} = 1 - \left(\frac{f(-\frac{1}{2})}{f(\frac{1}{2})}\right)^2 = 1 - \frac{1}{(2N + 1)^2}.$$

Also solved by N. T. Seeley, Jr., O. E. Stanaitis, and G. Szegö (two additional solutions).

Equilateral Hyperbola Inscribed in a Triangle

4393 [1950, 265]. *Proposed by Victor Thébault, Tennie, Sarthe, France*

Show that: (1) the center ω of an equilateral hyperbola $(\mathcal{H})$ inscribed in a triangle ABC, the orthocenter H of this triangle, and the center N of the nine-point circle (N) of the triangle of contact $\alpha\beta\gamma$ of $(\mathcal{H})$ with the sides of ABC are collinear. (2) The polar circle of triangle ABC is tangent to the circle (N).

(3) If BC, CA, AB; $\beta\gamma$, $\gamma\alpha$, $\alpha\beta$ respectively intersect one of the asymptotes of $(\mathcal{H})$ in A', B', C'; α', β', γ', we have

$$\omega N/\omega H = \omega\alpha' \cdot \omega\beta' \cdot \omega\gamma' / \omega A' \cdot \omega B' \cdot \omega C'.$$

Solution by Roscoe Woods, State University of Iowa. Without loss of generality, the equation of $(\mathcal{H})$ may be taken as $xy=1$. Its center ω is then the origin O of coördinates. By the "point t" on $(\mathcal{H})$ we shall mean the point whose coordinates are (t, t^{-1}). Let t_1, t_2, t_3 be the points α, β, γ of contact of $(\mathcal{H})$ with the sides of triangle ABC. The equations of the tangents to $(\mathcal{H})$ at α, β, and γ are then $x+t_i^2y=2t_i$, $i=1, 2, 3$. Then, if C is the point of intersection of the tangents at α and β, its coördinates are

$$\left(\frac{2t_1t_2}{t_1+t_2}, \quad \frac{2}{t_1+t_2}\right).$$

The coördinates of A and B may be found by cyclic permutation of the subscripts.

For convenience put $S_1=t_1+t_2+t_3$, $S_2=t_2t_3+t_3t_1+t_1t_2$, $S_3=t_1t_2t_3$. It is then found that the coördinates of H and N are respectively

$$\left(\frac{-2+2S_1S_3}{S_1S_2-S_3}, \quad \frac{2S_2-2S_3^2}{S_1S_2-S_3}\right), \qquad \left(\frac{-1+S_1S_3}{4S_3}, \quad \frac{S_2-S_3^2}{4S_3}\right).$$

Since the slopes of OH and ON are equal, property (1) is proved.

To demonstrate (2), use is made of the following well known facts: (a) the center of an equilateral hyperbola circumscribing a triangle lies on the nine-point circle of the triangle, (b) the center of the polar circle of a triangle coincides with the orthocenter, and (c) the square of the radius of the polar circle is the product of the segments in which the orthocenter divides an altitude. The demonstration is completed by a short calculation which shows that the square of the radius of the polar circle of ABC is $\overline{OH}^2$.

The squares of the distances ON and OH are readily found to be $T^2/16S_3^2$ and $4T^2/(S_1S_2-S_3)^2$, respectively where $T^2=(1+t_1^2t_2^2)(1+t_2^2t_3^2)(1+t_3^2t_1^2)$. The products of the intercepts of the sides of the triangles $\alpha\beta\gamma$ and ABC on the x-axis (one of the asymptotes of $(\mathcal{H})$) are, respectively, $S_1S_2-S_3$ and $8S_3$. Thus property (3) is established.

Also solved by E. D. Camier, Joseph Langr, and the Proposer.

Polynomials Related to Stirling Polynomials

4396 [1950, 342]. *Proposed by D. H. Browne, Buffalo, New York*

With the notation

$$P(s, t) = \prod_{m=0}^{s} (1+mt),$$

let

$$S(n, t) = P(n, t)\sum_{k=0}^{n} k!t^k/P(k, t).$$

Show that for $n\geqq 1$

$$S(n, t) \equiv P(n-1, t) \pmod{n+1},$$

and that for $n>2$, $t\leqq n$,

$$S(n, t) \equiv 0 \pmod{t+1}.$$

Solution by N. T. Seely, Jr., Agricultural, Mechanical and Normal College, Pine Bluff, Arkansas. From the hypothesis we have

$$\begin{aligned}
S(n, t) &- P(n-1, t)\\
&= P(n, t)\sum_{k=0}^{n} \frac{k!t^k}{P(k, t)} - P(n-1, t)\\
&= P(n, t) - P(n-1, t) + \sum_{k=1}^{n} k!t^k \frac{P(n, t)}{P(k, t)}\\
&= ntP(n-1, t) + \sum_{k=1}^{n-1} k!t^k\left\{\frac{P(n-1, t)}{P(k-1, t)} + (n-k)t\frac{P(n-1, t)}{P(k, t)}\right\} + n!t^n\\
&= (n+1)t\frac{P(n-1, t)}{P(0, t)} + (n+1)\sum_{k=1}^{n-2} k!t^{k+1}\frac{P(n-1, t)}{P(k, t)} + (n+1)(n-1)!t^n.
\end{aligned}$$

(The last summation is obtained from the preceding one by combining the second term of each summand with the first term of the next.) Thus,

$$S(n, t) - P(n-1, t) = (n+1)tS(n-1, t)$$

which is divisible by $n+1$, $n\geqq 1$.

Since $S(n, t)$ is a polynomial in t, the remainder on division by $t+1$, that is, $S(n, -1)$, must be shown to be divisible by $t+1$ for $n>2$, $t\leqq n$. Thus

$$\begin{aligned}
S(n, -1) &= \sum_{k=0}^{n} (-1)^k k!\frac{P(n, -1)}{P(k, -1)} = \sum_{k=0}^{n-1} (-1)^k k!\sum_{m=k+1}^{n} (1-m) + (-1)^n n!\\
&= \sum_{k=0}^{n-1} (-1)^n k!k(k+1)\cdots(n-1) + (-1)^n n!\\
&= (-1)^n(n-1)!\sum_{k=0}^{n-1} k + (-1)^n n!\\
&= (-1)^n(n-1)!\frac{n}{2}(n-1) + (-1)^n n!\\
&= (-1)^n(n+1)!/2,
\end{aligned}$$

which is divisible by $t+1$ for $n>2$, $t\leqq n$.

Also solved by the Proposer.

RECENT PUBLICATIONS

EDITED BY E. P. VANCE, Oberlin College

All books for review should be sent directly to the editor of this department, American Mathematical Monthly, 80 Waterman Street, Providence 6, Rhode Island, and not to any of the other editors or officers of the Association.

The Arithmetic Theory of Quadratic Forms. By B. W. Jones. Carus Mathematical Monograph No. 10. The Mathematical Association of America. (Distributed by John Wiley and Sons, Inc., New York.) 1950. x+212 pages. $3.00.

Let f and g be quadratic forms in n and m variables ($n\geqq m\geqq 1$), with coefficient matrices A and B respectively. The monograph deals principally with the conditions under which the form f represents the form g, or equivalently the existence of a matrix, X such that $X^TAX=B$. A and B have elements in a given field or ring, and X is to have elements in an assigned field or ring Starting with A, B, and X all having elements in the real field, the successive chapters narrow the fields or rings until finally the classic case where all elements are rational integers is treated.

A more detailed idea of the contents of the monograph is given by the chapter titles and principal topics in each chapter. I. Forms with real coefficients. Congruent forms. Canonical forms. Representation conditions. Automorphs and number of representations. II. Forms with p-atic coefficients. Definition and properties of p-atic numbers. Relation between rational representation modulo p^t, and p-atic representation. Hilbert and Hasse symbols. Application to representation problems. Universal forms. III. Forms with rational coefficients. Equivalence and reduction. Fundamental theorems on zero forms, and on rational congruences. Representation of one form by another. IV. Forms with coefficients in $R(p)$. Equivalence. Canonical forms. Numbers represented by forms. Conditions for equivalence. Forms with given invariants. Zero and universal forms. Automorphs. Binary forms. V. Genera and semi-equivalence. Definitions. Representation without essential denominator. Forms with integral coefficients and given invariants. VI. Representations by forms. Siegel's representation function. Asymptotic results. A representation function. VII. Binary forms. Automorphs. Number of representations. Quadratic ideals. Correspondence between ideal classes and classes of forms. Composition of classes. Genera. Reduced, and zero, definite forms. Reduced indefinite forms. Number

of classes. VIII. Ternary forms. Numbers represented by ternary genera. Number of representations. There is also a short bibliography of more recent work and an excellent set of problems, which will enable the reader to test his grasp in the material treated.

As has already been pointed out by Kloosterman, some of the proofs in Ch. II are incomplete. The essential difficulty is the passage from an infinite sequence of matrices, S_t, in $R(p)$, which satisfy $S_t^T A S_t \equiv B$ modulo p^t for each integer $t \geqq t_0$, to a matrix X satisfying $X^T A X = B$. Minor misprints were noted on pages 17 and 22.

The monograph is essentially self contained. It is presupposed only that the reader knows the fundamentals of matric theory and number theory. The style is clear, the presentation compact. More difficult concepts are clarified by well chosen examples. Starting with the basic ideas, in less than 200 pages we are led to the most recent developments in the subject, which heretofore have not been available in book form. Professor Jones' monograph is warmly recommended as an introduction to the arithmetic theory of quadratic forms.

J. D. ELDER

1. *Plane Trigonometry*. J. J. Corliss and W. V. Berglund. Houghton Mifflin Co. 1950, 12+388 pages. $3.00.
2. *Plane Trigonometry with Tables*. Gordon Fuller. McGraw-Hill, 1950, 11+270 pages. $2.75.
3. *College Trigonometry*. W. L. Hart, Heath and Co. 1951, 8+230+130 pages. $3.50.
4. *Plane Trigonometry—Alternate Edition with Tables*. E. R. Heineman. McGraw-Hill, 1950, 14+184+75 pages. $2.50.
5. *Trigonometry with Tables*. C. T. Holmes. McGraw-Hill, 1951, 9+246 pages. $3.00.
6. *Essentials of Plane Trigonometry*. J. B. Rosenbach, E. A. Whitman, and David Moskovitz. Ginn and Co. 1950, 8+168+15 pages.
7. *Plane and Spherical Trigonometry with Tables*. J. Shibli. (Third Edition), Ginn and Co. 1949, 12+262+94 pages.

All of these textbooks appear to be written carefully and should prove satisfactory in the trigonometry courses for which they are respectively intended. (3), (5) and (7) cover at least the fundamentals of spherical trigonometry with applications, while the others as their titles indicate cover only plane trigonometry. Except for (6) they all include the basic tables needed in trigonometry and all include the answers to at least one-half of the problems.

Except for (7) they all begin with the introduction of a coordinate system in the plane and the definition of the trigonometric functions of the general angle. (7) begins at a more elementary level with a discussion of measurement,

some historical remarks, a review of a few relevant theorems from high school plane geometry, and the definition of the trigonometric functions of an acute angle. (5) derives the formula $d=\sqrt{(x_2-x_1)^2+(y_2-y_1)^2}$ and applies it in two different coordinate systems to give an analytic proof of $\cos(\alpha-\beta)=\cos\alpha\cos\beta+\sin\alpha\sin\beta$ for all α, β. (6) proves the addition formula for α, β acute by expressing the area of a triangle in two ways, while the other books prove the addition formula in the standard way first for acute α, β and then for all angles.

Except for (6) they all derive the properties of logarithms from scratch before taking up the logarithmic solution of triangles. The authors of (6) evidently assume that many students have had a previous exposure to logarithms, but they do include a final chapter VII on logarithms for students with inadequate preparation. All except (6) include a chapter on complex numbers including De Moivre's theorem. (5) includes a section on the series representation of functions and one on Euler's formula as well, while (7) includes without proof the Maclaurin expansions of $\sin x$, $\cos x$, e^x, Euler's formula and the definitions of the hyperbolic functions.

(1) and (3) devote some space to the general concepts of "variable" and "function" while (5) gives a brief explanation of "function." The other authors limit their discussion to trigonometric functions without explaining what a function is.

(1) and (5) introduce polar coordinates explicitly, while (2), (3), (4) and (7) discuss the polar form of a complex number. All the books discuss vectors while all except (2), (4) and (5) discuss projections. On the whole (1) and (3) seem strongest on applications of plane trigonometry while (5) seems weakest. All except (6) devote a fair amount of space to significant figures and the accuracy of results. Apparently (6) has only a few footnotes on the subject. All the books have a reasonable number of illustrative examples and problems for the student.

I shall end the review by mentioning a few additional features of each of the texts.

The authors of (1) are careful to emphasize concepts and topics such as functions, variables and polar coordinates which are important in other mathematics courses. There is a good discussion of computation with approximate numbers and of interpolation. There is a good chapter on applications. The reviewer was surprised to notice a slip on page 206 where the authors do not justify the disappearance of $\pm$. There are four tables, squares, 4 place trig and 5-place log and log trig.

(2) is a rather conservative trigonometry book including good sections on approximate numbers, significant figures and rounding off. There are 5-place log and log trig tables.

(3) is a comprehensive book on trigonometry for mature students. Especially noteworthy are the numerous problems, the comprehensive review of plane trigonometry, the appendices to plane trigonometry including ones on the logarithmic scale, logarithms of functions near 0° and 90°, plane surveying

and the use of reversible operations, the very thorough treatment of spherical trigonometry with the use of haversines, and the 17 tables including squares and square roots, 3-place log and log trig, 4-place and 5-place log, trig and log trig, auxiliary table for angles near 0° or 90°, natural logs, logs for compound interest, and haversines.

(4) like (2) is conservatively written. There are 4 tables, 4-place trig, 5-place log and log trig, and squares, cubes, square roots, cube roots and reciprocals.

(5) is especially good at relating trigonometry to analytic geometry and calculus. The proof for cos $(\alpha-\beta)$, the definition of variables and functions, and the series representation of functions have been mentioned. There is a discussion of reversible operations in working with identities, common algebraic errors, the law of natural growth, e as a limit and the logarithmic and exponential functions as inverse functions. There are 3 tables, 5-place log, log trig and trig.

(6) is the shortest of the books and hence does not cover some of the material included in the others. It includes some historical notes, many illustrative examples with the analysis and computation in parallel columns, and a collection of general exercises at the end. In a pocket inside the back cover there are a ruler-protractor and a card with proportional parts for differences from 1 to 105 and a table of angles corresponding to certain rational values of the trigonometric functions.

As already pointed out (7) begins at an elementary level suitable for high school but still it covers as much material as most of the other texts by the end. It is very clear and spherical trigonometry with applications is covered quite well in 32 pages. There are 9 tables (many 5-place) including squares, square roots and reciprocals, natural logarithms, and hyperbolic functions.

L. J. BURTON

NEW BOOKS RECEIVED

Synthesis of Electronic Computing and Control Circuits, Annals 27. Harvard University Computation Laboratory. Cambridge, Harvard University Press, 1951. 278 pp.

Platon et la Recherche Mathematique de son epoque. Charles Mugler. Strasbourg. Editions. P. H. Heitz, 1948.

L'Acces Aux Principes de la Geometrie Euclidienne. By G. Bouligand. Paris, Librairie Vuibert, 1951, vii+87 pp. 320 fr.

General Homogeneous Coordinates in Space of Three Dimensions. By E. A. Maxwell. Cambridge University Press, 1951. xiv+169 pp. $2.75.

Programs for an Electronic Digital Computer. By Wilkes, Wheeler, Gill, Cambridge, Mass., Addison Wesley, 1951. 167 pp.

Mathematics of Statistics, Revised Edition. By J. F. Kenney and E. S. Keeping. New York, Van Nostrand. xiii+429 pp.

Mathematics for Technical Students, Book 2. By J. D. N. Gasson. Cambridge University Press, 1951. x+431 pp.

CLUBS AND ALLIED ACTIVITIES

EDITED BY L. F. OLLMANN, Hofstra College

Material for this department should be sent to Professor H. D. Larsen, Albion College, Albion, Michigan.

CLUB REPORTS, 1950–51

Sigma Phi Mu, Montclair, New Jersey State Teachers College

The papers presented to the membership of *Sigma Phi Mu*, the mathematics club at State Teachers College, Montclair, New Jersey, included the following:

The mathematics major in the elementary school, by Anna May Martin
Meteorology and mathematics, by William Lone
Mathematics and the slow student in high school, by Max A. Sobel
Probability, by Edward Molino.

The first meeting of the year was devoted to an initiation and welcoming of the Freshmen. Other social activities consisted of an annual Christmas Party with *Kappa Mu Epsilon*, a farewell party for the Seniors, and the annual May picnic.

A major project for the year was the initial publication of a club newspaper, "*The limit" as Sigma Pi Mu approaches the news*. This paper was written, edited and printed by members of the club. It contained scientific and feature articles, department and club news, editorials on current events in the department and the field of mathematics, a book review and various types of problems.

The officers for 1951–52 are: President, Joseph Sommer; Vice-President, Howard Mion; Secretary, Barbara Pearson; Treasurer, John Rowley.

Mathematics Club, University of Colorado

Included among the talks presented at the meetings of the *Mathematics Club* of the University of Colorado were:

Topics in topology, by A. L. Blakers
Meyers theorem and quadratic forms, by B. W. Jones
Consecutive powers and the form of binomial coefficients, by Paul Erdös
Infinite abelian groups, by K. A. Hirsch
The product of three linear forms, by Jurt Mahler
Mappings of spheres and disks, by A. Grudin
Point lattices and Farey sequences, by D. DeVol
Representations of primes, by N. C. Ankeny
Partitions and generating functions, by W. E. Briggs
Lattice points and hyperbolas, by B. Hunt
Operational calculus and applications to number theory, by J. R. Britton
Iteration of mappings of a metric space onto itself, by Albert Edrei
Crooked dice, by D. Hawkins.

The officers for the year were: President, B. Hunt; Secretary-Treasurer, R. G. Buschman.

Pi Mu Epsilon, University of California

The *California Alpha* chapter of *Pi Mu Epsilon* had ten meetings during the year, of which two were purely business meetings, two were initiation meetings held in private homes, and six were lecture meetings. The lectures were:

Theory of games, by Irving Glicksberg

Theory of braids, and a graphic approach to permutation groups, by James Jackson

Problem types in plasticity, by George Zizicus

Some results related to the fixed-point theorem—and stuff, by Dr. Robert Steinberg

Automatic computing machinery, by Dr. Harold Luxemberg

Transforms and tautochrones, by Dr. G. Milton Wing.

The Annual Calculus Prize examination was won by George S. Rasmussen and W. T. Root (tie).

Officers for 1951–52 are: Director, James Jackson; Vice-Director, Sharla Rita Perrine; Secretary, Mervin Muller; Faculty Adviser, Dr. Phil Hodge.

Mathematics Society, Massachusetts Institute of Technology

The *Mathematics Society* of Massachusetts Institute of Technology met weekly. The following were among the lectures given:

The existence of non-measurable sets on the sphere, by Prof. W. Hurewicz

Ergodic theory and probability, by Prof. W. Ambrose

Theory of elasticitity, by Prof. E. Reissner

Theory of characteristics and supersonic flow, by Prof. Lin

Asymptotic expansions, by Prof. Hildebrand

Fourier series, by Dr. Rudin

Geometry of differential equations, by Dr. Coddington

Maximum aspects of eigen problems in algebra and integral equations, by R. B. Davis

The Laplace transform, by W. Shields

Transcendental numbers, by H. Shapiro

Chess and mathematics, by H. Davis

Three proofs of Weierstrass' polynomial approximation theorem, by R. B. Kellogg

Klein's geometrical theory of algebraic equations, by W. Baily

Internal symmetric transforms, by R. Preisendorfer.

The Society published two issues of its *Bulletin* giving reviews of some of the lectures.

The officers for 1950–51 were: President, Bruce Kellogg; Vice-President, Rudolph Preisendorfer; Secretary-Treasurer, David Berkowitz; Bulletin Editor, Don Aronson.

Mathematics Club, Swarthmore College

The activities of the Swarthmore *Mathematics Club* during the past year

include lectures, problem contests, and participation in the William Lowell Putnam Mathematics contest.

The following talks were presented:

Characteristic properties of the circle, by R. A. Rosenbaum

Number partitions, by Arthur Mattuck

Number theory, by Oystein Ore

Properties of isogonal conjugates, by Louis Winer

Diophantine approximations, by H. W. Brinkmann

A mathematical model in the social sciences, by Lotte Lazarsfeld

Greek mathematics (in conjunction with the Classics Club), by Arnold Dresden

Some aspects of differential geometry in the large, by C. B. Allendoerfer

The prime number theorem, by Walter Leser

Continued fractions, by Joan Berkowitz.

Prizes for the contests sponsored by the club went to James VanderWeen, Arthur Mattuck, Louis Winer and Joan Berkowitz. Edward Prenowitz won the Putnam contest.

President-elect is Barbara Wolff.

Pi Mu Epsilon, University of Illinois

In addition to the usual business meetings, the Illinois *Alpha* chapter of *Pi Mu Epsilon*, University of Illinois, held two meetings during the school year 1950–51. They were:

A demonstration lecture on *The analog computer*, by Prof. A. T. Nordsieck and the Spring initiation meeting at which ninety-nine new members were taken into the fraternity. Prof. R. H. Bing of the University of Wisconsin addressed the meeting on *Some theorems in topology which are valid in two-space, but not in three-space.* Awards were given to Prof. J. W. Peters, G. E. Modesitt and Lloyd R. Welch.

The new officers for 1951–52 are: President, Thomas Elfe; Vice-President, John W. Toole; Secretary, Beverly Marshall; Treasurer, Richard Priest.

Zeno Club, Alfred University

The *Zeno Club* of Alfred University held eight regular meetings throughout the school year of 1950–51, and the following phases of mathematics were reported on:

Mathematics and world maps, by Dr. C. E. Rhodes

Equally likely, by Robert Lober

Arithmetic in the past, by Prof. V. Nevins

Magic squares, by Ralph Beals

Irrational numbers, by John Ging

Rank bi-serial coefficient of correlation, by Dr. Stephen Clark and Prof. John Freund

Projective number system, by Barbara Fischer

Poles and polars, by Dr. A. E. Whitford.

A dinner party was given by the faculty members and their wives for all mathematics majors and members of the Zeno Club.

In the spring the students in the club gave a picnic for the faculty and their families.

Officers of the club for the year were: President, Barbara Fischer; Vice-President, Robert Hultquist; Secretary-Treasurer, Jane Bette; Faculty Adviser, Dr. C. E. Rhodes.

Pi Mu Epsilon, Bucknell University

The following papers were presented to the *Pennsylvania Beta* chapter of *Pi Mu Epsilon* during the past year:

The background of necessary and sufficient, by Emil Polak

Diophantine problems, by Ralph Jones

Dimensional analysis, by Francis Huber

The cross ratio, by F. S. McFeely.

Officers elected for 1951–52 are: Director, Prof. W. K. Smith; Vice-Director, James E. Hole; Secretary, Rosemary Scheerer; Treasurer, C. Jerome Sechrist.

Kappa Mu Epsilon, Albion College

The *Michigan Alpha* chapter of *Kappa Mu Epsilon* at Albion College met monthly during 1950–51. The following programs were presented:

Arithmetic revisited, by Dr. H. D. Larsen

Wolfgang and Johann Bolyai, by Vinod Doshi

Gottfried Wilhelm Leibniz, by Robert Hooper

Neils Hendrick Abel, by James Young

Linkages, by Richard Burrows

A day of practice teaching, by Patricia Collins

Cryptography, by Milton Ehlert

The Laplace transform, by Albert Foster

The national convention of Kappa Mu Epsilon, by William Fryer

The planimeter, by William Horn.

The year's activities also included the presentation of the play, *It Can't Happen Here*, with the cast composed of initiates. The chapter also sponsored an open house in the Mathematics Department during Albion's annual "Meet The College" Day.

Officers elected for the year 1951–52 are: President, Paul Jones; Vice-President, William Ferguson; Secretary-Treasurer, Shirley Coffron; Faculty Sponsor, Dr. H. D. Larsen.

Kappa Mu Epsilon, Mount St. Scholastica College

The *Kansas Gamma* chapter of *Kappa Mu Epsilon* held bi-monthly meetings during the year 1950–51 following the chosen theme, *The development of mathematical algorithms*. The papers presented were:

Mathematics in the Orient, by Jeanne Culivan and Frances Donlon

Greek mathematics, by Ruth Link and Teresita Breitenbach

Hindu, Arabic and Persian developments, by Anne Robben and Jill Sullivan

Mathematics in the middle ages, by Margot Acree

Mathematics in the renaissance, by Elaine Barnes

Seventeenth century geometry, by Patti Shideler and Dorothy McManus

Development of the calculus, by Margaret McBride

The development of mathematics in the eighteenth century, by Mary Clare Brenwald.

A special feature in this year's program was the emphasis placed on visual aids for better understanding of principles involved.

The social program for the year consisted of a get-together picnic held for the pledges, the traditional English Wassail party at Christmas, and the formal initiation and banquet in the Spring. At the latter meeting four new members were initiated.

Special programs were the presentation of a skit *Mathematical relations* by the pledges and *It can't happen here* as a joint effort with the music society on the campus. A special contribution was made by Ruth Link in the form of an article, *Mysticism and mathematics*, which was published in the Mt. St. Scholastica quarterly magazine.

At the annual initiation Anne Robben was the recipient of the Hypatian award. This award is given each year for the oustanding contribution to the chapter. Miss Robben merited it because she presented a paper, *Some properties of the Simson line*, at the eighth biennial convention of the fraternity. Kathleen Feldhausen was presented with the underclassman award.

Officers elected to serve for 1951–52 are: President, Margaret McBride; Vice-President, Mary Brenwald; Secretary, Jill Sullivan; Treasurer, Anne Robben; Publicity, Elizabeth Loughlin; Musician, Kathleen Feldhausen. Sister Helen Sullivan will continue as Corresponding Secretary and Faculty Sponsor.

James G. White Mathematics Club, University of Kentucky

The following is a summary of the activities of the University of Kentucky *Mathematics Club* for 1950–51. The talks presented were:

The purpose of a mathematics club, by Dr. H. H. Downing

Mathematical squares, by Bill West

A story in mathematical terms, by Annette Siler

Mathematics for the million, by Gene Hagan

The mathematical aspects of weather, by Prof. Cordell Moore

Magic tricks, by Garland Dummitt

Topics on the early history of mathematics, by Dr. Aughtum Howard

Tisserand's Criterion, Gegenschein, and Trojan Asteroids, by Dr. Morris Davis.

Mathematics and logic, by Carl Faith.

The club joined with *Pi Mu Epsilon* in sponsoring a picnic for students and faculty.

The officers for the year are: President, Bill West; Vice-President, Roy Ellis; Secretary-Treasurer, Annette Siler; Faculty Advisor, Dr. James A. Ward.

Mathematics Club, Carleton College

The monthly meetings of the Carleton College *Mathematics Club* for the year 1950–51 included the following:

Theory of finite differences, by Mr. George Francis

The fourth dimension, by Norman Johnson

The mathematics of radiochemistry, by Dr. A. J. Miller

Introduction to spectral theory, by Mr. Winston Crum

Radiative transfer in stellar atmospheres, by A. F. Cook.

Dr. Warren Loud of the University of Minnesota spoke at the Annual banquet in January. His subject was *Curves of pursuit*.

The April meeting was conducted by the students with the theme *Mathematic for recreation*. Mathematical puzzles, charades, games, and a mathematics spelldown were included in the evening's program.

Officers elected for 1951–52 are: President, Gordon Grant; Vice-President-Treasurer, Norman Johnson; Secretary, Helen Heinzen; Faculty Adviser, Mr. Winston Crum.

NEWS AND NOTICES

EDITED BY EDITH R. SCHNECKENBURGER, University of Buffalo

Readers are invited to contribute to the general interest of this department by sending news items to Edith R. Schneckenburger, University of Buffalo, Buffalo 14, New York. Items must be submitted at least two months before publication can take place.

PERSONAL ITEMS

Professor R. M. Foster, head of Department of Mathematics of Polytechnic Institute of Brooklyn, represented the Association at the inauguration of President J. H. Davis of Stevens Institute of Technology on October 12, 1951.

Assistant Professor V. L. Klee, Jr., of the University of Virginia has been appointed National Research Fellow and is spending the year at the Institute for Advanced Study.

The following have been appointed to positions at the Armed Forces Security Agency, Washington, D. C.: Mr. Joseph Blum, previously mathematician at the National Bureau of Standards, as analyst; Mr. F. T. Dresser of the University of Virginia as analyst; and Associate Professor W. R. Murray, Wagner College, as mathematician.

Dr. C. M. Ablow of Brown University, Associate Professor R. E. Gaskell of Iowa State College of Agriculture and Mechanic Arts, and Assistant Professor W. M. Stone, Oregon State College, have accepted positions at Boeing

Airplane Company, Seattle, Washington.

The following appointments have been made at the Los Alamos Scientific Laboratory, New Mexico: Assistant Professor O. W. Rechard of Ohio State University and Associate Professor Andrew Sobczyk, who is on leave of absence from Boston University, have been appointed Staff Members; Graduate Student I. J. Cherry of Oregon State College, Mr. Stewart Schlesinger, formerly a student at Illinois Institute of Technology, and Mr. F. J. Wall, previously a mathematical analyst, have been appointed to research assistantships.

At Northrop Aircraft, Incorporated, Hawthorne, California: Mr. G. W. Fairchild, previously a graduate student at the University of California at Los Angeles, has been appointed an electronics laboratory analyst; Dr. G. E. Gourrich has been appointed Research Engineer.

The Woodward Governor Company, Rockford, Illinois announces: Miss Beverly Marshall of the University of Illinois and Mr. Howard Capwell have been appointed Mathematicians; Dr. Rufus Oldenburger is serving as Chief Mathematician.

Mr. W. R. Allen of the Undergraduate Division of the University of Illinois has a position as Mathematician at the University of Chicago.

Assistant Professor R. D. Anderson of the University of Pennsylvania is a member of the Institute for Advanced Study for the year 1951–52.

Graduate Assistant Jean M. Baldwin of the University of Oklahoma has been appointed to an instructorship at Lamar State College of Technology.

Associate Professor D. H. Ballou of Middlebury College is on leave of absence for the year 1951–52 and has received an appointment as Visiting Associate Professor at Yale University in the program of Internships in General Education sponsored by the Carnegie Foundation.

Assistant Professor Joshua Barlaz of Rutgers University has been promoted to an associate professorship.

Mr. Alexander Basil, formerly a student at Hofstra College, is now an engineer at the Airborne Instruments Laboratories, Mineola, New York.

Mr. R. W. Beals, Jr., of Alfred University is teaching at Cattaraugus Central School, New York.

Associate Professor R. F. Bell of Eastern Washington College of Education has been appointed Head of the Department of Mathematics and Physics.

Mr. K. S. Bergman, previously a student at Gonzaga University, has been appointed a field assistant with the United States Geological Survey, Spokane, Washington.

Dr. D. W. Blackett, graduate assistant at Princeton University, has been promoted to the position of Research Associate.

Mr. H. F. Blasch, formerly a student at Hofstra College, is now in military service.

Instructor G. M. Bloom of Northwestern University has been appointed to an assistant professorship at Miami University.

Miss Dorothy C. Breynaert, who has been a student at the University of

New Hampshire, is now an actuarial student at Monarch Life Insurance Company, Springfield, Massachusetts.

Graduate Student E. B. Bridgforth of the University of Chicago has been appointed Statistician in the Nutrition Division of the Medical School, Vanderbilt University.

Assistant Professor N. A. Brigham of the University of Maryland has been appointed Mathematician in the Applied Physics Laboratory, Johns Hopkins University.

Mr. H. H. Brown, who has been a research engineer at Franklin Institute Laboratories, Philadelphia, Pennsylvania, has accepted a position at the Armour Research Foundation of Illinois Institute of Technology.

Mr. R. G. Brown, formerly research engineer at the Willow Run Research Center, University of Michigan, is now Project Engineer.

Graduate Student M. O. Burrell of Emory University is now a civilian trainer in the Department of the Air Force, Warner Robins Air Force Base, Georgia.

Professor C. H. Butler of Western Michigan College of Education has been appointed Head of the Department of Mathematics.

Mr. R. K. Butz has been appointed to an instructorship at the University of Georgia.

Assistant Professor E. L. Canfield of Drake University has been promoted to an associate professorship.

Associate Professor W. R. Cashen of the University of Alaska has been promoted to a professorship.

Dr. G. Y. Cherlin, who has been an instructor at Rutgers University, is now an actuarial student at Mutual Benefit Life Insurance Company, Newark, New Jersey.

Assistant Professor Nathaniel Coburn of the University of Michigan has been promoted to an associate professorship.

Assistant Professor Esther Comegys of the University of Maine has been promoted to an associate professorship.

Mr. W. V. Cressy, formerly a student at the University of California at Los Angeles, has accepted a position at Douglas Aircraft Company, El Segundo, California.

Assistant Professor A. W. Davis of Iowa State College has been promoted to an associate professorship.

Miss Anne F. Downey, previously a student at Regis College, has a position as Rate Specialist with the Liberty Mutual Insurance Company, Boston, Massachusetts.

Professor W. I. Dykes, head of the Department of Mathematics of South Texas Junior College, has succeeded the Reverend G. D. Pickens as Dean of the College.

Mr. R. L. Ely, who has been a research engineer with the Pittsburgh-Des Moines Company, is now in military service.

Teaching Fellow J. L. Ericksen of Oregon State College has received an appointment as Mathematician at the Naval Research Laboratory, Washington, D. C.

Professor G. M. Ewing is on leave of absence from the University of Missouri and is serving as Mathematician at Sandia Corporation, Albuquerque, New Mexico.

Assistant Professor A. B. Farnell is on leave of absence from the University of Colorado and has returned to military service.

Mr. Daniel Finkel has a position as Junior Actuary, New York State Insurance Fund, New York, New York.

Mr. E. L. Friedman, formerly a student at Massachusetts Institute of Technology, has accepted a position at the Chandler-Evans Division of the Niles-Bement-Pond Company, West Hartford, Connecticut.

Mr. A. L. Gilmore, Jr., has been appointed Instructor of Mathematics and Physics at Pearl River Junior College, Poplarville, Mississippi.

Mr. R. D. Glauz has been appointed to a research assistantship at Brown University.

Assistant Professor H. E. Goheen of Syracuse University has been appointed to an assistant professorship in the Department of Electrical Engineering of the University of Pennsylvania.

Mr. Isidore Goldman, formerly a mechanical engineer in the United States Naval Shipyard, has accepted a position as Design Engineer at Greer Hydraulics, Incorporated, Brooklyn, New York.

Assistant Professor N. A. Goldsmith of Illinois Wesleyan University has been appointed Head of the Department of Mathematics of Henderson State Teachers College, Arkansas.

Mr. N. E. Golovin has been appointed Assistant Director for Administration of the National Bureau of Standards, Washington, D. C.

Assistant Professor D. B. Goodner of Florida State University has been promoted to an associate professorship.

Mr. D. A. Gorsline, who has been teaching at Cambridge Central School, New York, has accepted a position with the Equitable Life Assurance Society, Albany, New York.

Instructor L. C. Graue of Sacramento State College has been promoted to a professorship.

Assistant Professor R. E. Graves of the University of Minnesota has a position at White Sands Proving Ground, Las Cruces, New Mexico.

Mr. F. D. Grogan, who has been teaching at Minden High School, Louisiana, has accepted a position as Chemical Corps Inspector, Dallas Chemical Procurement District, Texas.

Dr. Emil Grosswald, formerly a special lecturer at the University of Saskatchewan, is now a member of the Institute for Advanced Study.

Assistant Professor S. W. Hahn of Wittenberg College has been appointed to a professorship at Winthrop College.

Associate Professor P. R. Halmos is on leave of absence from the University of Chicago and is at Instituto de Matematica y Estadistics, Montevideo, Uruguay.

Mr. R. C. Haseltine of Swarthmore Center, Pennsylvania State College, has accepted a position as Assistant Research Engineer with the Burroughs Adding Machine Company, Philadelphia, Pennsylvania.

Graduate Assistant Melvin Henriksen of the University of Wisconsin has been appointed to an assistant professorship at the University of Alabama.

Mr. D. B. Hoagland of Trenton Junior College has been appointed Correspondence Instructor at the University of Missouri.

Dr. A. J. Hoffman, who has been a member of the Institute for Advanced Study, is now at the National Bureau of Standards, Washington, D. C.

Instructor R. E. Horton of Los Angeles City College has been recalled to active duty in the United States Air Force.

Mr. J. F. Hudson, previously a graduate student at the University of Tennessee, has a position as Statistical Assistant at the Carbide and Chemical Corporation, Oak Ridge, Tennessee.

Professor G. B. Huff of the University of Georgia gave a series of lectures entitled "a^n" at a colloquium of the University of Kentucky on July 31, August 1–2, 1951.

Miss Grace M. Hyder has accepted a position as Senior Actuarial Clerk for the Commonwealth of Massachusetts, Boston, Massachusetts.

Mr. M. A. Hyman, formerly a mathematician at the Naval Ordnance Laboratory, Washington, D. C., has received an appointment as Fulbright Scholar for the year 1951–52 and is located at Technical University, Delft, Netherlands.

Assistant Professor S. J. Jasper of Kent State University has been appointed to an associate professorship at East Tennessee State College.

Dr. Fritz John, who has been a mathematician at the Institute for Numerical Analysis, has been appointed to a professorship at New York University.

Assistant Professor G. K. Kalisch of the University of Minnesota has been promoted to an associate professorship.

Mr. Sidney Kaplan, a statistician for the Bureau of Employment Security, United States Department of Labor and a lecturer at Catholic University of America, has accepted a position as a research mathematician in the Office of the Comptroller of the Army, Washington, D. C.

Mr. Raymond Kassler, formerly a student at Brooklyn College, has been appointed mathematician at Evans Signal Laboratory, Belmar, New Jersey.

Mr. C. E. Kelley, who has been teaching in the Texas Public Schools, has accepted a position with the New York Life Insurance Company, New York, New York.

Lecturer E. H. Kingsley of Roosevelt College has been appointed mathematician in the Aerial Measurements Laboratory, Technical Institute, Northwestern University.

Mr. R. J. Kohlmeyer of Pratt Institute has been appointed to an instructorship at Lafayette College.

Instructor F. A. Kros of New Mexico College of Agriculture and Mechanic Arts is now a graduate student at the University of Colorado.

Mr. Charles Kurland, formerly a research engineer at Locomotive Development Company, Dunkirk, New York, has accepted a position as Research Engineer, Bell Aircraft Corporation, Niagara Falls, New York.

Graduate Student P. J. Leonard of Boston College has a position as Mathematician in the Applied Mechanics Branch, Watertown Arsenal, Massachusetts.

Research Assistant S. L. Levy of Brown University has been promoted to a research associateship.

Instructor H. D. Lipsich of the University of Cincinnati has been promoted to an assistant professorship.

Assistant Professor B. J. Lockhart of the United States Naval Postgraduate School has been promoted to an associate professorship.

Mr. A. L. Mayerson, previously an actuarial assistant at the Institute for Life Insurance, New York City, has accepted a position as Principal Actuary for the New York State Insurance Department.

Assistant Professor J. J. McCarthy of Rutgers University has a position as Project Engineer at Sperry Gyroscope Company, Great Neck, New York.

Miss Mary L. McLaughlin, formerly a student at Regis College, is now a graduate student at Boston University.

Assistant Professor F. A. McMahon of Manhattan College has accepted a position with the Sperry Gyroscope Company, Lake Success, New York.

Mr. J. O. Neilson, formerly a student at Augustana College, is now in the United States Air Force.

Associate Professor Albert Newhouse of the University of Houston has been promoted to a professorship.

Mrs. Theresa M. Oehmke has accepted a position as a computer in the Air Weapons Research Center of the University of Chicago.

Instructor Daniel Orloff of Southern Illinois University has been promoted to an assistant professorship.

Consulting Physicist N. G. Parke, III, has been named President of Parke Mathematical Laboratories, Incorporated, Concord, Massachusetts.

Mr. J. M. Patterson of New York University has accepted a position as Mathematician at Bendix Aviation Corporation, Detroit, Michigan.

Dr. W. D. Peoples, Jr., formerly a graduate assistant at the University of Georgia, has been appointed to an assistant professorship at Howard College.

Instructor K. S. Phelan of St. John's University has been promoted to an assistant professorship.

Assistant Professor J. W. Ponds of West Virginia State College is now a mathematician in the United States Air Force.

Professor C. A. Reagan of Friends University has retired with the title of Professor Emeritus.

Instructor C. L. Riggs of Kent State University has been appointed to an assistant professorship at East Texas State Teachers College.

Associate Professor Helen G. Russell of Wellesley College has been promoted to a professorship.

Mr. W. T. Sharp has a position as Assistant Research Officer of the National Research Council of Canada.

Mr. R. L. Shively, previously a teaching fellow at the University of Michigan, has been appointed to an instructorship at Western Reserve University.

Associate Professor F. C. Smith of the College of St. Thomas has been promoted to a professorship.

Mr. W. I. Steinkamp, formerly a student at Gonzaga University, is now in military service.

Reverend R. J. Swords of Weston College is a student at Rathfornham Castle, Dublin, Ireland.

Dr. D. L. Thomsen, Jr., research fellow at the California Institute of Technology, has been appointed a research engineer at the Jet Propulsion Laboratory, California Institute of Technology.

Assistant Professor C. W. Topp has a year's leave of absence from Fenn College and is a research associate at Case Institute of Technology.

Associate Professor H. L. Turrittin has returned to the University of Minnesota after spending a sabbatical year on the staff of Princeton University.

Professor Jacob Wolfowitz of Columbia University has been appointed to a professorship at Cornell University.

Instructor L. J. Zimmerman of Goshen College has been promoted to an assistant professorship.

Professor Emeritus W. T. Burns of Xavier University died on July 23, 1951.

THE MATHEMATICAL ASSOCIATION OF AMERICA

Official Reports and Communications

THIRTY-SECOND SUMMER MEETING OF THE ASSOCIATION

The thirty-second summer meeting of the Mathematical Association of America was held at the University of Minnesota, Minneapolis, Minnesota on Monday and Tuesday, September 3–4, 1951, in conjunction with the summer meetings of the American Mathematical Society, the Institute of Mathematical Statistics, the Econometric Society, and Section A of the American Association for the Advancement of Science. A total of six hundred and twenty-three persons were registered, including the following two hundred and sixty-nine members

of the Association:

A. A. Albert, E. W. Anderson, R. D. Anderson, H. A. Antosiewicz, K. J. Arnold, C. F. Barr, M. A. Basoco, W. D. Baten, Helen P. Beard, E. G. Begle, J. H. Bell, Theodore Bennett, Brother Bernard Alfred, R. H. Bing, H. D. Block, K. H. Bracewell, J. W. Bradshaw, R. W. Brink, Charles Brumfield, R. C. Buck, L. J. Burton, L. E. Bush, J. H. Bushey, Jewell H. Bushey, W. H. Bussey, E. A. Cameron, R. H. Cameron, E. J. Camp, C. S. Carlson, Elizabeth Carlson, R. E. Carr, Abraham Charnes, Harold Chatland, S. S. Chern, E. W. Chittenden, A. G. Clark, F. Marion Clarke, H. D. Colson, C. H. Cook, T. F. Cope, A. H. Copeland, H. M. Cox, W. F. Crum, J. F. Daly, Marian E. Daniells, P. H. Daus, B. K. Dickerson, Flora Dinkines, M. H. Dipert, P. S. Dwyer, J. M. Earl, W. F. Eberlein, P. D. Edwards, H. P. Evans, F. D. Faulkner, H. E. Fettis, A. M. Feyerherm, C. H. Fischer, I. C. Fischer, Walter Fleming, E. E. Floyd, R. S. Fouch, H. D. Friedman, J. S. Frame, Abraham Franck, W. B. Fulks, W. R. Fuller, A. E. Gault, H. M. Gehman, Landis Gephart, C. B. Germain, Gladys Gibbens, W. M. Gilbert, R. E. Gilman, J. W. Givens, V. D. Gokhale, Michael Goldberg, D. B. Goodner, S. H. Gould, Arthur Grad, L. M. Graves, L. J. Green, V. G. Grove, Edwin Halfar, Mary E. Haller, P. C. Hammer, Frank Harary, W. L. Hart, Charles Hatfield, Charles Hatfield, Jr., C. E. Heilman, Ruth E. Henning, I. N. Herstein, M. R. Hestenes, A. J. Hoffman, R. V. Hogg, F. E. Hohn, D. L. Holl, Carl Holtom, E. Marie Hove, H. M. Hughes, Ralph Hull, M. Gweneth Humphreys, W. A. Hurwitz, W. H. Ito, C. G. Jaeger, G. H. Jaeger, D. A. Johnson, L. W. Johnson, E. R. Johnston, B. W. Jones, W. C. Kalinowski, G. K. Kalisch, Irving Kaplansky, Samuel Karlin, J. L. Kelley, L. M. Kells, D. E. Kibbey, E. C. Kiefer, V. L. Klee, Jr., Fulton Koehler, Max Kramer, W. H. Kruskal, O. E. Lancaster, R. E. Langer, E. H. Larguier, Rose Lariviere, L. S. Laws, W. G. Leavitt, J. R. Lee, D. H. Lehmer, F. C. Leone, C. H. Lindahl, Lee Lorch, W. S. Loud, C. C. MacDuffee, Saunders MacLane, H. M. MacNeille, Morris Marden, Margaret P. Martin, Ella Marth, Kenneth May, J. R. Mayor, Dorothy McCoy, S. W. McCuskey, W. C. McDaniel, W. R. McEwen, R. B. McHugh, J. V. McKelvey, Martha M. McKelvey, J. C. C. McKinsey, E. J. McShane, L. E. Mehlenbacher, Paul Meier, B. E. Meserve, D. M. Mesner, E. J. Mickle, J. J. Miller, W. H. Mills, Deane Montgomery, F. C. Mosteller, H. T. Muhly, Sigurd Mundhjeld, O. M. Nikodym, M. L. Norden, M. A. Nordgaard, E. A. Nordhaus, M. J. Norris, Rufus Oldenburger, Arthur Ollivier, J. M. H. Olmsted, Emma J. Olson, T. G. Ostrom, Margaret Owchar, O. J. Peterson, B. J. Pettis, H. P. Pettit, C. G. Phipps, George Piranian, A. E. Pitcher, J. C. Polley, George Polya, J. E. Powell, G. B. Price, G. C. Priester, F. M. Pulliam, A. L. Putnam, Tibor Rado, Henry Rainbow, J. F. Randolph, Ruth B. Rasmusen, L. T. Ratner, G. E. Raynor, M. O. Reade, W. T. Reid, Haim Reingold, H. B. Ribeiro, L. A. Ringenberg, D. D. Rippe, E. K. Ritter, J. H. Roberts, Fred Robertson, H. A. Robinson, P. C. Rosenbloom, Arthur Rosenthal, A. E. Ross, E. H. Rothe, Evelyn C. Rusk, Hans Samelson, A. C. Schaeffer, Henry Scheffe, Edith R. Schneckenburger, Nathan Schwid, R. R. Seeber, Jr., E. B. Shanks, C. R. Sherer, L. W. Sheridan, Harold Shniad, Sister M. Mercedes, Sister M. Prudentia, Sister M. Seraphim, Sister M. Teresine, M. F. Smiley, A. J. Smith, F. C. Smith, Ernst Snapper, W. S. Snyder, Andrew Sobczyk, Elizabeth Sokolnikoff, E. J. Specht, Abraham Spitzbart, R. C. Staley, W. L. Stamey, O. E. Stanaitis, H. E. Stelson, R. C. Stephens, M. H. Stone, Irwin Stoner, E. B. Stouffer, J. V. Talacko, P. Y. Tani, H. P. Thielman, G. H. M. Thomas, J. M. Thomas, Marian W. Thornton, Ella Thorp, C. B. Tompkins, Leonard Tornheim, J. I. Tracey, A. W. Tucker, J. W. Tukey, H. L. Turrittin, E. P. Vance, V. J. Varineau, E. C. Varnum, Bernard Vinograde, John von Neumann, J. A. Ward, S. E. Warschawski, K. W. Wegner, W. J. Wells, R. L. Westhafer, Louise A. Wolf, G. N. Wollan, R. S. Wollan, Y. K. Wong, J. W. T. Youngs, R. A. Zemlin, Antoni Zygmund.

Sessions of the Association were held on Monday afternoon and Tuesday morning in Room 150, Physics Building of the University of Minnesota. Vice-President L. M. Graves presided at the first session and President Saunders MacLane at the second session. The Program Committee for the meeting consisted of M. A. Basoco, Chairman, R. C. F. Bartels, and L. E. Bush.

FIRST SESSION OF THE ASSOCIATION

General Topic: "The Teaching of Undergraduate Collegiate Mathematics."

Address: "Let Us Teach Guessing," by Professor George Polya, Stanford University.

Panel discussion on the content of the undergraduate mathematical curriculum. Members of the panel: Professor M. R. Hestenes, University of California (Los Angeles) and National Bureau of Standards; Professor B. W. Jones, University of Colorado; Professor G. K. Kalisch, University of Minnesota; and Professor A. L. Putnam, University of Chicago.

Discussion and questions from the floor.

SECOND SESSION OF THE ASSOCIATION

"The Seven Functions of Ramanujan," by Dr. D. H. Lehmer, National Bureau of Standards.

"Applications of the Concept of Fairness of a Game to the Theory of Probability," by Professor J. L. Doob, University of Illinois.

Retiring Presidential Address: "The Things I Should Have Done, I Did Not Do," by Professor R. E. Langer, University of Wisconsin.

MEETING OF THE BOARD OF GOVERNORS

The Board of Governors of the Association met on Monday evening in the Conference Room of Pioneer Hall. Nineteen members of the Board were present. Among the more important items of business transacted were the following:

The resignation of E. N. Oberg as a member of the Board of Governors from the Iowa Section was accepted and W. M. Davis of Cornell College was elected to serve the balance of Professor Oberg's term. Professor W. B. Carver was reelected member of the Finance Committee for the four-year term, 1952–1955.

It was voted that the bequest to the Association by the late Professor Otto Dunkel be used to establish the "Dunkel Fund" of the Association. A committee was authorized to proceed with the publication of a book of problems from the MONTHLY to appear as a supplement to the MONTHLY. The cost of this supplement is to be charged against the Dunkel Fund. It was deemed appropriate that the book of problems be published as a memorial to Professor Dunkel because of his long association with the Problems Section of the MONTHLY.

It was voted to hold the 1952 summer meeting at Michigan State College, East Lansing, Michigan on September 1–2, 1952.

The following resolution was adopted:

"RESOLVED: The Board of Governors of the Association affirms its steady intention to conduct the scientific meetings, social gatherings, and other affairs of the Association so as to promote the interests of Mathematics without discrimination as to race, creed, or color. The President of the Association is hereby authorized and requested to determine the best means for avoiding dis-

crimination, by consultation on this subject with the various chairmen and secretaries of sections and other appropriate members of the Association, and to report the results of this consultation to the Board."

MEETING OF SECTION OFFICERS

A meeting of officers of the Sections of the Association was held on Tuesday evening in the Conference Room of Pioneer Hall. Representatives were present from twenty-two of the twenty-five Sections of the Association.

Professor H. A. Robinson described the programs of the Southeastern Section of the Association. Several other section officers described the programs of their respective sections. Comments were made regarding reports of section meetings and arrangements for scheduling meetings of the sections.

The relationship of a sectional governor to his section was discussed. It was the consensus of the group that if a sectional governor moves permanently from the geographical area of the section from which he was elected, it would be best for him to resign as a sectional governor.

Special activities of several sections were discussed. Brother Bernard Alfred reported on the contests for high school students conducted by the Metropolitan New York Section. Professor R. L. Westhafer described the plan of the Southwestern Section for sponsoring a traveling lecturer.

MEETINGS OF OTHER ORGANIZATIONS

The sessions of the American Mathematical Society began on the afternoon of Tuesday, September 4, and continued through Friday afternoon, September 7. The colloquium lectures on "Topological Transformation Groups" were delivered by Professor Deane Montgomery of the Institute for Advanced Study. Invited addresses were given by Professor R. H. Bing of the University of Wisconsin on "Partitioning Continuous Curves" and by Professor G. W. Whitehead of the Massachusetts Institute of Technology on "Homotopy Theory."

The Institute of Mathematical Statistics and the Econometric Society met jointly and separately beginning on Tuesday morning and continuing through Friday afternoon. At a joint session on Thursday afternoon, Professor Harold Hotelling of the University of North Carolina delivered the Rietz Memorial Lecture on the subject: "The Behavior of Standard Statistical Tests under Non-Standard Conditions." Three joint sessions were devoted to a "Symposium on Games, Decision Problems and Related Topics."

Section A of the American Association for the Advancement of Science held a single session on Wednesday morning at which Professor E. J. McShane of the University of Virginia delivered his retiring vice-presidential address on "Order Preserving Mappings of Partially Ordered Spaces."

ARRANGEMENTS, ENTERTAINMENT AND RECREATION

The Committee on Arrangements for the meeting consisted of J. M. H. Olmsted, Chairman, R. H. Cameron, H. M. Gehman, W. L. Hart, G. K.

Kalisch, Fulton Koehler, W. S. Loud, S. E. Warschawski, and J. W. T. Youngs.

Registration headquarters was in the lobby of Pioneer Hall. Rooms in Pioneer and Centennial Halls of the University of Minnesota were available to all attending the meetings and to their families from Sunday afternoon, September 2, until noon on Saturday, September 8. Meals were served in the cafeteria of Coffman Memorial Union.

On Tuesday evening an informal reception was held at the Campus Club under the sponsorship of the mathematical organizations and the College of St. Thomas. Arrangements were made for dancing, bridge and billiards during the evening.

On Wednesday evening a banquet for the mathematical organizations was held in the main ballroom of Coffman Memorial Union. Professor W. A. Hurwitz of Cornell University acted as toastmaster. President Saunders MacLane presented the greetings of the Association and spoke on the general topic of fashions in mathematics. President John von Neumann spoke on behalf of the American Mathematical Society. President P. S. Dwyer responded for the Institute of Mathematical Statistics and Professor Gerhard Tintner for the Econometric Society. Professor Ralph Hull of Purdue University presented a resolution expressing the thanks of the mathematical organizations to the officers of the University of Minnesota for making the facilities of the University available for the 1951 summer meeting.

On Thursday evening a recital of early keyboard music was given by Emilie Pray, concert pianist and instructor at Macalester College. The recital was held in Scott Hall on the University of Minnesota Campus.

In addition to the entertainment described above, special arrangements were made for the women attending the meetings. On Tuesday afternoon the Betty Crocker kitchens of the General Mills Corporation were visited. On Thursday a luncheon was held at the country club house of the Automobile Club of Minneapolis, overlooking the valley of the Minnesota River.

A particular debt of gratitude is due to the local members of the Committee on Arrangements for their untiring efforts to care for the visiting mathematicians.

H. M. GEHMAN, *Secretary-Treasurer*

MARCH MEETING OF THE MICHIGAN SECTION

The March meeting of the Michigan Section of the Mathematical Association of America was held on March 24, 1951, in the new Mathematics-Physics Building at Michigan State College in East Lansing in conjunction with the meetings of the Michigan Academy of Science, Arts, and Letters. Professor D. C. Morrow, Chairman of the Section, presided at both morning and afternoon meetings as well as at the business meeting held immediately following the luncheon in the Union. Professor J. H. Bell acted as local assistant chairman, helping the secretary make local arrangements.

The Michigan State College Mathematics Department served refreshments

in their seminar and meeting room, and their over-all activity as good hosts was recognized in a unanimously passed vote of thanks at the conclusion of the afternoon meeting.

A total of sixty-five persons attended the meeting including the following forty-eight members of the Association, N. H. Anning, J. W. Baldwin, J. M. Barbour, R. C. F. Bartels, W. D. Baten, F. A. Beeler, J. H. Bell, J. E. Bellardo, C. H. Butler, R. V. Churchill, C. J. Coe, P. C. Cox, J. W. Coy, C. C. Craig, D. A. Darling, D. E. Deal, P. W. Edmonson, Meta M. Ewing, V. G. Grove, H. H. Hannon, Gerald Harrison, Fritz Herzog, E. H. C. Hildebrandt, T. H. Hildebrandt, E. E. Ingalls, L. G. Johnson, P. S. Jones, Wilfred Kaplan, L. M. Kelly, H. D. Larsen, K. B. Leisenring, W. J. McClintock, L. E. Mehlenbacher, D. C. Morrow, H. W. Nace, A. L. Nelson, J. D. Novak, Mrs. Mary H. Payne, George Piranian, J. E. Powell, G. Y. Rainich, M. O. Reade, C. C. Richtmeyer, T. H. Southard, E. M. Steinbach, Leonard Tornheim, C. P. Wells, J. L. Wilson.

At the business meeting the reports of the Secretary and Treasurer were read and approved. The latter report included a payment of ten dollars to the CARE book fund for European university libraries, and a note that the Section had made a profit of approximately fifty dollars on sales of its college guidance pamphlet and charts. This profit did not include charges for the first printing of 2,000 copies which were distributed free.

The nominating committee, which consisted of Professors L. E. Mehlenbacher, V. G. Grove, J. W. Baldwin, reported the names of Professor N. H. Anning for Chairman and Professor P. S. Jones for Secretary-Treasurer for 1951–52. They were elected unanimously. It was announced that Professor J. S. Frame had been elected Governor for his region in the election conducted by the Secretary of the Association. The 1952 meeting will be held on April 12 at the University of Michigan in Ann Arbor, Michigan.

The following papers were presented at the morning and afternoon sessions. Due to illnesses and other difficulties, papers 12, 13 and 17 were presented by title.

1. *Variances of the difference between means when there are two missing values*, by Professor W. D. Baten, Michigan State College.

This paper demonstrated how to determine the variances between "treatment" means when there are two missing values in a randomized block design. Estimates of the missing values were obtained by minimizing the error sum of square. Variances of the various differences of two means were obtained by employing the variances of the individual plot values in such a way as to eliminate dependence between the terms of these differences. The variances were obtained between means containing estimates of the missing values and those that do not contain either one of these estimates, as well as the variances of the differences between two means when each contains an estimate of a missing plot value. These variances were compared with variances when there were no missing data.

2. *An 18th century approximation to the equally tempered scale*, by Professor J. M. Barbour, Music Department, Michigan State College.

Strähle's rough approximation to the powers of the 12th root of 2 featured the intersections of a transversal with 13 lines dropped from an external point of equally spaced points upon a line.

With a change in the angle of the transversal, this becomes an excellent approximation.

3. *Conformal mappings and Peano curves*, by Professor George Piranian, Dr. C. J. Titus, Professor G. S. Young, University of Michigan.

Salem and Zygmund [*Duke Math. J.* 12 (1945), 569–578] have proved that certain gap series $\sum a_k z^{n_k}$ with $\sum |a_k| < \infty$ map the unit circle into a Peano curve. The authors construct a class of functions whose Taylor series converge uniformly but not absolutely on the unit circle, and which map the unit circle onto the closure of any preassigned bounded domain.

4. *An abelian semi-group of transformations of Hausdorff type*, by Mr. George Brauer, student, University of Michigan, introduced by Professor George Piranian.

This paper dealt with the semi-group of Loeplitz matrices of the form $\epsilon^{-1}\nu\rho$ where ν is an arbitrary diagonal matrix and where the Euler matrix ρ, used by Hausdorff, has been replaced by the Cesaro matrix of order one. Regularity conditions and convergence fields of these matrices were discussed.

5. *A survey of the industrial subsidy of the study of applied mathematics*, by Professor T. H. Southard, Wayne University and the Industrial Mathematics Society.

In the spring of 1950 the Industrial Mathematics Society's Education Committee asked twenty-odd industrial concerns and technical government agencies for information on the kinds of courses in "applied mathematics" (including mathematics courses beyond the calculus as well as "refresher" calculus, and such courses as hydrodynamics, elasticity, etc.) which they were offering their employees on the premises or elsewhere. Information was also sought concerning any subsidies (in the form of time-off, money, tuition, etc.) the concerns offered their employees while attending these courses. This was a report on the 20 replies received.

6. *An inverse two dimensional problem in elasticity*, by Mr. L. G. Johnson, Research Laboratories Division, General Motors Corporation.

In determining the residual stresses in a plate, the usual procedure is to remove thin strips from the plate and then to determine the residual stresses in the strips. However, the strip stresses will be different from the original plate stresses, and in order to convert strip stresses into plate stresses, it becomes necessary to solve a boundary value problem involving the biharmonic partial differential equation. The author discussed this boundary value problem and a method of arriving at some satisfactory solutions.

7. *The order of approximation of real numbers by rationals*, by Professor Fritz Herzog, Michigan State College.

For real x and positive α, let $K(x:\alpha) \equiv \liminf q^{\alpha} |x-p/q|$, taken over all rational $p/q \neq x$ with $q>0$. For each x, there is a unique value γ such that $K(x:\alpha)=0$ for $\alpha<\gamma$ and $K(x:\alpha)=\infty$ for $\alpha>\gamma$. This number $\gamma=\gamma(x)$ is called the order of approximation of x. If x is itself rational then $\gamma(x)=1$. If x is irrational let b_n denote the nth partial quotient (denominator) and A_n/B_n the nth convergent of the simple continued fraction for x. Then

$$\gamma(x) = 2 + \limsup \frac{\log b_{n+1}}{\log B_n}$$

This formula enables us to study the function $y=\nu(x)$. From the set-theoretical viewpoint, the function is highly "pathological"; it assumes any given value of y, $2 \leqq y \leqq \infty$, on a set of values x which has the power of the continuum in any real interval. On the other hand, from the measure theoretical viewpoint, the function is extremely "normal"; in fact, $\gamma(x)=2$ for almost all x.

8. *Normalized coordinate vectors as a shorthand for Descartes*, by Dr. K. B. Leisenring, University of Michigan.

The point calculus of Grassman as applied to geometry by Forder (*Calculus of Extension*, Cambridge, 1941) is, in part, in the case of two dimensions formally identical with the algebra of a three-dimensional vector space, using vector and scalar products. Metrical properties, however, require a considerable elaboration of concepts not so familiar. If we treat ordinary homogeneous coordinates for the plane as coordinate vectors we can obtain a neat handling of metrical properties simply by suitable normalization.

A point vector is normal if its third component is unity; a line vector is normal if the sum of the squares of the first two components is unity. The line through A, B is represented by the vector product $[AB]$; the point of intersection of p and q by $[p\ q]$. The basic metric relations are: if A and B are normal then $\pm[A\ B]$ divided by the distance $\overline{AB}$ is normal; if p and q are normal then $[p\ q]/\sin(p\ q)$ is normal.

If A and B are normal, $P=(\lambda A+\mu B)/(\lambda+\mu)$ is normal and gives a barycentric scale on the line. A-B is not normalizable and represents the point at infinity on AB which gives a simple handling of parallelism. The vector $S=(0, 0, 1)$ representing the unnormalizable line at infinity is useful in handling perpendicularity. The perpendicular to AB through P has a vector proportional to $A-B-S\ (A\cdot P-B\cdot P)$, A, B, P being normal.

A number of metrical theorems were shown to be equivalent to identities in these normalized vectors and two fundamental theorems on conics were obtained.

9. *On a theorem of Borsuk*, by Professor L. M. Kelly, Michigan State College.

In 1932 Borsuk proved that any decomposition of a sphere of diameter one in n-dimensional euclidean space into n mutually exclusive sets, leaves at least one of the "pieces" of diameter one. This note offered an elementary proof of this result in E_3. The proof was based on a map coloring idea (probably not new) and does not generalize to E_N.

10. *What shall we do with mathematics?* by Professor E. H. C. Hildebrandt, Northwestern University.

In his invited one hour address, Professor Hildebrandt listed as some of the most serious problems in mathematics education, including the postponement of the beginning of elementary work, the cutting off at the top of the more advanced secondary school courses, policies of excessive regular chronological promotion, the lack of equipment and teaching aids in mathematics classrooms, inadequately varied teaching methods, inadequately trained teachers especially on the elementary level.

Professor Hildebrandt suggested that the Association can work to improve this situation by seeking and stimulating the better students through junior academies of science and contests, by improving the guidance work done with all students through publication of information about mathematics and its uses, by improving teaching through preparation of books, lectures, teaching aids, and by working for improved teacher training programs and requirements for the certification of teachers.

A lively discussion followed Professor Hildebrandt's talk.

11. *The solving systems of linear equations in college algebra*, by Professor Gerald Harrison, Wayne University.

It was proposed that in the course commonly called college algebra the method of elimination for solving systems of any number of linear equations in any number of unknowns be carried through without the intervention of determinants. The step-wise method of elimination can be easily justified to the student. He should be given the appropriate notation of "arrays" or "matrices" and shown how to systematically apply the elementary row operations to the array to ob-

tain the solution of any system of linear equations. This manner of solving system of linear equations has the advantages of unifying the subject, of being easily understandable to the student, of being a useful computational method in many cases superior to the determinantal method, and of convincing the student that a system of many linear equations in many unknowns is not in any essential respect more difficult to solve than a system to a few equations in a few unknowns.

12. *By scissors from n squares to one*, by Professor B. M. Stewart, Michigan State College.

For any given positive integer, there exists a positive integer k (for example, $k=8^n$), presumably dependent on n, such that if each of n unit squares is dissected in exactly the same way into k convex parts, then the kn pieces thus obtained may be composed, under rotation and translation, but excluding reflection, into one square of area n. The problem proposed, but still open, is that of deciding whether there is a k independent of n. The best result found is that $k=3$ will suffice if n is the sum of two squares of integers. Many other special results are obtained.

13. *A simple device for empirical sampling from an arbitrary distribution*, by Professor H. W. Alexander, Adrian College.

By the use of an appropriately constructed circular scale and a freely-spinning pointer, random samples may be drawn from an arbitrary population. This device may be used to illustrate the genesis of various well-known distributions derived from the normal distribution, and to study empirically various methods of industrial quality control.

14. *A problem of geometry suggested by G. B. Price*, by Professor Wilfred Kaplan, University of Michigan.

The following question was raised by Professor G. B. Price in a letter to Professor Kaplan: Is every homeomorphism of the disc $x^2+y^2<1$ into the plane which maps line segments onto line segments necessarily a collineation? This is answered affirmatively by the following reasoning. Let ABC be a triangle in the given disc and let D, E, F be the midpoints of AB, AC and BC. The given homeomorphism T can be followed by a collineation T_1 such that T_1T leaves A, B, C, D, E, fixed. Further, T_1T takes line segments to line segments. From these properties one deduces that T_1T must also leave F fixed, as well as the midpoints of AD, DB, AE, EC. By induction one now establishes that T_1T leaves fixed each point of a set dense on the periphery of ABC; hence, by continuity, T_1T leaves every point on the periphery fixed. From this one concludes that T_1T leaves all points of the disc fixed. Thus T is simply the inverse of T_1, is hence a collineation.

15. *On a method of averaging*, by Professor D. A. Darling, University of Michigan.

Some convexity arguments were used to define a general method of averaging, and to deduce conditions under which the averages can be ordered. Some of the classical inequalities appeared as special cases.

16. *Matrix equations*, by Professor J. H. Bell, Michigan State College.

This paper summarized the algorithm of M. H. Ingraham for the solution of the unilateral matrix equation together with a generalization to the case in which the coefficients are not square. The methods were then extended to solve a unilateral direct product equation and to indicate a method of solution of simultaneous unilateral equations.

17. *Configuration problems in distance spaces*, by Professor E. A. Nordhaus, Michigan State College.

Configurations having specified properties arising from point sets in metric or semi-metric

spaces were studied. A generalization of a theorem due to Erdös and Anning was given, and related questions and unsolved problems were discussed.

P. S. JONES, *Secretary*

APRIL MEETING OF THE OHIO SECTION

The thirty-fifth annual meeting of the Ohio Section of the Mathematical Association of America was held at the Ohio State University, Columbus, Ohio, on Saturday, April 21, 1951. Professor V. C. Stechschulte, Chairman of the Section presided at the morning and afternoon sessions.

One hundred seven persons registered attendance, including the following seventy-nine members of the Association: J. E. Adney, Jr., C. E. Amos, W. E. Anderson, P. R. Annear, D. F. Atkins, Grace M. Bareis, H. M. Beatty, William Beck, Theodore Bennett, Foster Brooks, C. D. Calhoon, Dorothy I. Carpenter, W. G. Clark, E. H. Clarke, W. F. Cornell, H. K. Crowder, Grace M. Cutler, Wayne Dancer, L. E. Davis, Jr., R. C. Davis, Violet B. Davis, R. E. Dowds, R. H. Downing, B. B. Dressler, P. L. Evans, H. E. Fettis, B. E. Gatewood, G. R. Glabe, E. L. Godfrey, S. W. Hahn, H. G. Harp, R. G. Helsel, I. N. Herstein, P. S. Herwitz, J. S. Hokanson, R. Y. Iwanchuk, S. J. Jasper, E. D. Jenkins, Margaret E. Jones, O. C. Juelich, John Kaiser, Chosaburo Kato, D. M. Krabill, D. H. Kraft, H. D. Lipsich, L. L. Lowenstein, R. H. Marquis, H. R. Mathias, C. C. Morris, O. M. Nikodym, Helen Olney, Emma J. Olson, S. R. Orr, C. F. Pinzka, H. S. Pollard, Tibor Rado, O. W. Rechard, Dorothy Renzema, R. F. Rinehart, D. T. Ross, S. A. Rowland, H. J. Ryser, K. C. Schraut, Samuel Selby, Sister M. Constantia, E. T. Stapleford, V. C. Stechschulte, Andrew Sterrett, Michael Tikson, H. E. Tinnappel, H. S. Toney, W. R. Transue, E. P. Vance, E. H. Wang, D. R. Whitney, R. B. Wildermuth, F. B. Wiley, C. O. Williamson, Alberta Wolfe.

The following officers were elected for the coming year: Chairman, R. F. Rinehart, Case Institute of Technology; Secretary-Treasurer, Foster Brooks, Kent State University; Member of the Executive Committee, L. H. Miller, The Ohio State University; Program Committee, H. R. Mathias, Bowling Green State University (Chairman), W. R. Transue, Kenyon College, L. L. Lowenstein, Kent State University.

The following papers were presented:

1. *Some mathematics in seismology*, by Professor V. C. Stechschulte, Xavier University.

Mathematical techniques of many kinds find application in various phases of seismology. Engineering seismology and instrumental seismology were mentioned as utilizing particularly the mathematics of vibrating systems. After a brief discussion of the various types of earthquake waves and of the character of the data supplied by seismograms, the contribution of statistical methods in seismology was described. Particular attention was given to the method wherein the solution of an integral equation (essentially of the kind involved in Abel's problem) leads to the determination of the velocity of earthquake waves at various depths within the earth. Some geometrical methods by which data from deep-focus earthquakes were made particularly useful for this velocity-depth problem were discussed.

2. *Canonical forms for matrices with elements in a finite field*, by Mr. Glenn Stahly, The Ohio State University, introduced by Professor D. R. Whitney.

An outline of the main results obtained in the study of matrices with elements in various fields was given. In particular, some results of L. E. Dickson's concerning congruence of matrices with elements in finite fields were put in more modern mathematical language. First, some definitions were given, after which congruence was discussed for matrices in various fields. Then number theory was considered, for some of these results were needed in succeeding work. Finally, canonical forms for matrices with elements in finite fields were obtained.

3. *Simple difference sets* (mod v) *for v less than* 2,560,000, by Mr. T. A. Evans and Professor H. B. Mann, The Ohio State University, introduced by the Chairman.

A set $a_1, a_2, \cdots, a_k$ of different residues (mod v) is called a difference set (v, k, λ) if the congruence $a_i - a_j \equiv d$ (mod v) has exactly λ solutions for every $d \not\equiv 0$ (mod v). Necessary and sufficient conditions for the existence of a difference set (v, k, λ) are not known. However Singer has demonstrated the existence of simple difference set $(v, k, 1)$ if $k-1=n$ is a prime power and $v=n^2+n+1$. The conjecture has been made that this is a necessary condition as well as a sufficient one for the existence of a difference set $(v, k, 1)$. By means of diverse theorems, this speaker has shown that the only simple difference sets existing for $v \leqq 2{,}560{,}000$ are those for which $k-1=n$ is a prime power and $v=n^2+n+1$.

4. *The ubiquitous golden section*, by Professor Wayne Dancer, University of Toledo.

The author pointed out the diverse places in the field of mathematics where the idea of the golden section occurs. This ubiquitous pattern is encountered not only in various kinds of geometry, but in arithmetic, algebra, trigonometry, and analysis as well. Sequences of integers first found in ancient Greek "ladders," and later known as the Fibonacci sequence, furnish approximations to the ratio of the segments involved in the golden section. These approximations are identical with the convergents of the continued fraction representing this ratio, that is, the positive root of the quadratic $x^2+x=1$. Both positive and negative powers of x reduce to binomials of the form $a+bx$, where a and b are successive elements of the sequence mentioned above. A rectangle or an isoceles triangle with sides in the ratio $x:1$ can be subdivided into similar rectangles and triangles whose vertices lie on a logarithmic spiral.

5. *The interactions of modern mathematics and modern physics*, by Professor Alfred Schild, Carnegie Institute of Technology.

This was an invited address. The interaction of mathematics and physics has provided valuable stimuli in both fields. This is illustrated in particular by tracing the central role played by one concept, namely invariance, in the twentieth century development of special relativity, of the theories of the Dirac electron and of elementary particles, of general relativity, and of non-Riemannian differential geometry.

This development has been limited by the small number of mathematicians involved. This is due not to a lack of interest, but rather to an ignorance arising from lack of opportunity. It is suggested that students of mathematics be encouraged to become familiar with the simple basic concepts of modern theoretical physics.

6. *Bounded variation for functions of two variables*, by Professor W. R. Transue, Kenyon College.

It was the purpose of this paper to describe the definitions of bounded variation for functions

of two variables given by Vitali and Frechet, and to discuss the properties of functions which have bounded variation according to these two definitions. A comparison was made of the two definitions, and examples were given to clarify their meanings. For functions of bounded Frechet variation, two-dimensional analogues of the following properties of functions of bounded variation of a single variable were presented: (a) The existence of a limit as a point is approached from the left or from the right; (b) Countability of the points of discontinuity of a function of bounded variation. These analogues are: (a) The existence of a limit as a point is approached from any one of the four open quadrants defined by lines through the point parallel to the coordinate axes; (b) Covering the points of discontinuity by a countable number of lines parallel to the coordinate axes.

7. *Preparation for graduate work in mathematics*, an informal discussion led by Professor D. R. Whitney, The Ohio State University.

8. *Classroom Notes*. Professor John Kaiser, Kent State University, presented a simple method for motivating the definition of e^{ix}.

FOSTER BROOKS, *Secretary*

CALENDAR OF FUTURE MEETINGS

Thirty-fifth Annual Meeting, Brown University, Providence, Rhode Island, December 29, 1951.

Thirty-third Summer Meeting, Michigan State College, East Lansing, Michigan, September 1–2, 1952.

The following is a list of the Sections of the Association with dates of future meetings as far as they have been reported to the Associate Secretary.

ALLEGHENY MOUNTAIN, Waynesburg College, Waynesburg, Pennsylvania, May 10, 1952.
ILLINOIS, Western Illinois State College, Macomb, May 9–10, 1952.
INDIANA, Indiana University, Bloomington, Spring, 1952.
IOWA, Coe College, Cedar Rapids, April 18–19, 1952.
KANSAS
KENTUCKY, University of Kentucky, Lexington.
LOUISIANA-MISSISSIPPI, Northwestern State College, Natchitoches, Louisiana, February 15–16, 1952.
MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA, National Bureau of Standards, Washington, D. C., December 8, 1951.
METROPOLITAN NEW YORK, Spring, 1952.
MICHIGAN, University of Michigan, Ann Arbor, April 12, 1952.
MINNESOTA
MISSOURI, Lindenwood College, St. Charles, Spring, 1952.
NEBRASKA
NORTHERN CALIFORNIA, Stanford University, Stanford, January 26, 1952.
OHIO, April 19, 1952.
OKLAHOMA
PACIFIC NORTHWEST, University of Oregon, Eugene, June 20, 1952.
PHILADELPHIA, University of Pennsylvania, Philadelphia, November 24, 1951.
ROCKY MOUNTAIN, Western State College, Gunnison, Colorado, May, 1952.
SOUTHEASTERN, Georgia Institute of Technology and Agnes Scott College, Atlanta, March 21–22, 1952.
SOUTHERN CALIFORNIA, Occidental College, Los Angeles, March 8, 1952.
SOUTHWESTERN, University of Arizona, Tucson, April 11–12, 1952.
TEXAS, East Texas State Teachers College, Commerce, April, 1952.
UPPER NEW YORK STATE, Hobart and William Smith Colleges, Geneva, May, 1952.
WISCONSIN

GEORGE BANTA PUBLISHING COMPANY, MENASHA, WISCONSIN

INDEX TO VOLUME 58, 1951

THE AMERICAN MATHEMATICAL MONTHLY

By L. J. GREEN, Case Institute of Technology

GENERAL MATHEMATICAL PAPERS

ALGEBRA

ANALYSIS

APPLIED MATHEMATICS

EDUCATION, HISTORY

GEOMETRY

UNCLASSIFIED

MATHEMATICAL NOTES

Edited by F. A. FICKEN, New York University

CLASSROOM NOTES

Edited by C. B. ALLENDOERFER, University of Washington

PROBLEMS AND SOLUTIONS

Edited by HOWARD EVES, State University of New York, and E. P. STARKE, Rutgers University

CORRECTIONS AND COMMENTS

Numbers in boldface type refer to problems, those in lightface to pages.

AUTHORS

Numbers refer to pages, boldface type indicating a problem solved and solution published; italics, a problem solved but the solution not published; ordinary type a problem proposed.

SOLUTIONS

Numbers in boldface type refer to problems, those in lightface to pages.

RECENT PUBLICATIONS

Edited by E. P. VANCE, Oberlin College

NEW BOOKS RECEIVED

REVIEWS

Names of authors are in ordinary type, those of reviewers in capitals.

CLUBS AND ALLIED ACTIVITIES

Edited by L. F. OLLMANN, Hofstra College

NEWS AND NOTICES

Edited by EDITH R. SCHNECKENBURGER, University of Buffalo

GENERAL INFORMATION

PERSONAL INFORMATION

The following persons presented papers at meetings of the Association and its Sections.

Personal Mention, This section contains the names of officers of the Association and its Sections, persons mentioned in the Department of News and Notices, and those conducting the business of the Association. The list does not include names of new members or of attendants at meetings.

NECROLOGY

REPORTS AND ANNOUNCEMENTS OF THE ASSOCIATION AND ITS SECTIONS

MEETINGS AND ANNOUNCEMENTS OF THE ASSOCIATION

MEETINGS OF ITS SECTIONS

THE AMERICAN MATHEMATICAL MONTHLY

THE OFFICIAL JOURNAL OF

THE MATHEMATICAL ASSOCIATION OF AMERICA, INC.

VOLUME 58 NUMBER 10

CONTENTS

DECEMBER 1951

The AMERICAN MATHEMATICAL MONTHLY

(Founded in 1894 by Benjamin F. Finkel)

EDITORIAL CORRESPONDENCE should be addressed to the Editor, C. V. Newsom, State Education Building, Albany 1, N. Y. Papers submitted for publication should be sent to C. B. Allendoerfer, Department of Mathematics, University of Washington, Seattle 5, Washington.

ADVERTISING CORRESPONDENCE should be addressed to J. F. Randolph, Morey Hall, University of Rochester, Rochester 3, N. Y.

NOTICE OF CHANGE OF ADDRESS by members of the Association as well as correspondence regarding subscriptions to the Monthly should be sent to the Secretary-Treasurer, H. M. Gehman, University of Buffalo, Buffalo 14, N. Y.

THIS IS THE OFFICIAL JOURNAL OF THE

MATHEMATICAL ASSOCIATION OF AMERICA, INC.

(Devoted to the Interests of Collegiate Mathematics)

Entered as second class matter at the post office at Menasha, Wis. Acceptance for mailing at special rate of postage provided for in the Act of February 28, 1925, embodied in Paragraph 4, Section 538, P. L. and R., authorized April 1, 1926.

Subscription Price: To Members, $4 a Year, To Others, $5 a Year.

Published by the Association at Menasha, Wisconsin, and Albany, N. Y. during the months of January, February, March, April, May, June-July, August-September, October, November, December.

A CLASS OF CONVEX CURVES RELATED TO THE CONIC SECTIONS

F. A. VALENTINE, University of California at Los Angeles

1. The problem. Consider two disjoint closed convex sets L and S in the euclidean plane E. For any point $x \epsilon E$ let $L(x)$ and $S(x)$ designate the shortest distances from x to L and from x to S, respectively. Let C denote the set of points $x \epsilon E$ for which

$$S(x) \leqq eL(x), \tag{1}$$

where e is a non-negative constant.

If L is a straight line and if S is a point, then we know that C is bounded by a conic section. In general, if S and L are arbitrary disjoint closed convex sets, the set C need not have convex components. However, if L is a straight line and if S is an arbitrary closed convex set disjoint with L, then each component of C is convex. To prove this, and other related theorems, we recall the fact* that for any closed convex set S the distance function $S(x)$ is a convex function of x. This means that for all $\alpha > 0$, $\beta > 0$, $\alpha + \beta = 1$, $x \epsilon E$, $y \epsilon E$, we have

$$\begin{aligned} S(\alpha x + \beta y) &\leqq \alpha S(x) + \beta S(y), \\ L(\alpha x + \beta y) &\leqq \alpha L(x) + \beta L(y). \end{aligned} \tag{2}$$

THEOREM 1. *If L is a straight line and if S is any closed convex set disjoint with L, then each component of the set of points C satisfying* (1) *is a convex set.*

Proof. Since $L \cdot S = 0$, we have $C \cdot L = 0$. Let C_1 and C_2 be the portions of C which lie on opposite sides of L. Choose $x \epsilon C_1$, $y \epsilon C_1$ so that we have

$$S(x) \leqq eL(x), \qquad S(y) \leqq eL(y). \tag{3}$$

Let $\alpha x + \beta y$ be any point on the line segment joining x and y so that $\alpha > 0$, $\beta > 0$, $\alpha + \beta = 1$. Since x and y lie on one side of L, and since L is a straight line, we have

$$L(\alpha x + \beta y) = \alpha L(x) + \beta L(y). \tag{4}$$

Conditions (2), (3) and (4) imply that

$$S(\alpha x + \beta y) \leqq eL(\alpha x + \beta y).$$

Hence $\alpha x + \beta y \epsilon C_1$, and thus C_1 is convex. Similarly C_2 is convex. We have also shown that C has at most two components.

* E. F. Beckenbach, Convex Functions, Bull. Amer. Math. Soc. vol. 54, 1948 pp. 439–460 (see page 446).

THEOREM 2. *In addition to the hypotheses of theorem* 1, *suppose S is bounded. If $e<1$, then C is a bounded convex set. If $e>1$, then C consists of two unbounded convex sets. If $e=1$, then C is an unbounded convex set.*

Proof. By definition, an outward normal N to S at a point p is a half-line which is perpendicular to some line of support T to S at p; moreover, N and S lie on opposite sides of T. We choose the line L as the x_2-axis, and choose an x_1-axis perpendicular to L. We also assume $x_1 \geqq 0$ in the half-plane which contains C_1. Now let N be an outward normal to S whose slope is m. Choose a (ξ_1, ξ_2) coordinate system with origin at the foot of N (namely $N \cdot S$), and with axes parallel to the corresponding axes of the (x_1, x_2) system. Also assume the equation of L in the (ξ_1, ξ_2) system is $\xi_1 = -a (a>0)$. The equation of the outward normal N is $\xi_2 = m\xi_1$. For any point $\xi = (\xi_1, \xi_2)$ on N we have

$$L(\xi) = | \xi_1 + a |, \qquad S^2(\xi) = \xi_1^2(1 + m^2).$$

In order for a point $\xi \epsilon N \cdot C_1$ to exist satisfying (1) with the equality sign it is necessary and sufficient that

$$|\xi_1| \sqrt{1 + m^2} = e(a + \xi_1), \qquad a + \xi_1 > 0.$$

This is equivalent to

$$\xi_1 = \frac{ae}{\sqrt{1 + m^2} - e} \qquad \text{if } \xi_1 > 0, \tag{5}$$
$$\xi_1 = \frac{-ae}{\sqrt{1 + m^2} + e} \qquad \text{if } \xi_1 < 0.$$

If $m = \pm \infty$, it is clear that there is always a point $\xi = (0, \xi_2) \epsilon N$ such that $S(\xi) = eL(\xi)$.

If a closed convex set C_1 is unbounded, there exists a direction such that for any point $x \epsilon C_1$, a half-line through x exists having this direction and lying in C_1. Notice that for $e \geqq 1$, the first of equations (5) has no solution for all values $m^2 \leqq e^2 - 1$. This together with the above remark implies that C_1 is unbounded when $e \geqq 1$. If $e<1$, then both of equations (5) have solutions for all values of m. This together with the opening remark in this paragraph implies that C_1 is bounded when $e<1$. To prove that C_2 must be unbounded when it exists one follows an argument similar to that given for C_1, in which, however, $\xi_1 < 0$, $a + \xi_1 < 0$. Moreover, it is easy to verify that the set C_2 is not null if and only if $e>1$.

In the remaining sections we assume the hypotheses of theorem 2.

2. Case $e>1$. Observe that when $e>1$, the first of equations (5) has no solution when $m^2 = e^2 - 1$.

THEOREM 3. *If $e>1$, then each component of C has two asymptotes whose slopes are $\pm\sqrt{e^2-1}$.*

Proof. We will establish the existence of an asymptote to C_1 with slope $\sqrt{e^2-1}$. Adopt the (x_1, x_2) coordinate system which has L as x_2-axis, and assume $x_1>0$ for all points in C_1. There exists at least one point $\alpha^*=(\alpha_1^*, \alpha_2^*)$ on the boundary of S for which an outward normal N exists having a slope $\sqrt{e^2-1}$ and lying in the half-plane $x_1>0$. Let T be the line of support to S at α^* which is perpendicular to N. Furthermore, for any point $\alpha=(\alpha_1, \alpha_2)$ in the half-plane $x_1>0$ let $\phi(\alpha)$ be the x_2-intercept of the line through α parallel to T. Thus

$$\phi(\alpha) = \frac{\alpha_1}{\sqrt{e^2-1}} + \alpha_2.$$

Since S is convex, it lies below its line of support T, so that for all $\alpha\epsilon S$ we have

$$\phi(\alpha) \leqq \phi(\alpha^*). \tag{6}$$

Consider the family of lines L_b parallel to N having b as x_2-intercept. Thus

$$L_b:\quad x_2 = \sqrt{e^2-1}\, x_1 + b. \tag{7}$$

We will prove $L_{\phi(\alpha^*)}$ is an asymptote to C_1.

To do this we first show that *if L_b intersects C_1, then $b<\phi(\alpha^*)$*. Let L_b intersect C_1; then L_b intersects the boundary of C_1 in, say, the point x. If α denotes the point of S nearest† to x, then since $x\epsilon L_b$, condition (7) implies

$$S^2(x) = (x_1-\alpha_1)^2 + (x_2-\alpha_2)^2 \equiv e^2L^2(x) - p(x, \alpha) + q(\alpha) \tag{8}$$

where, by definition,

$$\begin{aligned} p(x, \alpha) &= 2x_1\sqrt{e^2-1}\,(\phi(\alpha) - b), \\ q(\alpha) &= \alpha_1^2 + (\alpha_2 - b)^2 > 0. \end{aligned} \tag{9}$$

Because x lies on the boundary of C_1, we have $S(x)=eL(x)$ with $x_1>0$, so that (8) implies $p(x, \alpha)=q>0$. Hence, conditions (6) and (9) imply that $b<\phi(\alpha) \leqq\phi(\alpha^*)$.

We next prove that *if $b<\phi(\alpha^*)$, then L_b intersects C_1*. Consider a point x on L_b. Since α^* lies in S, we have

$$S^2(x) \leqq (x_1-\alpha_1^*)^2 + (x_2-\alpha_2^*)^2 \equiv e^2L^2(x) - p(x, \alpha^*) + q(\alpha^*).$$

Clearly $p(x, \alpha^*)\rightarrow\infty$ as $x_1\rightarrow\infty$ since $b<\phi(\alpha^*)$. Therefore $S(x)\leqq eL(x)$ for all points x of L_b with sufficiently large x_1. This shows that if $b<\phi(\alpha^*)$, then L_b intersects C_1 in a half-line. The latter two italicized statements prove $L_{\phi(\alpha^*)}$ to be an asymptote of C_1.

In the same manner one can prove that the line

$$x_2 = -\sqrt{e^2-1}\, x_1 - \frac{\beta_1^*}{\sqrt{e^2-1}} + \beta_2^*$$

† Since S is a closed convex set, there is a unique point of S nearest to x.

is an asymptote to C_1, where β^* is a point on S where an outward normal to S exists having a slope $-\sqrt{e^2-1}$ and lying in the half-plane $x_1>0$.

The pair of asymptotes to C_2 is obtained by considering normals to S which have slopes $\pm\sqrt{e^2-1}$, and which intersect L. The reasoning is similar to the above.

3. Case $e<1$. In this section we find an interpretation for the equation $c=ea$ when $e<1$.

LEMMA 1. *Let L_1 be a line of support to S which is parallel to L, and let $\xi \epsilon L_1 \cdot S$. The outward normal to S at ξ which is perpendicular to L_1 intersects the boundary of C at a point z. Then the line L_2 through z parallel to L_1 is a line of support to C.*

Proof. Suppose the lemma were false. Then there exists a point $y \epsilon C$ such that y and S are on opposite sides of L_2. Since C is convex, and since $z \epsilon C$, $y \epsilon C$, there exists a point $x \epsilon C \cdot L_2$ such that $S(x)<eL(x)=ex_1$. Since S and x are on opposite sides of L_1, $S(x)\geqq S(z)$. But we have $S(z)=eL(z)=eL(x)=ex_1$, which contradicts $S(x)<ex_1$. Hence, the lemma is true.

THEOREM 4. *Let L_x and L_y be the two lines of support to C which are parallel to L, and let L^* be the line parallel to L which is midway between L_x and L_y. Similarly let L_ξ and L_η be the two lines of support to S parallel to L, and let L' be the corresponding line midway between L_ξ and L_η. Let c be the distance between L^* and L', and let $2a$ be the distance between L_x and L_y.*

Then $c=ea$.

Proof. Without loss of generality, suppose the following lines are listed in increasing distance from L, namely, L_y, L_η L_ξ, L_x. By virtue of Lemma 1, there exist points $x \epsilon L_x \cdot C$, $\xi \epsilon L_\xi \cdot S$, $y \epsilon L_y \cdot C$, $\eta \epsilon L_\eta \cdot S$ such that $x_1 \geqq \xi_1$, $x_2=\xi_2$, $\eta_1 \geqq y_1$, $\eta_2=y_2$. By definition

$$c = \frac{x_1+y_1}{2} - \frac{\xi_1+\eta_1}{2} \equiv \frac{x_1-\xi_1}{2} - \frac{\eta_1-y_1}{2}. \tag{10}$$

Since $x_2=\xi_2$, $\eta_2=y_2$, we have $S(x)\equiv x_1-\xi_1=ex_1$, $S(y)\equiv\eta_1-y_1=ey_1$. Hence (10) implies that

$$c = \frac{e(x_1-y_1)}{2} \equiv ea.$$

No doubt there are further properties of the set C analogous to those of the conic sections. The definitions can be generalized to higher dimensions. It would be of interest to determine necessary and sufficient conditions on L and S so that the components of C be convex.

NOTES ON THEON OF SMYRNA

G. C. VEDOVA, Newark College of Engineering

1. Historical background and meaning. In "The mathematical rules necessary for the study of Plato"* Theon of Smyrna (circa 140 A.D.) states the following rule:

> Let two units be laid out, of which we take one as the side and the other as the diagonal . . . add to the side the diagonal and to the diagonal two sides . . . the diagonal is now 3 and the side 2. Again, to the side add the diagonal and to the diagonal twice the side . . . the diagonal is now 7 and the side 5 . . . and the addition being thus continuously made the ratio alternates, the square on the diagonal being now one more now one less than twice the square on the side . . . therefore the squares of all the diagonals are the double of the squares of all the sides, alternately exceeding or falling short by one. . . .

Symbolically, if s_n, d_n, denote the side and diagonal numbers obtained at the nth step, then Theon's rule gives the formulas

$$s_{n+1} = s_n + d_n,$$
$$d_{n+1} = 2s_n + d_n,$$

which yield the sequence of ratios d_n/s_n.

$$(1) \qquad 1/1,\ 3/2,\ 7/5,\ 17/12,\ 41/29,\ 99/70, \cdots$$

beginning with $d_1=1$, $s_1=1$. In this sequence, Theon seems to say, the ratio d_n/s_n alternates so that d_n^2 is now $2s_n^2+1$, now $2s_n^2-1$.

J. Dupuis** sees in this the solution, in integers, of the indeterminate equation $y^2-2x^2=\pm 1$; for, suppose $y=a$, $x=b$ is such a solution. Then $a^2-2b^2 = \pm 1$. Form by Theon's rule, the values $a'=a+2b$, $b'=a+b$. We have

$$a'^2 - 2b'^2 = (a + 2b)^2 - 2(a + b)^2 = - a^2 + 2b^2 = - 1,$$

that is, a', b' is a solution also. Now, $a=1$, $b=1$ is a solution; hence Theon's rule gives an infinity of solutions.

However, T. L. Heath† sees in Theon's rule an algorithm for successive approximations to the square root of 2. For, from $y^2-2x^2=\pm 1$ we get $y/x = \sqrt{2\mp 1/x^2}$ whence the limit $y/x=\sqrt{2}$, which shows that the term d_n/s_n of the sequence converges to $\sqrt{2}$, being alternately less and greater.

Theon gave neither a derivation nor a proof of his algorithm. But, from Proclus'‡ assertion that the Pythagoreans discovered the "side and diagonal" numbers, and that a proof of their characteristic property was given by "him" in the second book of the "Elements," and taking the "him" in question to be Euclid, Heath reconstructs the following derivation (Book II, Proposition 10):

* Theon de Smyrne, ed. Jaques Dupuis, Paris 1892.

** *Op. cit.*, p. 72.

† T. L. Heath, History of Greek Mathematics, Vol. 1, Oxford, 1927. See also G. J. Allman, Greek Geometry from Thales to Euclid, Dublin, 1889.

‡ T. L. Heath, *op. cit.*, Vol. 1.

> If a straight line be bisected, and a straight line be added to it in a straight line, the square on the whole with the added straight line and the square on the added straight line both together are double of the square on the half and of the square described on the straight line made up of the half and the added straight line as on one straight line,

which means that if AB is bisected at C, and the whole line is extended to D, as in the line below,

then $\overline{AD}^2+\overline{BD}^2=2(\overline{AC}^2+\overline{CD}^2)$, or, setting $\overline{AC}=\overline{CB}=x$, and $\overline{BD}=y$,

$$(2x+y)^2+y^2=2[x^2+(x+y)^2]$$

whence, by transposition,

$$(2x+y)^2-2(x+y)^2=2x^2-y^2,$$

an identity.

From this it is clear, says Heath, that if x_1, y_1, satisfy either of

$$2x^2-y^2=1, \qquad 2x^2-y^2=-1,$$

then $x_2=x_1+y_1$, $y_2=2x_1+y_1$ satisfy the other.

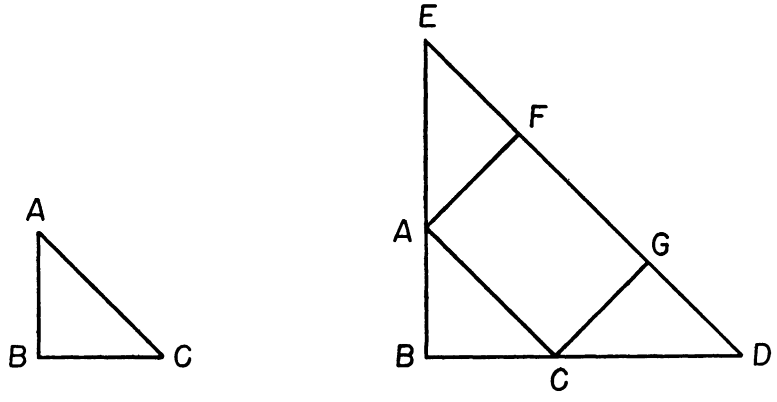


FIG. 1

A curious derivation of Theon's formulas, due to P. Bergh, is given by M. Cantor.* The proof as given by Bergh is quoted below, with reference to Figure 1:

> If we start with an isosceles right triangle, ABC, of side s_n and diagonal d_n, and extend each of the legs a distance d_n, to the points E and D, and complete the triangle EBD, then the new diagonal will be $2s_n+d_n$, and the new side s_n+d_n. If the perpendiculars AF, CG are drawn the proof is obvious.

* M. Cantor, Vorlesungen über Geschichte der Mathematik, I, p. 437, Leipzig, 1880. Also T. L. Heath, The Elements of Euclid, I., p. 401. Cambridge, 1924.

It is implied in this construction that the sides and diagonals obtained in the way described are Theon's "side" and "diagonal" numbers; that they cannot be, however, becomes immediately evident when it is noticed that for any pair s_n, d_n and s_m, d_m of Bergh's numbers we always have $d_n/s_n = d_m/s_m$, an equality which is never true of Theon's numbers.

It is probably far-fetched to see in this algorithm of Theon's either a deliberate attempt to solve* the indeterminate equation $y^2 - 2x^2 = \pm 1$, or one to find *successive rational approximations to the square root of 2 though, obviously,* considerable success was achieved, incidentally, in both. A clew to what seems to have been Theon's purpose is provided by a statement** he prefixes to his rule:

> As unity is the principle of all figures, according to the highest and generating ratio, so also is the ratio of the diagonal to the side found in the unit.

This strongly suggests that Theon, a known Pythagorean, is still trying, some 600 years after the death of Pythagoras, to defend the Master's doctrine that the unit is the constituent element of all numbers and the point that of all figures.

It has already been said that Proclus definitely assigned the "side" and "diagonal" numbers to the Pythagoreans. But it can also be shown that a derivation of Theon's numbers, proceeding entirely on Pythagorean considerations is entirely feasible. Consider Figures 2, 3, and 4. That $d^2 - 2s^2 = 0$ has no solutions in integers was known. Hence, that the ratio $d/s = \sqrt{2}$ could not be

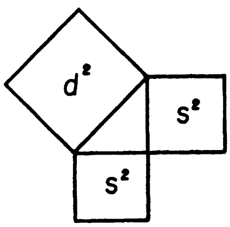


FIG. 2

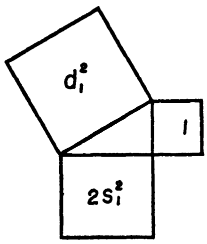


FIG. 3

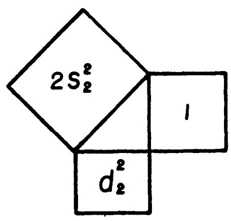


FIG. 4

generated by the unit was obvious, since d and s are incommensurable. This contradiction of the Pythagorean doctrine was to be combatted. The fact that $x^2 + y^2 = z^2$ had solutions in integers had been shown both by Pythagoras and by Plato. Hence, that the relations,

$$2s_1^2 + 1 = d_1^2,$$

$$d_2^2 + 1 = 2s_2^2,$$

* This would be not only an anticipation of Diophantus by over a hundred years, but also a vast improvement over the paucity of his solutions.

** J. Dupuis, *op. cit.*, p. 72.

might have solutions in integers was not impossible. These are the cases of Figures 3 and 4 above. Euclid's theorem (Bk. II, prop. 10) would then come to mind. The ratio d_n/s_n could still be found in the unit, though by a limiting process. The day was saved.

It should be noticed that the identity of Book II, proposition 10, would make the solution of the more general case, $y^2-2x^2=\pm k$ possible, for any number k. Thus, for $k=2$, if x_1, y_1 satisfy either of

$$y^2 - 2x^2 = 2, \qquad y^2 - 2x^2 = -2$$

then x_1+y_1, $2x_1+y_1$ satisfy the other. Now $x_1=1$, $y_1=2$ satisfy the first of these. Hence,

$$2/1,\ 4/3,\ 10/7,\ 24/17,\ \cdots \tag{2}$$

is another sequence of rational approximations to $\sqrt{2}$. But now the ratio d_n/s_n alternates so that the square of the diagonal is now 2 more, now 2 less than twice the square on the side. The unit does not play now the all-important role of "generating principle."

2. Some generalizations. A generalization of Theon's Rule to cover the more general case

$$y^2 - 2x^2 = \pm k$$

has been pointed out. It will be observed, of course, that the bigger the number k the slower is the convergence of d_n/s_n to $\sqrt{2}$. But here is another, more useful generalization.

Starting with the equation,

$$y^2 - Kx^2 = C,$$

and supposing that (x_1, y_1) is a solution, one may seek numbers m, n, r, s, which would make

$$x_2 = mx_1 + ny_1,$$
$$y_2 = rx_1 + sy_1,$$

also a solution, that is, such that

$$y_2^2 - Kx_2^2 \equiv (rx_1 + sy_1)^2 - K(mx_1 + ny_1)^2 = C.$$

This would give

$$(rx_1 + sy_1)^2 - K(mx_1 + ny_1)^2 \equiv y_1^2 - Kx_1^2 \tag{A}$$

whence

$$(s^2 - Kn^2)y_1^2 + 2(rs - Kmn)x_1y_1 + (r^2 - Km^2)x_1^2 \equiv y_1^2 - Kx_1^2.$$

If, now, m, n, r, s are chosen to satisfy the conditions,

$$\text{(B)} \qquad \begin{aligned} s^2 - Kn^2 &= 1, \\ rs - Kmn &= 0, \\ r^2 - Km^2 &= -K, \end{aligned}$$

then (A) will become an identity and (x_2, y_2) a solution.* This would give the recursion formula,

$$\begin{aligned} y_2 &= rx_1 + sy_1, \\ x_2 &= mx_1 + ny_1, \end{aligned}$$

for the sequence y_1/x_1, y_2/x_2, y_3/x_3, $\cdots$ of successive rational approximations to $\sqrt{K}$, from above if $C>0$, from below if $C<0$, the convergence being rapid or slow according as $|C|$ is small or large.

If successive rational approximations alternately from above and below were desired, then numbers m, n, r, s could be sought such that if x_1, y_1 satisfied either of $y^2-Kx^2=C$, $y^2-Kx^2=-C$, then

$$\begin{aligned} x_2 &= mx_1 + ny_1, \\ y_2 &= rx_1 + sy_1, \end{aligned}$$

would satisfy the other. This would lead to the condition

$$(rx_1 + sy_1)^2 - K(mx_1 + ny_1)^2 = -y_1^2 + Kx_1^2,$$

and hence to the system

$$\text{(C)} \qquad \begin{aligned} s^2 - Kn^2 &= -1, \\ rs - Kmn &= 0, \\ r^2 - Km^2 &= K, \end{aligned}$$

which, if solvable for m, n, r, s, would lead to the required approximations.

Thus, to find rational approximations to $\sqrt{5}$, from above, we may start with $y^2-5x^2=1$, taking $x_2=4$, $y_1=9$ as a first solution. Then, from (B), with $K=5$, satisfied by $m=9$, $n=4$, $r=20$, $s=9$, we would get

$$\begin{aligned} x_2 &= 9x_1 + 4y_1, \\ y_2 &= 20x_1 + 9y_1, \end{aligned}$$

as a recursion formula, and from this the sequence

$$\text{(3)} \qquad 9/4,\ 161/72,\ 2889/1292,\ \cdots$$

of rational approximations to $\sqrt{5}$ from above.

Again, starting with $y^2-5x^2=-1$, taking $x_1=1$, $y_1=2$, as a first solution

---

* The charge is often made against the ancients that they failed to make (at times obvious) generalizations. But here the solution of $y^2-2x^2=k$ would not serve Theon's purpose except for $k=1$, and the solution of the more general $y^2-Kx^2=C$ was altogether beyond him since the only identity (A) available to him was the one for the case $K=2$ (Eucl. BK II, 10).

and proceeding as above we would find the same recursion formula (it may be seen, from (B) or (C) that m, n, r, s are independent of the constant C of the equations $y^2-Kx^2=\pm C$) which yields the sequence

$$(4) \qquad 2/1,\ 38/17,\ 682/305,\ \cdots$$

of rational approximations to $\sqrt{5}$ from below.

Finally, starting with $y^2-5x^2=-1$, and the first solution $x_1=1$, $y_1=2$, we get, from

$$(rx_1+sy_1)^2-5(mx_1+ny_1)^2=-(y_1^2-5x_1^2),$$

the conditions

$$s^2-5n^2=-1,$$
$$rs-5mn=0,$$
$$r^2-5m^2=5,$$

satisfied by $m=2$, $n=1$, $r=5$, $s=2$, and thus the recursion formula,

$$x_2=2x_1+y_1,$$
$$v_2=5x_1+2y_1,$$

which yields the sequence

$$(5) \qquad 2/1,\ 9/4,\ 38/17,\ 682/305,\ \cdots$$

of rational approximations to $\sqrt{5}$, alternately from below and above.

We inquire now into the necessary and sufficient conditions under which the method outlined above is effective. Suppose rational approximations to $\sqrt{K}$, K not a perfect square, are to be found. We find a solution of $y^2-Kx^2=C$ (this can always be done by choosing C suitably)* and then seek numbers m, n, r, s, to satisfy conditions (B) or (C), above, for a monotonic or an alternating sequence, respectively.

Conditions (B) lead to

$$K=rs/mn, \quad s=m=\sqrt{rn+1}=\sqrt{Kn^2+1}, \quad \text{whence} \quad m^2-Kn^2=1.$$

Now $m^2-Kn^2=1$ is always solvable in integers;** hence the monotonic sequence of rational approximations can always be found, the convergence being from below or above according as $C<0$ or $C>0$.

Conditions (C) lead to:

$$K=rs/mn, \quad s=m=\sqrt{rn-1}=\sqrt{Kn^2-1}, \quad \text{whence} \quad m^2-Kn^2=-1.$$

This last equation is solvable in integers only if the number of quotients in

* *For example*, we can take $x=1$, $C=t^2-K$, where t^2 is the smallest square greater than K; this yields $y=t$. Thus $x=1$, $y=t$ is a solution.

** See G. Chrystal, Algebra, Part II, p. 450. Edinburgh, 1889.

the period of the development of $\sqrt{K}$ as a simple continued fraction is odd;* hence, the alternating sequence of rational approximations can be found only if the period of $\sqrt{K}$ is odd.

This method does not seem to be capable of extension to the finding of rational approximations to roots of higher index, that is, to $\sqrt[\nu]{K}$, $\nu > 2$. For, to find the νth root of K one begins with

$$y^\nu - Kx^\nu = C,$$

and the solution $x = x_1$, $y = y_1$, which may be found for any K by a suitable choice of C. Then, assuming

$$x_2 = mx_1 + ny_1,$$
$$y_2 = rx_1 + sy_1,$$

one seeks rational values of m, n, r, s that will make

$$(rx_1 + sy_1)^\nu - K(mx_1 + ny_1)^\nu = y_1^\nu - Kx_1^\nu$$

an identity. This leads to the conditions:

$$r^\nu - Km^\nu = -K, \tag{1}$$
$$r^{\nu-1}s - Km^{\nu-1}n = 0, \tag{2}$$
$$r^{\nu-2}s^2 - Km^{\nu-2}n^2 = 0, \tag{3}$$
$$\cdots\cdots\cdots\cdots$$
$$r^2s^{\nu-2} - Km^2n^{\nu-2} = 0, \tag{$\nu-1$}$$
$$rs^{\nu-1} - Kmn^{\nu-1} = 0, \tag{ν}$$
$$s^\nu - Kn^\nu = +1. \tag{$\nu+1$}$$

If non-zero solutions of this system exist, then from (2) and (3) we find $r/s = m/n$ where $nr/ms = 1$; then from (3) we find

$$K = \frac{r^{\nu-2}s^2}{m^{\nu-2}n^2} = \frac{r^{\nu-2}s^2}{m^{\nu-2}n^2}\cdot\frac{n^2r^2}{m^2s^2} = \frac{r^\nu}{m^\nu}\,.$$

Hence, a necessary condition for such a solution to exist is that K be a perfect νth power of a rational number. But in such a case rational approximations to $\sqrt[\nu]{K}$ are unnecessary.

We examine now the possibility of zero solutions. We find:

(a) If $r = 0$, then $m = 1$, $n = 0$, $s = 1$ and (D) becomes $x_2 = x_1$, $y_2 = y_1$, which yields no convergent sequence.

(b) If $n = 0$, the same result is obtained.

(c) If $m = 0$, then $K = -(r)^\nu$, a perfect νth power.

(d) If $s = 0$, then $K = -(1/n)^\nu$, a perfect νth power again.

Hence, under any assumptions, no useful results are obtained.

* G. Chrystal, *op. cit.*, p. 451.

3. Relationships to continued fractions. It has probably been noticed by the reader that sequences (1) and (5), above, are the convergents of the simple continued fraction expansions of $\sqrt{2}$ and $\sqrt{5}$, respectively, and that sequences (3) and (4) are, respectively, the even and odd convergents of the same expansion of $\sqrt{5}$. In a manner that is consistent with continued fraction theory, they converge to $\sqrt{5}$ from above and below. The terms of sequence (2), however, are not the successive convergents of the simple continued fraction expansion of $\sqrt{2}$, nor, in general, are the successive alternating rational approximations

$$\text{(E)} \qquad y_1/x_1,\ y_2/x_2,\ y_3/x_3,\ \cdots,\ y_i/x_i\ \cdots$$

to $\sqrt{K}$, obtained by this method from the first solution x_1, y_1 of the equation $y^2-Kx^2=C$, the convergents of the simple continued fraction expansion of $\sqrt{K}$, though they are the convergents of a continued fraction expansion of $\sqrt{K}$ of a more general type. For, if y_1/x_1, y_2/x_2, y_3/x_3, $\cdots$, are the convergents of the continued fraction

$$\text{(F)} \qquad a_1 + \frac{b_2}{a_2+}\ \frac{b_3}{a_3+}\ \cdots$$

then

$$\text{(G)} \qquad \begin{aligned} y_i &= a_iy_{i-1} + b_iy_{i-2}, \\ x_i &= a_ix_{i-1} + b_ix_{i-2}, \end{aligned}$$

with the initial conditions $y_0=1$, $y_1=a_1$; $x_1=1$, $x_2=a_2$. Hence, since $y_{i-1}/x_{i-1} \neq y_{i-2}/x_{i-2}$, we can solve for a_i and b_i. In other words, the terms of (E) above are the convergents of some general continued fraction (F).

Again, if (F) is to be a simple continued fraction, then $b_i=1$, for $i=2, 3, \cdots$, and (G) becomes

$$\text{(H)} \qquad \begin{aligned} y_i &= a_iy_{i-1} + y_{i-2}, \\ x_i &= a_ix_{i-1} + x_{i-2}, \end{aligned}$$

whence it follows that

$$\text{(I)} \qquad y_ix_{i-1} - y_{i-1}x_i = (-1)^i.$$

Hence, if the terms of (E) are to be the convergents of the simple continued fraction expansion of $\sqrt{K}$, then, from (I),

$$\begin{aligned} (-1)^i = y_ix_{i-1} - y_{i-1}x_i &= x_{i-1}(rx_{i-1} + sy_{i-1}) - y_{i-1}(mx_{i-1} + ny_{i-1}) \\ &= rx_{i-1}^2 - ny_{i-1}^2 + sx_{i-1}y_{i-1} - mx_{i-1}y_{i-1} \\ &= rx_{i-1}^2 - ny_{i-1}^2, \text{ since } s = m, \\ &= n(Kx_{i-1}^2 - y_{i-1}^2), \text{ since } K = rs/mn = r/n. \end{aligned}$$

Hence, for $i=2$,

$$1 = n(Kx_1^2 - y_1^2) = -nC, \text{ since } x_1, y_1 \text{ is a solution of } y^2 - Kx^2 = C$$

by hypothesis. Now n and C are integers, and n is positive; therefore, $n=1$, $C=-1$.

We have found that K must satisfy the condition $m^2-Kn^2=-1$ for an alternating sequence (E) to exist. If $n=1$ this gives $K=m^2+1$. We conclude then that the necessary conditions for the terms of (E) to be the successive convergents of the simple continued fraction expansion of $\sqrt{K}$ are:

(1) The period of $\sqrt{K}$ must be odd,
(2) K must have the form m^2+1,
(3) x_1, y_1 must be a solution of $y^2-Kx^2=-1$.

It is easy to show that these conditions are also sufficient. But the first of the above conditions is implied by the second; for if $K=m^2+1$ then it is easily found that

$$K = m + \frac{1}{2m +} \cdots,$$

a recurring continued fraction of period 1. Hence, (E) will represent the convergents of the simple continued fraction expansion of $\sqrt{K}$ if and only if K is of form m^2+1 and x_1, y_1 is a solution of $y^2-Kx^2=-1$. It will be seen now that sequences (1) and (5), above, are instances of (E) for $K=2=1^2+1$ and $K=5=2^2+1$.

4. **Remarks**. The main usefulness of this method lies in the ease with which it provides rational approximations to pure quadratic surds. Thus, to find an alternating sequence of rational approximations to $\sqrt{13}$ we may proceed thus:

We have: $K=r/n=13/1=26/2=39/3=52/4=65/5= \cdots$.

We stop at $65/5$, for $5\times 65-1=324$, a perfect square.

Hence, $r=65$, $n=5$, $s=m=18$, and thus

$$y_2 = 65x_1 + 18y_1,$$
$$x_2 = 18x_1 + 5y_1.$$

We solve now equation $y^2-13x^2=C$.

Take $x_1=1$; $y_1^2=13+C$; take $C=3$; this gives $y_1=4$.

Hence, $y/x=4/1$, $137/38$, $4936/1369$, $177833/49322$, $\cdots$ is a sequence of rational approximations to $\sqrt{13}$ from above and from below alternately.

The third term of this sequence agrees with its limit to six decimal places. For it is easy to see that the error ϵ_n, made in taking the nth term for the limit of the general alternating sequence, satisfies the condition

$$\epsilon_n < \left| y_n/x_n - y_{n+1}/x_{n+1} \right|.$$

This reduces, by (I) above, to the well-known form

$$\epsilon_n < \left| 1/x_n x_{n+1} \right|$$

if the sequence is the sequence of convergents of a simple continued fraction.

PECULIARITIES OF POLYHEDRA*

S. S. CAIRNS, University of Illinois

1. Introduction. Mathematical impossibilities. Mathematical reasoning, even of an elementary nature, frequently carries one far beyond the limits of his unaided imagination. The results of such reasoning can be divided into proofs of (1) the correctness of certain statements which one would be inclined to believe anyway, (2) the falsity or impossibility of something which one would expect off-hand to be true or possible, and (3) conclusions the very nature of which would not be suspected without mathematical analysis.

A special interest attaches to proofs that some plausible statements are false. To this category belong the demonstrations that the circle cannot be squared nor the angle trisected nor the cube "duplicated" with straightedge and compass under the classical restrictions. The false statement that these constructions are possible is plausible, partly because one can, with little difficulty, construct arbitrarily close approximations to the desired angles and areas. The proofs of the impossibility are far more interesting than successful constructions would have been. Such proofs have the added advantage of revealing fundamental properties of the real number system. In the case of trisecting the angle, the most direct proof involves (1) a study of what can be done with straightedge and compass and (2) a demonstration that the trisection problem falls outside the category of what thus can be done. Such discussions are accessible in the literature. They should be comprehensible to any one who has met with good success in high school algebra and has gone far enough beyond to master the solution of the general cubic. Nevertheless, angle trisectors are always with us. Some of them are probably incapable of comprehending the proofs that they are attempting the impossible. Others have never looked at such proofs, being uninformed as to their existence or without the initiative to seek them out in the literature; also, perhaps, being like a certain college freshman who once, with a sweet and confident smile, informed the writer that he "didn't like to think anything was impossible."

It is not the present purpose to add to the already ample literature on trisecting the angle, but, rather, to reveal a few apparently simple things which one cannot do with a lump of modeling clay. Suppose the clay is originally in the form of a solid sphere, and one sets out to investigate the various polyhedral shapes into which it can be deformed. In the course of such deformations it is understood that there shall be no pulling apart or sticking together of the clay. This rules out, for example, the punching of a hole, like a tunnel, all the way through the lump and also the formation of a handle by pulling some of the clay out in a long piece and sticking the free end back onto the main portion.

A polyhedral region is a solid whose boundary, referred to as a *polyhedron*, is made up of a finite number of faces, edges, and vertices. Certain restrictions

* An address delivered before the Missouri Section of the Mathematical Association of America on March 24, 1950.

on the relationships among the faces, edges and vertices can best be clarified by first considering convex polygons and convex polyhedra. A *convex polygonal region* in a plane is a finite region which can be defined as the intersection of a finite number of distinct half-planes, where a *half-plane* is definable by a condition of the form,

$$ax + by + c \geqq 0, \qquad |a| + |b| > 0, \tag{1.1}$$

in terms of some rectangular cartesian coordinate system (x, y). Thus a set of conditions,

$$a_i x + b_i y + c_i \geqq 0, \qquad (i = 1, \cdots, \alpha), \tag{1.2}$$

defines such a region $\bar{p}$. It will be assumed that (1) the interior,

$$p: \quad a_i x + b_i y + c_i > 0, \qquad (i = 1, \cdots, \alpha), \tag{1.3}$$

of this region is non-vacuous and (2) that none of the α conditions in (1.2) can be deleted without changing the region $\bar{p}$.

In similar fashion, and with precisely analogous restrictions, a *convex polyhedral region* $\bar{p}$ in euclidean three-dimensional space is defined by a set of inequalities,

$$\bar{p}: \quad a_i x + b_i y + c_i z + d_i \geqq 0, \qquad (i = 1, \cdots, F), \tag{1.4}$$

the interior of $\bar{p}$ being

$$p: \quad a_i x + b_i y + c_i z + d_i > 0, \qquad (i = 1, \cdots, F). \tag{1.5}$$

As a result of the restrictions, each of the F planes,

$$a_i x + b_i y + c_i z + d_i = 0, \tag{1.6}$$

contains a convex polygonal region on $\bar{p}$ called a *face* of $\bar{p}$ or of the boundary *polyhedron* $\Pi = \bar{p} - p$. There are therefore F faces of Π. Any two of these faces are distinct or have in common either a single point, called a *vertex* of Π, or else a line-segment (called an *edge* of Π) joining two vertices.

(A) The following facts present no difficulties:

(1) *Each face has at least three edges.*
(2) *Each vertex belongs to at least three edges.*
(3) *The vertices on any face can be cyclically ordered so that each is joined to its successor by an edge.*
(4) *The faces containing any vertex can be cyclically ordered so that each has an edge in common with its successor.*

(B) Many of the results will apply to the more general situation in which the surface S, of the lump of modeling clay is divided into a map of imaginary countries (the "*faces*" of S) covering the entire surface, subject to the following conditions. (1) There are at least two countries. (2) The boundary of each country is a simple closed contour with at least one point, called a vertex, on it,

each such contour being divided by its vertices into arcs called the "*edges*" of S. (3) If two countries have in common a boundary point other than a vertex, then they have in common the entire edge through that boundary point.

2. Restrictions on a polyhedron Π. The symbol Π will denote an arbitrary polyhedron subject to §1(A) and to the added conditions: (1) that two faces have in common at most a single vertex or a single edge and (2) that Π bound a region satisfying the condition imposed on the modeling clay; that is, no "handles" and no "tunnels." The following statements give some of the possibly surprising restrictions* on Π.

(A) *The polyhedron* Π *cannot have exactly seven edges, though six edges and any number higher than seven are possible.*

(B) *It is impossible that either* (1) *every face of* Π *have more than five vertices or* (2) *every vertex belong to more than five faces. In fact, at least four faces must have fewer than six vertices, and at least four vertices belong to fewer than six faces.*

(C) *It is impossible that, simultaneously, every face have more than three vertices and every vertex belong to more than three faces. In fact, the number of triangular faces plus the number of vertices on three faces is at least eight.*

(D) *It is impossible that there be either* (1) *an odd number of odd-sided faces or* (2) *an odd number of vertices each belonging to an odd number of faces.*

(E) (1) *If every vertex belongs to just three faces and every face is pentagonal or hexagonal, then exactly twelve faces are pentagonal.* (2) *If every face is triangular and each vertex belongs to either five or six faces, then exactly twelve vertices belong to five faces each.*

It is recommended to all practising and potential angle trisectors that, whenever their life mission begins to pall, they take a large hunk of modeling clay and try to disprove any or all of the foregoing statements. This task would equal the other in futility and possibly in fascination; and it has the advantage, for him who undertakes it, that he need go no further in this article, for, from here on, a logical establishment of the above results is presented and such demonstrations spoil the fun of the whimsical Don Quixotes on the fringes of mathematics.

3. Proofs. The following notation will be employed:

$$\begin{aligned} V &= \text{the number of vertices of } \Pi, \\ E &= \text{the number of edges}, \\ F &= \text{the number of faces}. \end{aligned} \tag{3.1}$$

Also

$$\begin{aligned} V_i &= \text{the number of vertices each belonging to exactly } i \text{ edges}, \\ F_i &= \text{the number of faces each having exactly } i \text{ edges} \qquad (i = 3, 4, 5, \cdots). \end{aligned} \tag{3.2}$$

* These results are to be found in the book, Vorlesungen über die Theorie der Polyeder by Ernst Steinitz, Springer, 1934, edited by Hans Rademacher.

Since each face has at least three and at most $V-1$ edges, and since each vertex belongs to at least three and at most $F-1$ edges,

$$\begin{aligned} V &= V_3 + V_4 + V_5 + \cdots + V_{F-1}. \\ F &= F_3 + F_4 + F_5 + \cdots + F_{V-1}. \end{aligned} \tag{3.3}$$

The results stated in §2 can now be symbolically expressed as follows:

(A) $E \neq 7$ *but* $E=6, 8, 9, \cdots$ *are all possible.*

(B) $F_3+F_4+F_5 \geqq 4$ *and* $V_3+V_4+V_5 \geqq 4$.

(C) $F_3+V_3 \geqq 8$.

(D) $(F_3+F_5+F_7+ \cdots)$ *and* $(V_3+V_5+V_7+ \cdots)$ *are both even numbers.*

(E) *If* $V=V_3$ *and* $F=F_5+F_6$, *then* $F_5=12$. *If* $F=F_3$ *and* $V=V_5+V_6$, *then* $V_5=12$.

It is noteworthy that all the above notations and results are symmetric in vertices and faces; in other words, that the F's and V's can be interchanged throughout results $(A)-(E)$ without any change of meaning.

(F) The interchangeability of the V's and F's is known as a *duality principle.* Two statements are *dual* to one another if each is carried into the other by the interchange of the V's and the F's.

For example, (B), (D), and (E) are each made up of a pair of dual results. Result (C) is *self-dual* and so, in a trivial way, is (A).

(G) The equations (3.3) are a dual pair. It may be observed, as the discussion continues, that all other equations on which the proofs of results (A) to (E) are based are also either self-dual or fall into dual pairs. This means that it will be sufficient to prove only one of each pair of dual results, since interchanging V's and F's would yield a proof of the other.

A geometric insight into the reason for duality can be obtained as follows. Consider the faces of Π as maps of countries on a polyhedral earth; or, what amounts to the same thing, let the faces, edges, and vertices of Π be mapped, preserving their relations to one another, onto a sphere S. Interior to each face select a vertex. These will be the vertices of a so-called dual "polyhedron" or map. If two faces f_1 and f_2 have an edge in common, let the dual vertices v_1' and v_2' on these two faces be joined by a new edge e', crossing e at a single point and meeting no other edge or vertex of either map. These new edges, such as e', belong to the dual map and subdivide S into "faces," each containing just one vertex of the original map. It is easily verified that if v is the vertex of Π on a given face f' of the dual map, then v belongs to as many faces of Π as f' has vertices in the dual map. Since the relations between the maps are symmetric, the duality principle follows.

EULER'S THEOREM. This result can be expressed by the equation

$$V + F = E + 2. \tag{3.4}$$

It is a self-dual relationship. Geometrically, it means that the number of vertices plus the number of faces of any polyhedron must be exactly two greater than

the number of edges.

For simplicity, Euler's Theorem is here stated and discussed only for a convex polyhedron Π, even though the theorem is of much more general applicability. Since Π is convex, it can be mapped by a central projection from an interior point O onto a sphere S with O for center. The edges of the map are geodesic arcs. If the radius of S is 1, then the so-called "spherical excess" for any face of the map equals the area of that face. The *spherical excess* is the amount by which the sum of the angles of an n-sided face exceeds $(n-2)\pi$ radians. Let the area formula be applied to all F of the faces, and let all the angles at each of the V vertices be summed, giving $2\pi V$ radians. The sum of the areas of the faces is the entire area of S, namely 4π. Hence, since each edge belongs to exactly two faces,

$$2\pi V - (2E - 2F)\pi = 4\pi, \tag{3.5}$$

which is equivalent to (3.4).

The above theorem, although named for Euler, was known, perhaps a hundred years earlier, to Descartes, whose argument employed the angles of the plane faces of a polyhedron. Euler's independent discovery of the relation was achieved through an inductive process. The proof presented above is to be found in A. M. Legendre, *Éléments de géométrie* (1809) and somewhat earlier, in M. Hirsch, *Sammlung geometrischer Aufgaben* (1807). For other proofs and further historical details, see §5 of Steinitz (*op. cit.*).

Since each edge belongs to exactly two faces,

$$2E = 3F_3 + 4F_4 + \cdots + (V-1)F_{V-1}. \tag{3.6}$$

By a dual argument,

$$2E = 3V_3 + 4V_4 + \cdots + (F-1)V_{F-1}. \tag{3.7}$$

Since the left sides of (3.6) and (3.7) are even, the right sides must be even, and (D) is a consequence.

From (3.3), (3.6), and (3.7),

$$\begin{aligned} &\text{(a)}\quad 3F \leqq 2E, \\ &\text{(b)}\quad 3V \leqq 2E, \end{aligned} \tag{3.8}$$

where the equality sign prevails in (3.8a) if and only if all faces are triangular, and in (3.8b) if and only if each vertex belongs to exactly three faces.

Substituting from (3.4) into (3.8), one finds

$$\begin{aligned} &\text{(a)}\quad 3E - 3V + 6 \leqq 2E, \\ &\text{(b)}\quad 3E - 3F + 6 \leqq 2E, \end{aligned} \tag{3.9}$$

or

$$\begin{aligned} &\text{(a)}\quad E + 6 \leqq 3V, \\ &\text{(b)}\quad E + 6 \leqq 3F. \end{aligned} \tag{3.10}$$

(I) The equality sign in (3.10a) holds if and only if all faces are triangular; that in (3.10b), if and only if each vertex belongs to just three faces.

From (3.10a) and (3.8b),

$$\frac{1}{3}E + 2 \leqq V \leqq \frac{2}{3}E. \tag{3.11}$$

If $E=7$, this implies

$$4\tfrac{1}{3} \leqq V \leqq 4\tfrac{2}{3}, \tag{3.12}$$

an absurdity establishing $E \neq 7$. It is more tedious to establish the comparatively dull fact, completing result (A), that $E=6, 8, 9, \cdots$ are all possible.

Next let equations (3.3) both be multiplied through by 6 and, from the results, let the respective equations (3.7) and (3.6) be subtracted. The resulting equation implies

$$\begin{aligned} &\text{(a)} \quad 6V - 2E \leqq 3V_3 + 2V_4 + V_5, \\ &\text{(b)} \quad 6F - 2E \leqq 3F_3 + 2F_4 + F_5. \end{aligned} \tag{3.13}$$

Result (B) follows from (3.10) and (3.13).

In relations (3.13b) the equality sign prevails if and only if $V_j = F_j = 0$, $(j=7, 8, \cdots, V-1)$, a condition which is implied by the hypotheses of the first part of (E), since the only non-zero terms on the right side of (3.13) are V_3 and F_5. Hence, using statement (I),

$$F_5 = 12. \tag{3.14}$$

This establishes the first part of result (E). The second part follows from a dual argument.

Next, following the method in Sommerville,* let equations (3.3) be multiplied by 4 and added together. From that sum, let the sum of equations (3.6) and (3.7) be subtracted, giving

$$4(V + F - E) = (F_3 + V_3) - (F_5 + V_5) - 2(F_6 + V_6) - \cdots . \tag{3.15}$$

Equations (3.4) and (3.15) imply result (C), thus completing the establishment of results (A)$-$(E).

* D. M. Y. Sommerville, Geometry of n Dimensions, London, 1929, p. 148.

FROM THE EDITOR

With this issue the term of the present editor of the MONTHLY comes to a close. Five years of meeting deadlines, evaluating manuscripts, and reading proof is a long period of time while in progress, but, in retrospect, it seems only a few months ago that plans for the editorship were being made—plans that in many respects were not to be realized. The MONTHLY, in common with most scholarly journals, is an organ addressed to those who want to stay abreast of their profession; at the same time, it represents an outlet for those who want to report upon their work and interests. There are occasions when this double purpose creates a dilemma that is difficult to resolve. Moreover, every ambitious editor soon learns that his policies are modified by the nature of the articles that can be solicited or are contributed; for example, sources for some kinds of mathematical papers seem to be virtually non-existent. The judgment and decisions of the retiring editor, controversial as they may be, are now a part of the record; it is his humble hope that the issues of this journal during the past five years have made a significant contribution to American mathematics.

The editor of the MONTHLY is assisted by a staff of associate editors, who, like himself, must carry out their responsibilities to the journal in their "after hours" and without tangible remuneration. To the members of this staff must be awarded major credit for such approbation as the journal may have received; they have been an unusually able and conscientious group of co-workers. The new editor, Carl Allendoerfer, has been a member of this staff; he has demonstrated a notable comprehension of the editorial problems faced by the MONTHLY and of the role that the magazine must play in the future of mathematics. It is a personal pleasure to wish him success in the five years ahead.

C. V. NEWSOM

ERRATA

The following errata in recent volumes have been called to the attention of the editors:

G. Y. Rainich, abstract of the paper *A vector proof of the Pascal theorem*, vol. 58.

p. 68, replace the third word of the third line of the abstract, namely, "of," by "off."

W. W. Boone, abstract of the paper *Effective processes and turing machines*, vol. 58.

p. 370, replace the first word of the next to the last line of the abstract, namely, "creative," by "effective."

MATHEMATICAL NOTES

EDITED BY F. A. FICKEN, *New York University*

Material for this department should be sent to F. A. Ficken, Department of Mathematics, University of Tennessee, Knoxville 16, Tenn.

A MEASURE-THEORETIC RELATION BETWEEN A FUNCTION AND ITS RECIPROCAL

S. STEIN, Columbia University

In this note the fundamental theorem on the limit of a sequence of summable functions is used to obtain a theorem establishing a measure-theoretic relation between a function and its reciprocal. Among other applications, this result sheds light on ideas related to Fubini's theorem on the evaluation of repeated integrals.

A "set" will be understood to be a "bounded subset of Euclidean n-space." The characteristic function of a set is Lebesgue integrable if and only if the set is Lebesgue measurable. If the characteristic function of a set is Riemann integrable, we say for simplicity that the set is Riemann measurable. We shall distinguish Lebesgue measure by the use of a prime, *i.e.* we shall write m' for Lebesgue measure, $\int'$ for the Lebesgue integral.

Let X and Y be two sets, possibly with common elements. Let 2^Y denote the set of subsets of Y. Let f be a function on X into 2^Y; that is, for each x in X, $f(x)$ is a subset of Y. We define the reciprocal function $f^{-1}(y) = \{x \mid x \in X, y \in f(x)\}$; that is, $f^{-1}(y)$ is the set of those x in X for which $y \in f(x)$.

The theorem and proof below are based on [1] in which the iterated integration of real functions is studied.

THEOREM: *Let f be a function on X into 2^Y and let f^{-1} be its reciprocal (on Y into 2^X). If $f(x)$ is Riemann measurable for each $x \in X$ and $f^{-1}(y)$ is Lebesgue measurable for each $y \in Y$, then the Lebesgue integral $\int'_X mf(x)dm'$ and the Riemann integral $\int_Y m'f^{-1}(y)dm$ exist and are equal.*

Proof: Let $F \subset X \times Y$ be the graph of f, that is, $F = \{(x, y) \mid y \in f(x)\}$. Let $k(x, y)$ be the characteristic function of F as a subset of $X \times Y$. Since $f(x)$ is Riemann measurable we have $mf(x) = \lim_{r\to 0} \sum_i k(x, y_i) m Y_i$ where the Y_i are subsets of Y with diameter less than r forming a partition for the Riemann approximating sum; y_i is an arbitrary point in Y_i.

But $k(x, y_i)$ as a function of x is summable over X since $f^{-1}(y_i)$ is Lebesgue measurable. Thus the function of x, $\sum_i k(x, y_i) m Y_i$, is summable over X. Moreover, since this function is bounded by $m(Y)$, by the fundamental theorem on the limit of summable functions we see that $mf(x)$ is summable, and indeed, that its Lebesgue integral is the limit of the integral of the summable functions. Therefore $\int'_X mf(x)dm' = \lim_{r\to 0} \int'_X \sum_i k(x, y_i) m Y_i dm' = \lim_{r\to 0} \sum_i \int'_X k(x, y_i) dm' m Y_i = \lim_{r\to 0} \sum_i m'f^{-1}(y_i) m Y_i$. Thus the Riemann integral of $mf^{-1}(y)$ over Y exists and equals the Lebesgue integral of $mf(x)$ over X. This ends the proof.

In terms of the notion of reciprocal functions, Fubini's theorem on the interchange of order of integration in a repeated integral may be stated this way: *If the graph of* f *is Lebesgue measurable then* $f(x)$ *and* $f^{-1}(y)$ *are measurable for almost all* x *and* y *and* $\int_X' m'f(x)dm' = m'F = \int_Y' m'f^{-1}(y)dm'$.

However it is possible for the two cross-sectional integrals to exist and equal each other although the graph F is not measurable. Sierpinski [2] gives an example with $X = Y =$ the unit interval. If the graph F of f is measurable then Fubini's theorem implies ours. In any event it suggests an intuitive way of looking at our theorem; namely, consider our two integrals as integrals of cross-sectional lengths, one horizontal and the other vertical.

Since Riemann measurability and integrability imply Lebesgue measurabilty and integrability we have the following corollary of our theorem.

COROLLARY: *Let* f *be a function on* X *into* 2^Y. *If* $f(x)$ *and* $f^{-1}(y)$ *are Riemann measurable for all* x *and* y *then the two Riemann integrals* $\int_X mf(x)dm$ and $\int_Y mf^{-1}(y)dm$ *exist and are equal.*

Whether this statement remains true when the word Riemann is replaced by the word Lebesgue does not seem to be known.

We present some applications of this corollary.

Application 1: Let $X = Y$ be a convex n-dimensional body. Let $f(x)$ be the largest subset of X symmetric with respect to the point x. The symmetric mean of X is defined as $1/mX\int_X mf(x)/mXdm$. By the corollary the symmetric mean equals $1/(mX)^2\int_X mf^{-1}(x)dm$. This latter integral can be computed with ease, for $mf^{-1}(x) = 2^{-n}m(X)$ independently of the point x. Indeed $f^{-1}(x)$ can easily be seen to be a magnification of X around x by the factor $\frac{1}{2}$.

Application 2: Let X be a Riemann measurable subset of the plane. Let L and L' be two lines intersecting in the point P. They break the plane into four sections. Let A and B be two opposite sections each including the portions of the lines L and L' forming their boundaries. Translate this configuration so that P coincides with the point $x \in X$. Denote by A' and B' the translates of A and B. Let $f(x) = X \cap A'$ and $g(x) = X \cap B'$. We assert that the average area of $f(x)$ equals the average area of $g(x)$.

To show this it is sufficient to prove $\int_X mf(x)dm = \int_X mg(x)dm$. But this follows from the corollary and the easily verified fact that $g(x) = f^{-1}(x)$.

Application 3: Let C denote "complement of." If f is a function on X into 2^X such that $f(x)$ and $f^{-1}(x)$ are Riemann measurable and $mf^{-1}(x)$ equals $mCf(x)$ for all x, then $\int_X mf(x)dm = (mX)^2/2$.

The proof is short.
$$\begin{aligned}(mX)^2 &= \int_X mXdm = \int_X m(f(x) + Cf(x))dm \\ &= \int_X mf(x)dm + \int_X mCf(x)dm = \int_X mf(x)dm + \int_X mf^{-1}(x)dm \\ &= \int_X mf(x)dm + \int_X mf(x)dm = 2\int_X mf\ (x)dm.\end{aligned}$$

The following functions defined on a Riemann measurable subset of the plane satisfy the hypotheses of this application. (1) $f(x)$ the set of points of X to the left of x. (2) $f(x)$ the set of points of X nearer than x to a fixed point x_0. (3) more generally let there be given level curves of a sufficiently continuous function g.

Take $f(x)$ the set of points of X for which g assumes values smaller than $g(x)$.

Application 4: For simplicity we state this application in geographical terms; a precise statement is not difficult to formulate. Let there be a lake with a smooth shore, perhaps containing islands. Then the average fraction of the shore visible from points of the lake equals the average fraction of the lake visible from points of the shore.

To begin the proof let X be the lake and Y be the shore. Let $f(x)$ be the set of points of the shore visible from the point x of the lake. Since visibility is a symmetric relation $f^{-1}(y)$ is the set of points of the lake visible from the point y of the shore. After making suitable assumptions on X and Y we apply the corollary to get $\int_X mf(x)dm = \int_Y mf^{-1}(y)dm$. Division of both sides by $mX \cdot mY$ yields $1/mX\int_X mf(x)/mY\,dm = 1/mY\int_Y mf^{-1}(y)/mX\,dm$. This ends the proof.

References

1. Leon Lichtenstein, Ueber die Integration eines bestimmten Integrals in Bezug auf einen Parameter, Göttinger Nachrichten, 1910, pp. 468–475.

2. Waclaw Sierpinski, Sur un problème concernant les ensembles mesurables superficiellement, Fundamenta Mathematicae, 1920, pp. 112–115.

A NOTE ON THE DENUMERABILITY OF THE RATIONAL NUMBERS

W. J. HARRINGTON, Pennsylvania State College

In a recent paper* Johnston has presented a one-to-one correspondence between the rational numbers and the positive integers, with the correspondence expressed constructively in terms of an algorithm. Niven** has given another constructive correspondence. The purpose of this note is to present a modification of the Johnston system which makes possible a simpler formulation of the algorithm.

We start with the following indicated array of positive rationals:

$$\begin{array}{llllllll}
1 \\
1/2, & 2 \\
1/3, & 2/3, & 3/2, & 3 \\
1/4, & 2/5, & 3/5, & 3/4, & 4/3, & 5/3, & 5/2, & 4 \\
\cdots
\end{array}$$

Each successive row is formed by placing below each fraction a/b the new fraction $a/(a+b)$ and then extending the new row by writing the reciprocals of the new fractions in reverse order. It is convenient to express this construction-pattern in terms of two transformations, S and T. Let a/b denote a positive rational number in lowest terms, and then, by definition, let

$$S(a/b) = a/(a+b),$$

and

* This MONTHLY, vol. 55 (1948), pp. 65–70.

** *Ibid.*, vol. 55 (1948), p. 358.

$$T(a/b) = (a + b)/b.$$

If a certain row of the above array is $x_1, x_2, \cdots, x_k$, (k being a power of 2), then the next row can be written as

$$S(x_1), S(x_2), \cdots, S(x_k), T(x_1), T(x_2), \cdots, T(x_k),$$

provided we show that for each j, $1 \leqq j \leqq k$, $T(x_{k-j})$ is the reciprocal of $S(x_j)$. This is easily seen, for if $x_j = c/d$, then $x_{k-j} = d/c$, and $S(x_j) = c/(c+d)$ while $T(x_{k-j}) = (d+c)/c$. Thus the first four rows of our array can be expressed as follows (using the usual notation to express a succession of transformations):

$$\begin{array}{llllllll}
1 \\
S(1), & T(1) \\
S^2(1), & ST(1), & TS(1), & T^2(1) \\
S^3(1), & S^2T(1), & STS(1), & ST^2(1), & TS^2(1), & TST(1), & T^2S(1), & T^3(1), \\
\cdots,
\end{array}$$

and the construction-pattern of the array can be formulated as follows. The first half of each row beyond the first results from the application of S to the successive elements of the preceding row, and the second half from the corresponding application of T.

It will be established below that each positive rational number appears exactly once in this array, and our system for counting the non-negative rationals is based on the array in the following fashion. Take 0 as the first non-negative rational, 1 as the second, $\frac{1}{2}$ and 2 as third and fourth, respectively, and so on, taking the numbers in each successive row in the order in which they appear, reading from left to right. In the nth row, there are 2^{n-1} rationals to be counted, and the total number counted, starting with 0 and continuing through the nth row will be

$$1 + 2^0 + 2^1 + 2^2 + \cdots + 2^{n-1} = 2^n.$$

It is convenient to label the 2^{n-1} successive positions (left to right) in the nth row with the 2^{n-1} successive integers, 0 through $(2^{n-1}-1)$, but with these integers expressed in the binary system (*i.e.* to the base 2) using *exactly* $(n-1)$ digital positions. For instance, in row 4, there are 2^3 positions to be labelled successively

$$000, 001, 010, 011, 100, 101, 110, 111.$$

If, with reference to these labels, we let 0 correspond to the transformation S and 1 to T, we can think of 000 as denoting S^3, 001 as S^2T, 010 as STS, and so on, obtaining thereby a one-to-one correspondence between the labels and the transformations exhibited in the array. This correspondence also holds in rows 2 and 3, and by induction is seen to hold throughout the array. For, assuming the correspondence in row n, $n \geqq 2$, we consider how row $(n+1)$ is obtainable from row n. The first 2^{n-1} elements are obtained by applying S to the 2^{n-1} elements

of row n. The labels of these 2^{n-1} elements of row $(n+1)$ agree in numerical significance with those of row n, and the placing of the 0, corresponding to S, at the left-hand end in the required extra digital position has precisely this effect. Each of the remaining 2^{n-1} elements of row $(n+1)$ is obtained by applying T to an element of row n, and its label with the extra 1 (corresponding to T) at the left-hand end exceeds in numerical significance that of the element of row n by exactly 2^{n-1}, as it should. Thus the correspondence holds throughout the array, and it enables us to present a one-to-one correspondence between the non-negative rationals and the positive integers in a fairly simple fashion.

First, we wish to show that every positive rational appears exactly once in the array. We employ inverses of S and T, to be denoted by S^{-1} and T^{-1}, but we choose to define these for restricted ranges only, as follows:

if $$0 < a/b < 1, \qquad S^{-1}(a/b) = a/(b-a),$$

and

if $$a/b > 1, \qquad T^{-1}(a/b) = (a-b)/b.$$

Note that if a/b is in lowest terms, then the fraction resulting from the application of one of S, T, S^{-1}, or T^{-1} is also in lowest terms. To an arbitrary positive rational number, a/b, ($a/b \neq 1$, and in lowest terms), S^{-1} or T^{-1}, but not both, is applicable, according to our restricted definitions of the inverses, and produces a positive rational, c/d, with $d<b$ or $c<a$. If $c/d \neq 1$, *either* S^{-1} *or* T^{-1} *is applicable*, and in a finite number of such steps (each uniquely determined and each resulting in a decreased denominator or numerator), the fraction 1/1, is obtainable, since at each stage the numerator and denominator are relatively prime positive integers. One obtains some such expression as

$$1 = S^{-h}T^{-i}S^{-j} \cdots S^{-p}T^{-q}(a/b),$$

and thus

$$a/b = T^{q}S^{p} \cdots S^{j}T^{i}S^{h}(1),$$

where $h, i, j, \cdots p, q$ denote positive integers. If $h+i+j+\cdots+p+q=r$, this locates a/b in row $(r+1)$, and when the corresponding label (determined by $T^qS^p \cdots S^jT^iS^h$) is set up and interpreted as an integer, s, $0 \leqq s < 2^r$, a/b is located as the $(s+1)$-st number (from the left) in row $(r+1)$. That a/b occurs *exactly* once in the array is implied by the fact that not both S^{-1} and T^{-1} are applicable at any stage of this process.

Our counting system constructs a one-to-one correspondence, $a/b \leftrightarrow N$, between the non-negative rational numbers a/b, and the positive integers, N, as follows:

I. *Given a/b, to find N:*

If $a/b=0$ or 1, $N=1$ or 2, respectively. In all other cases, apply transformations S^{-1} and T^{-1} as outlined above so as to locate a/b as the $(s+1)$-st element in row $(r+1)$. Then $N=2^r+s+1$.

II. *Given* N, *to find* a/b:

If $N=1$ or 2, $a/b=0$ or 1, respectively. If $N\geqq 3$, determine n and k such that $N=2^{n-1}+k$, with $0<k\leqq 2^{n-1}$. This locates a/b as the kth number in row n. Express $(k-1)$ in the binary system using exactly $(n-1)$ digital positions. Set up the corresponding transformation and apply it to 1 to obtain a/b.

Before illustrating the above numerically, it is worthwhile to note a few simple properties of S, T, S^{-1}, and T^{-1}:

1) $S^k(a/b)=a/(ka+b)$;

2) $T^k(a/b)=(a+kb)/b$;

3) If $1\leqq a<b$, and $b=ka+c$, $0<c<a$, then starting with a/b, S^{-1} is applicable k successive times and $S^{-k}(a/b)=a/c$;

4) If $1\leqq b<a$, and $a=kb+c$, $0<c<b$, then starting with a/b, T^{-1} is applicable k successive times and $T^{-k}(a/b)=c/b$.

To illustrate I and II above, we consider the following cases:

I. Let $a/b=17/30$; to find N: Since $0<17/30<1$, S^{-1} is applicable, and we obtain successively $S^{-1}(17/30)=17/13$; $T^{-1}(17/13)=4/13$; $S^{-3}(4/13)=4/1$, and $T^{-3}(4/1)=1$. Thus $T^{-3}S^{-3}T^{-1}S^{-1}(17/30)=1$, and $STS^3T^3(1)=17/30$. The corresponding label is 01000111 and this corresponds to $2^6+2^2+2+1=71$. Thus 17/30 appears as the 72nd number in row 9, and $N=2^8+72=328$.

II. Let $N=743$; to find a/b: Since $743=2^9+231$, we locate a/b as the 231st number in row 10. Writing $230=2^7+2^6+2^5+2^2+2$, we obtain the label 011100110, using the 9 required digital positions, and thus the transformation $ST^3S^2T^2S$. To determine $ST^3S^2T^2S(1)$, five simple steps are needed: $S(1)=1/2$; $T^2(1/2)=5/2$; $S^2(5/2)=5/12$; $T^3(5/12)=41/12$; and $S(41/12)=41/53$. Thus $a/b=41/53$.

This correspondence can be extended to include the negative rationals in the same manner as that used by Johnston. If $a/b\leftrightarrow N$ in the above arrangement, let the new ordinal $(2N-1)$ be assigned to a/b, and if $N>1$, let $(2N-2)$ be assigned to $(-a/b)$.

CLASSROOM NOTES

EDITED BY C. B. ALLENDOERFER, University of Washington

All material for this department should be sent to G. B. Thomas, Department of Mathematics, Massachusetts Institute of Technology, Cambridge 39, Mass.

ON THE DIRECT SOLUTION OF BERNOULLI'S EQUATION

A. N. AHEART, West Virginia State College

In searching a number of texts on differential equations and a number of texts on differential and integral calculus with chapters devoted to differential

equations, it was found that the solution of the Bernoulli Equation was canonically performed by making the transformation of variable, $z=y^{-n+1}$, thereby reducing it to the linear type $(dy/dx)+Py=Q$. However, such a reduction is not necessary since it is relatively easy to derive by a direct approach a formula for the complete solution. Hence, it may be of some interest to describe the direct method of solving the Bernoulli Equation and to also state in readily available form a formula for the complete solution.

Consider the equation

$$\frac{dy}{dx}+Py=Qy^n \qquad (n\neq 0,\ n\neq 1), \tag{1}$$

where P and Q are constants or functions of x alone.

To integrate (1), let

$$y=uz, \tag{2}$$

where $u=f(x)$ and $z=g(x)$ to be determined. Differentiating (2), we have

$$\frac{dy}{dx}=u\frac{dz}{dx}+z\frac{du}{dx}\cdot \tag{3}$$

Substituting from (2) and (3) in (1), we get

$$u\frac{dz}{dx}+z\frac{du}{dx}+Puz=Qu^nz^n, \quad \text{or}$$

$$u\frac{dz}{dx}+\left(\frac{du}{dx}+Pu\right)z=Qu^nz^n. \tag{4}$$

Determine u by integrating

$$\frac{du}{dx}+Pu=0 \tag{5}$$

in which the variables u and x are separable.

$$\int\frac{du}{u}+\int Pdx=\ln C,$$

where $\ln C$ is the constant of integration.

For the sake of simplicity and without the loss of any generality assume the particular value of zero for the integration constant $\ln C$, i.e., $C=1$. Whence,

$$\ln u=-\int Pdx \quad \text{or} \quad u=e^{-\int Pdx}. \tag{6}$$

Using the value of u from (6), we find z by solving

$$u\frac{dz}{dx} = Qu^n z^n, \tag{7}$$

in which the variables x and z are now separable.

$$\int \frac{dz}{z^n} = \int Qu^{n-1}dx + K.$$

On performing this integration we have $z^{n-1} = F(x)$. The solution follows immediately. The solution may be stated as

$$y^{n-1} = u^{n-1}z^{n-1}.$$

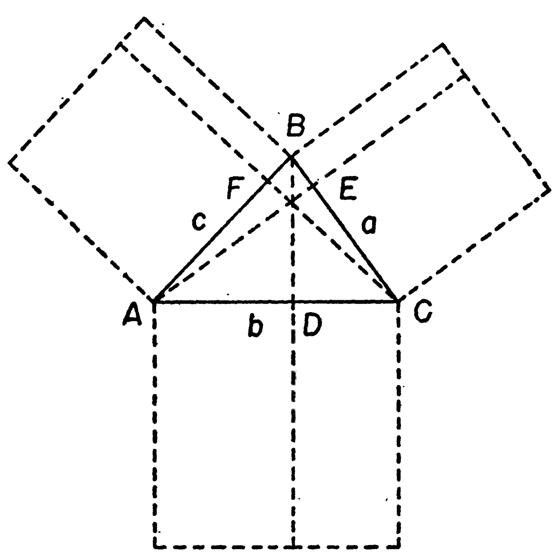
Thus, we have as the complete solution of (1)

$$(1 - n)y^{n-1}\int Qe^{-(n-1)\int Pdx}dx + K(1 - n)y^{n-1} - e^{-(n-1)\int Pdx} = 0. \tag{8}$$

NOTE ON THE LAW OF COSINES

S. L. Thompson, Alabama Polytechnic Institute

Most trigonometry texts use the theorem of Pythagoras to prove the Law of Cosines. Since the Pythagorean theorem is a special case of the Law of Cosines, it seems a little more satisfactory to prove the Law of Cosines directly.



This proof may be based on the definition of the cosine of an angle. In fact one may represent squares and products as areas which reminds the student of the familiar drawing for the theorem of Pythagoras and shows him how this differs from the Law of Cosines.

Our proof may be presented as follows; In triangle ABC (with sides a, b and c) draw altitudes AE, BD and FC. Assign a positive direction to each side so

that in passing counter-clockwise around the triangle one is always moving in a positive direction.

$$\begin{aligned} b^2 &= b(AD + DC) \\ &= cb \cos A + ab \cos C = c(FA) + a(CE) \\ &= c(c - a \cos B) + a(a - c \cos B) \\ &= c^2 + a^2 - 2ac \cos B. \end{aligned}$$

The Law of Cosines needs some emphasis due to the fact that it is used in mathematics courses following trigonometry. However with the extensive use of desk calculators or the like, by those doing much computing, this is a practical formula for solving triangles.

ELEMENTARY PROBLEMS AND SOLUTIONS

EDITED BY HOWARD EVES, State University of New York

Send all communications concerning Elementary Problems and Solutions to Howard Eves, Mathematics Department, Champlain College, Plattsburg, New York. This department welcomes problems believed to be new, and demanding no tools beyond those ordinarily furnished in the first two years of college mathematics. To facilitate their consideration, solutions should be submitted on separate, signed sheets, within three months after publication of problems.

PROBLEMS FOR SOLUTION

E 991. *Proposed by C. S. Ogilvy, Columbia University*

Two ships leave different points at the same time and steam on straight courses at constant but unequal speeds. Find the condition under which the slower ship can intercept (catch) the faster.

E 992. *Proposed by Kaidy Tan, Anglo-Chinese College, Amoy, China*

Draw a straight line which will bisect both the area and the perimeter of a given quadrilateral.

E 993. *Proposed by F. M. Carpenter, Colorado School of Mines*

Find the orthogonal trajectories of the family of cycloids

$$x = a(\theta - \sin \theta),$$
$$y = a(1 - \cos \theta).$$

E 994. *Proposed by C. S. Venkataraman, Trivandrum, South India*

Prove that $2!4! \cdots (2n)! > (n+1)!^n$.

E 995. *Proposed by L. C. Hsu, National Tsing-Hua University, Peiping, China*

Find the sum of the finite series

$$\sum_{n=1}^{N} (n/2^{2n})\binom{2n}{n}.$$

SOLUTIONS

An April First Quickie

E 961 [1951, 259]. *Proposed by Leo Moser, Texas Technological College*

Find the probability that if the digits 0, 1, 2, $\cdots$, 9 be placed in random order in the blank spaces of

5_383_8_2_936_5_8_203_9_3_76

the resulting number will be divisible by 396. (By permission of Prof. E. P. B. Umbugio, April 1, 1951.)

Solution by Prasert Na Nagara, College of Agriculture, Thailand. Since 76 is divisible by 4, the sum of all digits (irrespective of how the blanks are filled in) is divisible by 9, and the difference between the sum of all even placed digits and the sum of all odd placed digits is divisible by 11, it follows that the number is divisible by 4, 9, and 11, and thus by 396. Therefore the probability is 1.

Also solved by Aaron Bakst, Martin Berman, W. E. Buker, P. L. Chessin, Monte Dernham, R. P. Eisinger, Louisa Grinstein, V. C. Harris, Vern Hoggatt, H. A. James, E. S. Keeping, J. D. E. Konhauser, W. G. McGavock, Richard Powds, L. B. Rall, L. A. Ringenberg, C. W. Trigg, G. W. Walker, and the proposer.

Trigg wondered if Professor Umbugio granted permission because he knew that he, the problem's number, and its palindrome are all perfect squares, and that $(961-169)/2=396$. Or did he consider that the unlucky aspect of the square root of the palindrome is counterbalanced by $(1.1)(1951-1591)=396$ and $[1591-7 \text{ (for luck)}]/[4 \text{ (for April)}]=396$?

Gergonne and Nagel Points

E 964 [1951, 260]. *Proposed by C. W. Trigg, Los Angeles City College*

(1) Determine the relationship between the sides of a triangle ABC if the line joining the Gergonne point P to the Nagel point Q is parallel to side c.

(2) Show that, given a and b, such a triangle can always be constructed with straightedge and compasses.

(3) Find the smallest scalene triangle of this type with integer sides.

Solution by the Proposer. (1) Denote the feet of the cevians through P and Q by A_P, B_P, C_P and A_Q, B_Q, C_Q. Then (*cf.* N. A. Court, *College Geometry*, sec. 245)

$$CP/PC_P = CA_P/A_PB + CB_P/B_PA = (s-c)/(s-b) + (s-c)/(s-a),$$
$$CQ/QC_Q = CA_Q/A_QB + CB_Q/B_QA = (s-b)/(s-c) + (s-a)/(s-c).$$

Equating the expressions for CP/PC_P and CQ/QC_Q (since PQ is parallel to BA) we obtain, after solving for c and simplifying,

$$c = (a^2 + b^2)/(a + b).$$

(2) A euclidean construction of c is easily effected by first finding, say, the hypotenuse d of the right triangle having legs a and b. Then c can be found as the third proportional to $a+b$ and d.

(3) All the triangles with integer sides may be obtained readily from $c=b-a+2a^2/(b+a)$. The smallest scalene triangle has sides $a=2$, $b=6$, $c=5$.

Also solved by C. V. Fronabarger and Joseph Langr.

Langr showed that the locus of a point P, for which the segment joining P and its isotonic conjugate Q is always parallel to side BA of triangle ABC, is an ellipse (E) passing through the centroid of ABC and touching sides CA and CB at A and B. The corresponding ellipses where PQ is parallel to AC or CB are congruent and homothetic to (E).

As an allied problem Langr proposed the construction of a triangle ABC, given the positions of vertex C and of a pair of isotomic conjugate points P and Q, such that side BA is parallel to PQ.

An Application of the Mean Value Theorem

E 965 [1951, 260]. *Proposed by R. K. Meany, Purdue University*

A function $f(x)$ is continuous in the open interval (a, b). Its derivative, $f'(x)$ exists at every point of (a, b) except perhaps at c, but $\lim_{x\to c} f'(x)$ exists and equals A. Show that then $f'(c)$ also exists and equals A.

Solution by F. Bagemihl, University of Rochester. By the mean value theorem

$$[f(c+h) - f(c)]/h = f'(\xi),$$

where ξ is between c and $c+h$. Taking the limit of both sides as $h\to 0$ we obtain $f'(c)=A$.

This problem is not new; see, *e.g.*, R. Courant, *Differential and Integral Calculus*, Nordemann, N. Y., 1938, vol. I, p. 200.

Also solved by Martin Berman, W. Fulks, B. F. Hadnot, David Mandelbaum, C. S. Ogilvy, L. B. Rall, L. A. Ringenberg, C. J. Standish, J. A. Tierney, Peter Treuenfels, and the proposer.

Treuenfels found a solution to the problem in Gustav Doetsch, *Theorie und Anwendung der Laplace-Transformation*, Hilfssatz 3, p. 399.

ADVANCED PROBLEMS AND SOLUTIONS

EDITED BY E. P. STARKE, Rutgers University

Send all communications concerning Advanced Problems and Solutions to E. P. Starke, Rutgers University, New Brunswick, New Jersey. All manuscript should be typewritten, with double spacing and margins at least one inch wide. Problems containing results believed to be new or extensions of old results are especially sought. Proposers of problems should also enclose any solutions or information that will assist the editor. In general, problems in well known textbooks or results in readily accessible sources should not be proposed for this department.

PROBLEMS FOR SOLUTION

4446 [1951, 422]. Corrected. *Proposed by Victor Thébault, Tennie, Sarthe, France*

In an equifacial tetrahedron the lines which join the vertices to the circumcenter (or the orthocenter) of the opposite faces form a hyperbolic system. If the lines are concurrent, the tetrahedron is regular and conversely.

4463. *Proposed by R. Deaux, Faculté Polytechnique, Mons, Belgium*

The lines AP, BP, CP drawn through the vertices and a given point P in the plane of triangle ABC meet the opposite sides in A_1, B_1, C_1. Let A', B', C' be the points which with A, B, C respectively separate harmonically the pairs (B_1, C_1), (C_1, A_1), (A_1, B_1). Prove that the pairs (A, A'), (B, B'), (C, C') belong to a Möbius involution.

4464. *Proposed by R. Kissling, University of California, Berkeley*

Consider the class of single-valued differentiable functions $f(x)$ on the interval $0 \leqq x < 1$ such that $f(0)=0$; $f(x) \to \infty$ as $x \to 1^-$. Let

$$K(x) = f'(x)/\{1 + [f(x)]^2\}^{3/2}, \qquad |K(x)| \leqq 1,\ 0 \leqq x < 1.$$

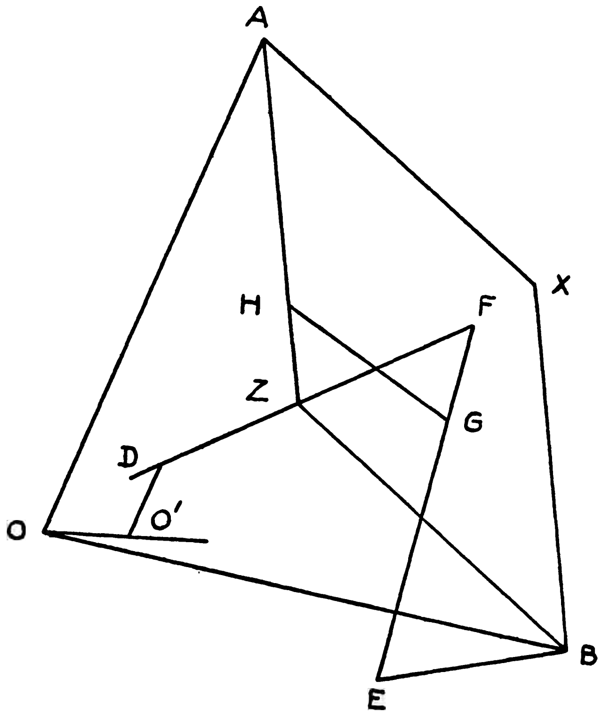
Prove that for any function of this class

$$\int_0^1 \{1 + [f(x)]^2\}^{1/2} dx = \pi/2.$$

4465. *Proposed by E. I. Gale, University of Saskatchewan*

$AXBZ$ is a jointed rhombus with sides of length $4a$. (See figure. For convenience a is taken considerably greater than half the unit.) AO and BO are bars of equal length. The fixed centers are O and O' with OO' half a unit. Let $O'D = \frac{1}{2}$, $FE = 4a$, $FZ = HG = BE = 2a$, $HZ = FG = a$. D is an adjustable set screw on bar FZD so that the length ZD can be set at pleasure.

Show that as D moves in a circle about O', X describes the general conic of eccentricity $1/ZD$.



4466. *Proposed by Leo Moser, University of Alberta*

Let

$$0 = a_1 < a_2 < \cdots < a_n < 1, \qquad 0 = b_1 < b_2 < \cdots < b_m < 1,$$

and suppose that $a_i + b_j = 1$ is not solvable. Prove that if the mn numbers $a_i + b_j$ are reduced (mod 1) then at least $m+n-1$ of the residues will be distinct.

4467. *Proposed by D. J. Newman, New York University*

Prove that αz^n is the only entire function whose modulus is constant for all $|z| = 1$.

SOLUTIONS

Sums of Greatest Integer Functions

4390′ [1950, 265]. *Proposed by Roshan Lal Gupta, Government College, Ludhiana, India*

If $[x]$ denotes the greatest integer function, prove that

$$\sum_{q \geqq 1} [(x/q^m)^{1/n}] = \sum_{q \geqq 1} [(x/q^n)^{1/m}].$$

Is there any relation between the terms of the two sums?

Solution by Hansraj Gupta, Government College, Hoshiarpur. Evidently, if r is any positive integer,

$$[(x/q^m)^{1/n}] \geqq r$$

if $[x/q^m] \geqq r^n$, that is, if $q \leqq [(x/r^n)^{1/m}]$. Therefore on the left side there are $[(x/r^n)^{1/m}]$ terms each not less than r. Since this is the rth term of the right side, the result follows immediately. If the common sum be denoted by S, it will be noticed that the two sides provide conjugate partitions of S.

Also solved by H. S. Zuckerman.

A Bounded Sum

4391 [1950, 265]. *Proposed by P. T. Bateman, University of Illinois*

Given a fixed positive integer k and a complex-valued function $f(n)$ defined on the positive integers and such that $f(n_1) = f(n_2)$ for $n_1 \equiv n_2 \pmod{k}$, $|f(n)| \leqq 1$ for all n, $f(n) = 0$ for $(n, k) > 1$, and $\sum_{n=1}^{k} f(n) = 0$. Show that

$$\left| \sum_{n=1}^{\infty} \frac{f(n)}{n} \right| < \log k.$$

Solution by Robert Breusch, Amherst College. For the purpose of this theorem, the following weaker conditions are sufficient:

$$|f(n)| \leqq 1, \tag{1}$$

$$f(k) = 0, \tag{2}$$

$$\sum_{\nu=pk+1}^{pk+k} f(\nu) = 0 \text{ for every integer } p \geqq 0. \tag{3}$$

Then for every $N = sk + t$,

$$\sum_{n=1}^{N} \frac{f(n)}{n} = \sum_{p=0}^{s-1} \sum_{r=1}^{k} \frac{f(pk + r)}{pk + r} + \sum_{n=sk+1}^{N} \frac{f(n)}{n},$$

or, by (1) and (3),

$$\sum_{n=1}^{N} \frac{f(n)}{n} = \sum_{p=0}^{s-1} \sum_{r=1}^{k} f(pk + r) \left[\frac{1}{pk + r} - \frac{1}{pk + k} \right] + \sum_{n=sk+1}^{N} \frac{f(n)}{n},$$

$$\left| \sum_{n=1}^{N} \frac{f(n)}{n} \right| \leqq \sum_{p=0}^{s-1} \left(\sum_{r=1}^{k} \frac{1}{pk + r} - \frac{k}{pk + k} \right) + \frac{k}{sk}$$

$$= \sum_{n=1}^{sk} \frac{1}{n} - \sum_{p=0}^{s-1} \frac{1}{p + 1} + \frac{1}{s} = \sum_{n=s+1}^{sk} \frac{1}{n} + \frac{1}{s} < \log sk - \log s + \epsilon,$$

for ϵ arbitrary and N large enough. Thus, by conditions (1) and (3),

$$\left| \sum_{n=1}^{\infty} \frac{f(n)}{n} \right| \leqq \log k.$$

If now, condition (2) is imposed, there results

$$\left|\sum_{r=1}^{k}\frac{f(r)}{r}\right| \leqq \sum_{r=1}^{k-1}\frac{1}{r}-\frac{k-1}{k-1}=\sum_{r=1}^{k}\frac{1}{r}-\frac{k}{k}-\frac{1}{k},$$

so that the previous result can be improved:

$$\left|\sum_{n=1}^{\infty}\frac{f(n)}{n}\right| \leqq \log k-\frac{1}{k}<\log k.$$

Also solved by Paul Erdös, Emil Grosswald, R. S. Lehman, and the Proposer.

Derivative at a Point of Maximum Modulus

4392 [1950, 265]. *Proposed by Paul Erdös, University of Aberdeen, Scotland*

Let $f(z)$ be analytic for $|z| \leqq 1$. Let z_0 be the point ($|z_0|=1$) where $|f(z)|$ assumes its maximum on the unit circle. Prove that $f'(z_0)\neq 0$.

Solution by George Piranian, University of Michigan. We shall prove a slightly stronger proposition: *Let $f(z)$ be regular at z_0; let k be the least natural number such that $f^{(k)}(z_0)\neq 0$; and let Δ be a domain bounded by a polygon which has at z_0 a vertex whose angle is greater than π/k. Then every neighborhood of z_0 contains a point z in Δ such that $|f(z)|>|f(z_0)|$.*

If $f(z_0)=0$, the result is trivial. If $f(z_0)\neq 0$, let

$$\frac{f^{(k)}(z_0)}{f(z_0)k!}=re^{i\phi},\qquad z-z_0=\rho e^{i\theta}.$$

Then, from the Taylor's expansion,

$$f(z)=f(z_0)\left[1+r\rho^k e^{i(\phi+k\theta)}+O(\rho^{k+1})\right]$$

throughout some neighborhood of z_0. But every neighborhood of z_0 contains a point z in Δ for which $\cos(\phi+k\theta)$ exceeds some positive constant independent of the neighborhood; and, for all such points that are near enough to z_0, $|f(z)|>|f(z_0)|$.

Also solved by Robert Breusch, W. B. Fulks, G. A. Panangat, W. Seidel, and the Proposer.

Definite Integral Evaluation

4394 [1950, 265]. *Proposed by H. F. Sandham, Trinity College, Ireland*

Evaluate

$$\int_0^{\infty}\frac{\log x}{e^x+1}\,dx.$$

Solution by Ernest Trost, Technikum Winterthur, Switzerland. We shall evaluate the integral

$$I(a) = \lim_{\delta \to 0} I(a, \delta) = \lim_{\delta \to 0} \int_\delta^\infty \frac{\log x}{1 + e^{ax}} dx,$$

where a is a positive constant. Integration by parts yields, because $\log t \log (1 \pm t) \to 0$ as $t \to 0$,

$$I(a, \delta) = \frac{1}{a} \log \delta \log (1 + e^{-a\delta}) + \frac{1}{a} \int_\delta^\infty \frac{dx}{x} \log (1 + e^{-ax}).$$

Further, we get

$$\begin{aligned}
\int_\delta^\infty \frac{dx}{x} \log (1 + e^{-ax}) &= \int_\delta^\infty \frac{dx}{x} \log (1 - e^{-2ax}) - \int_\delta^\infty \frac{dx}{x} \log (1 - e^{-ax}) \\
&= \int_{2\delta}^\infty \frac{du}{u} \log (1 - e^{-au}) - \int_\delta^\infty \frac{du}{u} \log (1 - e^{-au}) \\
&= - \int_\delta^{2\delta} \frac{du}{u} \log (1 - e^{-au}) = - \int_1^2 \frac{dy}{y} \log (1 - e^{-a\delta y}) \\
&= - \int_1^2 \frac{dy}{y} \log a\delta y \\
&\quad - \int_1^2 \frac{dy}{y} \log \left\{1 - \frac{a\delta y}{2!} + \frac{(a\delta y)^2}{3!} - \cdots \right\} \\
&= - \log \delta \log 2 - \tfrac{1}{2} \log^2 2a + \tfrac{1}{2} \log^2 a - \epsilon(\delta),
\end{aligned}$$

where $\epsilon(\delta) \to 0$ as $\delta \to 0$. Now we have

$$I(a, \delta) = \frac{1}{a} \log \delta \log \frac{1 + e^{-a\delta}}{2} - \frac{\log 2 \log 2a^2}{2a} - \frac{1}{a} \epsilon(\delta).$$

Finally, for $\delta \to 0$ we obtain

$$I(a) = - \frac{1}{2a} \log 2 \log 2a^2, \quad \text{and} \quad I(1) = - \tfrac{1}{2} \log^2 2.$$

The special case $a=1$ has been solved previously as problem 26, in *Wiskundige Opgaven met de Oplossingen*, Deel XV (Groningen 1933).

Also solved by Robert Breusch, H. E. Fettis, W. B. Fulks, Eric Groth, D. J. Newman, E. H. Sondheimer, O. E. Stanaitis, R. E. Wild, and the Proposer.

A Ratio of Infinitesimals

4397 [1950, 342]. *Proposed by Albert Wilansky, Lehigh University*

Suppose $f''(0)$ exists, is finite, and is different from 0. Assume also $f(0)=0$. For sufficiently small $h>0$ we may write

$$\frac{f(-h)}{-h} = f'(\xi), \qquad \frac{f(h)}{h} = f'(\zeta),$$

with $-h<\xi<0<\zeta<h$. Prove that

$$\lim_{h\to 0} \frac{\zeta - \xi}{h} = 1.$$

Solution by G. Lumer, Facultad de Ingenieria, Montevideo, Uruguay. We shall obtain a somewhat more general result. Let $f(x)$ have, at the point a, right-hand derivatives of all orders $\leqq n+m$, where $f^{(n+m)}$ is supposed to be merely defined at the point a. (In the following the usual symbols for derivatives indicate right-hand derivatives only.) Then, if θ is defined by

$$(1) \qquad f(a+h) = f(a) + hf'(a) + \cdots + \frac{h^n}{n!} f^{(n)}(a+\theta h), \qquad h > 0,$$

and if $f^{(n+m)}$, the first derivative following $f^{(n)}$ which is different from 0 at a, exists and is finite at a, we shall show that

$$(2) \qquad \lim_{h\to 0} \theta = \binom{n+m}{m}^{-1/m}.$$

In fact, we have

$$(3) \qquad \begin{aligned} f(a+h) = f(a) + hf'(a) + \cdots + \frac{h^n}{n!} f^{(n)}(a) + \cdots \\ + \frac{h^{n+m}}{(n+m)!} \left\{ f^{(n+m)}(a) + \epsilon \right\}, \end{aligned}$$

where $\epsilon \to 0$ as $h \to 0$. On the other hand we have

$$(4) \qquad f^{(n)}(a+\theta h) = f^{(n)}(a) + \frac{\theta^m h^m}{m!} \left\{ f^{(n+m)}(a) + \epsilon' \right\},$$

since $f^{(n+1)}(a) = f^{(n+2)}(a) = \cdots = f^{(n+m-1)}(a) = 0$.

Now from (1), (3) and (4) we obtain

$$\frac{\theta^m}{n!m!} \left\{ f^{(n+m)}(a) + \epsilon' \right\} = \frac{1}{(n+m)!} \left\{ f^{(n+m)}(a) + \epsilon \right\}.$$

As $f^{(n+m)}(a)$ is finite and different from 0, this gives

$$\lim_{h\to 0} \frac{f^{(n+m)}(a) + \epsilon}{f^{(n+m)}(a) + \epsilon'} = 1,$$

whence

$$\lim_{h\to 0} \frac{\theta^m(m+n)!}{n!m!} = 1,$$

which establishes (2).

In the original proposal we had $n=m=1$, so that $\lim \theta=\frac{1}{2}$. Then, with $\zeta=\theta h$, $\lim \zeta/h=\frac{1}{2}$. Since the left-hand derivative of $f(x)$ exists at $x=0$ we have, similarly, $\lim -\xi/h=\frac{1}{2}$, so that $\lim (\zeta-\xi)/h=1$.

Also solved by F. Bagemihl and W. Seidel, Joshua Barlaz, Sol Ciolkowski, R. S. Margulies, Norman Miller, Lawrence Ringenberg, John Samoloff, N. T. Seely, Jr., O. E. Stanaitis, A. E. Taylor, J. H. Wahab, and the Proposer.

RECENT PUBLICATIONS

EDITED BY E. P. VANCE, Oberlin College

All books for review should be sent directly to the editor of this department, American Mathematical Monthly, 80 Waterman Street, Providence 6, Rhode Island, and not to any of the other editors or officers of the Association.

Mathematical Snapshots. By Hugo Steinhaus. New York, Oxford University Press, 1950. 6+266 pages. $4.50.

This book is a welcome addition to the literature of recreational and popularized mathematics. The author's purpose is "to visualize mathematics" by illustrated discussions of a miscellany of topics such as polyhedra, dissections and reticulations, various loci, geometric projections, and the topology of networks and surfaces; a few subjects are less pictorial. The number of illustrations—photographs and drawings—averages more than one per page.

In the items in problem form, the solution usually follows the problem without paragraphing. Readers may prefer to pause after the statement of the problem in order to work on the solution themselves. The discussion is at the level of the interested amateur, with more difficult proofs omitted, although such a question as the unanswered "how?" on p. 7 is more than an "exercise for the reader." References are given.

The first edition of this book was printed in Poland before the war (this MONTHLY, 46 (1939), p. 354). The present edition has a fifty percent increase in text and number of illustrations, and the binding is better. The anaglyphs (red-green three dimensional drawings) of the first edition have been replaced by more satisfactory photographs, and the separate gadgets have been omitted. The one real loss is that the self-erecting dodecahedron which popped out of the earlier book is now only illustrated. Any gadgeteer will want to build one.

Interesting new material in the new edition includes a counterfeit coin problem; the fair division of a cake or of an estate; curves and strategies of pursuit;

the dissection of a square into unequal squares; and a dissected cube to be reassembled.

This book should be pleasant reading for amateur and professional mathematicians, and an occasional source-book for high-school and college mathematics clubs.

BRYANT TUCKERMAN

Elementary Theory of Equations. By Samuel Borofsky. New York, The Macmillan Company, 1950. 10+302 pages. $4.25.

This text is devoted to the usual material (*cf.* L. E. Dickson's *First Course in the Theory of Equations*). The calculus is not assumed as a prerequisite and the necessary properties of derivatives of polynomials are derived algebraically assuming the fundamental theorem of algebra. The term *field* is used for *subfield of the complex field*. More than the usual emphasis is given to the divisibility properties of polynomials and to the expression of a complex number in terms of radicals relative to a given "field." Perhaps one may regard these innovations as "a bridge between the ideas of elementary algebra . . . and . . . modern algebra." This reviewer has serious doubts that this text will do more than implant false ideas of the concepts of modern algebra in the immature reader's mind. The respect with which the author views these concepts can be judged by his treatment of determinants (there is almost no distinction observed between *determinant* and *matrix* and the catless cat-grin floats over these pages). The author regards the fundamental idea of *polynomial* as "too abstract" for the "average beginner" but expects this same "average beginner" to absorb his clumsy and involved notions of "Budan sequence" and "Sturm sequence." This reviewer found the proofs offered to be in the same spirit of dogged refusal to see the beauty of mathematical argument. We can only conclude that this text will not furnish the student of pure mathematics with a proper basis for further study and that it is grossly inadequate in fundamental ideas and techniques (such as matric algebra, criteria for stability, and computation of complex roots) to be of value to students of applied mathematics.

M. F. SMILEY

CLUBS AND ALLIED ACTIVITIES

EDITED BY L. F. OLLMANN, Hofstra College

All material for this department should be sent to Professor H. D. Larsen, Albion College, Albion, Michigan.

CLUB REPORTS, 1950–51

Mathematics Club, University of Wisconsin

The papers presented at the various *Mathematics Club* meetings at the Uni-

versity of Wisconsin during the academic year were:

Mathematical education in Austria and Germany, by Dr. Richard Huka
Mathematical education in France and England, by Dr. L. C. Young
General introduction to the theory of divergent sequences, by Dr. Konrad Knopp
The word problem for abstract algebras, by Trevor Evans
Abstract Galois theory, by Dr. Marc Krasner
Certain queer sets in the plane, by Dr. R. H. Bing
On a general concept of rank in ring theory, by Dr. Oswald Wyler
Group algebras, by Dr. R. C. Buck
Latin squares, by Dr. R. H. Bruck
On the scarcity of alternative division rings, by Erwin Kleinfeld
Functional spaces and functional completion, by Dr. Nachman Aronszajn
On the ideal structure of the ring of entire functions, by Melvin Henriksen.

Pi Mu Epsilon, University of Wisconsin

The *Wisconsin Beta* chapter of *Pi Mu Epsilon* heard the following program of talks during 1950–51:

An introduction to alternative rings, by Erwin Kleinfeld
The ergodic theorem, by William Donoghue
The propositional calculus, by James Renno
Theory of games, by Prof. R. C. Buck
Weird functions, by Lee Rubel
The regular solids, by Robert Buehler
Weird fields, by Melvin Henriksen.

An initiation and open house were held in the fall, while an initiation banquet and picnic were held in the spring.

Officers for the year 1950–51 were: President, William Ames; Vice-President, Benjamin Mitchell; Secretary-Treasurer, Robert San Soucie; Faculty Advisor, Prof. R. E. Fullerton.

Mathematics Club, University of Kansas

The *Mathematics Club* of the University of Kansas presented the following programs during the year 1950–51:

Careers in mathematics, by Dr. G. B. Price
Curve fitting, by John Forman
Report on the International Congress, by Alan Showalter
Mathematical puzzles, by Dr. Clarence Grothaus
Theory of games, by Dr. W. R. Scott
Mathematics in India and China, by Dr. S. Chowla and K. Heu
Theorems associated with triangles and circles, by Dr. G. W. Smith
A mean value theorem, by Dr. Robert Schatten.

The annual picnic was held in the spring.

Officers for 1951–52 are: President, John Gerriets; Vice-President, Irvin Gaston; Secretary-Treasurer, Patricia White.

Kappa Mu Epsilon, Alabama College

Papers presented to the *Alabama Gamma* chapter of *Kappa Mu Epsilon* during the past year included:

The calender, by Sarah Peppenhorst and Rose Floyd

Time-telling throughout the ages, by Dorothy Champion and Irene Pace

Mathematics and art, by Betty Crow

Symbolism, by Gay Penn and Dean Ingram.

The first meeting of the year was a party to which all upper class men majoring in mathematics were invited. In the Spring six new members were initiated. A banquet followed the initiation.

The officers elected for 1951–52 are: President, Irene Pace; Vice-President, Dorothy Champion; Secretary, Rose Floyd; Treasurer, Elizabeth Cauley; Program Chairman, Joan Gregory; Faculty Sponsor, Mamie Braswell; Corresponding Secretary, Rosa Lea Jackson.

NEWS AND NOTICES

EDITED BY EDITH R. SCHNECKENBURGER, University of Buffalo

Readers are invited to contribute to the general interest of this department by sending news items to Edith R. Schneckenburger, University of Buffalo, Buffalo 14, New York. Items must be submitted at least two months before publication can take place.

EXAMINATION QUESTIONS OF THE METROPOLITAN NEW YORK SECTION

The Committee on Contests and Awards of the Metropolitan New York Section announces that it has extra copies of the examination questions used in the 1951 Contest for High School Students. It will forward copies of these questions to any mathematics teacher, department chairman, or high school principal from whom a request is received. All requests should be sent to Professor W. H. Fagerstrom, City College, New York 31, New York.

ACKNOWLEDGMENT

The editor of the MONTHLY wishes to make grateful acknowledgment of the services rendered by the following persons who have refereed papers: C. B. Allendoerfer, R. A. Beaumont, Garrett Birkhoff, L. M. Blumenthal, H. S. M. Coxeter, Orrin Frink, Jr., J. S. Frame, Caspar Goffman, L. M. Graves, G. E. Hay, A. S. Householder, R. L. Jeffery, S. A. Jennings, Wilfred Kaplan, M. S. Knebelman, D. H. Lehmer, C. C. MacDuffee, H. B. Mann, A. B. Mewborn, G. B. Price, J. F. Randolph, Gabor Szegö, Olga Taussky, R. W. Wagner, D. L. Webb, G. T. Whyburn, R. L. Wilder, E. M. Wright.

THE MATHEMATICAL ASSOCIATION OF AMERICA

Official Reports and Communications

CALENDAR OF FUTURE MEETINGS

Thirty-fifth Annual Meeting, Brown University, Providence, Rhode Island, December 29, 1951.

Thirty-third Summer Meeting, Michigan State College, East Lansing, Michigan, September 1–2, 1952.

The following is a list of the Sections of the Association with dates of future meetings so far as they have been reported to the Associate Secretary.

ALLEGHENY MOUNTAIN, Waynesburg College, Waynesburg, Pennsylvania, May 10, 1952.
ILLINOIS, Western Illinois State College, Macomb, May 9–10, 1952.
INDIANA, Indiana University, Bloomington, Spring, 1952.
IOWA, Coe College, Cedar Rapids, April 18–19, 1952.
KANSAS
KENTUCKY, University of Kentucky, Lexington.
LOUISIANA-MISSISSIPPI, Northwestern State College, Natchitoches, Louisiana, February 15–16, 1952.
MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA, National Bureau of Standards, Washington, D. C., December 8, 1951.
METROPOLITAN NEW YORK, Spring, 1952.
MICHIGAN, University of Michigan, Ann Arbor, April 12, 1952.
MINNESOTA, College of St. Catherine, St. Paul, May 10, 1952.
MISSOURI, Lindenwood College, St. Charles, Spring, 1952.
NEBRASKA
NORTHERN CALIFORNIA, Stanford University, Stanford, January 26, 1952.
OHIO, April 19, 1952.
OKLAHOMA
PACIFIC NORTHWEST, University of Oregon, Eugene, June 20, 1952.
PHILADELPHIA
ROCKY MOUNTAIN, Western State College, Gunnison, Colorado, May, 1952.
SOUTHEASTERN, Georgia Institute of Technology and Agnes Scott College, Atlanta, March 21–22, 1952.
SOUTHERN CALIFORNIA, Occidental College, Los Angeles, March 8, 1952.
SOUTHWESTERN, University of Arizona, Tucson, April 11–12, 1952.
TEXAS, East Texas State Teachers College, Commerce, April, 1952.
UPPER NEW YORK STATE, Hobart and William Smith Colleges, Geneva, May, 1952.
WISCONSIN

THE RHIND MATHEMATICAL PAPYRUS

Only twenty-five copies of the Rhind Papyrus are now on hand. When these have been sold, only second hand copies will be available through book dealers who may be able occasionally to procure them for sale.

The Mathematical Association of America has been enabled to distribute the Rhind Mathematical Papyrus through a gift from the late Arnold Buffum Chace, Chancellor of Brown University. This exposition of one of the very oldest mathematical documents in the world is of value to all students of mathematics and of the Egyptian civilization of 4,000 years ago. Volume I, $11\frac{1}{4}$ by 8 inches, $8+210$ pages, contains the Free Translation, Commentary, and Bibliography of Egyptian Mathematics; Volume II, $11\frac{1}{4}$ by $14\frac{1}{4}$ inches, contains 140 photographic plates in original colors, black and red, with Text and Introductions,

and Literal Translation. The price to members of the Association is $20 for the set of two volumes; to non-members the price is $25 for the set.

Members of the Association may purchase sets through the office of the Secretary of the Mathematical Association of America, University of Buffalo, Buffalo 14, New York. Non-members must purchase copies from the Open Court Publishing Company, La Salle, Illinois.

THE AMERICAN MATHEMATICAL MONTHLY

THE OFFICIAL JOURNAL OF

THE MATHEMATICAL ASSOCIATION OF AMERICA

(INCORPORATED)

LIST OF OFFICERS AND MEMBERS

DECEMBER 1951

THIS LIST WAS PREPARED BY
THE OFFICE OF THE SECRETARY-TREASURER,
THE MATHEMATICAL ASSOCIATION OF AMERICA, Inc.,
UNIVERSITY OF BUFFALO
BUFFALO 14, N.Y.

OFFICERS OF THE MATHEMATICAL ASSOCIATION OF AMERICA

President, SAUNDERS MACLANE, University of Chicago (1951–1952)
Honorary President, W. D. CAIRNS, Oberlin College
First Vice-President, L. M. GRAVES, University of Chicago (1950–1951)
Second Vice-President, JEWELL H. BUSHEY, Hunter College (1951–1952)
Editor, C. V. NEWSOM, University of the State of New York (1947–1951)
C. B. ALLENDOERFER, University of Washington (1952–1956)
Secretary-Treasurer, H. M. GEHMAN, University of Buffalo (1948–1952)
Associate Secretary, EDITH R. SCHNECKENBURGER, University of Buffalo (1948–1952)

ADDITIONAL MEMBERS OF THE BOARD OF GOVERNORS

Ex-Presidents

C. C. MACDUFFEE, University of Wisconsin (1947–1952)
L. R. FORD, Illinois Institute of Technology (1949–1954)
R. E. LANGER, University of Wisconsin (1951–1956)

Governors at Large

C. B. ALLENDOERFER, University of Washington (1949–1951)
R. J. WALKER, Cornell University (1949–1951)
M. R. HESTENES, University of California at Los Angeles (1950–1952)
MARIE J. WEISS, Newcomb College (1950–1952)
H. S. M. COXETER, University of Toronto (1951–1953)
B. W. JONES, University of Colorado (1951–1953)

Sectional Governors (July 1, 1949–June 30, 1952)

Kansas, R. G. SANGER, Kansas State College
Missouri, G. M. EWING, Sandia Corporation
Ohio, F. B. WILEY, Ashland College
Pacific Northwest, M. S. KNEBELMAN, State College of Washington
Southeastern, TOMLINSON FORT, University of Georgia
Southwestern, EARL WALDEN, New Mexico College of A. and M. A.
Upper New York State, E. B. ALLEN, Rensselaer Polytechnic Institute
New England Region, R. E. GILMAN, Brown University

Sectional Governors (July 1, 1950–June 30, 1953)

Illinois, A. E. GAULT, Bradley University
Iowa, E. N. OBERG, University of Iowa (Resigned September 3, 1951)
W. M. DAVIS, Cornell College (Elected September 3, 1951)
Louisiana-Mississippi, T. A. BICKERSTAFF, University of Mississippi
Maryland-District of Columbia-Virginia, G. R. CLEMENTS, U. S. Naval Academy
Michigan, J. S. FRAME, Michigan State College
Minnesota, R. H. CAMERON, University of Minnesota
Philadelphia, G. E. RAYNOR, Lehigh University
Southern California, C. G. JAEGER, Pomona College
Texas, E. H. HANSON, North Texas State College

Sectional Governors (July 1, 1951–June 30, 1954)

Allegheny Mountain, J. B. ROSENBACH, Carnegie Institute of Technology*
Indiana, J. C. POLLEY, Wabash College
Kentucky, AUGHTUM S. HOWARD, Kentucky Wesleyan College
Metropolitan New York, T. F. COPE, Queens College
Nebraska, J. M. EARL, University of Omaha
Northern California, E. B. ROESSLER, University of California at Davis
Oklahoma, J. C. BRIXEY, University of Oklahoma
Rocky Mountain, C. F. BARR, University of Wyoming
Wisconsin, H. P. PETTIT, Marquette University

* Died, November 6, 1951.

STANDING COMMITTEES OF THE ASSOCIATION

FINANCE COMMITTEE

W. B. CARVER (1948–1951), J. F. RANDOLPH (1950–1953), H. M. GEHMAN, *ex officio*

EDITORIAL COMMITTEE ON CARUS MONOGRAPHS

PHILIP FRANKLIN, *Chairman* (1946–1951), J. L. SYNGE (1947–1952), H. S. M. COXETER (1948–1953), KARL MENGER (1949–1954), N. H. MCCOY (1950–1955), I. S. SOKOLNIKOFF (1951–1956).

COMMITTEE ON THE ARNOLD BUFFUM CHACE FUND

R. C. ARCHIBALD, R. W. BRINK, W. D. CAIRNS, W. R. LONGLEY

COMMITTEE ON PLACES OF MEETINGS

R. W. BRINK, *Chairman* (1949–1951), J. M. THOMAS (1950–1952), ORRIN FRINK, JR. (1951–1953)

COMMITTEE ON THE PUTNAM PRIZE COMPETITION

B. H. BROWN, *Chairman* (July 1949–June 1952), E. P. STARKE (July 1950–June 1953), R. G. SANGER (July 1951–June 1954), L. E. BUSH, Director (July 1948–June 1953)

COMMITTEE ON SECTIONS

FRED ROBERTSON (1949–1952), C. O. OAKLEY (1951–1954), EDITH R. SCHNECKENBURGER, *ex officio*

COMMITTEE ON SLAUGHT MEMORIAL PAPERS

W. T. REID. *Chairman* (1950–1952), C. V. NEWSOM (1950–1951), DEANE MONTGOMERY (1951–1953), B. W. JONES (1951, 1952–1954).

REPRESENTATIVES OF THE ASSOCIATION

On the Policy Committee for Mathematics:
C. V. NEWSOM (1949–1951), SAUNDERS MACLANE, *ex officio*, H. M. GEHMAN, *ex officio*
On the National Research Council:
A. W. TUCKER (July 1, 1950–June 30, 1953)
On the Council of the American Association for the Advancement of Science:
J. R. MUSSELMAN (1950–1951), W. H. GOTTSCHALK (1951–1952)
On the American Council on Education:
SAUNDERS MACLANE, C. V. NEWSOM, H. M. GEHMAN, *ex officiis*
On the A.A.A.S. Cooperative Committee on the Teaching of Mathematics and Science:
J. R. MAYOR (1950–1952)
On the Committee on Definitions of Electrical Terms:
S. A. SCHELKUNOFF
On the Committee on the Mathematical Training of Social Scientists:
F. L. GRIFFIN, E. P. NORTHROP

ASSOCIATE EDITORS OF THE AMERICAN MATHEMATICAL MONTHLY

C. B. ALLENDOERFER, E. F. BECKENBACH, L. M. BLUMENTHAL, N. B. CONKWRIGHT, H. S. M. COXETER, HOWARD EVES, F. A. FICKEN, L. J. GREEN, G. E. HAY, CAROLINE A. LESTER, W. T. MARTIN, N. H. MCCOY, L. F. OLLMAN, EDITH R. SCHNECKENBURGER, E. P. STARKE, E. P. VANCE

SECTIONS OF THE ASSOCIATION

Allegheny Mountain

Morris Ostrofsky, Duquesne University, *Chairman*
F. H. Steen, Allegheny College, *Secretary-Treasurer*

Illinois

S. S. Cairns, University of Illinois, *Chairman*
E. W. Schreiber, Western Illinois State College, *Vice-Chairman*
E. C. Kiefer, James Millikin University, *Secretary-Treasurer*

Indiana

M. W. Keller, Purdue University, *Chairman*
Florence Long, Earlham College, *Vice-Chairman*
J. C. Polley, Wabash College, *Secretary-Treasurer*

Iowa

L. E. Ernsdorff, Loras College, *Chairman*
L. A. Knowler, State University of Iowa, *Vice-Chairman*
Fred Robertson, Iowa State College, *Secretary-Treasurer*

Kansas

Anna Marm, Bethany College, *Chairman*
P. M. Young, Kansas State College, *Vice-Chairman*
Laura Z. Greene, Washburn University, *Secretary-Treasurer*

Kentucky

J. A. Ward, University of Kentucky, *Chairman*
G. G. Roberts, Berea College, *Secretary-Treasurer*

Louisiana-Mississippi

Arthur Ollivier, Mississippi State College, *Chairman*
C. G. Killen, Northwestern State College, *Vice-Chairman for Louisiana*
Virginia Felder, Mississippi Southern College, *Vice-Chairman for Mississippi*
F. A. Rickey, Louisiana State University, *Secretary-Treasurer*

Maryland-District of Columbia-Virginia

E. R. Sleight, University of Richmond, *Chairman*
C. H. Frick, Mary Washington College, *Secretary-Treasurer*

Metropolitan New York

James Singer, Brooklyn College, *Chairman*
L. F. Ollmann, Hofstra College, *Vice-Chairman*
E. I. Shapiro, Abraham Lincoln High School, *Vice-Chairman*
H. S. Kieval, Brooklyn College, *Secretary*
Aaron Shapiro, Midwood High School, *Treasurer*

Michigan

N. H. Anning, University of Michigan, *Chairman*
P. S. Jones, University of Michigan, *Secretary-Treasurer*

Minnesota

H. D. Colson, Bemidji State Teachers College, *Chairman*
F. C. Smith, College of St. Thomas, *Secretary-Treasurer*

Missouri

Francis Regan, St. Louis University, *Chairman*
P. B. Burcham, University of Missouri, *Vice-Chairman*
S. Louise Beasley, Lindenwood College, *Secretary-Treasurer*

Nebraska

W. G. Leavitt, University of Nebraska, *Chairman*
Morris Dansky, Creighton University, *Vice-Chairman*
Edwin Halfar, University of Nebraska, *Secretary-Treasurer*

NORTHERN CALIFORNIA

D. H. LEHMER, University of California, *Chairman*
C. D. OLDS, San Jose State College, *Vice-Chairman*
MARJORIE L. HOFFMAN, San Mateo Junior College, *Secretary-Treasurer*

OHIO

R. F. RINEHART, Case Institute of Technology, *Chairman*
FOSTER BROOKS, Kent State University, *Secretary-Treasurer*

OKLAHOMA

C. M. PIRRONG, JR., Oklahoma City University, *Chairman*
R. R. MURPHY, Panhandle A. and M. College, *Vice-Chairman*
J. C. BRIXEY, University of Oklahoma, *Secretary-Treasurer*

PACIFIC NORTHWEST

R. D. JAMES, University of British Columbia, *Chairman*
A. S. MERRILL, Montana State University, *Vice-Chairman*
J. M. KINGSTON, University of Washington, *Secretary-Treasurer*

PHILADELPHIA

P. A. CARIS, University of Pennsylvania, *Chairman*
C. O. OAKLEY, Haverford College, *Secretary-Treasurer*

ROCKY MOUNTAIN

C. H. COOK, Western State College, *Chairman*
B. W. JONES, University of Colorado, *Vice-Chairman*
J. R. BRITTON, University of Colorado, *Secretary-Treasurer*

SOUTHEASTERN

B. G. CLARK, Vanderbilt University, *Chairman*
H. K. FULMER, Georgia Institute of Technology, *Vice-Chairman*
H. A. ROBINSON, Agnes Scott College, *Secretary-Treasurer*

SOUTHERN CALIFORNIA

W. H. GLENN, JR., Pasadena City Schools, *Chairman*
C. W. TRIGG, Los Angeles City College, *Vice-Chairman*
P. H. DAUS, University of California at Los Angeles, *Secretary-Treasurer*

SOUTHWESTERN

CHARLES WEXLER, Arizona State College, *Chairman*
J. H. BUTCHART, Arizona State College, *Vice-Chairman*
R. L. WESTHAFER, New Mexico College of A. and M. A., *Secretary-Treasurer*

TEXAS

D. W. STARR, Southern Methodist University, *Chairman*
C. B. WRIGHT, East Texas State Teachers College, *Vice-Chairman*
C. R. SHERER, Texas Christian University, *Secretary-Treasurer*

UPPER NEW YORK STATE

C. W. MUNSHOWER, Colgate University, *Chairman*
J. F. RANDOLPH, University of Rochester, *Vice-Chairman*
N. G. GUNDERSON, University of Rochester, *Secretary-Treasurer*

WISCONSIN

P. E. MEADOWS, Carroll College, *Chairman*
LOUISE A. WOLF, University of Wisconsin in Milwaukee, *Secretary-Treasurer*

PERIODS OF SERVICE OF FORMER OFFICERS OF THE ASSOCIATION

(Except for the offices of President and Secretary-Treasurer, this list includes only the names of those who have held office since January 1, 1943. For information about preceding years, consult previous editions of the List of Officers and Members)

Presidents

E. R. Hedrick....1916
Florian Cajori....1917
E. V. Huntington....1918
H. E. Slaught....1919
D. E. Smith....1920
G. A. Miller....1921
R. C. Archibald....1922
R. D. Carmichael....1923
H. L. Rietz....1924
J. L. Coolidge....1925
Dunham Jackson....1926
W. B. Ford....1927–1928
J. W. Young....1929–1930
E. T. Bell....1931–1932
Arnold Dresden....1933–1934
D. R. Curtiss....1935–1936
A. J. Kempner....1937–1938
W. B. Carver....1939–1940
R. W. Brink....1941–1942
W. D. Cairns....1943–1944
C. C. MacDuffee....1945–1946
L. R. Ford....1947–1948
R. E. Langer....1949–1950

Vice-Presidents

Tomlinson Fort....1942–1943
C. C. MacDuffee....1943–1944
W. M. Whyburn....1944–1945
W. F. Cheney, Jr....1945–1946
W. L. Ayres....1946–1947
C. B. Allendoerfer....1947–1948
Saunders MacLane....1948–1949
N. H. McCoy....1949–1950

Secretary-Treasurer

W. D. Cairns....1916–1942
W. B. Carver....1943–1947

Associate Secretary

B. W. Jones....1943–1947
Harry Pollard....1947

Editor

L. R. Ford....1942–1946

Governors

(arranged alphabetically)

C. R. Adams....1948–1950
R. P. Agnew....1942–1944
W. L. Ayres....1942–1944, 1948–1950
H. M. Bacon....1941–1943
R. H. Bardell....1948–1951
Walter Bartky....1945–1947
H. M. Beatty....1946–1948
L. M. Blumenthal....1942–1944
J. W. Bradshaw....1947–1950
H. E. Bray....1943–1945, 1947–1950
M. C. Brown....1948–1951
L. E. Bush....1947–1950
C. C. Camp....1948–1951
W. F. Cheney, Jr....1942–1944
N. A. Court....1945–1947, 1948–1951
H. S. M. Coxeter....1945–1947
L. L. Dines....1945–1947
H. L. Dorwart....1948–1951
Arnold Dresden....1943–1945
W. L. Duren....1947–1950
P. D. Edwards....1948–1951
H. J. Ettlinger....1941–1943
Cornelius Gouwens....1941–1943
D. W. Hall....1943–1945, 1947–1950
E. S. Hammond....1946–1949
E. H. C. Hildebrandt....1947–1950
R. C. Huffer....1945–1947
Ralph Hull....1946–1948
C. G. Jaeger....1943–1945
W. C. Krathwohl....1941–1943
Gillie A. Larew....1945–1947
J. W. Lasley, Jr....1944–1946
C. G. Latimer....1942–1944
D. H. Lehmer....1947–1949
A. J. Lewis....1948–1951
Saunders MacLane....1943–1945
H. F. MacNeish....1944–1946
W. T. Martin....1947–1949
Sophia L. McDonald....1945–1947
E. J. McShane....1941–1943
A. S. Merrill....1946–1949
F. H. Miller....1948–1951
W. E. Milne....1942–1944
W. L. Miser....1944–1946
Deane Montgomery....1946–1948
F. R. Morris....1948–1951
E. J. Moulton....1943–1945
A. L. Nelson....1943–1945
C. V. Newsom....1946–1948
F. S. Nowlan....1944–1946
F. W. Owens....1941–1943
W. V. Parker....1943–1945
E. J. Purcell....1944–1946
W. R. Ransom....1944–1946
O. H. Rechard....1942–1944
A. W. Recht....1946–1948
H. A. Robinson....1942–1944
S. T. Sanders....1941–1943
R. G. Sanger....1944–1946
G. W. Smith....1944–1946
H. L. Smith....1945–1947
I. S. Sokolnikoff....1947–1950
E. P. Starke....1947–1950
H. P. Thielman....1947–1950
Morgan Ward....1944–1946
C. W. Watkeys....1946–1948
K. W. Wegner....1943–1945
R. L. Wilder....1942–1944
K. P. Williams....1945–1947
W. L. Williams....1946–1949

MEMBERS OF THE MATHEMATICAL ASSOCIATION OF AMERICA

This list gives the following information, in so far as it is available, for each person who is a member of the Association as of October 15, 1951: (1) Surname with initials, or the given name when there is but one, except that a given name of every woman member is given in full. In case of a married woman, the husband's initials are given. (2) Highest academic degree received, with name of institution conferring the degree. (3) Official title or position. (4) Name and address of institution or organization. (5) Address to be used for mail if different from (4). The name of city and postal zone are not repeated if they are the same as in (4).

ABBEDUTO, L. J., B.S. (Illinois Tech.) Instr., Hardin County Schools, Kenton, Ohio. *324 N. Main*
ABBEY, JANET E., B.A. (Wm. Smith) Teacher, Griffith Inst. and Central School, Springville, N.Y. *2470 South Ave., Niagara Falls, N.Y.*
ABBOTT, J. C., Ph.D. (Notre Dame) Asso. Prof., U. S. Naval Academy, Annapolis, Md. *99 Monticello Ave.*
ABLETT, C. B., M.S. (Southern Illinois) Asst. Prof., Lebanon Valley Coll., Annville, Pa.
ABLOW, C. M., Ph.D. (Brown) Research Engr., Boeing Airplane Co., Seattle, Wash. *1221 Taylor Ave. (19)*
ABRAMOWITZ, WALTER, B.S. (Brooklyn Poly.) Research Fellow, Poly. Inst. of Brooklyn, Brooklyn, N.Y. *180 Grafton St. (12)*
ABRAMSON, L. R., Student, Columbia Coll., New York, N.Y. *315 E. 206 St. (67)*
ACKERSON, R. H., A.M. (Columbia) Asst. Prof., Catawba Coll., Salisbury, N.C.
ADAMS, B. T., A.M. (Baylor) Training Specialist, Veteran's Administration, Wichita Falls, Tex. *2172 Avenue I*
ADAMS, C. R., Ph.D. (Harvard) Prof., Brown Univ., Providence 12, R.I. *60 Intervale Rd. (6)*
ADAMS, E. P., Ph.D. (Harvard) Emeritus Prof., Physics, Princeton Univ., Princeton, N.J. *Walpole, Mass.*
ADAMS, LOUISE, A.M. (North Carolina) Asst. Prof., High Point Coll., High Point, N.C.
ADAMS, L. J., A.M. (Southern California) Head of Dept., Santa Monica City College., Santa Monica, Calif. *227 25th St.*
ADAMS, O. S., D.Sc. (Kenyon) Principal Mathematician, Retired, U. S. Coast and Geodetic Survey, Washington 25, D.C. *R.F.D. 4, Mt. Vernon, Ohio*
ADKINS, J. B., Ed.M. (Harvard) Teacher, Phillips Exeter Acad., Exeter, N.H.
ADKINS, L. K., M.S. (Chicago) Emeritus Prof., State Teachers Coll., La Crosse, Wis. *1810 Cass St.*
ADKISSON, V. W., Ph.D. (Pennsylvania) Dean, Grad. School, Univ. of Arkansas, Fayetteville, Ark. *236 Buchanan St.*
ADLER, CLAIRE F. (Mrs. E. H.), Ph.D. (N.Y.U.) Asso. Prof., New York Univ., New York, N.Y. *189-21 Tioga Dr., St. Albans, N.Y.*
ADNEY, J. E., Jr., M.A. (Ohio State) Asst., Ohio State Univ., Columbus 10, Ohio
AGARD, H. L., Ph.D. (Yale) Emeritus Prof., Williams Coll., Williamstown, Mass. *The Knolls*
AGGARWAL, KANHAYALAL, B.A. (Panjab) Proprietor, Messrs. Hariram, Kanhayalal, Aggarwal, Majith Mandi, Amritsar, Panjab, India
AGNEW, R. P., Ph.D. (Cornell) Prof., Cornell Univ., Ithaca, N.Y.
AHEART, A. N., A.M. (Harvard) Asst. Prof., West Virginia State Coll., Institute, W. Va.
AHLFORS, L. V., Ph.D. (Helsingfors) Prof., Harvard Univ., Cambridge, Mass. *45 Yale St., Winchester, Mass.*
AHLIN, J. T., M.A. (Southern California) Research Engr., Douglas Aircraft Co., Santa Monica, Calif. *635 W. 77th St., Los Angeles 44, Calif.*
AHRENDT, M. H., M.A. (Wichita) Executive Secretary, National Council of Teachers of Mathematics, 1201 Sixteenth St., N.W., Washington 6, D.C.
AILARA, R. C., B.S. (Illinois Tech.) Junior Systems Engr., Electronics Sect., Aerophysics Group, Chance Vought Aircraft, Dallas, Tex.
AINSWORTH, EDITH W., (Mrs. O. R.), M.A. (California) *20 Dubois Ter., Tuscaloosa, Ala.*
AISSEN, M. I., B.S. (C.C.N.Y.) Instr., Univ. of Pennsylvania, Philadelphia 4, Pa.
AKERS, L. W., M.A. (Kansas) Asso. Prof., Central Missouri Coll., Warrensburg, Mo.
ALBERT, A. A., Ph.D. (Chicago) Prof., Univ. of Chicago, Chicago 37, Ill.
ALBERT, A. H., M.S. (Michigan) Instr., Kansas State Teachers Coll., Emporia, Kans.
ALBERT, DAVID, B.S. (Connecticut) Prod. Supervisor, Talon, Inc., Hamden, Conn. *24 Carmel St., New Haven, Conn.*
ALBERT, O. W., Ph.D. (U. of Washington) Prof., Univ. of Redlands, Redlands, Calif. *629 Buena Vista St.*
ALBERT, R. G., B.S. (Brooklyn) Grad. Student, Brown Univ., Providence 12, R.I. *62 Meeting St. (6)*
ALBERTI, FURIO, B.S. (Chicago) Instr., Univ. of Illinois, Navy Pier, Chicago 11, Ill. *Rt. 3, Paw Paw, Mich.*
ALBISER, Rev. H. B., M.S. (Notre Dame) Instr., St. Michael's Coll., Winooski, Vt.

ALDEN, H. H., Ph.D. (Ohio State) Asso. Prof., Ohio State Univ., Columbus 10, Ohio. *317 Northridge Rd. (2)*
ALDER, H. L., Ph.D. (California) Asst. Prof., Univ. of California at Davis, Davis, Calif.
AL-DHAHIR, M. W., M.A. (Columbia) Grad. Student, Columbia Univ., New York, N.Y. *255 W. 108th St., Apt. 6-D (25)*
ALEXANDER, H. W., Ph.D. (Princeton) Prof., Adrian Coll., Adrian, Mich.
ALEXANDER, J. W., Ph.D. (Princeton) Prof., Inst. for Advanced Study, Princeton, N.J. *29 Cleveland Lane*
ALFIERI, F. A., B.S. (C.C.N.Y.) Asst. Supervisor, Metropolitan Life Insurance Co., 1 Madison Ave., New York, N.Y. *11 Summer Lane, Hicksville, L.I., N.Y.*
AL-GHITA, M. K., M.S. (Michigan) Grad. Student, Univ. of Michigan, Ann Arbor, Mich. *507 Benjamin St.*
ALKIRE, DON, M.A. (South Dakota) Instr., Univ. of South Dakota, Vermillion, S.D.
ALLEN, BESS E., Ph.D. (Cincinnati) Asst. Prof., Wayne Univ., Detroit 1, Mich.
ALLEN, E. B., Ph.D. (Rensselaer) Prof., Rensselaer Poly. Inst., Troy, N.Y. *4 Sheldon Ave.*
ALLEN, E. F., Ph.D. (Missouri) Prof., Oklahoma A. and M. Coll., Stillwater, Okla. *1409 College Ave.*
ALLEN, FLORENCE E., Ph.D. (Wisconsin) Asst. Prof., Univ. of Wisconsin, Madison 6, Wis. *219 Lathrop St.*
ALLEN, W. R., M.S. (Northwestern) Mathematician, Univ. of Chicago, Chicago 37, Ill. *5822 Blackstone Ave.*
ALLENDOERFER, C. B., Ph.D. (Princeton) Prof., Univ. of Washington, Seattle 5, Wash.
ALLHANDS, TYLER, M.A. (Arizona) *2204 26th Ave., Forest Grove, Ore.*
ALLIOT, EUGENE, Lic. es Sc. (Sorbonne) Trustee, St. Michael's Coll., Winooski Park, Vt.
ALMAN, J. E., M.A. (Claremont) Instr., Boston Univ., Boston 15, Mass.
AL-SALAM, W. A., B.S. (California) *27/5/58 Hariri St., Baghdad, Iraq*
AMMANN, W. H., M.S. (St. Louis) Grad. Student, St. Louis Univ., St. Louis, Mo. *5344a Terry Ave. (20)*
AMOS, C. E., Ph.D. (Ohio State) Asso. Prof., Univ. of Toledo, Toledo, Ohio. *3027 W. Bancroft St. (6)*
ANDERSON, A. D., M.A. (Oregon) Mathematician, Naval Research Lab., Washington, D.C. *38 Galveston Pl., S.W.*
ANDERSON, DOROTHY J., B.A. (Macalester) Instr., Macalester Coll., St. Paul 5, Minn. *1568 Summit Ave.*
ANDERSON, E. W., Ph.D. (Iowa S.C.) Prof., Iowa State Coll., Ames, Iowa
ANDERSON, FLORENCE R., M.A. (U.C.L.A.) Research Lab. Analyst A, Northrop Aircraft Inc., Hawthorne, Calif. *4924 Fourth Ave., Los Angeles 43, Calif.*
ANDERSON, H. M., Ph.M. (Wisconsin) Asst. Prof., Gustavus Adolphus Coll., St. Peter, Minn. *716 W. Broadway*
ANDERSON, J. M., Student, Univ. of South Dakota, Vermillion, S.D. *Box 761, Custer, S.D.*
ANDERSON, P. H., Ph.D. (Illinois) *7605 Wildwood Dr., Takoma Park 12, Md.*
ANDERSON, R. D., Ph.D. (Texas) Member, Inst. for Advanced Study, Princeton, N.J.
ANDERSON, R. E., M.S. (Chicago) Asst. Prof., Northern Illinois State Teachers Coll., De Kalb, Ill.
ANDERSON, R. LUCILE, Ph.D. (Bryn Mawr) Asst. Prof., Hunter Coll., New York 21, N.Y.
ANDERSON, S. A., M.A. (George Peabody) Asso. Prof., Memphis State Coll., Memphis, Tenn. *414 Holmes Circle*
ANDERSON, W. E., Ph.D. (Pennsylvania) Emeritus Prof., Miami Univ., Oxford, Ohio. *112 E. Walnut St.*
ANDERTON, ETHEL L., Ph.D. (Yale) Teacher, High School, West Haven 16, Conn. *215 Park Terrace Ave.*
ANDREE, R. V., Ph.D. (Wisconsin) Asst. Prof., Univ. of Oklahoma, Norman, Okla.
ANDREWS, J. J., A.M. (St. Louis) Lecturer, St. Louis Univ., St. Louis, Mo. *155 S. Sappington Rd., Kirkwood 22, Mo.*
ANDRUS, W. E., Jr., A.B. (Syracuse) Technical Engr., International Business Machines Corp., Endicott, N.Y. *10 Allan Village, Watson Blvd.*
ANNEAR, P. R., Ph.D. (Michigan) Prof., Baldwin-Wallace Coll., Berea, Ohio. *280 Eastland Rd.*
ANNING, NORMAN, M.A. (Queen's U.) Asst. Prof., Univ. of Michigan, Ann Arbor, Mich. *1925 Packard St.*
ANNIS, EDITH W. (Mrs. L. E.), A.B. (Hunter) Head of Dept., Mamaroneck Senior High School, Mamaroneck, N.Y. *333 Florence St.*
ANSELONE, P. M., M.S. (Puget Sound) Mathematician, General Electric Co., Richland, Wash. *514 Wright Ave.*
ANTHONY, M. L., *Gunsaulus Hall, Apt. 904, 3140 S. Michigan, Chicago 16, Ill.*
ANTOSIEWICZ, H. A., Ph.D. (Vienna) Asso. Prof., Montana State Coll., Bozeman, Mont.
APOSTOL, T. M., Ph.D. (California) Asst. Prof., California Inst. of Tech., Pasadena 4, Calif.

APPLEBAUM, F. H., B.A. (Pennsylvania) Asst. Life Actuary, Insurance Dept., Commonwealth of Pennsylvania, 121 N. Broad St., Philadelphia, Pa. *4837 N. Franklin St. (20)*
APPUHN, W. E. F., A.M. (Columbia) Prof., St. Francis Coll., Brooklyn, N.Y. *548 State St. (17)*
ARCHIBALD, J. T., Student, St. John's Coll., Brooklyn, N.Y. *16 First Ave., New York 3, N.Y.*
ARCHIBALD, R. C., Dr. (Univ. Padua) Emeritus Prof., Brown Univ., Providence, R.I.
ARCHIBALD, R. G., Ph.D. (Chicago) Prof., Queens Coll., Flushing, N.Y.
ARENA, F. J., A.M. (Michigan) Instr., North Dakota Agr. Coll., State College, N.D.
AREND, J. S., S.B. (M.I.T.) Capt., U. S. Air Force. *General Delivery, Randolph Air Force Base, Tex.*
ARENS, HELEN C. (Mrs. Richard), A.M. (Radcliffe) Mathematician, Inst. for Numerical Analysis, Los Angeles 24, Calif. *12436 Deerbrook Lane (49)*
ARENSON, D. L., M.S. (Illinois Tech.) Research Engr., Armour Research Foundation, Technology Center, Chicago 16, Ill. *6116 N. Winthrop Ave. (40)*
ARMSTRONG, BEULAH M., Ph.D. (Illinois) Asst. Prof., Univ. of Illinois, Urbana, Ill.
ARMSTRONG, J. D., B.S.E. (Florida) Teacher, Landon High School, Jacksonville, Fla. *1129 N. 2nd Ave., Jacksonville Beach, Fla.*
ARMSTRONG, P. N., Student, Univ. of Nebraska, Lincoln, Neb. *1106 S. Ninth St.*
ARNDT, W. F. C., Ph.D. (Göttingen) Prof., Univ. Coll. of the Orange Free State, Bloemfontein, Union of South Africa
ARNOFF, E. L., Ph.D. (C.I.T.) Aeron. Research Scientist, National Advisory Committee for Aeronautics, Cleveland, Ohio. *14200 Shaw Ave., East Cleveland, Ohio*
ARNOLD, H. A., Ph.D. (C.I.T.) Asst. Prof., Univ. of California at Davis, Davis, Calif.
ARNOLD, H. C., M.S. (Carnegie) Tech. Director, Federal Enamel and Stamping Co., McKees Rocks, Pa. *76 Standish Blvd., Pittsburgh 16, Pa.*
ARNOLD, H. E., Ph.D. (Yale) Prof., Wesleyan Univ., Middletown, Conn.
ARNOLD, K. J., Ph.D. (M.I.T.) Asst. Prof., Univ. of Wisconsin, Madison 6, Wis.
ARNOLD, L. G., M.A. (Michigan) Eastman Kodak Co., Navy Ordnance, Rochester, N.Y. *50-A Highland Manor, South Ave.*
ARNOLD, P. H., A.B. (Nebraska Wesleyan) Grad. Asst., Kansas State Coll., Manhattan, Kan. *1622 Humboldt*
ARNOLD, W. C., M.S. (Chicago) Asso. Prof., DePauw Univ., Greencastle, Ind.
AROIAN, L. A., Ph.D. (Michigan) Research Physicist, Research and Development Labs., Hughes Aircraft Co., Culver City, Calif.
ARONOFSKY, J. S., M.S. (Stevens) Senior Research Engr., Magnolia Petroleum Co., Dallas, Tex. *2127 Barberry Dr*
ARTIN, EMIL, Ph.D. (Leipzig) Prof., Princeton Univ., Princeton, N.J.
ARZT, SHOLOM, M.S. (N.Y.U.) Instr., New York Univ., New York 53, N.Y.
ASHBAUGH, LAURA M. (Mrs. F. R.), A.M. (Pennsylvania) Grad. Student, Lehigh Univ., Bethlehem, Pa. *115 Wall St.*
ASHBURN, A. W., Ph.D. (Virginia) Asso. Prof., Texas State Coll. for Women, Denton, Tex.
ASHCRAFT, E. S., M.A. (North Carolina) Asst. Prof., Stetson Univ., DeLand, Fla. *245 S. Stone St.*
ASHCRAFT, T. B., Ph.D. (Johns Hopkins) Emeritus Prof., Colby Coll., Waterville, Maine *34 Pleasant St.*
ASPREY, WINIFRED A., Ph.D. (Iowa) Asso. Prof., Vassar Coll., Poughkeepsie, N.Y. *2434 N. Park Blvd., Santa Ana, Calif.*
ASTRACHAN, MAX, Ph.D. (Brown) Prof., Coll. of Industrial Administration, U.S.A.F. Inst. of Tech., Wright-Patterson Air Force Base, Dayton, Ohio. *1513 Cory Dr. (6)*
ATCHISON, W. F., Ph.D. (Illinois) Asst. Prof., Univ. of Illinois, Urbana, Ill.
ATKIN, EDITH I., A.M. (Columbia) Emeritus Asso. Prof., Illinois State Normal Univ., Normal, Ill. *815 S. Fell Ave.*
ATKINS, D. F., Ph.D. (Kentucky) Asst. Prof., Bowling Green State Univ., Bowling Green, Ohio
ATKINS, H. P., Jr., Ph.D. (Rochester) Asst. Prof., Univ. of Rochester, River Campus, Rochester 3, N.Y.
ATKINSON, ROBERT, A.M. (Haverford) Business Manager, The Shipley School, Bryn Mawr, Pa.
ATTRIDGE, R. F., Student, McMaster Univ., Hamilton, Ont., Canada. *33 Mapes Ave.*
AUDE, H. T. R., D.Sc. (Colgate) Emeritus Prof., Colgate Univ., Hamilton, N.Y.
AURORA, SILVIO, A.B. (Columbia) Research Fellow, Columbia Univ., New York, N.Y. *30-58 83rd St., Jackson Heights, N.Y.*
AYER, MIRIAM C., Ph.D. (Ohio State) Asst. Prof., Wellesley Coll., Wellesley 81, Mass.
AYLOR, M. W., M.S. (Virginia) Asst. Prof., Univ. of Virginia, Charlottesville, Va. *Madison, Va.*
AYOUB, R. G., M.Sc. (McGill) 6848 Drolet St., Montreal, P.Q., Canada
AYRE, H. G., Ph.D. (George Peabody) Prof., Western Illinois State Coll., Macomb, Ill.
AYRES, FRANK, Jr., Ph.D. (Chicago) Prof., Dickinson Coll., Carlisle, Pa. *233 Walnut St.*
AYRES, H. C., Ph.D. (California) *245 Pearl St., Eugene, Ore.*
AYRES, W. L., Ph.D. (Pennsylvania) Dean, School of Science, Purdue Univ., Lafayette, Ind.

BABBITT, A. E., A.M. (Illinois) Vice Pres. and Actuary, Lamar Life Ins. Co., Box 880, Jackson 107, Miss.
BABCOCK, L. E., M.S. (Illinois) Asso. Prof., Valdosta State Coll., Valdosta, Ga.
BABCOCK, R. W., Ph.D. (Wisconsin) Dean, School of Arts and Science, Kansas State Coll., Manhattan, Kan.
BABCOCK, WEALTHY, Ph.D. (Kansas) Asso. Prof., Univ. of Kansas, Lawrence, Kan. *Route 6*
BABIKIAN, SOUREN, B.S. (London) Instr., Aleppo Coll., Aleppo, Syria
BACHILLER, T. R., Dr.Sc. (U. of Madrid) Catedratico de la Fac. de Ciencias de la Universidad de Madrid, Madrid, Spain
BACON, H. M., Ph.D. (Stanford) Prof., Stanford Univ., Stanford, Calif.
BACON, R. P., M.A. (Miami U.) Tab. Analyst, Douglas Aircraft Co., El Segundo, Calif. *338 W. Plymouth St., Inglewood, Calif.*
BADE, W. G., Ph.D. (U.C.L.A.) Instr., Univ. of California, Berkeley, Calif.
BADGLEY, W. H., Jr., M.A. (Columbia) Grad. Student, Vanderbilt Univ., Nashville, Tenn. *3515 Byron Ave.*
BAEUMLER, H. W., M.A. (Buffalo) Instr., Univ. of Buffalo, Buffalo 14, N.Y. *113 Winspear Ave. (14)*
BAEZ, A. V., Ph.D. (Stanford) Prof., Dept. of Physics, Univ. Coll., Baghdad, Iraq
BAGBY, L. C., A.M. (Kansas) Prof., Lawrence Inst. of Tech., Detroit, Mich. *62 Glendale Ave., Highland Park 3, Mich.*
BAGEMIHL, FREDERICK, M.A. (Columbia) Asst. Prof., Univ. of Rochester, Rochester 3, N.Y.
BAGNETTO, LUCIEN, M.S. (Louisiana) Asst. Prof., Harding Coll., Searcy, Ark.
BAHNG, J. D. R., Student, St. Norbert Coll., West De Pere, Wis. *2221 16th St., Monroe, Wis.*
BAIDAFF, B. I., Dr. (Buenos Aires) Prof., Univ. Nac., Buenos Aires, Argentina. *Ave. de Mayo 560*
BAILEY, A. H., Ph.D. (Ohio State) Prof., Georgia Inst. of Tech., Atlanta, Ga.
BAILEY, E. A., M.S. (Emory) Dean, LaGrange Coll., LaGrange, Ga.
BAILEY, H. W., Ph.D. (Illinois) Asso. Dean, Lib. Arts and Science, Univ. of Illinois, Navy Pier, Chicago 11, Ill.
BAILEY, J. J., M.A. (Michigan) Acting Head of Dept., Shattuck School, Faribault, Minn.
BAILEY, P. M., M.S. (Iowa) Actuary, Great Eastern Mutual Life Insurance Co., Denver, Colo. *2105 S. Irving*
BAILEY, R. P., Ph.D. (Pennsylvania) Asso. Prof., U.S. Naval Acad., Annapolis, Md.
BAILEY, W. L., JR., Student, Massachusetts Inst. of Tech., Cambridge, Mass. *105 First National Bank Bldg., Waynesburg, Pa.*
BAKER, FRANCES E., Ph.D. (Chicago) Prof., Vassar Coll., Poughkeepsie, N.Y.
BAKER, G. A., Ph.D. (Illinois) Asso. Prof., Univ. of California at Davis, Davis, Calif. *606 C St.*
BAKER, R. E., M.A. (Texas) Asso. Prof., East Texas State Teachers Coll., Commerce, Tex. *2314 Monroe St.*
BAKER, S. R., M.E.M. (North Carolina S.C.) Instr., North Carolina State Coll., Raleigh, N.C. *2740 Rosedale Ave.*
BAKST, AARON, Ph.D. (Columbia) *135-12 77th Ave., Flushing, N.Y.*
BALDWIN, JEAN M., M.A. (Oklahoma) Instr., Lamar State Coll. of Tech., Beaumont, Tex.
BALDWIN, J. W., A.M. (Michigan) Prof., Wayne Univ., Detroit, Mich. *16191 Roselawn Ave. (21)*
BALL, N. H., Ph.D. (M.I.T.) Asso. Prof., U.S. Naval Acad., Annapolis, Md. *R.F.D. 1, Box 235*
BALL, R. W., Ph.D. (Illinois) Instr., Univ. of Washington, Seattle 5, Wash.
BALLANTINE, J. P., Ph.D. (Chicago) Prof., Univ. of Washington, Seattle 5, Wash. *1802 Ravenna Blvd.*
BALLARD, JO M. (Mrs. R. H.), B.S. (Howard) Instr., Howard Coll., Birmingham, Ala. *5500 6th St., S.*
BALLARD, RUTH M. (Mrs. F. K.), Ph.D. (Chicago) *6739 Bosworth Ave., Chicago 26, Ill.*
BALLOU, D. H., Ph.D. (Harvard) Asso. Prof., Middlebury Coll., Middlebury, Vt. *On leave 1951–52:* Yale Univ., New Haven, Conn.
BALOF, C. A., M.S. (Iowa) Business Manager, Lincoln Coll., Lincoln, Ill. *623 N. Union St.*
BAMFORTH, F. R., Ph.D. (Chicago) Prof., Otterbein Coll., Westerville, Ohio. *64 S. Vine St.*
BANCROFT, T. A., Ph.D. (Iowa State) Dir. of Statistics Lab., Iowa State Coll., Ames, Iowa
BANES, A. V., M.S. (Oklahoma A. and M.) Asst. Prof., Trinity Univ., San Antonio, Tex. *118 Overhill Dr. (1)*
BANHAGEL, E. W., M.A. (Wayne) Mathematician, Research Labs., Bendix Aviation, Detroit, Mich. *8212 Rathbone St. (9)*
BANKIER, J. D., Ph.D. (Rice) Asso. Prof., McMaster Univ., Hamilton, Ont., Canada
BANKOFF, Leon, D.D.S. (Southern California) *5410 Wilshire Blvd., Los Angeles 36, Calif.*
BANKS, G. B., Ph.D. (Niagara) Prof., Physical Sci., Niagara Univ., Niagara University, N.Y.
BANKS, J. H., Ph.D. (George Peabody) Asso. Prof., George Peabody Coll. for Teachers, Nashville 4, Tenn.
BARBOUR, J. M., Mus.D. (Toronto) Asso. Prof., Music, Michigan State Coll., East Lansing, Mich. *607 Division St.*

BARDELL, R. H., Ph.D. (Chicago) Prof., Univ. of Wisconsin in Milwaukee, Milwaukee 3, Wis.
BAREIS, GRACE M., Ph.D. (Ohio State) Emeritus Asst. Prof., Ohio State Univ., Columbus, Ohio. *164 13th Ave.*
BARLAZ, JOSHUA, Ph.D. (Cincinnati) Asso. Prof., Rutgers Univ., New Brunswick, N.J.
BARNARD, R. F., M.S. (Washington S.C.) Instr., North Central High School, Spokane, Wash.
BARNARD, R. W., Ph.D. (Chicago) Asso. Prof., Univ. of Chicago, Chicago 37, Ill.
BARNES, J. C., B.S. (North Georgia C.) Head of Dept., North Georgia Coll., Dahlonega, Ga.
BARNES, W. E., Ph.D. (Cornell) Mathematician, U.S. Naval Proving Ground, Dahlgren, Va.
BARNETT, H. H., M.A. (Kansas) Acting Asst. Prof., Florida State Univ., Tallahassee, Fla.
BARNETT, I. A., Ph.D. (Chicago) Prof., Univ. of Cincinnati, Cincinnati 21, Ohio
BARNETT, JOSEPH, JR., A.M. (Columbia) Emeritus Asso. Prof., Oklahoma A. and M. Coll., Stillwater, Okla. *Route 1, c/o Wm. McLaughlin, Country Club Rd., Clarksburg, W. Va.*
BARNHART, ESTHER P., M.A. (Michigan) *303-3 Chome, Hyakunin Machi, Shinjuku Ku, Tokyo, Japan*
BAROTZ, NATHAN, B.S. in I.E. (N.Y.U.) Instr., Brooklyn Coll., Brooklyn, N.Y. *733 E. 51st St. (3)*
BARR, C. F., M.S. (Chicago) Prof., Univ. of Wyoming, Laramie, Wyo.
BARRAL-SOUTO, JOSE, Sc.D. Prof., Statistics, Univ. of Buenos Aires, Buenos Aires, Argentina. *Cordoba 1459*
BARRER, D. Y., M.A. (Northwestern) Instr., Northwestern Univ., Evanston, Ill. *1636 Roberts Lane, Falls Church, Va.*
BARRICK, D. L., M.S. (Oklahoma) Asst. Prof., Univ. of Colorado, Boulder, Colo. *750 Eighth St.*
BARRON, J. J., Ph.D. (Wisconsin) Prof., Marshall Coll., Huntington, W. Va.
BARROW, D. F., Ph.D. (Harvard) Prof., Univ. of Georgia, Athens, Ga. *260 Cherokee Ave.*
BARROW, F. L., A.M. (Missouri) Asst. Prof., Central Coll., Fayette, Mo. *615 W. Ninth St., Lawrence, Kans.*
BARTELS, R. C. F., Ph.D. (Wisconsin) Asso. Prof., Univ. of Michigan, Ann Arbor, Mich.
BARTELS, W. H., B.S. in E.E. (Michigan S.C.) Asst. Elec. Engr., Board of Water and Electric Light Commission, Lansing, Mich. *512 Albert Ave., East Lansing, Mich.*
BARTKY, WALTER, Ph.D. (Chicago) Asso. Dean, Physical Sci., Univ. of Chicago, Chicago 37, Ill.
BARTLEY, E. F., B.A. (Scranton) Asst. Prof., Univ. of Scranton, Scranton, Pa.
BARTMAN, F. L., M. A. (Michigan) Research Engr., Willow Run Research Center, Ypsilanti, Mich. *1419 Sudbury, Willow Run, Mich.*
BARTON, D. C., M.A. (Rochester) Instr., Univ. of Rochester, Rochester 3, N.Y. *3 Orchard St., North Chili, N.Y.*
BARTON, HELEN, Ph.D. (Johns Hopkins) Prof., Woman's Coll., Univ. of North Carolina, Greensboro, N.C. *1027 Spring Garden St.*
BARTON, M. Q., B.A. (Central C., Mo.) Teaching Asst., Univ. of Illinois, Urbana, Ill.
BARTON, R. S., M.S. (Iowa) *701 N. Louise St., Glendale 6, Calif.*
BARTOO, G. C., A.M. (Michigan) Emeritus Prof., Western Michigan Coll., Kalamazoo, Mich. *1341 Hillcrest Ave.*
BARTRAM, H. G. H., M.A. (Cornell) *1134 Pleasant St., Boulder, Colo.*
BARTZ, R. D., Student, Univ. of Wisconsin, Madison 6, Wis. *3534 E. Cudahy Ave., Cudahy, Wis.*
BASIL, ALEXANDER, B.A. (Hofstra) Engr., Airborne Instruments Labs., Old Country Rd., Mineola, N.Y. *2124 Stuart Lane, East Meadow, Hempstead, L.I., N.Y.*
BASOCO, M. A., Ph.D. (C.I.T.) Prof., Univ. of Nebraska, Lincoln 8, Nebr.
BASS, T. J., JR., A.M. (California) Instr., City Coll. of San Francisco, San Francisco, Calif. *342 Moncada Way (27)*
BASYE, R. E., Ph.D. (Texas) Asso. Prof., Texas A. and M. Coll., College Station, Tex.
BATCHELDER, P. M., Ph.D. (Harvard) Asso. Prof., Univ. of Texas, Austin, Tex. *910 W. 22nd St. (21)*
BATEMAN, P. T., Ph.D. (Pennsylvania) Asst. Prof., Univ. of Illinois, Urbana, Ill.
BATEN, W. D., Ph.D. (Michigan) Prof., Michigan State Coll., East Lansing, Mich. *26 University Dr.*
BATES, GRACE E., Ph.D. (Illinois) Asst. Prof., Mount Holyoke Coll., South Hadley, Mass.
BATES, M. R., M.S. (Cornell) Mathematician, Bell Aircraft Corp., Niagara Falls, N.Y. *411 Wabash Ave., Kenmore 17, N.Y.*
BATES, O. K., Sc.D. (M.I.T.) Prof., St. Lawrence Univ., Canton, N.Y. *44 E. Main St.*
BATHO, E. H., B.S. (Fordham) Teaching Asst., Univ. of Wisconsin, Madison 6, Wis.
BATT, J. R., B.A. (Buffalo) *560 Cambridge Ave., Buffalo 15, N.Y.*
BATTIG, LEON, A.M. (Wisconsin) Instr., Univ. of Wisconsin in Milwaukee, Milwaukee 3, Wis. *2535 Elizabeth St., Sheboygan, Wis.*
BATTIN, I. L., Ph.D. (N.Y.U.) Asso. Prof., Drew Univ., Madison, N.J.
BAUER, L. M., A.M. (New Mexico) Dean, Menaul School, Albuquerque, N.M.
BAUM, W. R., Dr. Sc. (Swiss Fed. Inst. of Tech.) Asst. Prof., Syracuse Univ., Syracuse 10, N.Y.

BAUMGART, J. K., A.M. (Michigan) Asst. Prof., Elmhurst Coll., Elmhurst, Ill. *1441 S. 13th Ave., Maywood, Ill.*
BAUMGARTNER, R. A., A.M. (Illinois) Head of Dept., Freeport High School, Freeport, Ill. *1321 W. Storer St.*
BAUS, RENE, JR., B.S. (Tulane) Grad. Student, Tulane Univ., New Orleans, La. *1435 Pine St.*
BAUSER, A. V., A.M. (N.Y.U.) Teacher, J. W. Cooper High School, Shenandoah, Pa. *107 N. Ferguson St.*
BAUSERMAN, THOMAS, M.A. (Washington S.C.) Instr., West Virginia Univ., Morgantown, W. Va. *749 College Ave.*
BAXTER, C. B., Partner, C. B. Baxter & Co., 408-9 Long Bldg., 928 Grand Ave., Kansas City 6, Mo.
BAXTER, MARY M., M.A. (Columbia) Instr., Junior Coll. of Kansas City, Kansas City, Mo. *4938 Wyandotte (2)*
BAYS, SÉVÉRIN, Prof., Univ. of Fribourg, Fribourg, Switzerland. *Route de Bertiguy 41*
BEACH, J. W., Ph.D. (Iowa State) Asst. Prof., Univ. of New Mexico, Albuquerque, N.M. *1208 N. Lafayette Ave.*
BEAL, JUNA L., A.M. (Chicago) Asst. Prof., Butler Univ., Indianapolis 7, Ind. *3262 Broadway (5)*
BEALS, R. W., Jr., M.Ed. (Alfred) Teacher, Cattaraugus Central School, Cattaraugus, N.Y. *257 S. Main St.*
BEARD, HELEN P., Ph.D. (M.I.T.) Asst. Prof., Newcomb Coll., New Orleans 18, La.
BEARMAN, J. E., Ph.D. (Minnesota) Asst. Prof., Univ. of Minnesota, Minneapolis, Minn. *On leave:* Kfar Blum, Upper Galilee, Israel
BEASLEY, S. LOUISE, Ed.M. (George Peabody) Asso. Prof., Lindenwood Coll. for Women, St. Charles, Mo.
BEATLEY, RALPH, A.M. (Columbia) Asso. Prof., Education, Harvard Univ., Cambridge 38, Mass.
BEATTY, H. M., A.M. (Ohio State) Asst. Prof., Ohio State Univ., Columbus, Ohio. *200 Tibet Rd.*
BEATTY, SAMUEL, Ph.D. (Toronto) Dean, Faculty of Arts, Univ. of Toronto, Toronto 5, Ont., Canada
BEATY, E. B., A.M. (California) Emeritus Prof., Oregon State Coll., Corvallis, Ore.
BEAUMONT, R. A., Ph.D. (Illinois) Asso. Prof., Univ. of Washington, Seattle 5, Wash.
BEAVER, R. A., Ph.D. (Cornell) Prof., New York State Coll. for Teachers, Albany 3, N.Y.
BEBERMAN, MAX, M.A. (Columbia) Instr., Univ. High School, Univ. of Illinois, Urbana, Ill.
BECHTOLSHEIM, LULU, Ph.D. (Zurich) Asso. Prof., Univ. of Redlands, Redlands, Calif. *330 Cajon St.*
BECK, V. W., M.S. (Minnesota) Elec. Engr., American Television and Radio, St. Paul, Minn. *724 Sims Ave. (6)*
BECK, W. R., M.A. (N.Y.U.) Instr., Rensselaer Poly. Inst., Troy, N.Y.
BECK, WILLIAM, B.A. (Kent) Grad. Asst., Univ. of Kentucky, Lexington, Ky.
BECKENBACH, E. F., Ph.D. (Rice) Prof., Univ. of California at Los Angeles, Los Angeles 24, Calif.
BECKER, H. W., *1214 N. 34th St., Omaha 3, Nebr.*
BECKMAN, F. S., A.M. (Columbia) Instr., Pratt Inst., Brooklyn, N.Y. *1419 Jesup Ave., Bronx 52, N.Y.*
BECKSTROM, AGNES J., M.A. (Northwestern) Head of Dept., State Teachers Coll., Minot, N.D.
BECKWITH, MABLE L., A.M. (Claremont) Instr., San Rafael High School, San Rafael, Calif.
BECKWITH, W. S., A.M. (Harvard) Asso. Prof., Univ. of Georgia, Athens, Ga. *731 Cobb St.*
BEEGLE, B. L., M.S. (Washington) Dean, Seattle Pacific Coll., Seattle 99, Wash.
BEELER, F. A., Ph.D. (Michigan) Prof., Western Michigan Coll., Kalamazoo 45, Mich. *1749 W. Michigan Ave., Apt. F-4*
BEEMAN, W. E., M.S. (N. Tex. S.C.) Instr., Texas A. and M. Coll., College Station, Tex. *Edge Apt. No. 5, Bryan, Tex.*
BEENKEN, MAY M., Ph.D. (Chicago) Head of Dept., Immaculate Heart Coll., Los Angeles 27, Calif. *1906 Park Ave. (26)*
BEERMAN, WALTER, B.S. in M.E. (Michigan) Consulting Engr., 332 W. 46th St., New York 19, N.Y.
BEESACK, P. R., B.A. (McMaster) Grad. Asst., Washington Univ., St. Louis, Mo. *5500 Cabanne Ave., Apt. E, (12)*
BEESLEY, E. M., Ph.D. (Brown) Asso. Prof., Univ. of Nevada, Reno, Nev.
BEGLE, E. G., Ph.D. (Princeton) Asso. Prof., Yale Univ., New Haven, Conn. *Cooper Rd., North Haven, Conn.*
BEHRNS, V. N., M.A. (Buffalo) Instr., Univ. of Buffalo, Buffalo 14, N.Y. *217 Kelvin Dr.*
BEINERT, R. L., A.B. (Hobart) Instr., Hobart and William Smith Colls., Geneva, N.Y.
BELL, ALICE K., M.A. (Michigan) Prof., Fresno State Coll., Fresno, Calif. *509 Ashlan*
BELL, C. B., JR., M.S. (Notre Dame) Grad. Student, Univ. of Notre Dame, Notre Dame, Ind. *2236 W. Washington Ave., South Bend, Ind.*

BELL, C. M., Jr., B.S. (Franklin and Marshall) Chemist, Naval Ammunition and Net Depot, Seal Beach, Calif. *5319 Brittain St., Long Beach 8, Calif.*
BELL, CLIFFORD, Ph.D. (California) Prof., Univ. of California at Los Angeles, Los Angeles 24, Calif. *16032 Chase St., Sepulveda, Calif.*
BELL, E. T., Ph.D. (Columbia) Prof., California Inst. of Tech., Pasadena 4, Calif.
BELL, J. H., Ph.D. (Wisconsin) Asso. Prof., Michigan State Coll., East Lansing, Mich. *3626 Marigold Ave., Box 22F4, Route 2, Lansing, Mich.*
BELL, P. O., Ph.D. (California) Prof., Univ. of Kansas, Lawrence, Kans. *2112 Vermont St.*
BELL, R. F., M.S. (Michigan) Asso. Prof., Eastern Washington Coll. of Education, Cheney, Wash.
BELLAMY, B. C., B.S.C.E. (Wyoming) Civil Engr., Box 37, Lamont, Wyo.
BELLARDO, J. E., M.S. (St. Bonaventure) Instr., Aquinas Coll., Grand Rapids, Mich. *12 Franklin St., S.E.*
BELLMER, W. J., M.A. (Catholic U.) Prof., Univ. of Dayton, Dayton 9, Ohio
BEMMELS, W. D., Ph.D. (Colorado) Dean, Engr. Studies, Ottawa Univ., Ottawa, Kans.
BENAC, T. J., Ph.D. (Yale) Asso. Prof., U.S. Naval Acad., Annapolis, Md.
BENDER, H. A., Ph.D. (Illinois) Asso. Prof., Rhode Island State Coll., Kingston, R.I.
BENDER, JOHN, Student, Rutgers Univ., New Brunswick, N.J. *304 Sutton St., Perth Amboy, N.J.*
BENNER, J. A., A.M. (Lafayette) Asso. Prof., Lafayette Coll., Easton, Pa. *2 E. Campus*
BENNETT, A. A., Ph.D. (Princeton) Prof., Brown Univ., Providence 12, R.I.
BENNETT, THEODORE, Ph.D. (Illinois) Prof., Marietta Coll., Marietta, Ohio
BENSON, A. I., M.S. (Wisconsin) Staff Member, Los Alamos Scientific Lab., Los Alamos, N.M. *2645C Gold*
BENSON, D. C., B.A. (Pomona) Teaching Asst., Stanford Univ., Stanford, Calif. *726 N. Curson Ave., Los Angeles 46, Calif.*
BENTLEY, A. F., Ph.D. (Johns Hopkins) *Route 2, Paoli, Ind.*
BERARU, JONAS, B.A. (U.C.L.A.) Mathematician, Rand Corp., Santa Monica, Calif. *600 N. Cummings St., Los Angeles 33, Calif.*
BERBERIAN, S. K., M.S. (Michigan S.C.) Instr., Southern Illinois Univ., Carbondale, Ill.
BERCOS, JAMES, B.S. (Illinois) *4808 Washington Blvd., Chicago 44, Ill.*
BERG, H.S., M.S. (North Dakota) Instr., State Teachers Coll., St. Cloud, Minn.
BERG, L. E., M.A. (Syracuse) Teacher, Fairmount Board of Education, Fairmount, Ind. *215 S. Mill St.*
BERG, W. D., Ph.D. (Iowa) Senior Tech. Asst., Mutual Life Insurance Co., Broadway and 55th St., New York 19, N.Y. *35 Lowell St., Westwood, N.J.*
BERGER, EDLA G., A.M. (Minnesota) Mathematician, Equitable Life Assurance Soc., New York, N.Y. *130 W. Twelfth St. (11)*
BERGER, E. J., M.A. (Colorado) Instr., Monroe High School and Coll. of St. Catherine, St. Paul, Minn. *771 W. Hoyt Ave. (3)*
BERGLUND, WINIFRED V., M.A. (Northwestern) Instr., Univ. of Illinois, Navy Pier, Chicago 11, Ill. *120 Green Bay Rd., Hubbard Woods, Ill.*
BERGMAN, K. S., B.A. (Gonzaga) Field Asst., U.S. Geological Survey, S. 157 Howard St., Spokane, Wash. *503 W. 21st Ave. (9)*
BERGMANN, H. G., Ph.D. (N.Y.U.) Instr., City Coll. of New York, New York 31, N.Y. *2995 Marion Ave. (58)*
BERKOFSKY, LOUIS, M.S. (N.Y.U.) *19 Wabon St., Roxbury 21, Mass.*
BERKOWITZ, JEROME, B.S. (C.C.N.Y.) Research Asst., New York Univ., New York 3, N.Y. *31-32 89th St., Jackson Heights, N.Y.*
BERKOWITZ, S. D., Student, Columbia Univ., New York, N.Y. *585 W. 204 St. (34)*
BERMAN, GERALD, Ph.D. (Toronto) Instr., Illinois Inst. of Tech., Chicago, Ill. *3140 S. Michigan Ave., Apt. 406 (16)*
BERMAN, MARTIN, M.S. (V.P.I.) Instr., Univ. of Cincinnati, Cincinnati, Ohio. *35 Landon Ct.*
BERNARD ALFRED, Brother (Welch), A.M. (Catholic U.) Prof., Manhattan Coll., New York 63, N.Y.
BERNARD, R. R., Ph.D. (Virginia) Instr., Yale Univ., New Haven, Conn.
BERNARDI, S. D., M.S. (Notre Dame) Instr., New York Univ., New York 3, N.Y.
BERNDT, A. F., Student, Cooper Union, New York, N.Y. *2685 University Ave. (63)*
BERNER, L. J., B.A. (St. Norbert) Instr., St. Norbert Coll., West De Pere, Wis. *137 Reid St.*
BERNHARD, H. A., B.S. (Newark C. of Engg.) Grad. Student, Columbia Univ., New York, N.Y. *218 Freeman Ave., Jersey City, N.J.*
BERNHARD, IDA M., M.A. (Texas) Supervisor, Southwest Texas State Teachers Coll., San Marcos, Tex. *331 W. Hopkins*
BERNHART, ARTHUR, Ph.D. (Michigan) Asso. Prof., Univ. of Oklahoma, Norman, Okla.
BERNSTEIN, B. A., Ph.D. (California) Emeritus Prof., Univ. of California, Berkeley, Calif. *2785 Shasta Rd. (8)*
BERNSTEIN, DOROTHY L., Ph.D. (Brown) Asso. Prof., Univ. of Rochester, Rochester 3, N.Y.

BERRY, A. C., Ph.D. (Harvard) Prof., Lawrence Coll., Appleton, Wis.
BERRY, E. M., Ph.D. (Iowa) Emeritus Prof., State Teachers Coll., Chadron, Nebr. *602 Mears St.*
BERRY, H. H., B.S. (Kentucky) *3501 Hycliffe Ave., Louisville, Ky.*
BERRY, T. E., B.S. (George Washington) Analyst, Social Security Board, Washington, D.C. *102 Urell Pl., N.E. (11)*
BERRY, W. J., A.M. (Harvard) Emeritus Prof., Poly. Inst. of Brooklyn, Brooklyn, N.Y. *The Berrypatch, P.O. Box 298, Stony Brook, N.Y.*
BERT, O. F. H., Sc.D. (Thiel) Emeritus Prof., Washington and Jefferson Coll., Washington, Pa. *28 N. Lincoln St.*
BESSELL, W. W., JR., C.E. (Rensselaer) Col. and Prof., U.S. Military Acad., West Point, N.Y.
BETTINGER, A. K., A.M. (Wisconsin) Asst. Prof., Creighton Univ., Omaha, Nebr.
BETTS, BARBARA B., A.M. (Radcliffe) Math. Editor, D. C. Heath and Co., 285 Columbus Ave., Boston 16, Mass.
BETZ, WILLIAM, A.M. (Rochester) Math. Specialist, Rochester Public Schools, Rochester, N.Y. *652 Melville St.*
BEVERIDGE, H. R., Ph.D. (Illinois) Prof., Monmouth Coll., Monmouth, Ill. *1043 E. Detroit Ave.*
BEY, D. R., A.M. (Illinois) Asst. Prof., Illinois State Normal Univ., Normal, Ill. *402 W. Locust St.*
BIBB, S. F., M.S. (Chicago) Asso. Prof., Illinois Inst. of Tech., Chicago 16, Ill.
BICKEL, R. J., M.A. (Northwestern) Asst. Prof., Drexel Inst. of Tech., Philadelphia 4, Pa. *516 Upland Rd., Havertown, Pa.*
BICKERSTAFF, T. A., Ph.D. (Michigan) Prof., Univ. of Mississippi, University, Miss.
BICKNELL, W. S., M.S. (Michigan) Instr., Iowa State Coll., Ames, Iowa. *140 Pammel Ct.*
BIESELE, F. C., Ph.D. (Texas) Asso. Prof., Univ. of Utah, Salt Lake City, Utah. *1876 S. Woodside St. (7)*
BIGELOW, W. W., B.S.C.E. (George Washington) Analyst, A. O. Smith Corp., Milwaukee, Wis. *Rockton, Ill.*
BIGGERS, RUTH E., M.A. (Duke) Instr., Emory and Henry Coll., Emory, Va.
BIGGERSTAFF, J. S., M.S. (Washington) *2814 N.W. Monte Vista, Portland 10, Ore.*
BING, R. H., Ph.D. (Texas) Asso. Prof., Univ. of Wisconsin, Madison 6, Wis.
BINGLEY, G. A., A.M. (Princeton) Tutor, St. John's Coll., Annapolis, Md.
BIRCHBY, W. N., A.M. (Colorado C.) Asst. Prof., California Inst. of Tech., Pasadena, Calif.
BIRCHENOUGH, HARRY, A.M. (Columbia) Prof., New York State Coll. for Teachers, Albany 3, N.Y.
BIRD, M. T., Ph.D. (Illinois) Asso. Prof., San Jose State Coll., San Jose 14, Calif. *45 Pala Ave. (27)*
BIRD, R. W., B.A. (Roosevelt) Financial Mathematician, Walter E. Heller & Co., 105 W. Adams, Chicago, Ill. *409 W. Irving Park Blvd., Itasca, Ill.*
BIRKHOFF, GARRETT, A.B. (Harvard) Prof., Harvard Univ., Cambridge, Mass. *45 Fayerweather St.*
BIRNBAUM, Z. W., Ph.D. (Lwow) Prof., Univ. of Washington, Seattle 5, Wash.
BISER, ERWIN, Ph.D. (Pennsylvania) Asst. Prof., Rutgers Univ., New Brunswick, N.J. *16 McAuliffe Dr.*
BISSINGER, B. H., Ph.D. (Cornell) Vice-Pres., Geo. Gillis Shoe Corp., Fitchburg, Mass.
BLACK, A. H., Ph.D. (Cornell) Asso. Prof., Southern Illinois Univ., Carbondale, Ill. *1201 Hill Ave., Route 4*
BLACK, FLORENCE L., Ph.D. (Kansas) Asso. Prof., Univ. of Kansas, Lawrence, Kans. *1300 Louisiana St.*
BLACK, L. G., A.M. (Colorado) Asst. Prof., Purdue Univ., West Lafayette, Ind. *129 S. Grant St.*
BLACK, L. T., A.M. (Michigan) Instr., Long Beach City Coll., Long Beach, Calif. *1230 Hungerford St. (5)*
BLACKALL, C. J., Ph.D. (Cornell) *North Baltimore, Ohio*
BLACKETT, D. W., Ph.D. (Princeton) Research Asso., Princeton Univ., Princeton, N.J.
BLACKISTON, NANETTE R. (Mrs.) M.A. (Columbia) Supervisor of Math., Dept. of Education, 3 E. 25th St., Baltimore 18, Md. *725 Gorsuch Ave.*
BLACKSTOCK, MAY C. (Mrs.) M.A. (Brown) Instr., Univ. of Tennessee, Knoxville, Tenn. *108 Arrowhead Trail*
BLACKWELL, R. C., Ph.D. (North Carolina) Prof., Furman Univ., Greenville, S.C. *322 University Ridge*
BLAIR, BARBARA B. (Mrs. R. L.) M.S. (Iowa) Part-time Instr., State Univ. of Iowa, Iowa City, Iowa
BLAIR, HAROLD, A.M. (Michigan) Prof., Western Michigan Coll., Kalamazoo, Mich. *1220 Academy St. (49)*
BLAIR, R. V., Ph.D. (George Peabody) Prof., Vanderbilt Univ., Nashville 4, Tenn. *2421 Kensington Pl. (5)*

BLAKE, ARCHIE, Ph.D. (Chicago) Treasurer, Mechanical Research Corp., Chicago, Ill. *794 George Ave., Aurora, Ill.*
BLAKE, R. G., M.A. (Florida) Asst. Prof., Univ. of Florida, Gainesville, Fla. *Route 5, Box 25*
BLAKELEY, LAURA, M.A. (George Peabody) Instr., Armstrong Junior Coll., Savannah, Ga.
BLAKELEY, W. W., B.A. (Macalester) *c/o Raymond Meyer, 110 13th Ave., N.E., North St. Paul, Minn.*
BLANCH, GERTRUDE, Ph.D. (Cornell) Mathematician, National Bureau of Standards, Los Angeles 24, Calif.
BLANCHE, E. E., Ph.D. (Illinois) Chief, Statistics Branch, Research & Development Div., Dept. of the Army, Washington, D.C. *9409 Montgomery Ave., North Chevy Chase, Md.*
BLANK, W. R., A.B. (Union) Instr., Union Coll., Lincoln, Nebr. *4537 Hillside Ave. (6)*
BLASCH, H. F., B.A. (Hofstra) Pvt., U.S. Army. *60-81 67th Ave., Ridgewood, Brooklyn 27, N.Y.*
BLATTNER, J. W., Student, Central Coll., Fayette, Mo. *1312 S. Jefferson, Mexico, Mo.*
BLAU, L. W., Ph.D. (Texas) Research Consultant, Humble Oil & Refining Co., Houston 1, Tex. *2027 Colquitt Ave.*
BLEICK, W. E., Ph.D. (Johns Hopkins) Asso. Prof., U.S. Naval Postgraduate School, Annapolis, Md. *8 Taney Ave.*
BLIGH, H. F., M.S. (Syracuse) Asst. Prof., Hobart and William Smith Colls., Geneva, N.Y.
BLINCOE, J. W., Ph.D. (Virginia) Prof., Randolph-Macon Coll., Ashland, Va.
BLOCK, DANIEL, A.M. (Columbia) Instr., Yeshiva Univ., New York, N.Y. *775 Garden St. (60)*
BLOCK, H.D., Ph.D. (Iowa State) Asst. Prof., Iowa State Coll., Ames, Iowa
BLOM, C. E., Editor, Elementa, Karlbergsv 16, Stockholm, Sweden
BLOOM, G. M., Ph.D. (Northwestern) Asst. Prof., Miami Univ., Oxford, Ohio
BLUE, A.H., Ph.D. (Iowa) Prof., Culver-Stockton Coll., Canton, Mo.
BLUM, JOSEPH, M.A. (George Washington) Analyst, Armed Forces Security Agency, Washington, D.C. *4108 N. Henderson Rd., Arlington, Va.*
BLUM, W. H., M.S. (West Virginia) *324 Cobun Ave., Morgantown, W. Va.*
BLUMBERG, J. O., Ph.D. (Pittsburgh) Asst. Prof., Univ. of Pittsburgh, Pittsburgh, Pa. *6920 McPherson Blvd. (8)*
BLUMENTHAL, L. M., Ph.D. (Johns Hopkins) Prof., Univ. of Missouri, Columbia, Mo. *On leave 1951–52:* Inst. for Numerical Analysis, Los Angeles 24, Calif.
BLYTH, C. R., Ph.D. (California) Asst. Prof., Univ. of Illinois, Urbana, Ill.
BOAK, I. S., M.A. (New York S.T.C.) Instr., New York State Agr. and Tech. Inst. Canton, N.Y.
BOARDMAN, H. C., D.E. (S. Dakota Schl. Mines) Dir. of Research, Chicago Bridge and Iron Co., Chicago, Ill. *10357 S. Hoyne Ave. (43)*
BOAS, R. P., JR., Ph.D. (Harvard) Prof., Northwestern Univ., Evanston, Ill. *1725 Orrington Ave.*
BOATMAN, A. O., A.M. (Indiana) Prof., Carthage Coll., Carthage, Ill.
BOBLAK, T. F., B.S. (Illinois Tech.) Grad. Student, Illinois Inst. of Tech., Chicago 16, Ill. *1644 W. 44th St. (9)*
BOBLETT, A. P., M.S. (Case) Instr., School of Medicine, Western Reserve Univ., Cleveland 6, Ohio. *1340 E. 110th St.*
BODNER, S. R., B.C.E. (Brooklyn Poly.) Instr., Poly. Inst. of Brooklyn, Brooklyn, N.Y. *2719 Laconia Ave., Bronx 67, N.Y.*
BOEDER, PAUL, Ph.D. (Göttingen) Dir., Bureau of Visual Science, American Optical Co., Southbridge, Mass.
BOEHM, FRANK, Manager, Life Insurance, 80 Maiden Lane, New York 7, N.Y.
BOGGS, A. B., M.A. (Michigan) Asst. Prof., Michigan Coll. of Mining and Tech., Houghton, Mich. *On leave:* 447 Grove St., East Lansing, Mich.
BOHNENBLUST, H. F., Ph.D. (Princeton) Prof., California Inst. of Tech., Pasadena 4, Calif. *3595 Thorndale Rd. (10)*
BOLD, E. W., B.S. (Dayton) Grad. Fellow, St. Louis Univ., St. Louis, Mo. *8325 Virginia Ave. (11)*
BOLDYREFF, A. W., Ph.D. (Michigan) Staff Member, Sandia Corporation, 609 N. Tulane Ave., Albuquerque, N.M.
BOLKS, STANLEY, M.S. (Iowa State) Asst. Prof., Purdue Univ., West Lafayette, Ind. *R.R. 10, Lafayette, Ind.*
BOLSER, F. C., M.A. (Peabody) Asst. Prof., Univ. of Tampa, Tampa, Fla.
BOLTON, GRACE L., A.M. (Columbia) Instr., Barnard Coll., New York 27, N.Y.
BOND, L. B., B.S. (U. of Washington) Instr., Seattle Coll., Seattle, Wash. *4306 11th Ave., N.E. (5)*
BONNELL, C. R., M.S. (Missouri Schl. of Mines) Research Engr., Minneapolis-Honeywell Co., Minneapolis, Minn. *5845 S. Blaisdell*
BOOSE, J. A., Brother, B.S. (St. Ambrose C.) Instr., Spalding Inst., Peoria 3, Ill.
BORGMAN, W. M., JR., Ph.D. (Chicago) Asst. Dean, Administration, Wayne Univ., Detroit 1, Mich.
BORNMANN, W. C., A.M. (Columbia) Actuarial Clerk, Compensation Insurance Rating Board, 125 Park Ave., New York, N.Y. *1382 Dean St., Brooklyn, N.Y.*

BOROFSKY, SAMUEL, Ph.D. (Columbia) Asso. Prof., Brooklyn Coll., Brooklyn, N.Y. *65 Lenox Rd. (26)*
BORON, L. F., M.A. (Michigan) Mathematician, Navy Dept., Washington, D.C. *1711 19th St., N.W.*
BORSUK, JACOB, B.A. (Brooklyn) Mathematician, Evans Signal Lab., Belmar, N.J. *702 Second Ave.*
BOSSLER, I. L., M.S. (Purdue) Grad. Asst., Purdue Univ., Lafayette, Ind. *609 Byram St., Pennside, Reading, Pa.*
BOSTICK, C. W., B.S. (M.I.T.) Grad. Student, Univ. of Illinois, Urbana, Ill. *192 Edgewood St., Hartford, Conn.*
BOSWELL, A. V., A.M. (Northwestern) Prof., Tennessee A. and I. State Coll., Nashville 8, Tenn.
BOSWELL, J. M., M.A. (Kentucky) President, Cumberland Coll., Williamsburg, Ky.
BOTHWELL, F. E., Ph.D. (M.I.T.) *1 Trowbridge Pl., Cambridge 38, Mass.*
BOTSFORD, J. L., Ph.D. (C.I.T.) Asso. Prof., Univ. of Idaho, Moscow, Idaho
BOTT, H. A., Student, Illinois Inst. of Tech., Chicago, Ill. *6309 Grace St. (34)*
BOTTS, T. A., Ph.D. (Virginia) Asst. Prof., Univ. of Virginia, Charlottesville, Va. *Montibello Circle*
BOURNE, S. G., Ph.D. (Johns Hopkins) Instr., Univ. of Connecticut, Storrs, Conn.
BOUTELLE, H. D., B.S.Ch.E. (Worcester Poly.) Asst. Prof., Univ. of Massachusetts, Amherst, Mass. *10 Allen St.*
BOWDEN, JOSEPH, Ph.D. (Yale) Emeritus Prof., Adelphi Coll., Garden City, N.Y. *21 Carleton Pl., Baldwin, N.Y.*
BOWER, JULIA W., Ph.D. (Chicago) Asso. Prof., Connecticut Coll., New London, Conn.
BOWERSOX, E. R., B.S. in E.E. (Iowa) Retired. *1505 6th St., S.W., Cedar Rapids, Iowa*
BOWKER, J. G., Ed.M. (Harvard) Prof., Middlebury Coll., Middlebury, Vt. *14 Adirondack View*
BOWLING, FLOYD, M.S. (Iowa) Asso. Prof., Lincoln Memorial Univ., Harrogate, Tenn.
BOYCE, FANNIE W., Ph.D. (Chicago) Asso. Prof., Wheaton Coll., Wheaton, Ill.
BOYCE, JESSIE W., A.M. (Minnesota) Prof., Nebraska State Teachers Coll., Wayne, Nebr.
BOYCE, M. G., Ph.D. (Chicago) Prof., Vanderbilt Univ., Nashville 4, Tenn.
BOYD, P. P., Ph.D. (Cornell) Emeritus Dean, Univ. of Kentucky, Lexington 36, Ky. *119 Waller Ave.*
BOYER, C. B., Ph.D. (Columbia) Asso. Prof., Brooklyn Coll., Brooklyn 10, N.Y.
BOYER, L. E., Ed.D. (Penna. State) Head of Dept., State Teachers Coll., Millersville, Pa. *406 N. George St.*
BOYNTON, HOLMES, Ed.D. (Columbia) Prof., Northern Michigan Coll., Marquette, Mich. *905 N. Front St.*
BRACEWELL, K. H., Ph.D. (Indiana) Prof., Hamline Univ., St. Paul 4, Minn.
BRADFIELD, G. F., M.S. (Northwestern) Instr., Tilden Tech. High School, Chicago, Ill. *1864 Sherman Ave., Evanston, Ill.*
BRADFORD, J. C., Student, North Texas State Coll., Denton, Tex.
BRADLEY, A. D., Ph.D. (Columbia) Asst. Prof., Hunter Coll., New York 21, N.Y. *66 Villard Ave., Hastings-on-Hudson 6, N.Y.*
BRADSHAW, C. L., B.S. (Tennessee Poly.) *1403 McCullough Ave., Huntsville, Ala.*
BRADSHAW, J. W., Ph.D. (Strassburg) Emeritus Prof., Univ. of Michigan, Ann Arbor, Mich. *1304 Cambridge Rd.*
BRADT, R. N., M.A. (Kansas) Grad. Asst., Stanford Univ., Stanford, Calif. *634 Homer Ave., Apt. 4, Palo Alto, Calif.*
BRADY, W. G., M.A. (Washington & Jefferson) Grad. Asst., Univ. of Pittsburgh, Pittsburgh, Pa.
BRAMBLE, C. C., Ph.D. (Johns Hopkins) Dir. of Computation and Ballistics, Naval Proving Ground, Dahlgren, Va.
BRAMBLETT, INA M., A.M. (Texas) Asst. Prof., Texas Christian Univ., Fort Worth 9, Tex.
BRAND, LOUIS, Ph.D. (Harvard) Prof., Univ. of Cincinnati, Cincinnati 21, Ohio
BRANDEBERRY, J. B., Ph.D. (Michigan) Dean, Coll. of Engg., Univ. of Toledo, Toledo 6, Ohio
BRANDNER, F. A., M.S. (Chicago) Asst. Prof., Iowa State Coll., Ames, Iowa. *2821 Lincoln Way*
BRANDT, ANGELINE J., Ph.D. (Michigan) Asst. Prof., Wheaton Coll., Wheaton, Ill.
BRANNON, MILDRED J., M.A. (Illinois) *1204 S. Carle Ave., Urbana, Ill.*
BRANSOŃ, J. W., M.S. (Purdue) President, New Mexico Coll. of A. and M. A., State College, N.M.
BRASWELL, MAMIE I., M.A. (George Peabody) Asso. Prof., Alabama Coll. for Women, Montevallo, Ala. *Collegeview Apts.*
BRAUER, A. T., Ph.D. (Berlin) Prof., Univ. of North Carolina, Chapel Hill, N.C. *410 Pattcrson Pl.*
BRAUER, G. U., M.A. (Michigan) Grad. Student, Univ. of Michigan, Ann Arbor, Mich. *1609 S. University*
BRAUER, RICHARD, Ph.D. (Berlin) Prof., Univ. of Michigan, Ann Arbor, Mich.

BRAUN, J. H., B.Sc. (Illinois Tech.) Mathematical Physicist, Naval Gun Factory, Washington, D.C. *1426 21st St., N.W., (6)*
BRAUNE, O. C., Student, St. Mary's Univ., San Antonio 1, Tex. *1043 McKinley Ave. (10)*
BRAUNSCHWEIGER, C. C., M.S. (Wisconsin) Research Asst., Univ. of Wisconsin, Madison, Wis. *104 E. Pearl, Wellsville, N.Y.*
BRAVERMAN, BENJAMIN, A.M. (Columbia) Head of Dept., Abraham Lincoln High School, Brooklyn 35, N.Y. *1309 Avenue L (30)*
BRAY, H. E., Ph.D. (Rice) Prof., Rice Inst., Houston, Tex.
BREDT, ANITA, Student, Univ. of Pennsylvania, Philadelphia, Pa. *1526 Nedro Ave. (41)*
BREGLIO, J. V., Student, George Pepperdine Coll., Los Angeles, Calif. *933 "A" S. Fir St., Inglewood, Calif.*
BREMER, H. O., B.A. (Hofstra) Formed Products, Inc., Elmhurst, N.Y. *46-21 156th St., Flushing, N.Y.*
BRENKE, W. C., Ph.D. (Harvard) Emeritus Prof., Univ. of Nebraska, Lincoln, Nebr. *1250 S. 21st St. (2)*
BRENNER, J. L., Ph.D. (Harvard) Asst. Prof., Washington State Coll., Pullman, Wash.
BRERETON, J. D., M.A. (Columbia) Vice-Principal, New Paltz High School, New Paltz, N.Y. *P.O. Box 35, Cottekill, N.Y.*
BREUSCH, ROBERT, Ph.D. (Freiburg, Germany) Asso. Prof., Amherst Coll., Amherst, Mass. *12 Walnut St.*
BREWSTER, J. P., A.M. (Duke) Asso. Prof., Clemson Coll., Clemson, S.C.
BREYNAERT, DOROTHY C., B.S. (New Hampshire) Actuarial Student, Monarch Life Insurance Co., 365 State St., Springfield, Mass. *80 Johnson St. (9)*
BRIANT, R. C., Ph.D. (Pittsburgh) Project Dir., Oak Ridge National Lab., Oak Ridge, Tenn. *637 Pennsylvania Ave.*
BRIDGER, C. A., M.S. (Oregon S.C.) Dir., Bureau of Vital Statistics, Jefferson City, Mo. *New Bloomfield, Mo.*
BRIDGFORTH, E. B., A.B. (Duke) Statistician, Nutrition Div., Vanderbilt Medical School, Nashville 4, Tenn. *Pickens, Miss.*
BRIDGLAND, T. F., JR., B.S. (Florida) *Address Unknown.*
BRIGGS, C. F., M.A. (Michigan) Asst. Prof., Emory Univ., Emory University, Ga.
BRIGGS, W. E., M.A. (Colorado) Instr., Univ. of Colorado, Boulder, Colo.
BRIGHAM, N. A., Ph.D. (Pennsylvania) Mathematician, Applied Physics Lab., Johns Hopkins Univ., Silver Spring, Md. *1001 Merrimack Dr.*
BRIGHT, H. F., M.S. (Rochester) Research Associate, Univ. of Texas, Austin, Tex. *1700 E. 38th St.*
BRIGHT, S. K., M.A. (Texas) Prof., Austin Peay State Coll., Clarksville, Tenn.
BRINK, R. W., Ph.D. (Harvard) Prof., Univ. of Minnesota, Minneapolis 14, Minn. *2243 Hoyt Ave., St. Paul 8, Minn.*
BRINKMANN, H. W., Ph.D. (Harvard) Prof., Swarthmore Coll., Swarthmore, Pa.
BRISTOW, LEONARD, Ph.D. (Illinois) Head of Dept., Wisconsin State Teachers Coll., Oshkosh, Wis. *191 Western Ave.*
BRITTON, J. R., Ph.D. (Colorado) Prof., Univ. of Colorado, Boulder, Colo. *929 15th St.*
BRIXEY, J. C., Ph.D. (Chicago) Prof., Univ. of Oklahoma, Norman, Okla. *927 S. Pickard St.*
BROCK, PAUL, Ph.D. (N.Y.U.) Mathematician, Reeves Instrument Corp., 215 E. 91st St., New York 28, N.Y. *7 Stuyvesant Oval (9)*
BRONSTEIN, SAMUEL, A.M. (N.Y.U.) Teacher, High School, Hartford, Conn. *271 Wethersfield Ave. (6)*
BROOKE, W. E., A.M. (Nebraska) Emeritus Prof., Univ. of Minnesota, Minneapolis 14, Minn. *3135-22nd Ave., S. (7)*
BROOKS, E. L., B.S. (Alabama Poly.) *2407 Gulf St., Gulfport, Miss.*
BROOKS, FOSTER, Ph.D. (Ohio State) Prof., Kent State Univ., Kent, Ohio. *51 North Blvd., Twin Lakes*
BROOKS, R. L., B.S. (W. Va. Wesleyan) Mathematician, National Bureau of Standards, Washington, D.C. *8310 Garland Ave., Takoma Park 12, Md.*
BROTHERS, W. H., JR., Ph.D. (Michigan) Prof., Talladega Coll., Talladega, Ala.
BROWN, ALBERT, *Address Unknown.*
BROWN, A. B., Ph.D. (Harvard) Asso. Prof., Queens Coll., Flushing, N.Y.
BROWN, BAILEY, M.A. (Princeton) Prof., Amherst Coll., Amherst, Mass. *19 Hitchcock Rd.*
BROWN, B. H., Ph.D. (Harvard) Prof., Dartmouth Coll., Hanover, N.H. *7 Ripley Rd.*
BROWN, B. K., A.M. (Colorado) Asso. Prof., James Millikin Univ., Decatur 24, Ill. *1451 W. Decatur St.*
BROWN, C. H., Ph.D. (Kansas) Asso. Prof., Central Missouri State Teachers Coll., Warrensburg, Mo. *406 S. Washington St.*
BROWN, CHARLOTTE, M.S. (N.Y.U.) Part-time Instr., Brooklyn Coll., Brooklyn, N.Y. *724 E. 27th St. (10)*

BROWN, D. M., Ph.D. (Illinois) Supervisor, Univ. of Michigan, Ann Arbor, Mich. *1209 Henry St.*
BROWN, E. C., A.M. (Maine) Prof., Worcester Poly. Inst., Worcester, Mass. *51 Grafton St., Shrewsbury, Mass.*
BROWN, F. R., M.A. (Columbia) Asst. Prof., Illinois State Normal Univ., Normal, Ill.
BROWN, G. M., D.Sc. (Michigan) Asso. Prof., Univ. of Georgia, Atlanta Div., Atlanta, Ga. *1316 Oakland Dr., S.W.*
BROWN, H. H., A.B. (Penna. State) Research Engr., Armour Research Foundation, Illinois Inst. of Tech., Chicago, Ill. *914 Sheldrake Apt., 4520 N. Clarendon Ave. (40)*
BROWN, HELEN E. (Mrs.), A.M. (Michigan) *Address Unknown.*
BROWN, H. K., Ph.D. (Michigan) Asso. Prof., Northeastern Univ., Boston, Mass. *87 Harvard Ave., Brookline, Mass.*
BROWN, H. S., M.S. (Lafayette) Emeritus Prof., Hamilton Coll., Clinton, N.Y. *115 Boston St., Guilford, Conn.*
BROWN, J. E., M.A. (Georgia) Acting Instr., Florida State Univ., Tallahassee, Fla. *I. & E. Office, Eglin Air Force Base, Fla.*
BROWN, J. W., B.S. (North Carolina) Instr., Clemson Coll., Clemson, S.C. *Calhoun Rd.*
BROWN, K. E., Ph.D. (Columbia) Asso. Prof., Univ. of Tennessee, Knoxville, Tenn.
BROWN, LILLIAN O., A.M. (Columbia) Prof., Hood Coll., Frederick, Md.
BROWN, M. C., A.M. (Kentucky) Asst. Prof., Univ. of Kentucky, Lexington 29, Ky. *448 Clifton Ave.*
BROWN, MYRTLE C., A.M. (Texas) Asso. Prof., North Texas State Coll., Denton, Tex. *1415 W. Oak St.*
BROWN, M.D., M.A. (George Peabody) Instr., Scott Air Force Base, Ill. *500 Park Ave., Belleville, Ill.*
BROWN, M. D. R., Student, Univ. of Kansas, Lawrence, Kans. *1423 S. 33rd St., Kansas City 3, Kans.*
BROWN, O. E., Ph.D. (Chicago) Prof., Case Inst. of Tech., Cleveland 6, Ohio. *1387 East Blvd.*
BROWN, R. C., JR., M.S. (West Virginia) Part-time Instr., Univ. of Kentucky, Lexington, Ky. *710 Donnelly Dr., Cooperstown*
BROWN, R. G., M.A. (Yale) Project Engr., Willow Run Research Center,. Ypsilanti, Mich. *9671 Woodland Ct.*
BROWN, R. K., M.S. (Rutgers) Grad. Student, Rutgers Univ., New Brunswick, N.J. *36 McLaren St., Red Bank, N.J.*
BROWN, T. H., Ph.D. (Yale) Prof., Business Statistics, Harvard Graduate School of Business Administration, Boston 63, Mass. *25 Meadow Way, Cambridge, Mass.*
BROWN, W. B., Ph.D. (Ohio State) Aeronautical Research Scientist, National Advisory Comm. for Aeronautics, Cleveland, Ohio. *13983 Clifton Blvd., Lakewood 7, Ohio*
BROWN, W. C., M.A. (Oklahoma) Second Lt., U. S. Army. *345 Circle Dr., Fort Collins, Colo.*
BROWNE, D. H., *10 Indian Orchard Pl., Buffalo 10, N.Y.*
BROWNE, E. T., Ph.D. (Chicago) Prof., Univ. of North Carolina, Chapel Hill, N.C. *730 E. Franklin St.*
BROWNE, MARJORIE L., Ph.D. (Michigan) Instr., North Carolina Coll., Durham, N.C.
BRUCE, C. W., Ph.D. (Virginia) Prof., Physics, Tennessee Poly. Inst., Cookeville, Tenn.
BRUCE, R. E., Ph.D. (Boston U.) Emeritus Prof., Boston Univ., Boston, Mass. *319 Buena Vista St., Redlands, Calif.*
BRUCK, R. H., Ph.D. (Toronto) Asso. Prof., Univ. of Wisconsin, Madison 6, Wis. *University Houses, Eagle Heights (5)*
BRUMFIEL, C. F., S.M. (Chicago) *306 N. Talley St., Muncie, Ind.*
BRUMFIELD, EMALOU, M.A. (Ohio State) Asst. Prof., Kent State Univ., Kent, Ohio
BRUNE, I. H., Ph.D. (Ohio State) Asso. Prof., Iowa State Teachers Coll., Cedar Falls, Iowa
BRUNK, H. D., Ph.D. (Rice) Mathematician, Systems Evaluation Dept., Sandia Corp., Sandia Base, Albuquerque, N.M.
BRYAN, N. R., Ph.D. (Columbia) Asso. Prof., Clemson Coll., Clemson, S.C.
BRYANT, B. F., M.A. (Peabody) Fellow, Vanderbilt Univ., Nashville, Tenn. *1905 Glen Echo Rd. (12)*
BRYSON, A. M., A.M. (Pittsburgh) Instr., Univ. of Pittsburgh, Pittsburgh, Pa. *McCutcheon Lane, Wilkinsburg, Pa.*
BUCHANAN, ANN S., A.M. (Louisiana) Asso. Prof., Southwestern Louisiana Inst., Lafayette, La. *1112 Lee Ave.*
BUCHANAN, H. E., Ph.D. (Chicago) Emeritus Prof., Tulane Univ., New Orleans 15, La. *Route 3, Prairie Grove, Ark.*
BUCHMAN, A. L., A.M. (Columbia) Teacher, Hutchinson-Central High School, Buffalo, N.Y. *52 Tioga St.*
BUCK, C. C., M.S. (Michigan) Instr., Univ. of Nebraska, Lincoln, Nebr.
BUCK, ELSIE M., Ph.D. (California) Instr., Boise Junior Coll., Boise, Idaho

BUCK, R. C., Ph.D. (Harvard) Asso. Prof., Univ. of Wisconsin, Madison 6, Wis. *3601 Sunset Dr., Shorewood Hills, (5)*
BUELL, C. E., Ph.D. (Washington U.) Asso. Prof., Univ. of New Mexico, Albuquerque, N.M. *Route 5, Box 736.*
BUELL, E. L., Ph.D. (M.I.T.) Mathematician, Aerial Measurements Lab., Northwestern Tech. Inst., Evanston, Ill.
BUELOW, ELSA M., A.M. (Loyola U., Ill.) Teacher, Chicago City Coll., Wright Branch, Chicago, Ill. *3641 W. Fullerton Ave. (47)*
BUIKSTRA, B. H., M.S. (Kansas S.C.) Asst. Prof., U. S. Naval Acad., Annapolis, Md.
BUKER, W. E., A.M. (Ohio State) Teacher, Public School, Pittsburgh, Pa. *3833 Oswego St.*
BULLARD, J. A., Ph.D. (Clark) Prof., Univ. of Vermont, Burlington, Vt. *110 Summit St.*
BULLIS, G. L., M.A. (Wisconsin) State Teachers Coll., Platteville, Wis.
BULLITT, W. M., B.S. (Princeton) Bullitt and Middleton, Kentucky Home Life Bldg., Louisville 2, Ky.
BULLOCK, R. C., Ph.D. (Chicago) Prof., North Carolina State Coll., Raleigh, N.C.
BUMER, C. T., Ph.D. (Ohio State) Prof., Clark Univ., Worcester, Mass. *121 Richmond Ave. (2)*
BUNCH, W. H., A.M. (Oregon) Teacher, Culver City High School, Culver, Oregon.
BUNYAN, L. H., Ph.D. (Wisconsin) Asso. Prof., Rutgers Univ., New Brunswick, N.J.
BURACK, M.D., M.S. (DePaul) Instr., Wilson Junior Coll., Chicago, Ill. *125 N. Humphrey Ave., Oak Park, Ill.*
BURCHAM, P. B., Ph.D. (Northwestern) Asst. Prof., Univ. of Missouri, Columbia, Mo. *209 S. Williams St.*
BURCKHARDT, J. J., Prof., Univ. of Zurich, Zurich, Switzerland. *Bergheimstrasse 4, Zurich 32, Switzerland*
BURDETTE, A. C., Ph.D. (Illinois) Asso. Prof., Univ. of California, Davis, Calif.
BURGER, C. G., JR., M.A. (Rensselaer) Instr., Rensselaer Poly. Inst., Troy, N.Y. *18 Peoples Dr. W., Carnegie Bldg.*
BURGESS, R. W., Ph.D. (Cornell) Chief Statistician, Western Elec. Co., 195 Broadway, New York 7, N.Y.
BURGESS, W. J., Morris Coll., Sumter, S.C.
BURINGTON, ARTHA J., M.S. (Maryland) Teacher, Mount Vernon High School, Route 4, Alexandria, Va. *5200 N. Carlin Spring Rd., Arlington, Va.*
BURINGTON, R. S., Ph.D. (Ohio State) Chief Mathematician, Bureau of Ordnance, Navy Dept., Washington, D.C. *5200 N. Carlin Spring Rd., Arlington, Va.*
BURK, J. D., B.A. (Toronto) Asso. Prof., Univ. of Toronto, Toronto 5, Ont., Canada
BURKART, MARY P., M.A. (Detroit) Instr., Trinity Coll., Washington, D.C. *4311 28th Pl., Apt. 4, Mt. Rainier, Md.*
BURKE, J. G., Rev., A.M. (Mt. St. Mary) Mount St. Mary Coll., Emmitsburg, Md.
BURKETT, F. J. H., Ph.D. (N.Y.U.) Asso. Prof., Union Coll., Schenectady, N.Y. *1030 Park Ave. (8)*
BURKHALTER, EVA, M.S. (Oregon) Teacher, Klamath Union High School, Klamath Falls, Ore.
BURKHART, SARAH M., M.A. (Kansas) Asst. Prof., Univ. of Tulsa, Tulsa, Okla. *501 N. Elwood (6)*
BURNAM, J. E., A.M. (Texas) Prof., Hardin-Simmons Univ., Abilene, Tex. *Route 4, Box 6*
BURNETTE, H. W., M.S. (Georgia) Instr., Alabama Poly. Inst., Auburn, Ala.
BURNS, G. C., B.S. (Oklahoma A. & M.) Grad. Fellow, Mary Hardin-Baylor Coll., Belton, Tex.
BURNS, H. E., Ph.D. (Northwestern) Dir., Indiana Univ. Ext., Calumet Center, East Chicago, Ind.
BURNS, W. F., Rev., M.S. (Boston C.) Asso. Prof., Coll. of the Holy Cross, Worcester, Mass.
BURR, I. W., Ph.D. (Michigan) Prof., Purdue Univ., West Lafayette, Ind. *265 Littleton St.*
BURRELL, M. O., A.M. (Emory) Trainer, Dept. of the Air Force, Warner Robins Air Force Base, Ga. *402 MacArthur Blvd., Warner Robins, Ga.*
BURRILL, B. A., B.A. (Buffalo) *184 S. Elmwood Ave., Buffalo 1, N.Y.*
BURROWS, R. A., B.A. (Albion) Grad. Student, Yale Univ., New Haven, Conn. *931 Elm St. (11)*
BURT, MARY M. (Mrs. L. N.), A.M. (Wisconsin) *800 24th Ave., N., St. Petersburg 4, Fla.*
BURTON, G. L., B.S. (M.I.T.) *338 Orchard, Elgin, Ill.*
BURTON, L. J., Ph.D. (Harvard) Asst. Prof., Bryn Mawr Coll., Bryn Mawr, Pa.
BURTON, L. P., A.M. (Alabama) Instr., Univ. of California, Davis, Calif.
BURWELL, W. R., Ph.D. (Oxford) Chairman, Brush Development Co., 3405 Perkins Ave., Cleveland 14, Ohio
BUSCHMAN, R. G., M.A. (Oregon) Part-time Instr., Univ. of Colorado, Boulder, Colo.
BUSCHMAN, W. O., M.Ed. (Oregon) Instr., Vanport Extension Center, Portland, Ore. *3817 S.E. Carlton St. (2)*
BUSEMANN, HERBERT, Ph.D. (Göttingen) Prof., Univ. of Southern California, Los Angeles 7, Calif.

BUSH, K. A., Ph.D. (North Carolina) Asso. Prof., Champlain Coll., Plattsburg, N.Y.
BUSH, L. E., Ph.D. (Ohio State) Prof., Coll. of St. Thomas, St. Paul 1, Minn.
BUSHEY, J. H., Ph.D. (Michigan) Asso. Prof., Hunter Coll., New York 21. N.Y.
BUSHEY, JEWELL H. (Mrs. J. H.), Ph.D. (Chicago) Asso. Prof., Hunter Coll., New York 21, N.Y. *501 W. 113th St. (25)*
BUSHYAGER, G. R., A.M. (Penna. State) Prof., Morningside Coll., Sioux City, Iowa. *1221 Morningside Ave. (20)*
BUSSEY, W. H., Ph.D. (Chicago) Emeritus Prof., Univ. of Minnesota, Minneapolis 14, Minn. *23 River Terrace Ct.*
BUTCHART, J. H., Ph.D. (Illinois) Prof., Arizona State Coll., Flagstaff, Ariz.
BUTLER, C. H., Ph.D. (Missouri) Prof., Western Michigan Coll., Kalamazoo, Mich. *1941 Stevens Ave. (37)*
BUTLER, E. A., M.A. (Columbia) Instr., New York State Coll. for Teachers, Albany, N.Y. *116 Main St., Altamont, N.Y.*
BUTLER, J. W., Asso. Engr., Argonne National Laboratory, Chicago, Ill. *P.O. Box 5207 (80)*
BUTLER, L. G., A.M. (Oregon) Instr., State Coll. of Washington, Pullman, Wash. *1904 B St.*
BUTTER, F. A., JR., Ph.D. (Stanford) Research Physicist, Hughes Aircraft Co., Culver City, Calif. *P.O. Box 621, Pacific Palisades, Calif.*
BUTTERFIELD, A. D., Dr. Sci. (Worcester Poly.) Emeritus Prof., Univ. of Vermont, Burlington, Vt. *479 Main St.*
BUTZ, R. K., M.S. (Georgia) Instr., Univ. of Georgia, Athens, Ga.
BUXTON, C. L., M.S. (Case) President, Paul Smith's Coll., Paul Smith's, N.Y.
BYE, NIKOLINE A., A.M. (Michigan) Asst. Prof., Central Michigan Coll., Mt. Pleasant, Mich.
BYRD, P. F., M.S. (Chicago) Aeronautic Research Scientist, National Advisory Comm. for Aeronautics, Moffett Field, Calif. *3235 Emerson St., Palo Alto, Calif.*
BYRNE, MARGARET C., A.M. (Columbia) Prof., St. Joseph's Coll. for Women, Brooklyn 5, N.Y. *431 Beach 142 St., Rockaway Park, N.Y.*
BYRNE, W. E., Ph.D. (Rensselaer) Prof., Virginia Military Inst., Lexington, Va.

CAIRNS, S. S., Ph.D. (Harvard) Prof., Univ. of Illinois, Urbana, Ill.
CAIRNS, W. D., Ph.D. (Göttingen) Honorary President and Honorary Life Member. Emeritus Prof., Oberlin Coll., Oberlin, Ohio. *1675 Kenneth Way, Pasadena 3, Calif.*
CALCAGNO, H. E., *Patria 715, Montevideo, Uruguay*
CALHOON, C. D., M.A. (Illinois) Asst. Prof., Univ. of Toledo, Toledo 6, Ohio
CALKINS, ELEANOR, A.M. (Michigan) Asst. Prof., Coll. of William and Mary, Williamsburg, Va. *608 Jamestown Rd.*
CALKINS, HELEN, Ph.D. (Cornell) Prof., Pennsylvania Coll. for Women, Pittsburgh 32, Pa.
CALLAHAN, C. M., B.S. (Lincoln Memorial U.) Lincoln Memorial Univ., Harrogate, Tenn.
CALLAHAN, ETHEL B., Ph.D. (Columbia) Prof., Hartwick Coll., Oneonta, N.Y. *122 Chestnut St.*
CALVERT, R. L., Ph.D. (Illinois) Component Engr., Sandia Corp., Albuquerque, N.M. *415 N. Adams*
CAMERON, E.A., Ph.D. (North Carolina) Prof., Univ. of North Carolina, Chapel Hill, N.C. *404 Laurel Hill Rd.*
CAMERON, J. M., M.S. (North Carolina S.T.) Mathematician, Statistical Engg. Lab., National Bureau of Standards, Washington 25, D.C.
CAMERON, R. H., Ph.D. (Cornell) Prof., Univ. of Minnesota, Minneapolis 14, Minn. *4401 15th Ave., S.*
CAMP, B. H., Ph.D. (Yale) Emeritus Prof., Wesleyan Univ., Middletown, Conn. *110 Mt. Vernon St.*
CAMP, C. C., Ph.D. (Cornell) Prof., Univ. of Nebraska, Lincoln 8, Nebr.
CAMP, E. J., Ph.D. (Chicago) Prof., Macalester Coll., St. Paul 5, Minn.
CAMPAIGNE, H. H., Ph.D. (Northwestern) Mathematician, Navy Dept., Washington, D.C. *7 Oxford St., Garrett Park, Md.*
CAMPBELL, D. F., Ph.D. (Harvard) Consulting Actuary, Room 1205, 188 W. Randolph, Chicago 1, Ill.
CAMPBELL, FRANCES L., Ph.D. (Michigan) Asso. Prof., George Pepperdine Coll., Los Angeles 44, Calif.
CAMPBELL, G. A., Ph.D. (Harvard) Retired Research Engr., American Tel. and Tel. Co., New York, N.Y. *129 Bellevue Ave., Upper Montclair, N.J.*
CAMPBELL, G. J., Student, Univ. of Detroit, Detroit, Mich. *212 S. Maple, Royal Oak, Mich.*
CAMPBELL, J. D., Ph.D. (Illinois) Asso. Prof., Rensselaer Poly. Inst., Troy, N.Y.
CAMPBELL, J. G., Student, Univ. of Kentucky, Lexington, Ky. *Box 426, Albany, Ky.*
CAMPBELL, J. W., Ph.D. (Chicago) Prof., Univ. of Alberta, Edmonton South, Alta., Canada
CAMPBELL, JESSIE R., A.B. (Syracuse) Retired. *12 The Strand, Hermosa Beach, Calif.*

CAMPBELL, L. G., A.M. (New Jersey S.T.C.) Instr., Teachers Coll., Columbia Univ., New York 27, N.Y.
CAMPBELL, RICHARD CRAWFORD, B.A. (Dartmouth) Grad. Asst., Univ. of Colorado, Boulder, Colo. *1502 Cascade Ave.*
CAMPBELL, RICHARD CROWLEY, A.M. (Pennsylvania) Asso. Prof., U. S. Naval Postgraduate School, Annapolis, Md. *1010 Forest Hills Dr.*
CAMPOPIANO, C. N., B.S. (Rutgers) Grad. Student, Rutgers Univ., New Brunswick, N.J. *734 Hamilton St.*
CANFIELD, E. L., Ph.D. (Iowa S.C.) Asso. Prof., Drake Univ., Des Moines, Iowa. *4110 S.W. Fifth St. (15)*
CANNING, JOSEPH, A.B. (Intermountain) Roundhouse Foreman, Northern Pacific R.R., Garrison, Mont.
CANTWELL, MARY L. (Mrs.), M.A. (Columbia) Teacher, St. Petersburg Junior Coll., St. Petersburg, Fla. *2730 Dartmouth Ave., N.*
CAPARÓ, J. A., Ph.D. (Notre Dame) Emeritus Prof., Electrical Engg., Univ. of Notre Dame, Notre Dame, Ind. *1024 Leeper Blvd., South Bend 17, Ind.*
CAPESIUS, JOHN, REV., A.M. (Alabama) Head of Dept., St. Bernard Coll., St. Bernard, Ala.
CARBERRY, J. P., M.S. (Carnegie) Actuarial Student, Mutual Benefit Life Insurance Co., Newark, N.J. *Apt. 9-K, 14 Marshall St., Irvington, N.J.*
CAREY, E. F. A., M.S. (California) Emeritus Asso. Prof., Montana State Univ., Missoula, Mont. *3808 Grande Rd., Albuquerque, N.M.*
CARIS, P. A., Ph.D. (Pennsylvania) Asso. Prof., Univ. of Pennsylvania, Philadelphia 4, Pa. *717 Shadeland Ave., Drexel Hill, Pa.*
CARIS, V. B., A.M. (Defiance) Asst. Prof., Ohio State Univ., Columbus 2, Ohio. *99 W. Weber Rd.*
CARL, I. A., M.A. (Columbia) Asst. Prof., New York Univ., New York 53, N.Y. *2400 Sedgewick Ave. (63)*
CARLAN AUDREY M. (Mrs. A. J.) B.A. (Brooklyn) Mathematician, Dept. of Defense, Pentagon, Washington, D.C. *116 Lynnwood St., Alexandria, Va.*
CARLAN, A. J., B.A. (Brooklyn) Second Lt., U. S. Marine Corps. *116 Lynwnood St., Alexandria, Va.*
CARLEN, MILDRED E., M.S. (Brown) Registrar, Grad. School, Brown Univ., Providence 12, R. I.
CARLITZ, LEONARD, Ph.D. (Pennsylvania) Prof., Duke Univ., Durham, N.C.
CARLOCK, VIRGINIA, M.S. (Alabama Poly.) *Kensington, Ga.*
CARLSON, C. S., M.S. (Iowa) Prof., St. Olaf Coll., Northfield, Minn.
CARLSON, ELIZABETH, Ph.D. (Minnesota) Asso. Prof., Univ. of Minnesota, Minneapolis 14, Minn. *3024 14th Ave., S. (7)*
CARLSON, J. A., M.S. (Washington) Prof., Whitworth Coll., Spokane 12, Wash.
CARLTON, L. VIRGINIA, A.M. (Tulane) Head of Dept., Wesleyan Coll., Macon, Ga.
CARMAN, M. G., Ph.D. (Illinois) Head of Dept., Murray State Coll., Murray, Ky.
CARMICHAEL, F. L., A.M. (Princeton) Prof., Statistics, Univ. of Denver, Denver, Colo. *2230 Colorado Blvd.*
CARMICHAEL, R. D., Ph.D. (Princeton) Emeritus Dean, Grad. School, Univ. of Illinois, Urbana, Ill. *Box 421, Griggsville, Ill.*
CARNES, W. F., A.M. (Harvard) *3705 W. 61st St., Los Angeles 43, Calif.*
CARPENTER, DOROTHY I., M.A. (Michigan) Instr., Denison Univ., Granville, Ohio. *321 N. Pearl St.*
CARPENTER, D. R., A.M. (Princeton) Prof., Roanoke Coll., Salem, Va.
CARPENTER, F. M., A.M. (Illinois) Asso. Prof., Colorado School of Mines, Golden, Colo.
CARPENTER, J. A., M.A. (North Carolina) Mathematician, Snow & Schule, Inc., Central Square Bldg., Cambridge 39, Mass.
CARPENTER, P. N., M.S. (U. of Washington) Prof., Grove City Coll., Grove City, Pa. *214 E. Pine St.*
CARR, C. R., A.M. (Michigan) Teacher, Indiana Tech. Coll., Fort Wayne, Ind.
CARR, R. E., Ph.D. (Iowa State) Asst. Prof., Michigan State Coll., East Lansing, Mich.
CARROLL, C. L., JR., Ph.D. (North Carolina) Asso. Prof., North Carolina State Coll., Raleigh, N.C. *609 Stacy St.*
CARROLL, I. S., A.M. (Columbia) Emeritus Prof., Syracuse Univ., Syracuse 10, N.Y. *118 Roosevelt Ave.*
CARROLL, R. E., M.S. (Illinois) Dynamics Engr., Bell Aircraft Corp., Niagara Falls, N.Y. *34 Ward Park Rd., Grand Island, N.Y.*
CARSCALLEN, G. E., A.M. (Illinois) Asso. Prof., Wabash Coll., Crawfordsville, Ind. *112 N. Barr St.*
CARSON, A. B., Ph.D. (Chicago) Asso. Prof., Army Air Forces Inst. of Tech., Wright Field, Dayton, Ohio

CARSON, T. C., A.M. (Duke) Prof., East Tennessee State Coll., Johnson City, Tenn.
CARSON, W. E., B.S. (Brooklyn Poly.) Part-time Instr., Poly. Inst. of Brooklyn, Brooklyn 2, N.Y. *66 Forest Rd., Dumont, N.J.*
CARSTOIU, ION, Ph.D. (Paris) Asst. Prof., Aeronautics, Johns Hopkins Univ., Baltimore 18, Md.
CARTER, H. N., M.S. (Colorado) Asst. Prof., Univ. of Tulsa, Tulsa 4, Okla.
CARTER, H. P., M.A. (Vanderbilt) *Iron City, Tenn.*
CARVER, W. B., Ph.D. (Johns Hopkins) Honorary Life Member, Emeritus Prof., Cornell Univ., Ithaca, N. Y. *204 Oak Hill Rd.*
CARVILLE, L. E., M.A. (Boston U.) Mathematician, Army Map Service, Washington, D.C. *1204 Delefield Pl., N.W.*
CARY, R. J., Student, Harpur Coll., Endicott, N. Y. *15 Mygatt St., Binghamton, N.Y.*
CASCIO, MONA, *171 Rogers Ave., Brooklyn, N.Y.*
CASE, J. H., B.S. (Alabama Poly.) Teaching Fellow, Alabama Poly. Inst., Auburn, Ala.
CASEY, ELLA F. (Mrs. N. E.), M.Ed. (U. of Rochester) *R. D., Romulus, N.Y.*
CASHEN, W. R., M.A. (U. of Washington) Prof., Univ. of Alaska, College, Alaska.
CASKEY, R. L., A.M. (Oklahoma) Asst. Prof., Oklahoma A. and M. Coll., Stillwater, Okla. *Route 4*
CASSEB, GEORGE, B.S. (Chicago) Instr., St. Mary's Univ., San Antonio, Tex. *1551 W. Rosewood (1)*
CATHEY, ELIZABETH C., M.S. (Louisiana) Instr., Univ. of Alabama, University, Ala.
CATO, B. R., JR., A.M. (Duke) Instr., Univ. of Arizona, Tucson, Ariz. *2032 Greenway Ave., Charlotte 4, N.C.*
CATON, W. B., Ph.D. (Yale) Asst. Prof., DePaul Univ., Chicago, Ill.
CAUFFMAN, P. F., M.A. (Lehigh) Head of Dept., Maryland State Teachers Coll., Salisbury, Md. *867 Hammond St.*
CAVALLI, S. F., D.Sc. (Milan) Mech. Engr. Mathematician, Frankford Arsenal, Philadelphia, Pa. *1357 Kenwood Ave., Camden 3, N.J.*
CAWLEY, JOHN, M.S. (Lafayette) Asso. Prof., Lafayette Coll., Easton, Pa. *621 Coleman St.*
CEBULA, RICHARD, REV., M.S. (Michigan) Asst. Prof., St. Martin's Coll., Olympia, Wash.
CEDERBERG, W. E., Ph.D. (Wisconsin) Emeritus Prof., Augustana Coll., Rock Island, Ill. *2542 22½ Ave.*
CELAURO, F.L., A.M. (N.Y.U.) *607 Shaler Blvd., Ridgefield, N.J.*
CELL, J. W., Ph.D. (Illinois) Prof., North Carolina State Coll., Raleigh, N.C.
CERTAINE, JEREMIAH, Ph.D. (Harvard) Asst. Prof., Howard Univ., Washington, D.C. *45 E. 135th St., New York, N.Y.*
CHAMBERS, L. H., Ph.D. (Cornell) Asso. Prof., U. S. Naval Acad., Annapolis, Md.
CHAMBERS, W. W., M.A. (Colorado S.C.) Instr., Purdue Univ., Extension Div., Indianapolis, Ind. *3349 N. Capitol Ave. (8)*
CHANLER, JOSEPHINE H., Ph.D. (Illinois) Asst. Prof., Univ. of Illinois, Urbana, Ill.
CHAPMAN, D. G., Ph.D. (California) Asst. Prof., Univ. of Washington, Seattle, Wash.
CHARLESWORTH, G. B., M.A. (Cambridge) Instr., Hofstra Coll., Hempstead, N.Y.
CHARLESWORTH, H. W., M.A. (Colorado S.C.) Chmn., East High School, Denver, Colo. *1546 Cook St. (6)*
CHARNES, ABRAHAM, Ph.D. (Illinois) Asso. Prof., Carnegie Inst. of Tech., Pittsburgh 13, Pa.
CHASE, L. R., Teacher, Rogers High School, Newport, R.I. *10 Boulevard Terr.*
CHASEN, S. H., B.C.E. (Georgia Tech.) Grad. Student, Emory Univ., Emory University, Ga. *2595 Powell Lane, Apt. 4, Decatur, Ga.*
CHATELAIN, VIRGINIA L. (Mrs.), M.S. (Kansas S.T.C.) Instr., Adela Hale Business Career School, Hutchinson, Kans. *1703 N. Van Buren*
CHATLAND, HAROLD, Ph.D. (Chicago) Prof., Montana State Univ., Missoula, Mont.
CHEATHAM, T. E., JR., B.S. (Purdue) Research Asst., Purdue Univ., Lafayette, Ind. *690 Waldron St., West Lafayette, Ind.*
CHELLEVOLD, J. O., A.M. (Northwestern) Asst. Prof., Lehigh Univ., Bethlehem, Pa. *1111 Maple St.*
CHEN, Y. W., Dr. (Göttingen) Asso. Prof., Univ. of Oklahoma, Norman, Okla. *1107 McNamee St.*
CHENEY, NANCY V. H. (Mrs. G. T.), B.A. (Carleton) Part-time Instr., Univ. of Colorado, Boulder, Colo. *458 Evergreen, East Lansing, Mich.*
CHENEY, W. F., JR., Ph.D. (M.I.T.) Prof., Univ. of Connecticut, Storrs, Conn.
CHERLIN, G. Y., Ph.D. (Rutgers) Actuarial Student, Mutual Benefit Life Insurance Co., Newark, N.J. *234 Davey St., Bloomfield, N.J.*
CHERN, S. S., D.Sc. (Hamburg) Prof., Univ. of Chicago, Chicago 37, Ill.
CHERNOFSKY, M. I., M.B.A. (N.Y.U.) Lecturer, City Coll. of New York, New York, N.Y. *P.O. Box 81, St. Johns Place Station, Brooklyn 13, N.Y.*
CHERRY, I. J., M.A. (Oregon S.C.) Research Asst., Univ. of California, Los Alamos Scientific Lab., Los Alamos, N.M. *2166-B 44th St.*

CHERRY, W. J., M.A. (Northwestern) Teacher, Morton High School and Junior Coll., Cicero, Ill. *2123 Oak Park Ave., Berwyn, Ill.*
CHESNA, JOHN, Physicist, Eastman Kodak Co., Rochester, N.Y. *12 Skylane Dr. (21)*
CHESSIN, P. L., M.A. (Wisconsin) Instr., Cooper Union Inst., 50 Astor Pl., New York, N.Y. *316 W. 84th St., New York 24, N.Y.*
CHESTON, H. J., Jr., M.A. (Maryland) Instr., Montgomery Junior Coll., Bethesda, Md. *2916 18th St., N.W., Washington 9, D.C.*
CHIAPPINELLI, B. A., A.B. (U.C.L.A.) Mathematician, Rand Corp., Santa Monica, Calif. *10790 Galvin St., Culver City, Calif.*
CHIN, LOUISE H., Ph.D. (California) Asst. Prof., Univ. of Arizona, Tucson, Ariz.
CHITTENDEN, E. W., Ph.D. (Chicago) Prof., Univ. of Iowa, Iowa City, Iowa. *1101 Kirkwood Ave.*
CHOUTEAU, C. C., A.B. (Kansas) Teacher, Bowring High School, Bowring, Okla.
CHOVITZ, B. H., M.A. (Harvard) Mathematician, Army Map Service, Washington 16, D.C. *2339 40th Pl., N.W. (7)*
CHOWLA, SARVADAMAN, Ph.D. (Cambridge) Visiting Prof., Univ. of Kansas, Lawrence, Kan.
CHRESTENSON, H. E., M.A. (Washington S.C.) Teaching Fellow, Univ. of Oregon, Eugene, Ore.
CHRISTENSEN, R. A., Student, Willamette Univ., Salem, Ore. *1325 Lee St.*
CHRISTIAN, ERMINE A., A.B. (Mercer) Mathematician, Naval Ordnance Lab., Silver Spring, Md. *Box 245, Indian Head, Md.*
CHRISTIAN, H. S., Jr., B.S. (Illinois Tech.) Engr., Sentinel Radio Corp., 2100 Dempster St., Evanston, Ill. *2737 Eastwood Ave.*
CHRISTIAN, R. S., M.S. (Washington U.) Asst. Prof., Georgia Inst. of Tech., Atlanta, Ga. *1302 W. Peachtree St., N.W., Apt. 323*
CHRISTIANO, J. G., Ph.D. (Pittsburgh) Asst. Prof., Univ. of Pittsburgh, Pittsburgh, Pa. *222 Dan Dr. (13)*
CHRISTIE, D. E., Ph.D. (Princeton) Asso. Prof., Bowdoin Coll., Brunswick, Maine. *R.F.D. 2, Bowdoinham, Maine*
CHRISTMAN, LAURA E., A.M. (Wisconsin) Teacher, Senn High School, Chicago, Ill. *1217 Elmdale Ave.*
CHU, R. H., Student, Wayne Univ., Detroit, Mich. *Detroit Memorial Hospital, 1420 St. Antoine St. (26)*
CHURCH, ELSIE T., M.A. (Kentucky) Instr., Northwestern State Coll., Natchitoches, La.
CHURCH, RANDOLPH, Ph.D. (Yale) Prof., U. S. Naval Postgraduate School, Annapolis, Md. *316 N. Glen Ave.*
CHURCHILL, R. V., Ph.D. (Michigan) Prof., Univ. of Michigan, Ann Arbor, Mich. *924 S. Forest Ave.*
CICCHESE, A. C., B.S. (Notre Dame) Grad. Student, St. Mary's Univ., San Antonio 1, Tex. *714 W. Russell Pl.*
CIOFFI, P. O., M.S. (New Mexico A.&M.A.) Research Engr., Massachusetts Inst. of Tech., Cambridge 38, Mass. *68 Dana St. (38)*
CIOLKOWSKI, S. I., Student, New York Univ., New York, N.Y. *72-25 153rd St., Kew Garden Hills, Flushing, N.Y.*
CIVIN, PAUL, Ph.D. (Duke) Asso. Prof., Univ. of Oregon, Eugene, Ore.
CLARK, A. G., A.M. (Colorado) Prof., Colorado A. and M. Coll., Fort Collins, Colo.
CLARK, B. B., M.A. (Oberlin) Capt., U. S. Air Force. *Box 302, Rantoul, Ill.*
CLARK, B. G., Ph.D. (Illinois) Asso. Prof., Vanderbilt Univ., Nashville, Tenn. *Hill Rd., Brentwood, Tenn.*
CLARK, C. E., Ph.D. (Cornell) Asso. Prof., Emory Univ., Emory University, Ga.
CLARK, C. L., Ph.D. (Virginia) Prof., Oregon State Coll., Corvallis, Ore.
CLARK, CHARLES R., A.M. (Michigan) Staff Member, Sandia Corp., Albuquerque, N.M. *3227-D 34th St.*
CLARK, CHESTER R., A.M. (Michigan) Teacher, McKinley High School, Washington, D.C. *2707 Adams Mill Rd., N.W.*
CLARK, F. E., Ph.D. (Duke) Asst. Prof., Rutgers Univ., New Brunswick, N.J.
CLARK, K. L., Ph.B. (Wisconsin) Mathematician, A. O. Smith Corp., Milwaukee, Wis. *2331 N. 25th St. (6)*
CLARK, WILLIE E. (Mrs.), M.A. (Atlanta) Head of Dept., Agr., Mech., and Normal Coll., Pine Bluff, Ark.
CLARK, W. G., Ph.D. (Kentucky) Asst. Prof., Mt. Union Coll., Alliance, Ohio
CLARKE, E. H., Ph.D. (Chicago) Prof., Hiram Coll., Hiram, Ohio
CLARKE, F. MARION, A.M. (Smith C.) Instr., Univ. of Nebraska, Lincoln 8, Nebr. *1748 F St., Apt. 4 (8)*
CLARKE, L. BEATRICE (Mrs.), A.M. (Michigan) Asst. Prof., Florida A. and M. Coll., Tallhassee, Fla.
CLARKE, W. B., President, W. B. Clarke & Co., P.O. Box 343, San Jose 2, Calif.

CLARKSON, HELEN E., A.M. (Duke) Instr., Creighton Univ., Omaha, Nebr. *528 S. 52nd St.*
CLARKSON, J. A., Ph.D. (Brown) Head of Dept., Tufts Coll., Medford, Mass. *25 Sagamore Park, West Medford, Mass.*
CLATWORTHY, W. H., M.A. (Kentucky) Student, Univ. of North Carolina, Chapel Hill, N.C.
CLAWSON, J. W., M.A. (New Brunswick) Dean, Ursinus Coll., Collegeville, Pa. *954 College Ave.*
CLAYTON, M. H., M.E. (North Carolina S.C.) Instr., North Carolina State Coll., Raleigh, N.C.
CLAYTOR, W. W. S., Ph.D. (Pennsylvania) Asso. Prof., Howard Univ., Washington 1, D.C.
CLELAND, W. E., Ph.D. (Princeton) Prof., Geneva Coll., Beaver Falls, Pa.
CLELLAND, R. C., A.M. (Columbia) Instr., Hamilton Coll., Clinton, N.Y. *11 Williams St.*
CLEMANS, K. G., M.A. (Minnesota) Instr., Univ. of Oregon, Eugene, Ore. *2270-1 Patterson Dr.*
CLEMENS, P. F., M.A. (New York S.T.C.) Instr., Rensselaer Poly. Inst., Troy, N.Y. *21 Green St., Rensselaer, N.Y.*
CLEMENT, MARY D., Ph.D. (Chicago) *1711 Ashwood Ave., Nashville 4, Tenn.*
CLEMENT, P. A., Ph.D. (U.C.L.A.) Instr., State Coll. of Washington, Pullman, Wash.
CLEMENTS, G. R., Ph.D. (Harvard) Retired Prof., U. S. Naval Acad., Annapolis, Md. *7 Thompson St.*
CLEVELAND, M. J., M.S. (Florida) Lt. (j.g.), U. S. Navy. Station 9½, Sullivan's Island, S.C.
CLEVELAND, W. H., M.S. (Alabama) Head of Dept., Meridian Junior Coll., Meridian, Miss. *4024 30th St.*
CLIFFORD, A. H., Ph.D. (C.I.T.) *Locust Hill, Glyndon, Md.*
CLIFFORD, P. C., A.M. (Columbia) Instr., New Jersey State Teachers Coll., Upper Montclair, N.J.
CLINTON, FLORENTINA M. (Mrs. J. C.), A.M. (Ohio State) Teacher, Chillicothe High School, Chillicothe, Ohio. *172 Vine St.*
CLOS, CHARLES, C.E. (N.Y.U.) Member, Tech. Staff, Bell Telephone Labs.; Instr., Pratt Inst., Brooklyn, N.Y. *6434 83rd St., Middle Village, N.Y.*
CLUCAS, HELEN S. (Mrs.), M.A. (Oregon) *610 S. Shelton, Santa Ana, Calif.*
CLUTTERHAM, D. R., M.S. (Arizona) Grad. Asst., Univ. of Illinois, Urbana, Ill.
COBB, R. N., A.M. (Harvard) Asso. Prof., Worcester Poly. Inst., Worcester, Mass. *26 Einhorn Rd. (2)*
COBB, REGINALD, M.S. (Florida) Statistician, Naval Air Tech. Training Center, Jacksonville, Fla. *6410 Pearl St.*
COBB, S. C., S.M. (Arizona) Teacher, Phillips Acad., Andover, Mass.
COBLE, A. B., Ph.D. (Johns Hopkins) Emeritus Prof., Univ. of Illinois, Urbana, Ill. *1907 Tondolea Lane, La Canada, Calif.*
COBURN, NATHANIEL, Ph.D. (M.I.T.) Asso. Prof., Univ. of Michigan, Ann Arbor, Mich. *1308 Harbrook Ave.*
COCHRAN, H. M., Col., U. S. Army, Retired. 101 Miramonte Ave., Palo Alto, Calif.
COCUZZA, P. J., B.S. (C.C.N.Y.) Grad. Student, Brooklyn Coll., Brooklyn, N.Y. *2571 W. 16th St. (14)*
CODY, W. J., Jr., Student, Elmhurst Coll., Elmhurst, Ill. *159 Elmhurst Ave.*
COE, C. J., Ph.D. (Harvard) Asst. Prof., Univ. of Michigan, Ann Arbor, Mich. *2022 Hill St.*
COFFEY, D. E., M.A. (Montana State U.) *Box 7, Neihart, Mont.*
COFFIN, E. W., B.S. (Maryland) Analytic Aide, Dept. of Defense, Washington, D.C. *1916 N. Van Buren St., Falls Church, Va.*
COFFIN, L. M., A.M. (Michigan) Emeritus Prof., Coe Coll., Cedar Rapids, Iowa. *633 S. Bixel St., Los Angeles 14, Calif.*
COFFMAN, M. L., M.A. (Oklahoma) Instr., Physics, East Texas State Teachers Coll., Commerce, Tex. *Box 721, College Station*
COGAN, E. J., M.A. (Wisconsin) Instr., Pennsylvania State Coll., State College, Pa.
COHEN, A. C., Jr., Ph.D. (Michigan) Asso. Prof., Univ. of Georgia, Athens, Ga.
COHEN, CARL, M.A. (Harvard) Lecturer, Wheaton Coll., Norton, Mass. *40 Prentiss St., Cambridge 40, Mass.*
COHEN, ECKFORD, Ph.D. (Duke) Inst. for Advanced Study, Princeton, N.J.
COHEN, H. J., Ph.D. (Wisconsin) Univ. of Paris, Inst. Henri Poincare, 11 Rue Pierre Curie, Paris 5, France
COHEN, K. J., B.A. (Reed) Rhodes Scholar, Queen's Coll., Oxford Univ., Oxford, England
COHEN, L. W., Ph.D. (Michigan) Asso. Prof., Queens Coll., Flushing, N.Y. *217-18 48th Ave., Bayside, N.Y.*
COHEN, TERESA, Ph.D. (Johns Hopkins) Prof., Pennsylvania State Coll., State College, Pa. *315 S. Atherton St.*
COHEN, WILLIAM, M.A. (George Washington) Dir., Town and Country Day School, 9401 Georgia Ave., Silver Spring, Md.
COHN, R. M., Ph.D. (Columbia) Instr., Rutgers Univ., New Brunswick, N.J. *905 West End Ave., New York 25, N.Y.*

COKER, R. L., A.M. (Alabama) Prof., Physics, Southern Coll. of Optometry, Memphis, Tenn. *1877 Cowden Ave. (4)*
COLE, NANCY, Ph.D. (Radcliffe) Asst. Prof., Syracuse Univ., Syracuse 10, N.Y.
COLE, R. H., Ph.D. (Wisconsin) Asso. Prof., Univ. of Western Ontario, London, Ont., Canada
COLEMAN, A. J., Ph.D. (Toronto) Lecturer, Univ. of Toronto, Toronto 5, Ont., Canada
COLEMAN, E. P., Ph.D. (Columbia) Math. Statistician, Hughes Aircraft Co., Culver City, Calif. *412 Waterview St., Playa Del Rey, Calif.*
COLEMAN, H. B., M.S. (Michigan) Research Engr., Univ. of Michigan, Ann Arbor, Mich. *15034 Ashton Rd., Detroit 23, Mich.*
COLEMAN, J. B., Ph.D. (California) Acting Prof., Presbyterian Coll., Clinton, S.C.
COLEMAN, W. W., B.A. (Cornell) Asst. Vice-President, Irving Trust Co., 1 Wall St., New York, N.Y. *71 E. 77th St. (21)*
COLLIER, MYRTIE, Ph.D. (Strasbourg) Emeritus Prof., Immaculate Heart Coll., Los Angeles, Calif. *225 Thurston Ave. (49)*
COLLINS, J. L., B.S. (St. Mary's) Asst., St. Louis Univ., St. Louis 3, Mo.
COLMENARES-CARRILLO, NICOLAS, Ing.Civ. (Univ. Nac. de Colombia) Engr., *Av. Mis. Encantos #9, Q. Villa Mercedes, Chacao, Caracas, Venezuela*
COLQUITT, L. A., Ph.D. (Ohio State) Asst. Prof., Texas Christian Univ., Ft. Worth, Tex. *2124 Fairmount Ave. (4)*
COLSON, H. D., Ph.D. (Minnesota) Asso. Prof., Bemidji State Teachers Coll., Bemidji, Minn.
COLVIN, B. H., Ph.D. (Wisconsin) Mathematician, Boeing Airplane Co., Seattle 14, Wash. *4758 Second Ave., N.E. (5)*
COMBELLACK, W. J., Ph.D. (Boston U.) Head of Dept., Colby Coll., Waterville, Maine
COMEGYS, ESTHER, Ph.D. (Radcliffe) Asso. Prof., Univ. of Maine, Orono, Maine. *32 Myrtle St.*
COMFORT, E. G. H., Ph.D. (Brown) Asst. Prof., Illinois Inst. of Tech., Chicago, Ill. *812 Prairie Ave., Wilmette, Ill.*
COMPTON, ESTHER A., M.A. (Indiana) Instr., Cumberland Coll., Williamsburg, Ky.
CONKLING, R. M., M.S. (Bucknell) Grad. Student, Univ. of Florida, Gainesville, Fla.
CONKWRIGHT, N. B., Ph.D. (Illinois) Asso. Prof., Univ. of Iowa, Iowa City, Iowa
CONNELLY, DAMIAN, Brother, Ph.D. (Catholic U.) Asst. Prof., La Salle Coll., Philadelphia 41, Pa.
CONNORS, J. L., B.S. (Rutgers) Grad. Student, Rutgers Univ., New Brunswick, N.J. *172 Bevier Rd.*
CONSTABLE, MARY L., A.M. (Pennsylvania) Teacher, Philadelphia High School for Girls, Philadelphia, Pa. *The Whittier, 140 N. 15th St. (2)*
CONWELL, G. M., Ph.D. (Princeton) Asso. Prof., Univ. of Georgia, Athens, Ga. *12-D Court Dr., Wilmington, Del.*
CONWELL, H. H., Ph.D. (Wisconsin) Dean, Beloit Coll., Beloit, Wis. *1621 Emerson St.*
COOK, A. E., A.B. (Georgetown C.) Asst. Prof., Georgetown Coll., Georgetown, Ky. *112 Chambers Ave.*
COOK, C. H., M.S. (Iowa) Asst. Prof., Western State Coll., Gunnison, Colo.
COOK, D. W., B.S. (California) Senior Instr., California State Poly. Coll., San Luis Obispo, Calif.
COOK, E. M., M.A. (Boston U.) Asst. Prof., Northeastern Univ., Boston 15, Mass.
COOK, G. S., A.M. (Kansas) Asso. Prof., Colorado School of Mines, Golden, Colo.
COOK, I. D., B.S. (Maryland) Sales Manager, Automotive Parts, System Brakes, 10 E. Lafayette Ave., Baltimore 2, Md. *942 Cator Ave. (18)*
COOKE, J. V., Ph.D. (Peabody) Asst. Prof., North Texas State Coll., Denton, Tex. *Route One, 2300 W. Oak St.*
COOKE, K. L., M.S. (Stanford) Instr., State Coll. of Washington, Pullman, Wash.
COOLEY, H. R., Ph.D. (N.Y.U.) Prof., New York Univ., New York 3, N.Y.
COOLEY, J. A., Ph.D. (Illinois) Prof., Univ. of Tennessee, Knoxville, Tenn.
COOLIDGE, J. L., Ph.D. (Bonn) Emeritus Prof., Harvard Univ., Cambridge 38, Mass. *27 Fayerweather St.*
COOPER, ELIZABETH M., Ph.D. (Illinois) Head of Dept., Hunter Coll. High School, New York, N.Y. *201 E. 71st St. (21)*
COOPER, LAKE C. (Mrs.), A.M. (Morehead) Instr., Univ. of Kentucky, Lexington, Ky. *Box 203, Hindman, Ky.*
COPE, T. F., Ph.D. (Chicago) Prof., Queens Coll., Flushing, N.Y. *Montrose, N.Y.*
COPELAND, A. H., Ph.D. (Harvard) Prof., Univ. of Michigan, Ann Arbor, Mich. *805 Oakdale, Barton Hills*
COPP, P. T., A.M. (Ohio State) Research Engr., Hays Corp., Michigan City, Ind. *Box 64, Porter, Ind.*
CORAL, MAX, Ph.D. (Chicago) Asso. Prof., Wayne Univ., Detroit, Mich.
CORBIN, C. E., A.M. (Northwestern) Emeritus Prof., Coll. of the Pacific, Stockton, Calif. *P.O. Box 115, Carmel, Calif.*

CORE, HELEN E., M.S. (Michigan S.C.) Research Asst., Univ. of Michigan, Ann Arbor, Mich. *Apt. 6, 1446 Washington Heights, Ann Arbor, Mich.*
CORLEY, G. J., M.A. (Columbia) Asst. Prof., Northwestern State Coll., Natchitoches, La.
CORLISS, J. J., Ph.D. (Michigan) Chm. of Undergraduate Div., Univ. of Illinois, Navy Pier, Chicago, Ill.
CORNELL, W. F., M.A. (Ohio State) Asso. Prof., Bowling Green State Univ., Bowling Green, Ohio. *301 W. Merry Ave.*
CORNWELL, J. W., III, B.A. (Dartmouth) Senior Engr., Teleregister Corp., New York 7, N.Y. *Box 391, Syosset, N.Y.*
CORY, J. P., Student, New York Univ., New York, N.Y. *98 Rockland Ave., Yonkers 5, N.Y.*
COSBY, BYRON, A.M. (Missouri) Retired. *1 Ridgeley Rd., Columbia, Mo.*
COSBY, BYRON, Jr., Ph.D. (Chicago) Asso. Prof., State Univ. of Iowa, Iowa City, Iowa
COTHRAN, F. E., B.S. (California) Capt., U. S. Army, Retired. *312 Grace St., Greenwood, S.C.*
COTHRAN, J. C., Ph.D. (Cornell) Prof., Chemistry, Duluth Branch, Univ. of Minnesota, Duluth 5, Minn.
COULTER, D. F., Jr., M.A. (Washington S.C.) Mathematician, U. S. Navy Mine Countermeasures Station, Panama City, Fla. *1413 Deer Ave., c/o General Delivery, St. Andrew, Fla.*
COURANT, RICHARD, Ph.D. (Göttingen) Prof., New York Univ., New York, N.Y. *142 Calton Rd., New Rochelle, N.Y.*
COURT, N. A., D.Sc. (Ghent) Prof., Univ. of Oklahoma, Norman, Okla.
COVEYOU, R. R., M.A. (Tennessee) Asso. Mathematician, Oak Ridge National Lab., Oak Ridge, Tenn. *129 Orchard Lane*
COWAN, R. W., Ph.D. (California) Asst. Prof., Univ. of Florida, Gainesville, Fla.
COWAN, W. D., M.A.E. (Florida) Instr., Chipola Junior Coll., Marianna, Fla.
COWELL, W. R., M.S. (Kansas State) Teaching Asst., Univ. of Wisconsin, Madison 6, Wis.
COWLES, W. H. H., A.M. (Columbia) Prof., Pratt Inst., Brooklyn, N.Y. *132 Joralemon St. (2)*
COWLING, V. F., Ph.D. (Rice) Asso. Prof., Univ. of Kentucky, Lexington 29, Ky.
COX, H. M., A.M. (Duke) Dir., Bureau of Instructional Research, Univ. of Nebraska, Lincoln 8, Neb.
COX, J. G., M.S. (Alabama Poly.) Instr., Alabama Poly. Inst., Auburn, Ala. *229 N. Ross St.*
COX, M. K., B.S. in I.E. (V.P.I.) Asst., Kent State Univ., Kent, Ohio. *334 S. Willow St.*
COX, P. C., A.M. (New Mexico) Asst. Prof., Albion Coll., Albion, Mich. *816 N. Berrien St.*
COXETER, H. S. M., Ph.D. (Cambridge) Prof., Univ. of Toronto, Toronto 5, Ont., Canada. *24 Strathearn Blvd. (10)*
COY, J. W., M.A. (New Mexico) Instr., Michigan State Coll., East Lansing, Mich. *616 Cherry Lane*
CRAIG, A. T., Ph.D. (Iowa) Prof., Univ. of Iowa, Iowa City, Iowa
CRAIG, C. C., Ph.D. (Michigan) Dir., Statistical Research Lab., Univ. of Michigan, Ann Arbor, Mich.
CRAIG, H. V., Ph.D. (Wisconsin) Prof., Univ. of Texas, Austin, Tex.
CRAIN, K. W., M.S. (Iowa) Asst. Prof., Purdue Univ., West Lafayette, Ind. *1844 Garden St.*
CRAMER, G. F., Ph.D. (Missouri) Mathematician, U. S. Navy, Washington, D.C. *112 Quincy St., Chevy Chase, Md.*
CRAMER, PAUL, A.M. (Illinois) Asst. Prof., Monmouth Coll., Monmouth, Ill. *732 E. 2nd Ave.*
CRAMLET, C. M., Ph.D. (Washington) Prof., Univ. of Washington, Seattle, Wash.
CRANE, R. E., B.S. (Harvard) American Tel. & Tel. Co., 195 Broadway, New York, N.Y. *120 Early St., Morristown, N.J.*
CRANE, RUFUS, A.M. (Ohio State) Asso. Prof., Ohio Wesleyan Univ., Delaware, Ohio. *269 W. William St.*
CRAWFORD, W. S. H., A.M. (Minnesota) Prof., Mount Allison Univ., Sackville, N.B., Canada
CREELY, J. W., M.S. (Pennsylvania) Research Chemist, American Cyanamid Co., Bound Brook, N.J. *245 D St., East Bound Brook, N.J.*
CRESSY, W. V., Designer, Aircraft, Douglas Aircraft, El Segundo, Calif. *11052½ W. Olympic Blvd., Los Angeles 64, Calif.*
CRISLER, E. H., M.A. (Michigan) Grad. Student, Univ. of Michigan, Ann Arbor, Mich. *1121 Southwick Ct., Willow Run, Mich.*
CRISPIN, J. W., Jr., M.S. (Michigan) Instr., Wayne Univ., Detroit, Mich. *363 Richton Ave. (3)*
CROCI, FRANCO, *3992 White Plains Ave., New York 66, N.Y.*
CROMWELL, J. W., Jr., A.M. (Dartmouth) Certified Public Accountant, Washington, D.C. *1815 13th St., N.W.*
CROUCH, R. B., M.S. (Illinois) Instr., New Mexico Coll. of A. and M. A., State College, N.M.
CROW, E. L., Ph.D. (Wisconsin) Head, Statistics Branch, U. S. Naval Ordnance Test Station, China Lake, Calif. *509-A Saratoga Ave.*
CROW, PAULINE E., M.A. (Pittsburgh) Instr., Carnegie Inst. of Tech., Pittsburgh, Pa. *125 N. Linden Ave.*
CROWDER, H. K., M.S. (Case) Instr., Case Inst. of Tech., Cleveland, Ohio. *2129 W. 101st St. (2)*

CROXTON, R. R., M.E. (South Carolina) Adjunct Prof., Univ. of South Carolina, Columbia, S.C. *Route 1, Box 125*
CRULL, H. E., Ph.D. (Illinois) Prof., Butler Univ., Indianapolis, Ind.
CRUM, W. F., M.A. (Minnesota) Student, Seabury-Western Theological Seminary, 600 Haven St., Evanston, Ill.
CUDE, DON, A.M. (Southwest Texas S.T.C.) Prof., Southwest Texas State Teachers Coll., San Marcos, Tex.
CULMER, ORPHA A., A.M. (Michigan) Prof., State Teachers Coll., Florence, Ala.
CULPEPPER, G. A., M.A. (Colorado) Mathematician, White Sands Proving Ground, Las Cruces, N.M. *208 E. Fleming*
CULWELL, P. R., M.A. (Texas) Head of Dept., Trinity Univ., San Antonio, Tex. *3115 W. Ashby St.*
CUMMING, JOANN M., M.S. (Marquette) Mathematician, Aberdeen Proving Ground, Md. *21 Rigdon Rd., Aberdeen, Md.*
CUMMINGS, MARY L., M.A. (Illinois) Instr., Univ. of Missouri, Columbia, Mo.
CUMMINS, K. B., M.A. (Bowling Green) Teacher, New Washington High School, New Washington, Ohio
CUNHA, S. H., B.Sc. (McGill) Elect. Engr., Quebec Hydro-Electric Commission, Montreal, P.Q., Canada. *1483 Atwater Ave., Apt. 14*
CUNNINGHAM, A. B., Ph.D. (West Virginia) Asst. Prof., West Virginia Univ., Morgantown, W.Va.
CUNNINGHAM, L. E., Ph.D. (Harvard) Asso. Prof., Astronomy, Univ. of California, Berkeley 4, Calif.
CURFMAN, L. E., M.S. (Colorado) Prof., Kansas State Teachers Coll., Pittsburg, Kan. *406 W. Adams St.*
CURRAN, P. M., M.A. (Columbia) Instr., Fordham Univ., New York, N.Y. *62-96 Wetherole St., Rego Park, N.Y.*
CURRIE, J. C., Ph.D. (Louisiana) Asso. Prof., Georgia Inst. of Tech., Atlanta, Ga. *259 Chelsea Dr., Decatur, Ga.*
CURRIER, A. E., Ph.D. (Harvard) Asso. Prof., U. S. Naval Acad., Annapolis, Md.
CURRY, DOROTHY S., M.S. (Chicago) Asso. Prof., Wilberforce Univ., Wilberforce, Ohio
CURRY, H. B., Ph.D. (Göttingen) Prof., Pennsylvania State Coll., State College, Pa. *228 E. Prospect Ave.*
CURTIS, H. B., Ph.D. (Cornell) Emeritus Prof., Lake Forest Coll., Lake Forest, Ill. *127 Oak Terr., Lake Bluff, Ill.*
CURTIS, MARJORY R., M.A. (Michigan) Teacher, Emerson Junior High School, Flint, Mich. *713 Liberty St. (3)*
CURTISS, D. R., Ph.D. (Harvard) Emeritus Prof., Northwestern Univ., Evanston, Ill. *1349 Monterey St., Redlands, Calif.*
CURTISS, J. H., Ph.D. (Harvard) Chief, National Bureau of Standards, Washington 25, D.C. *4802 Bradley Blvd., Chevy Chase, Md.*
CUTHILL, ELIZABETH H. (Mrs. J. R.), M.S. (Brown) Instr., Purdue Univ., Lafayette, Ind. *510 Wood St., West Lafayette, Ind.*
CUTLER, E. H., Ph.D. (Harvard) Asso. Prof., Lehigh Univ., Bethlehem, Pa.
CUTLER, GRACE M., M.A. (Toledo) Asst. Prof., Univ. of Toledo, Toledo, Ohio. *2656 Algonquin Pkwy. (6)*

DADOURIAN, H. M., Ph.D. (Yale) Emeritus Prof., Trinity Coll., Hartford, Conn. *177 N. Main St., West Hartford 7, Conn.*
DALAL, R. D., Retired, Stock Broker, London, England
DALY, D. L., M.A. (Michigan) Instr., Univ. of Maine, Orono, Maine. *14 Elm St.*
DALY, J. F., Ph.D. (Princeton) Statistician, Bureau of the Census, Washington, D.C. *4213 18th St., N.E. (18)*
DANCER, WAYNE, Ph.D. (Michigan) Prof., Univ. of Toledo, Toledo 6, Ohio
DANCEY, L. S., A.M. (Illinois) Prof., Carroll Coll., Waukesha, Wis. *125 N. Charles St.*
DANESE, A. E., A.M. (Harvard) Instr., Univ. of Rochester, Rochester, N.Y.
DANIELLS, MARIAN E., M.S. (Iowa State) Asst. Prof., Iowa State Coll., Ames, Iowa
DANSKY, MORRIS, M.A. (Michigan) Instr., Creighton Univ., Omaha, Neb. *5124 Underwood Ave. (3)*
DANZL, ARTHUR, Rev., A.M. (Columbia) Prof., St. John's Univ., Collegeville, Minn.
D'ARCO, PAUL, M.S. (DePaul) Instr., DePaul Univ., Chicago, Ill. *1206 Lexington St. (7)*
DARK, H. J., Ph.D. (Peabody) Head of Dept., David Lipscomb Coll., Nashville 4, Tenn.
DARKOW, MARGUERITE D., Ph.D. (Chicago) Asso. Prof., Hunter Coll., New York 21, N.Y. *16 E. 82nd St. (28)*
DARLING, D. A., Ph.D. (C.I.T.) Asst. Prof., Univ. of Michigan, Ann Arbor, Mich.
DARLING, F. W., A.B. (Cornell) Retired Mathematician, U. S. Coast and Geodetic Survey, Washington, D.C. *45 Kenilworth Ave., Garrett Park, Md.*

DARLING, J. F., A.B. (Cornell) Retired. *11 W. Grant St., Woodstown, N.J.*
DARRAUGH, J. E., A.M. (Brooklyn) Instr., Case Inst. of Tech., Cleveland 6, Ohio. *2917 Glenwood Rd., Brooklyn 10, N.Y.*
D'ATRI, A. J., C.E. (Brooklyn Poly.) Civil Engr., Dept. of Public Works, New York, N.Y. *1821 Madison Pl., Brooklyn 29, N.Y.*
DAUGHERTY, J. D., A.M. (Pennsylvania) Head of Dept., Eastside High School, Paterson, N.J.
DAUGHTRY, A. C., Student, Alabama Poly. Inst., Auburn, Ala.
DAUM, J. A., Ph.D. (Nebraska) Asso. Prof., Texas A. and M. Coll., College Station, Tex.
DAUS, P. H., Ph.D. (California) Prof., Univ. of California at Los Angeles, Los Angeles 24, Calif.
DAVIDSON, D. O., B.S. (Oregon S.C.) *1833 5th St., Manhattan Beach, Calif.*
DAVIES, ROBERT, Ph.D. (Wisconsin) Asso. Engr., Rand Corp., Santa Monica, Calif. *15213 DePauw St., Pacific Palisades, Calif.*
DAVIS, A. F., Rev., A.B. (Catholic U.) Grad. Student, Atonement Seminary, Washington 17, D.C.
DAVIS, A. W., Ph.D. (Iowa State) Asso. Prof., Iowa State Coll., Ames, Iowa
DAVIS, CONSTANCE H. (Mrs. Leslie), *238 Audley St., South Orange, N.J.*
DAVIS, D. J., B.S. (Harvard) Mathematician, Balistic Research Lab., Aberdeen Proving Ground, Md. *13 Defense Dr., Aberdeen, Md.*
DAVIS, D. R., Ph.D. (Chicago) Prof., New Jersey State Teachers Coll., Upper Montclair, N.J. *43 College Ave.*
DAVIS, GEORGIE T., M.S. (V.P.I.) Instr., Virginia Poly. Inst., Blacksburg, Va.
DAVIS, H. A., Ph.D. (Cornell) Prof., West Virginia Univ., Morgantown, W.Va. *307 Duquesne Ave.*
DAVIS, J. B., A.M. (Columbia) Dean, Amarillo Coll., Amarillo, Tex. *1605 Travis St.*
DAVIS, JAMES EDGAR, A.M. (Ohio State) Asst. Prof., Pharmacy, Univ. of Illinois, 808 S. Wood St., Chicago 12, Ill.
DAVIS, JAMES ELMER, A.M. (Wisconsin) Prof., Drexel Inst. of Tech., Philadelphia 4, Pa.
DAVIS, L. E., B.Sc. (Ohio State) Research Asst., Research Cryogenic Lab., Ohio State Univ., Columbus, Ohio. *938 Heyl Ave.*
DAVIS, MAMIE M., M.A. (Louisiana) Instr., Francis T. Nicholls Junior Coll., Thibodaux, La.
DAVIS, MARGARET R., Ed.D. (Columbia) Prof., Southeastern Louisiana Coll., Hammond, La.
DAVIS, R. C., B.S. (Akron) Instr., Univ. of Akron, Akron 9, Ohio. *1002 Berwin St., (10)*
DAVIS, RUTH M., A.B. (American U.) Grad. Student, American Univ., Washington 16, D.C. *1390 Tewkesbury Pl., N.W. (12)*
DAVIS, VIOLET B., M.A. (Toledo) Asst. Prof., Univ. of Toledo, Toledo, Ohio. *716 Vandalia St. (11)*
DAVIS, W. M., Ph.D. (Chicago) Prof., Cornell Coll., Mount Vernon, Iowa. *616 7th Ave. N.*
DAY, A. S., Ph.D. (Yale) Asst. Prof., Wesleyan Univ., Middletown, Conn.
DAY, S. L., M.E. (Pratt) Senior Draftsman, Century Projector Corp., Long Island City, N.Y. *87-04 75th St., Woodhaven 21, N.Y.*
DEAL, D. E., A.M. (Michigan) Teaching Fellow, Univ. of Michigan, Ann Arbor, Mich. *1314 Iroquois Dr.*
DEAL, R. B., Jr., M.A. (Oklahoma) Instr., Univ. of Oklahoma, Norman, Okla.
DEAN, ALICE C., A.M. (Rice) Librarian Emerita, Rice Inst., Houston, Tex. *1109 Milford (6)*
DEAN, MILDRED W. (Mrs.), Ph.D. (Johns Hopkins) Instr., Queens Coll., Flushing, N.Y. *260-21 Pembroke Ave., Great Neck, N.Y.*
DEAN, T. S., M.S. (North Texas S.C.) President, Design Associates, Inc., Sherman, Tex. *5712 Goliad, Dallas, Tex.*
DEARBORN, D. C., Ph.D. (Duke) Dean, Catawba Coll., Salisbury, N.C.
DEAUX, ROLAND, D.Sc. (Gand) Prof., Faculté Polytechnique, Mons, Belgium. *47 Chaussée de Binche*
DEBAGGIS, H. F., Rev., Ph.D. (Notre Dame) Asst. Prof., Univ. of Notre Dame, Notre Dame, Ind.
DECHERD, MARY E., A.M. (Texas) Emeritus Asst. Prof., Univ. of Texas, Austin, Tex. *2313 Nueces St. (21)*
DECICCO, JOHN, Ph.D. (Columbia) Prof. De Paul Univ., Chicago 1, Ill. *6340 Blackstone Ave., Apt. 303 (37)*
DECK, L. J., A.M. (Pennsylvania) Prof., Muhlenberg Coll., Allentown, Pa.
DECLEENE, L. A. V., Rev., Ph.D. (Catholic U.) Head of Dept., St. Norbert Coll., West DePere, Wis. *1015 S. Monroe Ave., Green Bay, Wis.*
DEDERICK, L. S., Ph.D. (Harvard) Asso. Dir., Ballistic Research Lab., Aberdeen Proving Ground, Md.
DEFRANCESCO, H. F., B.E.E. (Virginia) *Address Unknown*
DE GREIFF B., LUIS, Civ. Eng. (Univ. Nac. de Colombia) Prof., Facultad de Minas, Medellin, Colombia

DEJONGE, M. W., M.A. (Illinois) Asst. Prof., Purdue Univ., West Lafayette, Ind.
DEKKER, D. B., Ph.D. (California) Asst. Prof., Univ. of Washington, Seattle 5, Wash.
DE LA GARZA, ELEUTERIO, *P.O. Box 2042, Brownsville, Tex.*
DELANY, B. PEARSON (Mrs.), B.S. (Illinois Tech.) Grad. Student, Illinois Inst. of Tech., Chicago, Ill. *Sutton Rd., Barrington, Ill.*
DE LA SALLE, LOUIS, Brother, Ph.D. (Catholic U.) Dean of Studies, St. Mary's Coll., Winona, Minn.
DELOACH, B. C., Jr., Student, Alabama Poly. Inst., Auburn, Ala. *1412 19th Pl. S.W., Birmingham 7, Ala.*
DE LOSADA Y PUGA, CRISTÓBAL, D.Sc. (Lima) Prof., Catholic Univ. of Peru, Lima, Peru. *Apartado 2708*
DEMING, R. M., M.S. (Iowa State) Prof., Upper Iowa Univ., Fayette, Iowa
DEMOSS, M. L., M.S. (Kansas S.T.C.) Instr., General Motors Inst., Flint, Mich. *2213 Flushing Rd.*
DENBOW, CARL, Ph.D. (Chicago) Prof., Ohio Univ., Athens, Ohio
DENISTON, R. F., A.M. (Washington) *1296 Sudbury Ct., Willow Run Village, Mich.*
DENNIS, F. L., Ph.D. (Illinois) Prof., Ursinus Coll., Collegeville, Pa. *930 Main St.*
DENNISON, C. H., B.S. (M.I.T.) Chemist, Archer Rubber Co., Milford, Mass. *4 Crescent St., Natick, Mass.*
DENNY, C. E., B.S. (U. S. Naval Acad.) Instr., Central Coll., Fayette, Mo. *105 Lucky St.*
DENTON, W. W., Ph.D. (Illinois) Asst. Prof., Univ. of Arizona, Tucson, Ariz. *519 N. Jacobs Ave.*
DEPEW, R. D., M.A. (Peabody) Asst. Prof., State Teachers Coll., Florence, Ala. *On leave:* Grad Fellow, Wesley Hall, Vanderbilt Univ., Nashville, Tenn.
DEPKOVICH, T. E., B.S. (Indiana S.T.C.) Asst. Prof., Coll. of Steubenville, Steubenville, Ohio. *1246 Oregon Ave.*
DE REGT, M. P., B.S. (Webb Inst.) Sales Engr., Pacific Gear & Tool Wks., 1035 Folsom, San Francisco, Calif.
DERFLINGER, R. B., B.S. (Geneva C.) Teacher, Chemistry, Turtle Creek High School, Turtle Creek, Pa. *1417 Swissvale Ave., Wilkinsburg, Pa.*
DERNHAM, M. A., LL.B. (California) Attorney-at-Law, 405 Montgomery St., Room 1114, San Francisco 4, Calif.
DERR, J. I., B.S. (Southern Methodist) Grad. Student, Southern Methodist Univ., Dallas, Tex. *902 N. Edgefield St.*
DERRY, DOUGLAS, Ph.D. (Göttingen) Asso. Prof., Univ. of British Columbia, Vancouver, B.C., Canada. *4593 W. 5th Ave.*
DEUTSCH, J. G., A.M. (Columbia) Head of Dept., Andrew Jackson High School, St. Albans, Queens, N.Y. *66 St. Pauls Pl., Brooklyn 26, N.Y.*
DEUTSCH, R. A., B.S. (Long Island U.) Design Engr., Brown & Matthews, New York, N.Y. *4672 Broadway (34)*
DEVOL, DAVID, M.A. (Colorado) Instr., Univ. of Colorado, Boulder, Colo. *482 Marine St.*
DIAMOND, A. H., Ph.D. (California) Visiting Prof., Univ. of Kansas, Lawrence, Kan.
DIAMOND, L. E., M.S. (Oklahoma) *Box 157, Milford, Tex.*
DIBELLA, ANGELO, B.S. (New Mexico A.&M.A.) Research Asst., Sandia Corp., Albuquerque, N.M. *3248-D A St., Sandia Base*
DICKERSON, B. K., M.A. (Northwestern) *2311 Woodlawn, Austin, Tex.*
DICKEY, H. E., M.S. (Iowa S.C.) Asst. Prof., Iowa State Coll., Ames, Iowa, *2136 Friley Rd.*
DICKINSON, ALICE B. (Mrs.), M.A. (Columbia) Instr., Pennsylvania State Coll., State College, Pa. *529 W. Foster Ave.*
DICKS, ANNIE L., A.M. (South Carolina) Prof., Columbia Coll., Columbia, S.C.
DICKSON, L. E., Ph.D. (Chicago) Honorary Life Member. Emeritus Prof., Univ. of Chicago, Chicago 37, Ill. *c/o Mrs. Harlow Higinbotham, R.F.D. 2, Joliet, Ill.*
DICKSON, R. J., Jr., B.S. (Stanford) Grad. Asst., California Inst. of Tech., Pasadena, Calif. *1201 E. California St. (4)*
DIDONATO, A. R., Student, Duquesne Univ., Pittsburgh, Pa. *325 Cedarville St. (24)*
DIESEN, C. E., M.A. (Minnesota) Dynamics Engr., Bell Aircraft Corp., Niagara Falls, N.Y. *100 Nash Rd., North Tonawanda, N.Y.*
DIETRICH, V. E., M.Sc. (Notre Dame) Instr., Alabama Poly. Inst., Auburn, Ala. *120 S. Ross St.*
DILLON, G. M., M.A. (Columbia) Capt., U. S. Army.
DILWORTH, R. P., Ph.D. (C.I.T.) Asso. Prof., California Inst. of Tech., Pasadena, Calif.
DIMICK, C. E., A.M. (Pennsylvania) Emeritus Prof., U. S. Coast Guard Acad., New London, Conn. *Box 806, Tryon, N.C.*
DINES, L. L., Ph.D. (Chicago) Emeritus Prof., Carnegie Inst. of Tech., Pittsburgh, Pa. *131 N. 14th St., Quincy, Ill.*
DINKINES, FLORA, A.M. (Michigan) Grad. Student, Univ. of Chicago, Chicago 37, Ill.
DIPERT, M. H., M.S. (Indiana) Prof., Westmar Coll., LeMars, Iowa, *R.R. 2*

DIPRIMA, R. C., B.S. (Carnegie) Part-time Instr., Carnegie Inst. of Tech., Pittsburgh, Pa. *5010 Morewood Pl. (13)*
DIX, L. E., M.S. (Norwich) Prof., Norwich Univ., Northfield, Vt. *5 Spring St.*
DIXON, GENEVIEVE H., M.A. (Buffalo) Computer, Bell Aircraft Corp., Niagara Falls, N.Y. *100 Willert Park, Buffalo 4, N.Y.*
DOBBINS, J. W., B.A. (St. Mary's U.) Grad. Student, Univ. of Texas, Austin, Tex. *459 Cloverleaf, Apt. 4, San Antonio, Tex.*
DOBSON, W. P., D.Sc. (Queen's) Dir. of Research, Hydro-Electric Power Commission of Ontario, 620 University Ave., Toronto 2, Ont., Canada. *62 Beaty Ave.*
DODES, I. A., Ph.D. (N.Y.U.) Teacher, Stuyvesant High School, 345 E. 15th St., New York 3, N.Y. *100-10 67th Rd., Forest Hills, N.Y.*
DODGE, D. A., Student, Univ. of Kansas, Lawrence, Kan. *1645 Tennessee*
DODSON, N. E., M.A. (Alabama) Asso. Prof., Lenoir Rhyne Coll., Hickory, N.C. *219 4th St. Pl., S.W.*
DOERINGSFELD, H. A., C.E. (Wisconsin) Prof., Univ. of Minnesota, Minneapolis 14, Minn. *4904 11th Ave., S. (17)*
DOERMANN, F. W., Ph.D. (Vienna) Retired.
DOLAN, W. W., Ph.D. (Oklahoma) Prof., Linfield Coll., McMinnville, Ore.
DOLCIANI, MARY P., Ph.D. (Cornell) Instr., Vassar Coll., Poughkeepsie, N.Y.
DOLNEY, E. L., M.S. (Notre Dame) Instr., Univ. of Alaska, College, Alaska
DOMINIC, Brother, M.A. (California) Asst. Prof., St. Mary's Coll., St. Mary's College, Calif.
DONALDSON, F. W., M.A. (Kentucky) Technical Staff, Los Alamos Scientific Lab., Los Alamos, N.M. *P.O. Box 1663*
DONER, R. D., Ph.D. (Illinois) Prof., Alabama Poly. Inst., Auburn, Ala. *477 E. Samford Ave.*
DONOHOE, H. T., M.A. (Baylor) Asst. Prof., Louisiana Coll., Pineville, La.
DOREMUS, MARY C., M.A. (Columbia) Head of Dept., North High School, Denver, Colo. *1115 Logan St., Apt. 201*
DORROH, J. L., Ph.D. (Texas) Asso. Prof., Texas Coll. of Arts and Industries, Kingsville, Tex.
DORSEY, F. F., LL.B. (Boston U.) Patent Attorney, Retired. *205 Garfield Pl., South Orange, N.J.*
DORWART, H. L., Ph.D. (Yale) Prof., Trinity Coll., Hartford 6, Conn.
DOSS, H. W., Jr., M.A. (Missouri) Instr., Univ. of Omaha, Omaha, Neb. *3920 Frederick St., Apt. 7 (5)*
DOSSEY, M. P., M.S. (Texas) Asso. Prof., Southwestern Louisiana Inst., Lafayette, La. *229 Westgate W., Cambridge 39, Mass.*
DOSTAL, B. F., A.M. (Indiana) Asso. Prof., Univ. of Florida, Gainesville, Fla.
DOTTERER, J. E., A.M. (Illinois) Prof., Manchester Coll., North Manchester, Ind.
DOUGHERTY, LUCY T., A.M. (Kansas) Retired. *1203 Oread, Apt. 9, Lawrence, Kan.*
DOUGLAS, E. G., A.M. (Mercer) Instr., Univ. of South Carolina, Columbia, S.C. *1520 S. Lumpkin St., Athens, Ga.*
DOUGLAS, JESSE, Ph.D. (Columbia) Lecturer, Columbia Univ., New York 27, N.Y. *2610 Glenwood Rd., Brooklyn 10, N.Y.*
DOUGLAS, WILLIAM, *P.O. Box 268, Courtenay, B.C., Canada*
DOUGLASS, R. D., Ph.D. (M.I.T.) Prof., Massachusetts Inst. of Tech., Cambridge, Mass. *18 Oak Ave., Belmont, Mass.*
DOWDS, R. E., B.S. (Kent) Mathematician, Naval Ordnance Plant, Indianapolis, Ind. *1848 20th St., Cuyahoga Falls, Ohio*
DOWNEY, ANNE F., A.B. (Regis C.) Rate Specialist, Liberty Mutual Insurance Co., Boston, Mass. *366 Broadway, Somerville 45, Mass.*
DOWNIE, E. J., A.B. (Colgate) Asst. Prof., Colgate Univ., Hamilton, N.Y. *17 Madison St.*
DOWNING, F. A., Asst. Dir. of Traffic, North Carolina Utilities Com., Raleigh, N.C. *B3-47 Twin Castles Apts., Winston-Salem, N.C.*
DOWNING, H. H., Ph.D. (Chicago) Prof., Univ. of Kentucky, Lexington 29, Ky. *214 State St. (44)*
DOWNING, R. H., Ph.D. (West Virginia) Prof., Army Air Forces Inst. of Tech., Wright Field, Dayton, Ohio. *4204 Royal Oak Dr. (9)*
DOYLE, T. C., Ph.D. (Princeton) Mathematician, Naval Research Lab., Code 3825, Washington 25, D.C.
DOYLE, W. C., Ph.D. (St. Louis) Asso. Prof., Rockhurst Coll., Kansas City 4, Mo.
DRAGOO, R. C., A.M. (Oklahoma) Asst. Prof., Univ. of Oklahoma, Norman, Okla. *625 S Lahoma St.*
DRAIM, N. A., M.Sc. (M.I.T.) Capt., U. S. Navy. *8581 Locust Hill Rd., Bethesda 14, Md.*
DRAKE, JOAN, Student, Seton Hill Coll., Greensburg, Pa. *85-08 215th St., Queens Village 8, New York, N.Y.*
DRAPER, OLIVE M., A.M. (Michigan) Prof., Taylor Univ., Upland, Ind.
DRAUDT, BEATRICE A., A.M. (Columbia) Instr., Hofstra Coll., Hempstead, N.Y. *80 Sunnyside Ave.*

DRESDEN, ARNOLD, Ph.D. (Chicago) Prof., Swarthmore Coll., Swarthmore, Pa. *606 Elm Ave.*
DRESSEL, F. G., Ph.D. (Duke) Asso. Prof., Duke Univ., Durham, N.C. *309 Francis St.*
DRESSER, F. T., B.S. (V.M.I.) Analyst, Armed Forces Security Agency, Washington, D.C. *4611 23rd Rd., N., Arlington, Va.*
DRESSLER, B. B., A.M. (Illinois) Asst. Prof., Kent State Univ., Kent, Ohio
DREW, J. W., A.M. (Cornell) Asst. Prof., Virginia Union Univ., Richmond 20, Va.
DRIBIN, D. M., Ph.D. (Chicago) Research Analyst, Army Security Agency, Washington, D.C. *101 S. Utah St., Arlington 4, Va.*
DRIVER, D. D., A.M. (Nebraska) Instr., Hesston Coll., Hesston, Kan.
DROUSSENT, LUCIEN, Ing. (Ecole Centrale des Arts et Manuf., Paris) Engr., *Rue de Laniscourt, Laon (Aisne), France*
DUARTE, F. J., Ing. Civ. (Univ. de Caracas) Dir. de Fronteras, Ministerio de Relaciones Exteriores. *Apartado 1855, Caracas, Venezuela*
DUBBERT, K. E., M.A. (Columbia) Instr., Rochester Junior Coll., Rochester, Minn.
DUBAY, G. H., M.A. (Loyola, Chicago) Prof., Univ. of St. Thomas, Houston, Tex. *5818 Southwell (21)*
DUBÉ, G. R., B.S. (M.I.T.) Asst., Yale Univ., New Haven, Conn.
DUBÉ, L. H., Ph.D. (Gregorian Univ., Rome) Prof., Ottawa Univ., Ottawa, Ont., Canada
DUBISCH, ROY, Ph.D. (Chicago) Asso. Prof., Fresno State Coll., Fresno, Calif. *1501 Palm Ave.*
DUBOIS, D. W., B.S. (Oklahoma) *Address Unknown*
DUERKSEN, J. A., A.B. (Bethel) Acting Chief, Gravity & Astronomy Section, U. S. Coast and Geodetic Survey, Washington, D.C. *3134 Monroe St., N.E. (18)*
DUGGAR, ANDERSON, Jr., Student, Univ. of Detroit, Detroit, Mich. *615 W. Eleven Mile Rd., Royal Oak, Mich.*
DUKE, W. M., Sc.M. (N.Y.U.) Asst. Dir., Cornell Aeronautical Lab., Buffalo 5, N.Y.
DUMONT, THOMAS, Lt. Comdr., U. S. Coast Guard. *4 Berncliffe Ave., Albany 3, N.Y.*
DUNCAN, D. C., Ph.D. (California) Head of Dept., East Los Angeles Junior Coll., Los Angeles 22, Calif.
DUNCAN, D. G., Ph.D. (Michigan) Asst. Prof., Univ. of Arizona, Tucson, Ariz.
DUNCAN, R. L., A.B. (Tulsa) Div. Landman, Carter Oil Co., Oklahoma City, Okla. *2945 N.W. 49th St.*
DUNFORD, NELSON, Ph.D. (Brown) Prof., Yale Univ., New Haven, Conn. *34 Fountain St.*
DUNKELBERGER, ESTHER S. (Mrs.), M.Litt. (Pittsburgh) Instr., Duquesne Univ., Pittsburgh, Pa. *7227 Kelly St. (8)*
DUNLAP, L. T., A.M. (Penna. State) Asso. Prof., Pennsylvania State Coll., State College, Pa.
DUNLEAVY, MARY T., A.M. (Columbia) Statistician, E. I. duPont de Nemours & Co., Inc., Wilmington, Del. *20-35 Crescent St., Astoria 5, N.Y.*
DUNNING, D. L., B.A. (North Central C.) Asst. Actuary, Warner-Watson, Inc., 135 S. LaSalle, Chicago, Ill. *6224 S. Kimbark Ave. (37)*
DURAND, JANET C., A.M. (Pennsylvania) *223 W. Sedgwick St., Philadelphia 19, Pa.*
DUREN, W. L., Jr., Ph.D. (Chicago) Prof., Tulane Univ., New Orleans 18, La. *3323 Joseph St.*
DURFEE, WALTER H., Ph.D. (Cornell) Prof., Hobart and William Smith Coll., Geneva, N.Y.
DURFEE, WILLIAM H., Ph.D. (Cornell) National Bureau of Standards, Applied Math. Labs., Washington 25, D.C.
DURHAM, GENEVA E., A.M. (Northwestern) Asst. Prof., Pacific Union Coll., Angwin, Calif.
DURIG, R. F., Student, Kent State Univ., Kent, Ohio. *R.D. 1, McKinley Heights, Girard, Ohio*
DUSTHEIMER, O. L., Ph.D. (Michigan) Pension Consultant, D. Miley Phipps & Assoc., 910 Hanna Bldg., Cleveland, Ohio. *4935 E. 71st St. (9)*
DWYER, P. S., Ph.D. (Michigan) Prof., Univ. of Michigan, Ann Arbor, Mich. *403 Lenawee Dr.*
DWYER, W. E., Jr., Student, Catholic Univ. of America, Washington, D.C. *704 Barrett Rd., Falls Church, Va.*
DYBVIK, RAGNAR, Cand.Real. (Oslo) Lektor, Levanger kommunale hogre almenskole, Levanger, Norway
DYE, L. A., Ph.D. (Cornell) Assoc. Prof., The Citadel, Charleston, S.C.

EAGLE, E. L., M.A. (Miami U.) Engr., Glenn L. Martin Aircraft Co., Middle River, Md. *3919 Frankford Ave., Baltimore 6, Md.*
EARHART, F. C., A.M. (Wisconsin) *Hopkinton, Iowa*
EARL, J. M., Ph.D. (Minnesota) Prof., Univ. of Omaha, Omaha 6, Neb. *528 S. 53rd St.*
EASTCOTT, JAMES, A.M. (Michigan) Research Scientist, Defense Research Board, Ottawa, Ont., Canada
EASTHAM, J. N., Ph.D. (Catholic U.) Asso. Prof., Cooper Union, New York, N.Y. *149-41 Hawthorne Ave., Flushing, N.Y.*
EATON, J. E., Ph.D. (Yale) Asst. Prof., Queens Coll., Flushing, N.Y. *67-50 B 188th St.*
EAVES, E. D., Ph.D. (Texas) Prof., Univ. of Tennessee, Knoxville 16, Tenn.

EAVES, J. C., Ph.D. (North Carolina) Asso. Prof., Alabama Poly. Inst., Auburn, Ala.
EBERHART, PAUL, Ph.D. (Brown) Prof., Washburn Univ., Topeka, Kan. *2068 Lane St.*
EBERLEIN, W. F., Ph.D. (Harvard) Asst. Prof., Univ. of Wisconsin, Madison 6, Wis.
ECKSTEIN, J. B., B.S. (Notre Dame) Grad. Student, Univ. of Detroit, Detroit, Mich. *1407 E. Harry St., Hazel Park, Mich.*
EDDINGTON, G. M., M.A. (George Peabody) Dean of Men, Tusculum Coll., Greeneville, Tenn.
EDDY, RUTH B., A.M. (Brown) Teacher, Hope High School, Providence, R.I. *666 Angell St.* (*6*)
EDELSON, SIDNEY, M.A. (N.Y.U.) Mathematician, U. S. Naval Observatory, Massachusetts Ave., N.W., Washington, D.C.
EDINGTON, W. E., Ph.D. (Illinois) Prof., DePauw Univ., Greencastle, Ind. *E. Franklin St.*
EDISON, BEATRICE G., A.M. (N.Y.U.) Asst. Prof., New York Univ., New York, N.Y. *Brewster, N.Y.*
EDISON, T. M., B.S. (M.I.T.) President, Calibron Products, Inc., 51 Lakeside Ave., West Orange, N.J. *Llewellyn Park*
EDMONSON, NAT, JR., Ph.D. (Rice) Principal Mathematician, Oak Ridge National Labs., Oak Ridge, Tenn. *P.O. Box 183, Kingston, Tenn.*
EDMONSON, P. W., M.A. (Michigan) Teacher, Dearborn Junior Coll., Dearborn, Mich. *3601 McKinley Ave.*
EDMUNDSON, H. P., M.A. (U.C.L.A.) Teaching Asst., Univ. of California at Los Angeles, Los Angels, Calif. *4553½ W. 18th St.* (*19*)
EDWARDS, MYRTLE, M.A. (Georgia) Teacher, Brenau Coll., Gainesville, Ga.
EDWARDS, P. D., Ph.D. (Indiana) Prof., Ball State Teachers Coll., Muncie, Ind.
EDWARDS, R. D., M.A. (Pittsburgh) Instr., Univ. of Pittsburgh, Pittsburgh, Pa. *1120 Wightman St.* (*17*)
EDWARDS, R. E., M.A. (Michigan) Asso. Actuary, Baltimore Life Insurance Co., Baltimore, Md. *600 Walker Ave.* (*12*)
EDWARDS, R. H., Ph.D. (Illinois Tech.) Research Physicist, Hughes Aircraft Co., Culver City, Calif. *120 Via Colusa, Hollywood Riviera, Redondo Beach, Calif.*
EGAN, F. P., B.A. (Manhattan C.) Instr., Niagara Univ., Niagara University, N.Y. *777 Van Rensselaer Ave., Niagara Falls, N.Y.*
EGGERS, H. C. T., Ph.D. (Michigan) Prof., Univ. of Minnesota, Minneapolis 14, Minn. *2100 31st. Ave., S.* (*6*)
EGGERT, O. E., Postal Clerk, U. S. Post Office Dept. *1916 Forest Ave., Morton, Pa.*
EGGETT, HAZEL E., A.M. (California) *6457 Regent St., Oakland 9, Calif.*
EIARDI, A. J., Rev., M.S. (Boston) Asso. Prof., Boston Coll., *Chestnut Hill 67, Mass.*
EIDE, KARL, B.A. (Emmanuel Missionary C.) Grad. Student, Univ. of Wisconsin, Madison, Wis. *150 Langdon*
EIDE, MARGARET C. (Mrs. R. B.) A.M. (Wisconsin) *1924 Greenwood Dr., Tallahassee, Fla.*
EILENBURG, SAMUEL, Ph.D. (Warsaw) Prof., Columbia Univ., New York 27, N.Y.
EILERTSEN, E. B., M.S. (California) Instr., City Coll. of San Francisco, San Francisco, Calif. *1763 Louvaine Dr., Colma 25, Calif.*
EISELE, CAROLYN, A.M. (Columbia) Instr., Hunter Coll., New York 21, N.Y. *257 W. 86th St.* (*24*)
EISEN, N. A., B.S.E. (Missouri) Chemist, Mid Continent Petroleum Corp., Tulsa, Okla. *1127 S. Braden St.* (*12*)
EISENBERG, BERNARD, B.E.E. (Brooklyn Poly.) Teaching Asst., Rutgers Univ., New Brunswick, N.J. *4304 15th Ave., Brooklyn 19, N.Y.*
EISENHART, L. P., Ph.D. (Johns Hopkins) Executive Officer, American Philosophical Society, Independence Square, Philadelphia, Pa. *25 Alexander St., Princeton, N.J.*
EISENMAN, R. L. P., M.A. (Connecticut) Instr., Univ. of Maryland, College Park, Md. *4316 Colesville Rd., Hyattsville, Md.*
EISINGER, R. P., A.B. (Swarthmore) Student, Coll. of Physicians and Surgeons, Columbia Univ., New York 32, N.Y. *40 W. 77th St.* (*24*)
EKMAN, W. E., A.M. (South Dakota) Prof., Univ. of South Dakota, Vermillion, S.D. *104 S. Yale St.*
EKSTROM, R. E., M.A. (Missouri) Mathematician, U. S. Naval Ordnance Plant, Indianapolis, Ind. *979 Layman Ave.* (*19*)
ELDER, MARSHALL, M.A. (U.C.L.A.) Instr., Los Angeles City Coll., Los Angeles, Calif. *6535 Ben Ave., North Hollywood, Calif.*
ELIEZER, D. F., M.A. (Lehigh) *Colonial Beach, Va.*
ELLIOTT, D. L., Student, Pomona Coll., Claremont, Calif. *118 W. "J" St., Ontario, Calif.*
ELLIOTT, W. W., Ph.D. (Cornell) Prof., Duke Univ., Durham, N.C.
ELLIS, DAVID, Ph.D. (Missouri) Asst. Prof., Univ. of Florida, Gainesville, Fla.
ELLIS, TERRELL (Miss), M.A. (Texas Christian) Asst. Prof., North Texas State Coll., Denton, Tex.
ELSTON, J. S., A.B. (Cornell) Asso. Actuary, Travelers Insurance Co., Hartford 15, Conn.
ELY, R. L., Sc.D. (C.I.T.) Major, U. S. Air Force, *216 Lamont Dr., Apt. 2, Dayton 9, Ohio*

EMERSON, JEROME, B.A. (Buffalo) *R.F.D. 4, Lockport, N.Y.*
EMICH, H. M., Student, Univ. of Rochester, Rochester, N.Y. *1250 Kensington Ave., Buffalo, N.Y.*
EMMERT, J. L., M.Ed. (Pittsburgh) Asst. Prof., Coll. of Steubenville, Steubenville, Ohio. *334 Richland Ave., Wheeling. W.Va.*
EMMONS, H. W., D.S. (Harvard) Asso. Prof., Grad. School of Engineering, Harvard Univ., Cambridge, Mass. *Concord Rd., Sudbury, Mass.*
ENGEBRETSON, HELEN, M.A. (Minnesota) Asst. Prof., State College, Brookings, S.D. *810 9th Ave.*
ENGEL, R. W., *Apartado 299, Guatemala, Guatemala*
ENGSTROM, H. T., Ph.D. (Yale) Vice President, Engineering Research Assoc., Washington, D.C. *507 18th St., S., Arlington, Va.*
ENGSTROM, P. G., A.B. (Bethany C., Kan.) Student, Augustana Theological Seminary, Rock Island, Ill.
EPPERSON, E. R., B.S.Ed. (Miami U.) Grad. Asst., Miami Univ., Oxford, Ohio. *343 N. Second St., Hamilton, Ohio*
EPSELL, MAX, B.S. (C.C.N.Y.) *698 Ashford St., Brooklyn 7, N.Y.*
EPSTEIN, BENJAMIN, Ph.D. (Illinois) Asso. Prof., Wayne Univ., Detroit 1, Mich.
ERDÉLYI, ARTHUR, D.Sc. (Edinburgh) Prof., California Inst. of Tech., Pasadena 4, Calif.
ERDÖS, PAUL, D.Sc. (Manchester) Visiting Lecturer, University of Aberdeen, Aberdeen, Scotland
ERICKSEN, J. L., M.A. (Oregon S.C.) Mathematician, Naval Research Lab., Washington 25, D.C. *1209 Valley Ave., S.E., Apt. 305 (20)*
ERICKSEN, W. S., Ph.D. (Wisconsin) Mathematician, U. S. Forest Products Lab., Madison, Wis. *1924 Rowley Ave. (5)*
ERICKSON, R. L., M.S. (Wisconsin) Prof., Lebanon Valley Coll., Annville, Pa. *2724 Waunona Way, Route 4, Madison 5, Wis.*
ERICKSON, R. W., Ph.D. (Minnesota) Prof., Psychology, Mississippi State Coll. for Women, Columbus, Miss.
ERIKSON, C. M., Ph.D. (Michigan) Prof., Michigan State Normal Coll., Ypsilanti, Mich. *101 Wallace Blvd.*
ERKILETIAN, D. H., Jr., A.M. (Illinois) Asst. Prof., School of Mines & Met., Univ. of Missouri, Rolla, Mo.
ERNSBERGER, IVA B., A.M. (Nebraska) Retired. *905 W. 73rd St., Los Angeles, Calif.*
ERNSDORFF, L. E., Rev., M.S. (Notre Dame) Asso. Prof., Loras Coll., Dubuque, Iowa
ERRERA, ALFRED, Prof., L'Univ. libre, Bruxelles, Belgium. *140 Avenue Moliere*
ERSKINE, A. R., M.A. (Michigan) Instr., Pennsylvania State Coll., State College, Pa. *961 Old Boalsburg Rd.*
ERWIN, M. C., Teacher ,Reynolds High School, Reynolds, Ind.
ESARY, J. D., A.B. (Whitman) Lt., Chemical Corps School, Ft. McClellan, Ala.
ETHRIDGE, J. O., M.S. (Georgia) Teaching Asst., Central High School, Jackson, Miss.
ETTER, D. O., B.A. (Texas Christian) Grad. Student, Texas Christian Univ., Ft. Worth, Tex. *3821 Mattison*
ETTINGER, W. J., B.S. in M.E. (Lewis Inst.) Designing Engr., Edison General Electric Appliance Co., 5600 W. Taylor St., Chicago, Ill.
ETTLINGER, H. J., Ph.D. (Harvard) Prof., Univ. of Texas, Austin, Tex. *3110 Harris Park Ave.*
EULENBERG, M. D., A.M. (Loyola U.) Instr., Wright Junior Coll., Chicago 34, Ill.
EVANS, G. C., Ph.D. (Harvard) Prof., Univ. of California, Berkeley 4, Calif.
EVANS, H. B., Ph.D. (Pennsylvania) Emeritus Prof., Univ. of Pennsylvania, Philadelphia, Pa. *88 Merbrook Lane, Merion Station, Pa.*
EVANS, H. P., Ph.D. (Wisconsin) Prof., Univ. of Wisconsin, Madison 6, Wis.
EVANS, M. B., B.S.E. (Michigan) Instr., Missouri School of Mines and Met., Rolla, Mo. *1265 Meadowmere, Springfield 4, Mo.*
EVANS, P. L., M.S. (Kansas S.C.) Asso. Prof., Kent State Univ., Kent. Ohio
EVERETT, H. S., Ph.D. (Chicago) Extension Prof., Univ. of Chicago, Chicago 37, Ill.
EVERETT, J. P., Ph.D. (Columbia) Emeritus Prof., Western Michigan Coll., Kalamazoo, Mich. *907 W. South St.*
EVERETT, J. R., A.M. (Wisconsin) Retired Asso. Prof., Colorado School of Mines, Golden, Colo. *1700 Washington St.*
EVES, HOWARD, Ph.D. (Oregon) Head of Dept., Champlain Coll, Plattsburg, N.Y.
EWING, G. M., Ph.D. (Missouri) Mathematician, Sandia Corp., Sandia Base, Albuquerque, N.M.
EWING, META M., M.S. (Michigan S.C.) Teacher, Bay City Junior Coll., Bay City, Mich. *704½ N. Grant Pl.*
EWY, D. J., M.S. (Stanford) Instr., Fresno State Coll., Fresno, Calif. *303 Vassar Ave.*

FADELL, A. G., M.A. (Buffalo) Grad. Student, Ohio State Univ., Columbus, Ohio. *1310 Whitney Ave., Niagara Falls, N.Y.*

FAGERSTROM, W. H., Ph.D. (Columbia) Asso. Prof., City Coll. of New York, New York 31, N.Y. *706 Riverside Dr.*
FAIRCHILD, G. W., B.A. (Arizona S.C.) Electronics Lab. Analyst, Northrop Aircraft, Hawthorne, Calif. *505 E. Broadway, Long Beach 2, Calif.*
FAITH, C. C., Student, Univ. of Kentucky, Lexington, Ky. *418 W. 16th St., Covington, Ky.*
FALKENSTERN, O. J., B.S. (Montana) *48 Balboa, San Jose, Calif.*
FALVEY, FRANCES E., Ed.D. (Columbia) Dir. of Student Affairs, James Millikin Univ., Decatur, Ill.
FAN, KY, D.Sc. (Paris) Asso. Prof., Univ. of Notre Dame, Notre Dame, Ind.
FARBER, LORRAINE W. (Mrs. Marvin), M.A. (Buffalo) *234 Wallace Ave., Buffalo 16, N.Y.*
FARNELL, A. B., Ph.D. (California) Major, U. S. Military Acad., West Point, N.Y.
FAULKNER, F. D., M.S. (Kansas S.C.) Asst. Prof., U. S. Naval Postgraduate School, Annapolis, Md.
FAULKNER, J. E., B.S. (Utah S.A.C.) Grad. Asst., Kansas State Coll., Manhattan, Kan.
FAY, E. A., A.M. (Harvard) Grad. Student, Univ. of California, Berkeley 4, Calif. *P.O. Box 163, Ridgecrest, Calif.*
FEDER, JOSEPHINE C. (Mrs. Ralph), M.A. (Bryn Mawr) Physicist, Pitman Dunn Lab., Frankford Arsenal, Philadelphia, Pa. *1213 Arrott St. (24)*
FEDERICO, P. J., A.M. (George Washington) Examiner-in-Chief, U. S. Patent Office, Washington 25, D.C. *3634 Joselyn St., N.W. (15)*
FEEMSTER, H. C., A.M. (Nebraska) Emeritus Prof., York Coll., York, Nebr. *812 E. 7th St.*
FEENEY, W. J., Rev., M.S. (Catholic U.) Grad. Student, Catholic Univ. of America, Washington 17, D.C. *1225 Otis St., N.E. (17)*
FEHR, H. F., Ph.D. (Columbia) Prof., Teachers Coll., Columbia Univ., New York 27, N.Y. *Seth Low, Apt. 63, 121st St. and Morningside Dr. (27)*
FEIDNER, JEAN B. (Mrs. G. F.), M.A. (Buffalo) Instr., Univ. of Buffalo, Buffalo 14, N.Y. *570 Parkside (16)*
FEIGE, RUDOLF, M.A. (Berlin) Instr., Univ. of Cincinnati, Cincinnati, Ohio. *1620 Madison Ave. (6)*
FEINSTEIN, I. K., M.A. (Northwestern) Instr., Univ. of Illinois, Undergraduate Div., Chicago 25, Ill.
FELD, J. M., Ph.D. (Columbia) Asst. Prof., Queens Coll., Flushing, N.Y. *65-30 Kissena Blvd.*
FELDER, VIRGINIA I., M.S. (Tulane) Asst. Prof., Mississippi Southern Coll., Hattiesburg, Miss.
FELDMAN, H. D., Student, Rutgers Univ., New Brunswick, N.J. *111-33 66th Ave., Apt. 2-C, Forest Hills, New York, N.Y.*
FELDMAN, H. M., Ph.D. (Washington U.) Teacher, Beaumont High School, University City, Mo. *765 Yale Ave. (5)*
FELLER, WILLIAM, Ph.D. (Göttingen) Prof., Princeton Univ., Princeton, N.J.
FELLING, W. E., M.S. (Louisiana) Instr., Parks Coll., East St. Louis, Ill.
FELTGES, EDNA M., A.M. (Wisconsin) Retired. *R.R. 7, Box 126, Orlando, Fla.*
FERGUSON, W. E., M.A. (Missouri) Instr., Connecticut Coll. for Women, New London, Conn.
FERNS, H. H., Ph.D. (Toronto) Prof., Univ. of Saskatchewan, Saskatoon, Sask., Canada
FERRY, F. C., Ph.D. (Clark) Emeritus President, Hamilton Coll., Clinton, N.Y. *324 Hart St., New Britain, Conn.*
FETTIS, H. E., A.B. (Wittenberg) Mathematician, Wright-Patterson Air Force Base, Dayton, Ohio. *931 Five Oaks Ave. (6)*
FEYERHERM, A. M., M.S. (Iowa) Instr., Iowa State Coll., Ames, Iowa
FICKEN, F. A., Ph.D. (Princeton) Prof., Univ. of Tennessee, Knoxville 16, Tenn. *On leave:* Inst. of Math. and Mechanics, New York Univ., New York, N.Y.
FIELD, FLOYD, A.M. (Harvard) Dean of Men, Georgia Inst. of Tech., Atlanta, Ga. *2865 Tupelo St., S.E.*
FIELD, S. E., A.M. (Michigan) Instr., Gogebic Junior Coll., Ironwood, Mich. *635 E. Cloverland Dr.*
FIELDS, R. I., M.A. (Arizona) Grad. Student, Univ. of Kentucky, Lexington, Ky. *431 N. Limestone St.*
FIELDS, W. L., A.M. (Indiana) Asso. Prof., Hampton Inst., Hampton, Va.
FINAN, E. J., Ph.D. (Ohio State) Prof., Catholic Univ. of America, Washington, D.C. *2228 Taylor St., N.E. (18)*
FINCH, J. V., Ph.D. (Chicago) Asst. Prof., Beloit Coll., Beloit, Wis.
FINCK, E. R., JR., Student, St. Mary's Univ., San Antonio, Tex. *Route 2, Box 225b*
FINDLAY, WILLIAM, Ph.D. (Chicago) Emeritus Prof., McMaster Univ., Hamilton, Ont., Canada. *32 S. Oval*
FINDLEY, G. B., B.S. (Florida) Grad. Asst., Univ. of Florida, Gainesville, Fla.
FINE, N. J., Ph.D. (Pennsylvania) Asst. Prof., Univ. of Pennsylvania, Philadelphia 4, Pa. *272 S. Felton St.*

FINIGAN, MARY T., Student, Regis Coll., Weston, Mass. *46 Lexington Rd., Concord, Mass.*
FINKBEINER, D. T., II, Ph.D. (C.I.T.) Asso. Prof., Kenyon Coll., Gambier, Ohio
FINKBEINER, E. R., Student, St. Norbert Coll., West De Pere, Wis. *414 W. Walnut St., Apt. 4, Green Bay, Wis.*
FINKEL, DANIEL, B.S. (C.C.N.Y.) Junior Actuary, N.Y. State Insurance Fund, 625 Madison Ave., New York 22, N.Y. *690 Hendrix St., Brooklyn 7, N.Y.*
FIRESTONE, C. D., Ph.D. (Cornell) Asst. Prof., Rutgers Univ., New Brunswick, N.J.
FIRL, D. H., B.A. (Gustavus Adolphus C.) Grad. Asst., Kansas State Coll., Manhattan, Kan. *Route 3, Box 34, Red Wing, Minn.*
FIRST, DOUGLAS, B.S. (Brooklyn) Electronics Engr., Material Lab., New York Naval Shipyard, Brooklyn, N.Y. *69-31 226th St., Bayside, N.Y.*
FISCH, A. L., M.S. (Wisconsin) Acting Instr., Ohio Univ., Athens, Ohio. *15 Church St.*
FISCH, F. N., M.A. (Colorado S.C.) Asst. Prof., Colorado State Coll. of Education, Greeley, Colo.
FISCHER, C. H., Ph.D. (Iowa) Prof., Univ. of Michigan, Ann Arbor, Mich.
FISCHER, F. J., REV., M.S. (DePaul) Asst. Prof., DePaul Univ., Chicago, Ill. *1010 Webster Ave. (14)*
FISCHER, I. C., M.S. (Marquette) Asst. Prof., Engg., Univ. of Minnesota, Minneapolis, Minn. *4252 Columbus Ave. (7)*
FISCHER, J. J., M.S. (Illinois Tech.) Research Engr., Aerophysics Lab., North American Aviation, Downey, Calif. *602 N. Magnolia Ave., Whittier, Calif.*
FISH, RUTH A., M.S. (Arizona) Teacher, Contra Costa Junior Coll., Richmond, Calif. *946 Arlington Ave., Berkeley 7, Calif.*
FISHBACK, W. T., Ph.D. (Harvard) Asst. Prof., Univ. of Vermont, Burlington, Vt. *916 Shelburne St.*
FISHER, ANSELM, A.M. (N.Y.U.) Instr., Central High School, Washington, D.C. *1723 Harvard St., N.W.*
FISHER, F. H., M.S. (West Virginia) Teacher, McDonald High School, McDonald, Pa. *96 S. Wade Ave., Washington, Pa.*
FISHMAN, MILTON, B.S.E. (Michigan) Auto. Design Engr., Detroit Arsenal, Centerline, Mich. *2680 Ewald Circle, Detroit 4, Mich.*
FITE, W. B., Ph.D. (Cornell) Emeritus Prof., Columbia Univ., New York, N.Y. *44 Morningside Dr. (25)*
FITZGERALD, E. L., REV., M.A. (Gonzaga) Missionary, Ateneo de Manila, 406 Padre Suara, Manila, Philippines
FITZPATRICK, BEN, JR., Student, Alabama Poly. Inst., Auburn, Ala. *226 Genelda Ave.*
FLAGG, ELINOR B., M.S. (Illinois) Asst. Prof., Illinois State Normal Univ., Normal, Ill. *29 Payne Pl.*
FLANAGAN, R. J., B.S. (New Mexico) Mathematical Analyst, Sandia Corp., Albuquerque, N.M. *2022 N. Cardenas Dr.*
FLANDERS, HARLEY, Ph.D. (Chicago) Instr., Univ. of California, Berkeley 4, Calif.
FLANDERS, R. L., M.C.E. (Cornell) Prof., Civ. Engg., Oklahoma A. and M. Coll., Stillwater, Okla.
FLEISHER, EDWARD, Ph.D. (N.Y.U.) Prof., Brooklyn Coll., Brooklyn, N.Y. *20 Plaza Pl. (17)*
FLEMING, J. R., Student, Univ. of Washington, Seattle, Wash. *740 11th Ave., N. (2)*
FLEMING, WALTER, Ph.D. (Minnesota) Asso. Prof., Mankato State Teachers Coll., Mankato, Minn.
FLESHLER, A. D., M.S. (N.Y.U.) Asst. Prof., Champlain Coll., Plattsburg, N.Y.
FLOGSTAD, IDA, M.S. (Iowa State) Head of Dept., State Teachers Coll., Superior, Wis. *1009 N. 18th St.*
FLORIS, ATHANASIUS, Grad. Engr. (Tech. Hochschule, Munich) Pacific Elec. Ry. Co., Los Angeles, Calif. *748 S. Kingsley Dr. (5)*
FLYNN, A. J., B.S. (Illinois State Normal) Grad. Asst., Illinois State Normal Univ., Normal, Ill. *1305½ S. Madison St., Bloomington. Ill.*
FOARD, C. W., Ph.D. (Iowa) Physicist, Eastman Kodak Co., Rochester 4, N.Y. *1 Chadwell Rd. (9)*
FOBES, M. P., Ph.D. (Harvard) Prof., Coll. of Wooster, Wooster, Ohio
FOLEY, J. F., B.Sc. (Chicago) Asst. Physicist, Gas & Electric Co., Baltimore, Md. *1311 Silverthorne Rd. (12)*
FOLLEY, K. W., Ph.D. (Toronto) Asso. Prof., Wayne Univ., Detroit, Mich. *19230 Gainsborough St. (23)*
FOLSOM, R. B., A.M. (Columbia) Asst. Prof., The Citadel, Charleston, S.C.
FONG, AH YOKE, Student, San Diego State Coll., San Diego, Calif. *2706 Monroe St. (4)*
FOOTE, J. R., Ph.D. (M.I.T.) Mathematician, Wright-Patterson Air Force Base, Dayton, Ohio. *1813 Brookline Ave. (10)*
FOOTE, KATHERINE S., M.S. (Louisiana) Critic Teacher, Mississippi Southern Coll., Hattiesburg, Miss. *204 4th Ave.*

FORD, CLARENCE, A.M. (Kentucky) Teacher, Male High School, Louisville 3, Ky. *239 Glendora Ave. (12)*
FORD, L. R., Ph.D. (Harvard) Emeritus Prof., Illinois Inst. of Tech., Chicago 16, Ill. *200 Cameron Lane, Charlottesville, Va.*
FORD, P. L., M.S. (Louisiana) *1005 Fifth St., Lake Charles, La.*
FORD, PEARL L., A.M. (Michigan) Asso. Prof., Western Michigan Coll. of Education, Kalamazoo, Mich. *302 Hunter St., Battle Creek, Mich.*
FORD, W. B., Ph.D. (Harvard) Emeritus Prof., Univ. of Mich., Ann Arbor, Mich. *Hayt Corners, Seneca Co., N.Y.*
FOREMAN, W. C., A.M. (Kansas) Instr., Baker Univ., Baldwin City, Kan.
FORMAN, J. W., B.A. (Omaha) Instr., Univ. of Kansas, Lawrence, Kan. *1130 Tennessee*
FORMAN, WILLIAM, A.M. (Brooklyn) Lecturer, Brooklyn Coll., Brooklyn, N.Y. *494 Powell St. (2)*
FORRAY, M. J., M.S. (N.Y.U.) Instr., Poly. Inst. of Brooklyn, Brooklyn, N.Y. *1396 E. 16th St.*
FORSETH, R. A., M.S. (North Dakota) Instr., State Teachers Coll., Mayville, N.D. *1317 Summit Ave., Detroit Lakes, Minn.*
FORSYTH, C. H., Ph.D. (Michigan) Retired. *71 Lebanon Rd., Hanover, N.H.*
FORSYTHE, G. E., Ph.D. (Brown) Mathematician, National Bureau of Standards, Los Angeles, Calif. *414 22nd St., Santa Monica, Calif.*
FORT, M. K., JR., Ph.D. (Virginia) Asst. Prof., Univ. of Illinois, Urbana, Ill. *1709 W. Park, Champaign, Ill.*
FORT, TOMLINSON, Ph.D. (Harvard) Prof., Univ. of Georgia, Athens, Ga.
FOSTER, A. E., M.A. (Tennessee) Instr., Univ. of Kentucky, Lexington, Ky.
FOSTER, MARY L., M.S. (Louisiana) Asso. Prof., Henderson State Teachers Coll., Arkadelphia, Ark.
FOSTER, R. M., B.S. (Harvard) Prof., Poly. Inst. of Brooklyn, Brooklyn, N.Y. *265 S. Harrison St., Apt. J-7, East Orange, N.J.*
FOUCH, R. S., M.S. (Chicago) Asst. Prof., Arizona State Coll., Tempe, Ariz.
FOULIS, D. J. R., Student, Univ. of Miami, Coral Gables, Fla. *P.O. Box 393, South Miami 43, Fla.*
FOUNTAIN, J. H., B.A. (Buffalo) Research Chemist, The Upson Co., Lockport, N.Y. *758 Walnut St.*
FOUST, J. W., Ph.D. (Michigan) Prof. and Asst. to President, Central Michigan Coll. of Education, Mt. Pleasant, Mich.
FOX, A. H., Ph.D. (Yale) Prof., Union Coll., Schenectady, N.Y. *1101 Millington Rd.*
FOX, H. H., M.A. (Wisconsin) Asst., Univ. of Illinois, Urbana, Ill. *Ct. E., 121-B, Stadium Terr., Champaign, Ill.*
FRAME, J. S., Ph.D. (Harvard) Prof., Michigan State Coll., East Lansing, Mich.
FRANCIOL, CHARLES, M.S. (Louisiana) Instr., Gillis High School, Lake Charles, La.
FRANCIS, G. C., M.S. (Minnesota) Instr., Carleton Coll., Northfield, Minn.
FRANCIS, S. A., A.M. (California) Head of Dept., San Mateo Junior Coll., San Mateo, Calif. *Route 1, Box 389, Los Altos, Calif.*
FRANCIS, T. C, Accountant, Hanley & Bird, Bradford, Pa. *65 State St.*
FRANCK, ABRAHAM, Ph.D. (Minnesota) Asst. Prof., Kansas State Coll., Manhattan, Kan.
FRANK, D. H., B.S. (C.C.N.Y.) Head of Dept., Forest Hills High School, Forest Hills, N.Y. *411 W. 114th St., New York 25, N.Y.*
FRANK, EVELYN, Ph.D. (Northwestern) Asso. Prof., Univ. of Illinois, Chicago, Ill. *2226 Sherman Ave., Evanston, Ill.*
FRANK, F. T., A.B. (Stanford) Asst. to Manager, Insurance Dept., Parrott & Co., San Francisco, Calif. *772 Kendall Ave., Palo Alto, Calif.*
FRANK, W. L., Student, Illinois Inst. of Tech., Chicago, Ill. *5459 Greenwood Ave. (15)*
FRANK, W. M., Student, Yeshiva Univ., New York, N.Y. *204 Wilson St., Brooklyn 11, N.Y.*
FRANKEL, E. T., B.S. (C.C.N.Y.) Budget Analyst, Health and Welfare Federation of Allegheny County, 519 Smithfield St., Pittsburgh 22, Pa.
FRANKEL, R. W., A.B. (Michigan) Michigan Bell Tel. Co., Ypsilanti, Mich. *1001 W. Water St., Elmira, N.Y.*
FRANKLIN, PHILIP, Ph.D. (Princeton) Prof., Massachusetts Inst. of Tech., Cambridge 39, Mass. *312 Pleasant St., Belmont 78, Mass.*
FRANZ, E. A., M.S. (Iowa) First Lt., U. S. Army. *2006 Central Ave., Apt. 8, Augusta, Ga.*
FRASER, G. C., A.M. (Pennsylvania) *1221 E. Edgemont St., Phoenix, Ariz.*
FRASER, W. C. G., Ph.D. (Toronto) Asst. Prof., Dartmouth Coll., Hanover, N.H.
FREAS, ELIZABETH, A.M. (Louisiana) Asst., Louisiana State Univ., Baton Rouge, La.
FRÉCHETTE, EMILE, REV., Lic. ès Sci. Math. (Montreal) Teacher, Seminary of Philosophy, 3880 Côte-des-Neiges Rd., Montreal 25, P.Q., Canada
FREDERICK, D. D., Student, George Pepperdine Coll., Los Angeles, Calif. *7023 "A" Plaska Ave., Huntington Park, Calif.*
FREE, N. S., M.A. (British Columbia) Lecturer, Univ. of California, Berkeley 4, Calif.

FREEBORN, R. M., A.B. (Whittier) Instr., Mark Keppel High School, Alhambra, Calif. *905 S. Abbot Ave., San Gabriel, Calif.*
FREELAND, D. E., Student, Northwestern Univ., Evanston, Ill. *2339 Sheridan Rd.*
FREEMAN, LEONA, B.A. (Hunter) Teacher, Stitt Junior High School, New York, N.Y. *270 Seaman Ave. (34)*
FREESE, FRANCES, A.M. (Radcliffe) Dean of Women, Mount Union Coll., Alliance, Ohio
FREITAG, HERTA T. (Mrs.), M.A. (Columbia) Asst. Prof., Hollins Coll., Hollins College, Va.
FREUND, J. E., M.A. (U.C.L.A.) Asso. Prof., Alfred Univ., Alfred, N.Y.
FRICK, C. H., Ph.D. (North Carolina) Prof., Mary Washington Coll., Fredericksburg, Va.
FRIEDLEN, D. M., B.S. (Illinois Tech.) Grad. Student, Illinois Inst. of Tech., Chicago 16, Ill. *3750 Lake Shore Dr. (13)*
FRIEDMAN, E. L., M.A. (Yale) Tester, Chandler-Evans Div., Niles-Bement-Pond Co., West Hartford, Conn. *81 Carpenter Ave., Meriden, Conn.*
FRIEDMAN, H. D., M.A. (Columbia) Grad. Student, Pennsylvania State Coll., State College, Pa. *3245 Colden Ave., New York 67, N.Y.*
FRIEDMAN, JOYCE B., B.A. (Wellesley) *1325 Locust Rd., N.W., Washington 12, D.C.*
FRINK, ORRIN, JR., Ph.D. (Columbia) Prof., Pennsylvania State Coll., State College, Pa. *706 Sunset Rd.*
FROM, W. H., B.A. (Virginia) Research Engr., Aircraft Armaments, Baltimore, Md. *2002 Ramblewood Rd., Apt. B, (14)*
FRONABARGER, C. V., Ph.D. (Missouri) Prof., Southwest Missouri State Coll., Springfield, 4 Mo.
FRY, CLEOTA G., Ph.D. (Purdue) Asst. Prof., Purdue Univ., Lafayette, Ind.
FRY, T. C., Ph.D. (Wisconsin) Asst. to President, Bell Telephone Labs., 463 West St., New York 14, N.Y.
FRY, W. J., M.S. (Penna. State) Asst. Prof., Elec. Engg., Univ. of Illinois, Urbana, Ill. *907 W. Eureka St., Champaign, Ill.*
FUCHS, W. H., Ph.D. (Cambridge) Asso. Prof., Cornell Univ., Ithaca, N.Y.
FUDGE, HELEN G., Ph.D. (Pennsylvania) Teacher, Lincoln High School, Philadelphia, Pa. *Rosemont, Pa.*
FULKS, W. B., Ph.D. (Minnesota) Asst. Prof., Univ. of Minnesota, Minneapolis 14, Minn.
FULLER, GORDON, Ph.D. (Michigan) Prof., Texas Tech. Coll., Lubbock, Tex.
FULLER, GRACE A., A.B. (Fresno S.C.) Head of Dept., Union High School, Madera, Calif.
FULLER, K. G., Ph.D. (Columbia) Prof., Teachers Coll. of Connecticut, New Britain, Conn.
FULLER, L. E., Ph.D. (Wisconsin) Sr. Development Engr., Goodyear Aircraft Corp., Akron, Ohio. *105 E. Cuyahoga Falls Ave. (10)*
FULLER, W. R., B.S. (Butler) *1536 E. 73rd St., Indianapolis, Ind.*
FULMER, H. K., A.M. (Columbia) Prof., Georgia Inst. of Tech., Atlanta, Ga.
FULTON, C. M., Ph.D. (Tech. Hochschule, Munich) Asst. Prof., Univ. of California, Davis, Calif.
FUNKHOUSER, H. G., Ph.D. (Columbia) Instr., Phillips Exeter Acad., Exeter, N.H.
FURMAN, ALBERT, M.S. (New Hampshire) Asst. Prof., Kansas State Coll., Manhattan, Kan.
FUTRAL, BARBARA A., Student, Agnes Scott Coll., Decatur, Ga. *718 McLaurin St., Griffin, Ga.*

GABA, M. G., Ph.D. (Chicago) Retired. *2755 Rathbone Rd., Lincoln 2, Nebr.*
GABRIEL, R. F., M.A. (Columbia) Instr., Rutgers Univ., New Brunswick, N.J. *242 Upson Lane, University Heights*
GADDUM, J. W., M.A. (Missouri) Instr., Univ. of Missouri, Columbia, Mo.
GAFFNEY, M. P., Jr., B.S. (Harvard) Research Asst., Univ. of Chicago, Chicago 37, Ill. *525 Ash St., Winnetka, Ill.*
GAGE, W. H., M.A. (British Columbia) Dean, Univ. of British Columbia, Vancouver, B.C., Canada
GAGER, W. A., Ph.D. (Peabody) Prof., Univ. of Florida, Gainesville, Fla.
GAINES, R. E., A.M. (Furman) Emeritus Prof., Univ. of Richmond, Richmond, Va.
GALBRAITH, M. G., M.S. (Rutgers) Asst. Prof., Rutgers Univ., New Brunswick, N.J. *121 Magnolia St.*
GALE, A. S., Ph.D. (Yale) Emeritus Prof., Univ. of Rochester, Rochester, N.Y. *93 Bellevue Dr. (7)*
GALE, E. I., A.M. (Columbia) *Norland, Ont., Canada*
GALLEGO-DIAZ, JOSÉ, Dr. Sci. (Madrid) Ingeniero Agronomo. *Alarcon 27, Madrid, Spain*
GALLOWAY, GERALDINE (Mr.), M.S. (Illinois) Head of Dept., Flat River Junior Coll., Flat River, Mo. *111 Northwest 10th St., Fairfield, Ill.*
GALVAN, P. JOAQUIN, Student, St. Mary's Univ., San Antonio, Tex. *830 N. St. Mary's*
GAMBILL, R. A., B.A. (Butler) Grad. Student, Butler Univ., Indianapolis, Ind. *4926 E. New York St.*
GAMMEL, J. L., Ph.D. (Cornell) *1976-E 42nd St., Los Alamos, N.M.*
GAMZON, W. V., M.A. (U.C.L.A.) Teacher, Woodrow Wilson High School, Long Beach, Calif. *3315 Vista St.*
GANDY, W. W., M.S. (Texas A. and M.) Instr., Texas A. and M. Coll., College Station, Tex.

GANTVOORT, N. C., M.S. (Iowa) Dean, Huron Coll., Huron, S.C. *238 9th St., S.W.*
GARABEDIAN, C. A., Ph.D. (Harvard) Prof., Wheaton Coll., Norton, Mass.
GARBER, H. N., Student, Univ. of Pennsylvania, Philadelphia, Pa. *5812 N. 12th St. (41)*
GARCIA, MARIANO, JR., Ph.D. (Virginia) Prof., Coll. of Agriculture, Univ. of Puerto Rico, Mayaguez, Puerto Rico
GARDNER, R. W., A.M. (Boston U.) Prof., Olivet Nazarene Coll., Kankakee, Ill.
GARFUNKEL, ROBERT, Student, Rutgers Univ., New Brunswick, N.J. *101 Kensington Ave., Jersey City. N.J.*
GARLAND, KATHERINE C., M.A. (Denver) Instr., Univ. of Denver, Denver, Colo. *356 Corona St. (3)*
GARNER, L. L., A.M. (North Carolina) Asso. Prof., Univ. of North Carolina, Chapel Hill, N.C. *507 North St.*
GARRETT, J. A., A.M. (Peabody) Prof., Arkansas A. and M. Coll., Monticello, Ark.
GARRETT, J. R., Ph.D. (Duke) Asst. Prof., Georgia Inst. of Tech., Atlanta, Ga. *Apt. 129, 251 Tenth St., N.W.*
GARRETT, W. H., Sc.D. (Illinois C.) Emeritus Prof., Baker Univ., Baldwin, Kan. *822 Jersey St.*
GARRISON, L. M., A.M. (Missouri), Ed.M. (Peabody) Asso. Prof., Louisiana Poly. Inst., Ruston, La. *821 Hergot*
GASAWAY, SADIE C., M.S. (Illinois) Instr., Tennessee A. and I. State Coll., Nashville, Tenn. *1211 Krayer St., Memphis, Tenn.*
GASKELL, R. E., Ph.D. (Michigan) Mathematician, Boeing Airplane Co., Seattle 14, Wash. *Route 1, Box 177, Bellevue, Wash.*
GASS, C. B., A.M. (Nebraska) Dean of Men, Nebraska Wesleyan Univ., Lincoln 4, Nebr.
GASS, S. I., M.A. (Boston U.) Mathematician, Aberdeen Bombing Mission, 751 S. Figueroa St., Los Angeles 14, Calif. *1153 N. Formosa Ave. (16)*
GATEWOOD, B. E., Ph.D. (Wisconsin) Prof., U.S.A.F. Inst. of Tech., Wright Field, Ohio. *919 Hampshire Rd., Dayton 9, Ohio*
GATT, LOUIS, B.A. (U.C.L.A.) Grad. Student, Univ. of California at Los Angeles, Los Angeles 24, Calif. *602 Gayley*
GAULT, A. E., M.S. (Chicago) Dean, Arts and Sci., Bradley Univ., Peoria 5, Ill.
GAUTHIER, ABEL, A.M. (Columbia) Prof., Univ. of Montreal, 2900 Blvd. Mont-Royal, Montreal, P.Q., Canada
GAYLORD, LESLIE J., M.S. (Chicago) Asst. Prof., Agnes Scott Coll., Decatur, Ga.
GAYTON, C. F., JR., B.S. (Nebraska) Instr., Electronic Radio Television Inst., Omaha, Nebr. *1910 Burt St.*
GEHMAN, H. M., Ph.D. (Pennsylvania) Prof., Univ. of Buffalo, Buffalo 14, N.Y. *163 Winspear Ave. (15)*
GEIS, R. W., M.A. (Northwestern) Capt., U. S. Army. *4600 Walter Way, El Paso, Tex.*
GELBART, ABE, Ph.D. (M.I.T.) Asso. Prof., Syracuse Univ., Syracuse 10, N.Y.
GELDER, H. M., M.A. (Missouri) Asst. Prof., Western Washington Coll. of Education, Bellingham, Wash.
GENTRY, F. C., Ph.D. (Illinois) Visiting Asso. Prof., Univ. of New Mexico, Albuquerque, N.M. *450 N. Sycamore*
GENTZLER, W. E., A.M. (Columbia) Asst. Provost, Columbia Univ., New York 27, N.Y.
GEORGES, J. S., Ph.D. (Chicago) Head of Dept., Wright Junior Coll., Chicago, Ill. *4515 N. Kildare Ave.*
GEPHART, LANDIS, M.A. (Dayton) Mathematician, Office of Air Research, Wright-Patterson Air Force Base, Dayton, Ohio. *2341 Smithville Rd. (10)*
GERE, B. H., Ph.D. (M.I.T.) Asso. Prof., Hamilton Coll., Clinton, N.Y.
GERGEN, J. J., Ph.D. (Rice) Prof., Duke Univ., Durham, N.C.
GERMAIN, C. B., B.A. (C. of St. Thomas) Grad. Student, Univ. of Wisconsin, Madison, Wis. *796 Cherokee Ave., St. Paul 7, Minn.*
GERMAN, F. C., M.A. (Illinois) Asso. Prof., Kansas State Teachers Coll., Pittsburg, Kan. *226 W. Adams*
GERMINO, E. J., M.A. (Columbia) Instr., St. John's Univ., Brooklyn, N.Y. *53-01 32nd Ave., Woodside, N.Y.*
GERST, F. J., REV., Ph.D. (Johns Hopkins) Prof., Loyola Univ., Chicago 26, Ill.
GERST, IRVING, Ph.D. (Columbia) Tech. Consultant, Army Service Forces. *1055 Clarkson Ave. Brooklyn 12, N.Y.*
GERTZ, H. E., B.S. (Illinois Tech.) Grad. Student, Illinois Inst. of Tech., Chicago, Ill. *7322 S. Phillips Ave. (49)*
GETCHELL, B. C., Ph.D. (Michigan) Research Analyst, Dept. of the Army, Washington, D.C. *13 Providence Terr., McLean, Va.*
GETMAN, K. G., A.B. (Alfred) Asso. Editor, John Wiley & Sons, Inc., 440 Fourth Ave., New York 16, N.Y.

GETTIG, R. E., M.S. (Pittsburgh) Instr., Univ. of Pittsburgh, Pittsburgh 13, Pa. *563 Montview Pl. (21)*
GHENT, K. S., Ph.D. (Chicago) Asso. Prof., Univ. of Oregon, Eugene, Ore. *2105 McMillan St.*
GIBBENS, GLADYS, Ph.D. (Chicago) Asso. Prof., Univ. of Minnesota, Minneapolis 14, Minn.
GIBNEY, ESTHER F., Ph.D. (Northwestern) Asso. Prof., Univ. of Houston, Houston, Tex. *3801 St. Bernard St. (4)*
GIBSON, F. W., A.A. (Los Angeles C.C.) Student, Los Angeles State Coll., Los Angeles 29, Calif. *10621 Butterfield Rd. (64)*
GIBSON, J. C., M.A. (Yale) Sales Trainee, Dept. of Applied Science, International Business Machines Corp., Albany, N.Y. *71 Lancaster St.*
GIBSON, R. W., Ph.D. (Illinois) *Wakeeney, Kan.*
GILBERT, R. C., A.B. (Harvard) Engg. Technician, Lockheed Aircraft Corp., Burbank, Calif. *10410 Lanark St., Sun Valley 3, Calif.*
GILBERT, W. M., M.A. (Oregon) Grad. Student, Princeton Univ., Princeton, N.J.
GILL, B. P., Ph.D. (Columbia) Prof., City Coll. of New York, New York, N.Y. *493 Warwick Ave., West Englewood, N.J.*
GILLAM, B. E., Ph.D. (Missouri) Prof., Drake Univ., Des Moines, Iowa
GILLESPIE, R. H., Student, Albion Coll., Albion, Mich. *925 N. Monroe St.*
GILLIS, M. E., A.M. (Chicago) Prof., Blue Mountain Coll., Blue Mountain, Miss.
GILLMAN, GLADYS E., B.A. (Skidmore) Teacher, Winsor School, Boston, Mass. *41 Carlton St., Brookline, Mass.*
GILMAN, R. E., Ph.D. (Princeton) Prof., Brown Univ., Providence, R.I. *44 E. Manning St.*
GILMORE, A. L., JR., B.S. (Mississippi Southern) Instr., Pearl River Junior Coll., Poplarville, Miss. *401 Dearborn St., Hattiesburg, Miss.*
GILVARRY, J. J., Ph.D. (Princeton) The Rand Corp., 1500 4th St., Santa Monica, Calif. *5443 Overdale Dr., Los Angeles 43, Calif.*
GINIVAN, F. J., REV., M.S. (Notre Dame) Teaching Fellow, Univ. of Notre Dame, Notre Dame, Ind.
GINSBURG, JEKUTHIEL, Sc.D. (Columbia) Prof., Yeshiva Univ., New York, N.Y. *610 W. 139th St. (31)*
GINTHER, L. B., B.E. (Toledo) Jr. Glass Tech., Libbey-Owens-Ford, Toledo, Ohio. *3106 Glanzman Rd., (14)*
GIORDMAINE, J. A., Student, Univ. of Toronto, Toronto, Ont., Canada. *97 Westminster Ave. (3)*
GIVEN, JACQUELINE, M.A. (Colorado C.) Head of Dept., Pueblo Junior Coll., Pueblo, Colo. *2414 Orman Ave.*
GIVENS, J. W., Ph.D. (Princeton) Prof., Univ. of Tennessee, Knoxville, Tenn.
GLABE, G. R., M.A. (Minnesota) *551 Hess, Apt. B, River Rd. Housing, Columbus 10, Ohio*
GLACKEN, W. T., Student, Purdue Univ., Lafayette, Ind. *690 Waldron, West Lafayette, Ind.*
GLANDER, HAROLD, B.S. (Marquette) Instr., Carroll Coll., Waukesha, Wis.
GLASS, R. L., M.Ed. (Peabody C.) Head of Dept., Texarkana Coll., Texarkana, Tex. *2020 Magnolia*
GLASS, W. E., B.A. (Texas) Mathematician, U. S. Air Force, San Antonio, Tex. *1801 W. French Pl.*
GLAUZ, R. D., M.S. (Michigan) Research Asst., Brown Univ., Providence, R.I. *117 Waterman Ave., East Providence, R.I.*
GLAZIER, HARRIET E., A.M. (Chicago) Emeritus Asst. Prof., Univ. of California at Los Angeles, Los Angeles, Calif. *1307 Lucile Ave. (26)*
GLEASON, A. M., B.S. (Yale) Lt. Comdr., U.S.N.R., Armed Forces Security Agency, Washington 25, D.C. *3040 Idaho Ave., N.W. (16)*
GLEISSNER, G. H., M.A. (Columbia) Mathematician, U. S. Naval Proving Ground, Dahlgren, Va.
GLENN, W. H., JR., A.M. (U.C.L.A.) Asst. Curriculum Coordinator, Pasadena City Schools, Pasadena, Calif. *1425 Beech St., South Pasadena, Calif.*
GLOVER, B. C., A.M. (Chicago) Emeritus Prof., Otterbein Coll., Westerville, Ohio. *220 Hiawatha Ave.*
GLUSMAN, SIDNEY, M.A. (Columbia) Teacher, New York City School System, New York, N.Y. *51 Bond St.*
GODDERZ, H. W., B.S. (Northern S.T.C.) Coll. of St. Thomas, St. Paul 1, Minn. *923 Linwood Ave. (5)*
GODFREY, E. L., A.M. (Indiana) Asso. Prof., Defiance Coll., Defiance, Ohio. *125 E. Sessions Ave.*
GOEN, F. MONICA, M.S. (Iowa) Asso. Prof., Mississippi State Coll., State College, Miss.
GOEPPER, F. S., JR., M.A. (Cincinnati) Dept. of Defense, Washington 25, D.C. *389 Terrace Ave., Cincinnati 20, Ohio*
GOHEEN, H. E., Ph.D. (Stanford) Asst. Prof., Elec. Engg., Univ. of Pennsylvania, Philadelphia 4, Pa.

GOING, MILDRED W. (Mr. R. F., Jr.), M.A. (Maryland) Mathematician, Corps of Engineers, Army Map Service, Washington, D.C. *2511 Arundel Rd., Apt. 4, Mount Rainier, Md.*
GOINGS, E. W., M.S. (Michigan) Asst. Prof., Michigan State Normal Coll., Ypsilanti, Mich. *2506 Burns Ave.*
GOINS, MARY A., A.M. (Michigan) Asst. Prof., Marshall Coll., Huntington, W. Va. *435 Ninth Ave.*
GOKHALE, V. D., Ph.D. (Chicago) Prof., Atlanta Univ., Atlanta, Ga.
GOLD, B. K., JR., M.A. (U.C.L.A.) Instr., Los Angeles City Coll., Los Angeles, Calif. *8043 Bellingham Ave., North Hollywood, Calif.*
GOLD, J. S., A.M. (Bucknell) Prof., Bucknell Univ., Lewisburg, Pa. *306 S. Third St.*
GOLDBECK, B. T., JR., M.A. (Texas Christian) Instr., Texas Christian Univ., Fort Worth, Tex. *3875 S. Hills Circle (9)*
GOLDBERG, MICHAEL, A.M. (George Washington) Principal Engr., Bureau of Ordnance, Navy Dept., Washington, D.C. *5823 Potomac Ave., N.W. (16)*
GOLDMAN, A. J., Student, Brooklyn Coll., Brooklyn, N.Y. *3100 Ocean Parkway (35)*
GOLDMAN, ISIDORE, B. of M.E. (Cooper Union) Design Engr., Greer Hydraulics, Inc., 454 18th St., Brooklyn 15, N.Y. *1320 E. 51st St. (34)*
GOLDMAN, I. L., A.M. (Columbia) *2515 8th St., East Meadow, N.Y.*
GOLDSMITH, N. A., Ph.D. (Illinois) Head of Dept., Henderson State Teachers Coll., Arkadelphia, Ark.
GOLDSTEIN, LEON, Staff Member, Div. of Industrial Cooperation, Massachusetts Inst. of Tech., Cambridge, Mass. *4936 25th St., S., Arlington 6, Va.*
GOLDSTINE, H. H., Ph.D. (Chicago) Asst. Project Dir., Electronic Computer Project, Inst. for Advanced Study, Princeton, N.J.
GOLDWATER, M. L., M.A. (U.C.L.A.) *401 N. Van Ness, Apt. 2, Los Angeles 4, Calif.*
GOLOMB, MICHAEL, Ph.D. (Berlin) Prof., Purdue Univ., Lafayette, Ind. *449 Maple St., West Lafayette, Ind.*
GOLOMSKI, W. A., M.S. (Marquette) Fellow, St. Louis Univ., St. Louis, Mo. *Custer, Wis.*
GOLUB, ABRAHAM, M.A. (Delaware) Statistician, Ballistics Research Lab., Aberdeen Proving Ground, Md.
GOLUB, M. A., M.S. (New Brunswick) Instr., Univ. of Missouri, Columbia, Mo.
GOMAN, E. G., M.S. (Oregon S.C.) Instr., Coll. of Puget Sound, Tacoma, Wash.
GONZÁLEZ, M. O., D.P.M.S. (Havana) Prof., Univ. of Havana, Havana, Cuba, *Escuela de Ciencias*
GOOD, J. M., M.Sc. (Brown) Manager, Jenkel-Davidson Optical Co., Am. Trust Bldg., Berkeley, Calif. *928 Lassen St., Richmond, Calif.*
GOOD, L. V., M.A. (Washington) Dean, Skagit Valley Junior Coll., Mount Vernon, Wash. *233 Laurence St.*
GOOD, R. A., Ph.D. (Wisconsin) Asst. Prof., Univ. of Maryland, College Park, Md.
GOODE, H. H., M.A. (Columbia) Supervisor, Aerophysics Group, Aeronautical Research Center, Univ. of Michigan, Ann Arbor, Mich. *1441 White St.*
GOODHUE, E. A., M.S. (Missouri Schl. of Mines) Asso. Prof., Missouri School of Mines & Met., Rolla, Mo. *656 Salem Ave.*
GOODMAN, A. W., Ph.D. (Columbia) Asso. Prof., Univ. of Kentucky, Lexington, Ky.
GOODMAN, N. R., Student, Illinois Inst. of Tech., Chicago, Ill. *1430 N. Hoyne Ave. (22)*
GOODNER, D. B., Ph.D. (Illinois) Asso. Prof., Florida State Univ., Tallahassee, Fla.
GOODRICH, M. T., A.M. (Clark U.) Asst. Prof., Keene Teachers Coll., Keene, N.H. *36 Wyman Way*
GOORMAGHTIGH, RENÉ-VICTOR, C.E. (Ghent) General Mgr., La Brugeoise Steelworks. *"White Lodge," 465 Zandstraat, St. André-lez-Bruges, Belgium*
GORDON, R. D., Ph.D. (Indiana) Asso. Prof., Hampton Inst., Hampton, Va.
GORDON, W. O., A.M. (Penn. State) Asso. Prof., Pennsylvania State Coll., State College, Pa.
GORE, G. D., Ph.D. (Chicago) Prof., Roosevelt Coll., Chicago, Ill. *430 S. Michigan Ave., 10th Floor*
GORMAN, J. R., A.M. (U.C.L.A.) Asst. Prof., U. S. Naval Acad., Annapolis, Md. *217 Norwood Rd.*
GORMSEN, S. T., M.S. (Florida) Instr., Univ. of Florida, Gainesville, Fla. *2216 N.W. 5th Pl.*
GORRELL, G. W., A.M. (Ohio State) Emeritus Prof., Univ. of Denver, Denver, Colo. *1809 Mariposa Ave., Boulder, Colo.*
GORSLINE, D. A., M.A. (Oklahoma) Special Agent, Equitable Life Assurance Society, Albany, N.Y. *Box 441, 7 Gilmore Ave., Cambridge, N.Y.*
GORSLINE, K. E., A.M. (Denver) Teacher, East High School, Denver, Colo. *1301 Fairfax St. (7)*
GOSS, R. N., Ph.D. (Iowa State) Mathematician, U. S. Navy Electronics Lab., San Diego 52, Calif.
GOTTLIEB, M. J., Ph.D. (Washington U.) Asst. Prof., Univ. of Chicago, Chicago 37, Ill.
GOTTSCHALK, W. H., Ph.D. (Virginia) Asst. Prof., Univ. of Pennsylvania, Philadelphia 4, Penn.

GOUGH, LILLIAN, M.A.(Buffalo) Instr., Univ. of Buffalo, Buffalo 14, N.Y. *35 Sussex Ave. (15)*
GOUGH, E. S. J., B.Ed.(Montreal) Prof., Jacques Cartier Normal School, 820 Convent St., Montreal, P.Q., Canada. *4415 Coronation Ave. (28)*
GOULD, ALICE B., A.B.(Bryn Mawr) *35 Congress St., Boston, Mass.*
GOULD, H. W., Pvt., U. S. Army. *422 N. 6th Ave., Portsmouth, Va.*
GOULD, S. H., Ph.D.(Yale) Asst. Prof., Purdue Univ., Lafayette, Ind.
GOURRICH, G. E., M.A.(U.C.L.A.) Research Engr., Northrop Aircraft, Hawthorne, Calif. *12830 Parkyns St., Los Angeles 49, Calif.*
GOUWENS, CORNELIUS, Ph.D.(Chicago) Prof., Iowa State Coll., Ames, Iowa
GOVE, H. E., M.S.(Washington U.) Elec. Engr., Union Electric Co. of Missouri, 315 N. 12th Blvd., St. Louis 1, Mo.
GRABBE, HYACINTH, REV., A.M.(Catholic U.) Prof., St. Joseph's Coll. and Military Acad., Hays, Kan.
GRABLE, E. S., A.M.(Washington and Jefferson) Asst. Prof., Univ. of Richmond, Richmond, Va.
GRABNER, CLEM, JR., M.A.(Indiana) Asst. Prof., Madison Coll., Harrisonburg, Va.
GRACE, A. G., JR., B.S.(Trinity) Instr., Trinity Coll., Hartford, Conn.
GRAD, ARTHUR, Ph.D.(Stanford) Mathematician, Office of Naval Research, Washington, D.C. *4217 Fourth St., S.E., (20)*
GRAESSER, R. F., Ph.D.(Illinois) Prof., Univ. of Arizona, Tucson, Ariz. *1648 E. Fifth St.*
GRAFF, F. G., Ph.D.(Pittsburgh) Asst. Prof., Oberlin Coll., Oberlin, Ohio. *148 W. College St.*
GRAHAM, BEULAH, M.A.(Kentucky) Head of Dept., Campbellsville Coll., Campbellsville, Ky.
GRAHAM, MARIA D., A.M.(Columbia) Emeritus Asso. Prof., East Carolina Teachers Coll., Greenville, N.C. *Warrenton. N.C.*
GRAHAM, P. H., A.M.(Virginia) Asso. Dean, Washington Square Coll., New York Univ., New York, N.Y.
GRAHAM, R. P., Professional Writer. *Box 272, Laguna Beach, Calif.*
GRAHAM, W. W., Ph.D.(George Peabody) Asso. Prof., Vanderbilt Univ., Nashville 4, Tenn.
GRAINGER, G. R., M.S.(Notre Dame) Teaching Fellow, Univ. of Notre Dame, Notre Dame, Ind.
GRANEY, E. P., M.S.(Notre Dame) Research Asst., Willow Run Research Center, Univ. of Michigan, Ypsilanti, Mich. *219 N. State St., Ann Arbor, Mich.*
GRANT, ALICE A., M.A.(Columbia) Principal, Miss Grant's School, Apt. 241, Cawthra Mansions, 215 College St., Toronto 2B, Ont., Canada
GRANT, H. S., Ph.D.(Pennsylvania) Prof., Rutgers Univ., New Brunswick, N.J.
GRAS, E. C., A.M.(Harvard) Asst. Prof., U. S. Naval Acad., Annapolis, Md.
GRAU, A. A., Ph.D.(Michigan) Asso. Prof., Univ. of Oklahoma, Norman, Okla. *304 S. Webster*
GRAUE, L. C., Ph.D.(Indiana) Asst. Prof., Sacramento State Coll., Sacramento 18, Calif.
GRAVATT, T. E., M.S.(Rutgers) Prof., Pennsylvania State Coll., State College, Pa. *344 E. College Ave.*
GRAVES, G. H., Ph.D.(Columbia) Asso. Prof., Purdue Univ., West Lafayette, Ind. *227 S. Grant St.*
GRAVES, L. M., Ph.D.(Chicago) Prof., Univ. of Chicago, Chicago 37, Ill.
GRAVES, R. E., Ph.D.(Minnesota) Mathematician, White Sands Proving Ground, Las Cruces, N.M. *504 N. Armijo St.*
GRAVES, W. L., A.M.(Pennsylvania) Head of Dept., Drury Coll., Springfield, Mo. *920 S. Fremont Ave.*
GRAY, D. O., M.B.A.(Houston) Instr., Univ. of Houston, Houston, Tex. *2926 Pine Gully Blvd. (17)*
GRAY, MARION C., Ph.D.(Bryn Mawr) Member, Tech. Staff, Bell Telephone Labs., Murray, Hill, N.J.
GRAY, MARY W. (Mrs. A. B.), A.M.(Connecticut C.) Instr., Barnard School for Girls, New York 33, N.Y. *87-40 Elmhurst Ave., Elmhurst, N. Y.*
GRAYBEAL, W. T., A.M.(North Carolina) Asst. Prof., Emory and Henry Coll., Emory, Va.
GREANEY, W. R., Postal Employee. *485 E. 188th St., New York 58, N.Y.*
GREELEY, J. B., M.S.(M.I.T.) Asso. Prof., Utica Coll., Oneida Sq., Utica, N.Y.
GREEN, E. E., B.A.(Buffalo) *153 Mariner St., Buffalo 1, N.Y.*
GREEN, J. W., Ph.D.(California) School of Mathematics, Inst. for Advanced Study, Princeton, N.J.
GREEN, L. J., Ph.D.(Chicago) Asst. Prof., Case Inst. of Tech., Cleveland 6, Ohio
GREEN, R. E., Student, Kansas State Teachers Coll., Pittsburg, Kan. *504 W. Forest St.*
GREENE, LAURA Z., M.S.(Chicago) Asst. Prof., Washburn Univ., Topeka, Kan. *1516 Boswell Ave.*
GREENE, R. L., B.E.E.(Clarkson) Junior Design Engr., Western Electric Corp., Sharon, Pa. *418 Morton St.*

GREENE, S. H., M.A. (Pennsylvania) Lt., U. S. Army.
GREENE, U. S., C.P.A., 304-6 Marine Midland Bldg., Binghamton, N.Y.
GREENLEAF, H. E. H., Ph.D. (Indiana) Prof., DePauw Univ., Greencastle, Ind. *1024 S. College Ave.*
GREENSPAN, DONALD, M.S. (Wisconsin) Instr., Univ. of Maryland, College Park, Md.
GREENWOOD, R. E., JR., Ph.D. (Princeton) Lt. Comdr., U. S. Navy. *3040 Idaho Ave., N.W., Apt. 206, Washington 16, D.C.*
GREER, CASSIE C., A.M. (Chicago) Teacher, Englewood High School, Chicago, Ill. *6056 S. Kimbark Ave. (37)*
GREER, E. V., M.A. (Illinois) Prof., Bethany-Peniel Coll., Bethany, Okla. *700 N. Donald St.*
GREGG, J. B., M.S. (National Northern) Retired. *1016 S. Second Ct., S.E., Fort Lauderdale, Fla.*
GREGG, MABEL E., B.S. (East Tennessee S.C.) Teacher, Church Hill School, Church Hill, Tenn. *1613 Blackmore St., Kingsport, Tenn.*
GREGORY, B. W., Student, Oregon State Coll., Corvallis, Ore. *P.O. Box 554, LaGrande, Ore.*
GREGORY, L. D., B.A. (Oklahoma) Instr., Univ. of Oklahoma, Norman, Okla.
GREGORY, R. T., M.S. (Iowa) Grad. Asst., Univ. of Illinois, Urbana, Ill. *908 W. Hill St.*
GREGORY, W. H., M.S. (Illinois) Teacher, Hayward Union High School, Hayward, Calif. *25567 Valley View Dr.*
GREKILA, R. B., B.S. (Ohio U.) Asst. Chief Chemist, Ball Bros. Co., Muncie, Ind. *2113 W. Washington St.*
GREVILLE, T. N. E., Ph.D. (Michigan) Chief, Actuarial Analysis Branch, Public Health Service, Washington 25, D.C. *P.O. Box 7363 (4)*
GRIESMER, J. H., Student, Univ. of Notre Dame, Notre Dame, Ind. *11472 Whithorn Ave., Detroit 5, Mich.*
GRIFFIN, F. L., Ph.D. (Chicago) Prof., Reed Coll., Portland 2, Ore.
GRIFFIN, HARRIET M., Ph.D. (N.Y.U.) Asso. Prof., Brooklyn Coll., Brooklyn, N.Y. *3609 Farragut Rd. (10)*
GRIFFIN, J. S., JR., B.S. (Auburn) Teaching Fellow, Tulane Univ., New Orleans, La. *7011 Jeanette St. (18)*
GRIFFITH, H. C., M.A. (Missouri) Teaching Asst., Univ. of Tennessee, Knoxville, Tenn. *208 13th St.*
GRIMES, A. C., A.M. (Mississippi) Asst. Prof., Mississippi State Coll., State College, Miss.
GRIMES, RUBY M., A.M. (Illinois) Asst. Prof., North Dakota Agr. Coll., State College, N.D.
GRINDALL, E. L., M.S. (Michigan S.C.) *520 Linden Rd., State College, Pa.*
GRINSTEIN, LOUISA S., B.A. (Buffalo) Grad. Student, Univ. of Buffalo, Buffalo 14, N.Y. *232 Colvin Ave. (16)*
GROGAN, F. D., M.A. (Oklahoma) Chemical Corps Insp., Dallas Chemical Procurement Dist., 1114 Commerce St., Dallas 2, Tex. *510 East & West St., Minden, La.*
GROSSMAN, BETTY, M.A. (Michigan) Teacher, Detroit Public Schools, Detroit, Mich. *2552 LaSalle Gardens, N. (6)*
GROSSMAN, GEORGE, A.M. (Columbia) Chm. of Dept., DeWitt Clinton High School, New York, N.Y.; Instr., Poly. Inst. of Brooklyn, Brooklyn, N.Y. *3425 Gates Pl., New York 67, N.Y.*
GROSSWALD, EMIL, Ph.D. (Pennsylvania) Member, Inst. for Advanced Study, Princeton, N.J. *7-B Goodman Rd.*
GROTTS, G. W., M.S. (Illinois) Instr., General Motors Inst., Flint, Mich. *2607 Brownell Blvd. (4)*
GROVE, C. C., Ph.D. (Johns Hopkins) Emeritus Prof., City College of New York, New York, N.Y. *143 Milburn Ave., Baldwin, N.Y.*
GROVE, V. G., Ph.D. (Chicago) Prof., Michigan State Coll., East Lansing, Mich. *438 Rosewood Ave.*
GROVER, BLANCHE B. (Mrs. R. M.), M.A. (Texas) Asst. Prof., Univ. of Houston, Houston, Tex. *4362 Blodgett (4)*
GRUNDMAN, ROSE A., M.S. (Northwestern) Instr., Univ. of Illinois, Chicago, Ill. *9701 Vanderpoel Ave. (43)*
GUARD, H. T., M.S. (Colorado) Prof., Colorado A. and M. Coll., Fort Collins, Colo.
GUENTHER, P. E., Ph.D. (Harvard) Asst. Prof., Case Inst. of Tech., Cleveland, Ohio. *1597 Westwood Ave., Lakewood 7, Ohio*
GUHSÉ, M. P., B.S. (Hamilton) Grad. Student, Univ. of Massachusetts, Amherst, Mass. *84 Main St., Northfield, Mass.*
GUILFORD, G. F., JR., M.S. (Syracuse) Asst. Prof., Rensselaer Poly. Inst., Troy, N.Y. *27 Bleeker Ave.*
GUNDER, D. F., Ph.D. (Wisconsin) Prof., Engg., Cornell Univ., Ithaca, N.Y.
GUNDERSON, N. G., Ph.D. (Cornell) Asst. Prof., Univ. of Rochester, Rochester 3, N.Y.
GUNDLACH, B. H., Ph.D. (Hamburg) Asst. Prof., Univ. of Arkansas, Fayetteville, Ark.
GUNNING, R. C., Student, Univ. of Colorado, Boulder, Colo. *945 Eleventh St.*

GURK, H. M., Student, Univ. of Pennsylvania, Philadelphia, Pa. *6431 N. 15th St. (26)*
GUSTIN, W. S., Ph.D. (U.C.L.A.) Asst. Prof., Indiana Univ., Bloomington, Ind.
GUTZMAN, R. R., M.S. (Iowa) Asst. Prof., Colorado School of Mines, Golden, Colo.
GUY, D. L., B.A. (Macalester) Student, Univ. of Chicago, Chicago, Ill. *c/o B. J. Bockelman, 1831 N. 49th St., Milwaukee 8, Wis.*
GUY, W. T., JR., Ph.D. (C.I.T.) Asst. Prof., Univ. of Texas, Austin 12, Tex.
GWIN, J. P., Student, Univ. of Tulsa, Tulsa, Okla. *1424 N. Main (6)*
GYSLAND, MARIAN S. (Mrs.), A.M. (California) Instr., Extension Div., Univ. of Colorado, Denver, Colo. *7000 W. 35th Ave., Wheatridge, Colo.*

HAAG, V. H., A.M. (Duke) Instr., Franklin and Marshall Coll., Lancaster, Pa. *1712 Linnwood Ave.*
HABER, SEYMOUR, B.A. (Yeshiva) Part-time Instr., Syracuse Univ., Syracuse 10, N.Y.
HACKER, S. G., Ph.D. (Princeton) Prof., State Coll. of Washington, Pullman, Wash.
HADLEY, J. R., B.S. (Ohio State) Mftg. Control Manager, The Richardson Co., Indianapolis, Ind. *2141 N. New Jersey St. (2)*
HADLOCK, E. H., Ph.D. (Cornell) Prof., Univ. of Florida, Gainesville, Fla. *1021 N.E. 4th St.*
HADNOT, B. F., M.S. (Georgia) *P.O. Box 324 Ag Hill, Athens, Ga.*
HAFNER, RALPH, M.A. (Michigan) Mathematician, Naval Ordnance Plant, Indianapolis, Ind. *3344 Arthington Blvd.*
HAFSTROM, J. E., M.A. (Minnesota) Instr., Univ. of Minnesota, Minneapolis, Minn.
HAGEN, BEATRICE L., Ph.D. (Chicago) Asso. Prof., Pennsylvania State Coll., State College, Pa.
HAGEN, G. R., Student, Univ. of Kentucky, Lexington, Ky. *Thompkinsville, Ky.*
HAGGERTY, G. B., M.A. (Bucknell) Asst. Prof., Rhode Island State Coll., Kingston, R.I.
HAHN, S. W., Ph.D. (Duke) Prof., Winthrop CollL, Rock Hill, S.C. *968 Eden Terr.*
HAILPERIN, THEODORE, Ph.D. (Cornell) Asst. Prof., Lehigh Univ., Bethlehem, Pa.
HAIMO, FRANKLIN, Ph.D. (Harvard) Asst. Prof., Washington Univ., St. Louis 5, Mo.
HAIR, C. L., A.M. (Duke) Col., The Citadel, Charleston, S.C.
HALBERG, C. J. A., JR., B.A. (Pomona C.) Grad. Student, Univ. of California at Los Angeles, Los Angeles, Calif. *2341 18th St., Santa Monica, Calif.*
HALFAR, EDWIN, Ph.D. (Iowa) Asst. Prof., Univ. of Nebraska, Lincoln, Nebr.
HALL, DILLA, M.S. (Chicago) Asst. Prof., Southern Illinois Univ., Carbondale, Ill. *820 S. Illinois Ave.*
HALL, D. W., Ph.D. (Virginia) Prof., Univ. of Maryland, College Park, Md. *Box 374, Beltsville, Md.*
HALL, MARSHALL, JR., Ph.D. (Yale) Prof., Ohio State Univ., Columbus, Ohio
HALL, N. A., Ph.D. (C.I.T.) Prof., Mech. Engg., Univ. of Minnesota, Minneapolis 14, Minn.
HALL, VIRGINIA M., A.M. (Boston U.) Instr., Simmons Coll., Boston 15, Mass.
HALLER, MARY E., Ph.D. (Washington) Asso. Prof., Univ. of Washington, Seattle 5, Wash.
HALLER, M. N., M.A. (Michigan) Jr. Equipment Engr., Southern Bell. Tel. & Tel. Co., 820 Poydras St., New Orleans 12, La.
HALLERBERG, A. E., A.M. (Illinois) Asso. Prof., Illinois Coll., Jacksonville, Ill.
HALLETT, W. N., Ph.D. (Pennsylvania) *324 Forest Ave., Ben Avon, Pittsburgh 2, Pa.*
HALMOS, P. R., Ph.D. (Illinois) Asso. Prof., Univ. of Chicago, Chicago 37, Ill. *On leave:* Instituto de Matematica y Estadistica, Facultad de Ingenieria, Av. Julio Herrera y Reissig, Montevideo, Uruguay
HALPERN, EDWARD, M.A. (Columbia) Instr., Univ. of Massachusetts, Amherst, Mass.
HAMILTON, H. J., Ph.D. (Brown) Asso. Prof., Pomona Coll., Claremont, Calif. *1269-C N. Harvard*
HAMILTON, O. H., Ph.D. (Texas) Prof., Oklahoma A. and M. Coll., Stillwater, Okla.
HAMILTON, PARKER, B.S. (Harvard) Asst. Prof., Antioch Coll., Yellow Springs, Ohio. *R.R. 1*
HAMILTON, W. C., B.S. (Okla. A. and M.) Grad. Student, Oklahoma A. and M. Coll., Stillwater, Okla. *503 Pine St.*
HAMMAR, HELEN V., B.S. (Illinois Tech.) Mathematician, David Taylor Model Basin, Navy Dept., Washington, D.C. *2817 Bellevue Terr.*
HAMMER, CARL, Ph.D. (Munich) Chairman, Div. of Tech. Education, Walter Hervey Junior Coll., New York 23, N.Y. *141-11 68th Dr., Flushing, L.I., N.Y.*
HAMMER, P. C., Ph.D. (Ohio State) Group Leader, Los Alamos Scientific Lab., Los Alamos, N.M. *4756 Trinity Dr.*
HAMMING, R. W., Ph.D. (Illinois) Member, Tech. Staff, Bell Telephone Labs., Murray Hill, N.J.
HAMMOND, E. S., Ph.D. (Princeton) Prof., Bowdoin Coll., Brunswick, Maine. *9 Thompson St.*
HAMMOND, J. R., A.M. (Harvard) Asso. Prof., U. S. Naval Acad., Annapolis, Md. *5 Thompson St.*
HANCOCK, CLARA L., A.M. (Iowa) *21 Midway, Waukon, Iowa*
HAND, HELEN J., M.S. (Fordham) Instr., D'Youville Coll., Buffalo 1, N.Y.

HANHAUSER, M. A., REV., A.B. (St. Bonaventure) Siena Coll., Loudonville, N.Y.
HANKERSON, K. L., M.S. (North Dakota) Asst. Prof., Univ. of North Dakota, Grand Forks, N.D.
HANNA, J. R., M.S. (Kansas S.T.C.) Asso. Prof., Univ. of Wichita, Wichita 6, Kan. *On leave, 1951–52:* Univ. of Colorado, Boulder, Colo. *1141 11th St., Boulder, Colo.*
HANNAN, C. V., III, Maintenance Technician, Civil Aeronautics Administration, 102 Richards Rd., Room 205, Kansas City, Mo. *10530 W. 59th Terr., Shawnee, Kan.*
HANNAN, D. J., B.A. (Loras) Teaching Fellow, Univ. of Detroit, Detroit 21, Mich. *930 Pearson, Ferndale, Mich.*
HANNON, H. H., M.A. (Michigan) Asst. Prof., Western Michigan Coll. of Education, Kalamazoo, Mich. *1420 Nassau St.*
HANSEN, A. G., M.S. (Purdue) Research Scientist, National Advisory Committee for Aeronautics, Cleveland, Ohio. *249 E. Bridge St., Berea, Ohio.*
HANSEN, J. B., Student, Harvard Univ., Cambridge, Mass. *27 Havana St., Roslindale 31, Mass.*
HANSMAN, MARGARET M., Ph.D. (Illinois) Asst. Prof., Colorado Coll., Colorado Springs, Colo. *2038 Armstrong Ave.*
HANSON, E. H., Ph.D. (Ohio State) Dir., Dept. of Math., North Texas State Coll., Denton, Tex. *619 W. Oak St.*
HANSON, W. R., M.S. (Chicago) Instr., City Coll. of San Francisco, San Francisco, Calif. *327 Judson Ave.*
HANZEL, P. C., B.S. (Duquesne) Research Asst., Cryogenic Lab., Ohio State Univ., Columbus, Ohio
HARARY, FRANK, Ph.D. (California) Research Asso., Research Center for Group Dynamics, Univ. of Michigan, Ann Arbor, Mich.
HARBISON, C. L., M.A. (Indiana) Asst. Prof., South Dakota School of Mines and Tech., Rapid City, S.D.
HARDEN, J. C., JR., M.A. (Tennessee) Asst. Prof., Clemson Coll., Clemson, S.C. *136 Wigington St.*
HARDIN, J. A., A.M. (Chicago) Dean, Centenary Coll., Shreveport 16, La.
HARDING, F. R., Student, Marquette Univ., Milwaukee, Wis. *2519 N. 67th St., Wauwatosa 10, Wis.*
HARDING, HOWARD, B.M.E. (Michigan) Rochester Gas and Electric Corp., Rochester, N.Y. *29 Kingston St.*
HARDY, F. J., B.A. (Duquesne) Grad. Asst., Duquesne Univ., Pittsburgh 19, Pa.
HARDY, H. M., M.A. (Sam Houston S.T.C.) Teacher, Uvalde High School, Uvalde, Tex. *815 N. High St.*
HARMAN, H. H., M.S. (Chicago) Chief, Statistical Research & Analysis Unit, Dept. of Army, Washington 25, D.C. *4111 Maryland Ave. (16)*
HARMON, ELIZABETH H. (Mrs. J. P.), M.S. (Washington U.) *Box 62, Duff Rd., R.D. 1, Pittsburgh 21, Pa.*
HARP, E. L., JR., A.M. (New Mexico) Auditor, Roswell Public Schools, Roswell, N.M.
HARP, H. G., M.S. (Chicago) Asso. Prof., Ohio Northern Univ., Ada, Ohio. *620 Union St.*
HARPER, F. S., Ph.D. (Iowa) Prof., Actuarial Science, Drake Univ., Des Moines, Iowa
HARRELL, E. G., Ph.D. (Iowa) Head of Dept., State Teachers Coll., Platteville, Wis. *414 S. Chestnut St.*
HARRINGTON, C. E., M.S. (Buffalo) Chief Engr., W. H. Smith, Inc., Springville, N.Y. *12½ Elk St.*
HARRINGTON, W. J., Ph.D. (Cornell) Asso. Prof., Pennsylvania State Coll., State College, Pa.
HARRIS, B. T., A.B. (Knox) Editor, The Macmillan Co., 60 Fifth Ave., New York 11, N.Y.
HARRIS, ISABEL, A.M. (Columbia) Retired, Westhampton Coll., Univ. of Richmond, Richmond, Va. *6411 Three Chopt Rd.*
HARRIS, J. I., Student, Univ. of Michigan, Ann Arbor, Mich. *3303 W. Outer Dr., Detroit 21, Mich.*
HARRIS, L. H., Student, Poly. Inst. of Brooklyn, Brooklyn, N.Y. *464-47th St. (20)*
HARRIS, V. C., Ph.D. (Northwestern) Asst. Prof., San Diego State Coll., San Diego, Calif. *5122 Campanile Dr. (15)*
HARRIS, W. R., JR., LL.B. (Southern Methodist) Attorney, 904 Rep. Bank Bldg., Dallas 6, Tex.
HARRISON, CLAIRE A., A.B. (Northeastern S.C.) Instr., Connors State Agr. Coll., Warner, Okla.
HARRISON, GERALD, Ph.D. (C.I.T.) Asst. Prof., Wayne Univ., Detroit 1, Mich.
HARRISON, R. A., Ph.D. (Cornell) Head of Dept., The Peddie School, Hightstown, N.J.
HARRJE, H. J., Civ. Engr. (C.C.N.Y.) Harrje Associates, Arch. & Engr., 4901 Willow St., Bellaire, Tex.
HARSHBARGER, FRANCES, Ph.D. (Illinois) Prof., Kent State Univ., Kent, Ohio
HART, BERTHA I., A.M. (Cornell) Mathematician, Ballistic Research Lab., Aberdeen Proving Ground, Md.
HART, J. J., M.A. (Alabama) Mathematician, Redstone Arsenal, Huntsville, Ala. *207 Longwood Ct.*

HART, W. L., Ph.D. (Chicago) Prof., Univ. of Minnesota, Minneapolis 14, Minn. *2738 W. River Rd. (6)*
HART, W. W., A.B. (Chicago) Retired Asso. Prof., Univ. of Wisconsin, Madison, Wis. *Box 189, Libertyville. Ill.*
HARTIG, H. E., Ph.D. (Minnesota) Prof., Communication Engg., Univ. of Minnesota, Minneapolis 14, Minn. *3940 Lake Curve, Robbinsdale 12, Minn.*
HARTLEY, M. C., Ph.D. (Illinois) Asst. Prof., Univ. of Illinois, Navy Pier, Chicago, Ill. *5540 S. Blackstone (37)*
HARTMAN, D. K., M.S. (Minnesota) Instr., Elec. Engg., Univ. of Minnesota, Minneapolis 14, Minn.
HARTMAN, H. K., B.S. (Bucknell) Claim Settlement Agent, Pennsylvania Dept. of Public Assistance, Harrisburg, Pa. *2816 Rudy Rd.*
HARTNELL, GEORGE, Retired, Magnetic Observ., U.S. Coast and Geodetic Survey. *Wyoming, N.Y.*
HARTRANFT, ROBERT, B.S. (Stanford) Mathematician, Naval Air Missile Test Center, Pt. Mugu, Calif. *2241 Cloyne, Oxnard, Calif.*
HARTUNG, M. L., Ph.D. (Wisconsin) Asso. Prof., Univ. of Chicago, Chicago 37, Ill. *Judd Hall*
HARTZLER, H. H., Ph.D. (Rutgers) Prof., Goshen Coll., Goshen, Ind.
HARVEY, G. G., Ph.D. (Washington U.) Asso. Prof., Physics, Massachusetts Inst. of Tech., Cambridge 39, Mass.
HARWOOD, MAY N., A.M. (Syracuse) Emeritus Asst. Prof., Syracuse Univ., Syracuse, N.Y. *R.F.D., Orwell, Vt.*
HASELTINE, R. C., M.A. (Penn. State) Asst. Research Engr., Burroughs Adding Machine Co., Philadelphia, Pa. *256 Washington Ave., Clifton Heights, Pa.*
HASKINS, E. E., Ph.D. (Boston) Asso. Prof., Fenn Coll., Cleveland, Ohio
HASKINS, ELIZABETH M., M.S. (M.I.T.) Asst. Prof., Fitchburg State Teachers Coll., Fitchburg, Mass.
HASLAM, M. B., M.A. (Indiana) Grad. Student, Indiana Univ., Bloomington, Ind.
HASSELL, C. L., JR, M.S. (Purdue) Staff Member, Sandia Corp., Albuquerque, N.M. *3234c A St., Sandia Base*
HASSLER, J. O., Ph.D. (Chicago) Prof., Univ. of Oklahoma, Norman, Okla. *425 S. Lahoma Ave.*
HASTINGS, CECIL, JR., B.S. (Florida) Asso. Mathematician, Rand Corp., Santa Monica, Calif. *907 Fifth St.*
HATCH, D. A., A.M. (Columbia) Emeritus Asso. Prof., Lafayette Coll., Easton, Pa. *705 High St.*
HATCHER, T. W., Ph.D. (Cornell) Prof., Virginia Poly. Inst., Blacksburg, Va.
HATFIELD, CHARLES, A.M. (Tennessee) Prof., Georgetown Coll., Georgetown, Ky. *301 Clayton Ave.*
HATFIELD, CHARLES, JR., Ph.D. (Cornell) Asst. Prof., Univ. of Minnesota, Minneapolis 14, Minn. *4612 14th Ave. S. (7)*
HATFIELD, F. C., M.S. (Minnesota) Instr., Physics, Blackburn Coll., Carlinville, Ill.
HATTAN, CORINNE R., Ph.D. (Illinois) Instr., Univ. of Illinois, Urbana, Ill.
HAUSMANN, B. A., REV., Ph.D. (Yale) Prof., West Baden Coll., West Baden Springs, Ind.
HAVILAND, E. K., Ph.D. (Harvard; Johns Hopkins) Asso. Prof., Johns Hopkins Univ., Baltimore 18, Md.
HAWTHORNE, FRANK, M.A. (Columbia) Instr., Hofstra Coll., Hempstead, N.Y. *40-C Broadfield Rd.*
HAY, G. E., Ph.D. (Toronto) Asso. Prof., Univ. of Michigan, Ann Arbor, Mich.
HAYES, C. A., JR., Ph.D. (California) Asst. Prof., Univ. of California at Davis, Davis, Calif.
HAYES, J. J., B.S. (Utah) Retired, Asst. Prof., Univ. of Utah, Salt Lake City, Utah. *1148 East First, South (2)*
HAYNES, NOLA L. A. (Mrs. E. S.), Ph.D. (Missouri) Asso. Prof., Univ. of Missouri, Columbia, Mo. *1408 Rosemary Lane*
HAYS, LENA L. (Mrs. A. Z.), M.E. (Texas Tech.) Instr., Abilene Christian Coll., Abilene, Tex.
HAYS, R. F., A.B. (Transylvania) Grad. Student, Univ. of Kentucky, Lexington, Ky. *310 Jefferson St., LaGrange, Ky.*
HAYWARD, LEOTA C. (Mrs. M. B.), M.S. (Colorado S.C.) Asst. Prof., Colorado A. and M. Coll., Fort Collins, Colo.
HAZARD, C. T., A.M. (Indiana) Prof., Purdue Univ., Lafayette, Ind. *344 N. Western Ave.*
HAZARD, KATHARINE E., Ph.D. (Chicago) Asst. Prof., New Jersey Coll. for Women, Rutgers Univ., New Brunswick, N.J.
HAZELTINE, B. A., A.M. (Columbia) Prof., Middlebury Coll., Middlebury, Vt.
HAZELTINE, L. A., A.M. (Columbia) Consultant. *15 Tower Dr., Maplewood, N.J.*
HAZLEWOOD, E. A., Ph.D. (Cornell) Prof., Texas Tech. Coll., Lubbock, Tex. *3218 20th St.*
H'DOUBLER, F. T., M.D. (Harvard) Surgeon. *406 Professional Bldg., Springfield, Mo.*
HEALY, P. W., Ph.D. (Kentucky) Asst. Prof., Univ. of New Mexico, Albuquerque, N.M.

HEARN, J. R., REV., A.M. (Woodstock) St. Joseph's Coll., Philadelphia 31, Pa.
HEASLET, M. A., Ph.D. (Stanford) Physicist, Dept. of Theoretical Aerodynamics, N.A.C.A., Moffett Field, Calif. *Box 402, Los Altos, Calif.*
HEATER, HELEN M., M.S. (West Virginia) Grad. Student, Univ. of Michigan, Ann Arbor, Mich. *407 Hamilton Pl.*
HEATH, MARY C., M.S. (Kentucky) Instr., Florida State Univ., Tallahasse, Fla.
HEATH, R. E., B.A. (Hastings) Grad. Student, Univ. of Nebraska, Lincoln 8, Nebr. *200 W. 5th St., Hastings, Nebr.*
HEATH, R. V., Member, New York Stock Exchange, New York, N.Y. *40 Wall St.*
HEBEL, I. L., M.S. (Colorado) Prof., Colorado School of Mines, Golden, Colo.
HECKROTH, E. E., B.A. (Buffalo) Computer, Bell Aircraft Corp., Niagara Falls, N.Y. *388 Elmwood Ave., Buffalo 22, N.Y.*
HEDBERG, E. A., Ph.D. (Missouri) Prof., Univ. of South Carolina, Columbia, S.C.
HEDBERG, MARGUERITE Z. (Mrs. E. A.), Ph.D. (Missouri) Asso. Prof., Univ. of South Carolina, Columbia, S.C.
HEDLUND, G. A., Ph.D. (Harvard) Prof., Yale Univ., New Haven, Conn.
HEID, C. J., REV., M.L. (Pittsburgh) Instr., St. Vincent Coll., Latrobe, Pa. *St. Vincent Archabbey*
HEIGL, VIVIAN L. (Mrs.), M.A. (Wisconsin) *205 Ninth Ave., Menomonie, Wis.*
HEILMAN, C. E., A.M. (Duke) Asso. Prof., Elizabethtown Coll., Elizabethtown, Pa. *Main and Chocolate Sts., Florin, Pa.*
HEIMANN, E. E., A.M. (Texas) Prof., East Central State Coll., Ada, Okla. *523 S. Highland St.*
HEINEMAN, E. R., A.M. (Wisconsin) Prof., Texas Tech. Coll., Lubbock, Tex. *3015 21st St.*
HEINKE, C. H., A.M. (Ohio State) Asst. Prof., Capital Univ., Columbus 9, Ohio
HEINS, M. H., Ph.D. (Harvard) Prof., Brown Univ., Providence, R.I.
HEINZMAN, W. P., A.M. (Illinois) Prof., New Mexico Coll. of A. and M.A., State College, N.M. *On leave 1951–52:* Univ. of North Carolina, Chapel Hill, N.C.
HELLERSTEIN, SIMON, Student, Yeshiva Univ., New York, N.Y. *399 Hinsdale St., Brooklyn 7, N.Y.*
HELLMAN, M. J., M.S. (C.C.N.Y.) Instr., Newark Colleges, Rutgers Univ., Newark 2, N.J.
HELLMICH, E. W., Ph.D. (Columbia) Prof., Northern Illinois State Teachers Coll., DeKalb, Ill.
HELME, G. C., M.S. (Washington U.) Asst. Prof., Pratt Inst., Brooklyn, N.Y. *161 Emerson Pl. (5)*
HELMS, C. B., B.S. (Temple) Asst. Prof., Pennsylvania Military Coll., Chester, Pa.
HELMS, L. L., Student, Bradley Univ., Peoria, Ill. *237 Proctor St.*
HELSEL, R. G., Ph.D. (Ohio State) Prof., Ohio State Univ., Columbus 10, Ohio
HELTON, CLINTON, Student, Eastern Kentucky State Coll., Richmond, Ky. *Mount Vernon, Ky.*
HELTON, F. F., Ph.D. (Illinois) Prof., Central Coll., Fayette, Mo.
HEMENWAY, L. D., A.M. (Harvard) Prof., Simmons Coll., Boston, Mass.
HENDERSON, ARCHIBALD, Ph.D. (North Carolina; Chicago) Emeritus Prof., Univ. of North Carolina, Chapel Hill, N.C. *721 E. Franklin St.*
HENDERSON, G. P., Ph.D. (Toronto) Lecturer, Univ. of Western Ontario, London, Ont., Canada
HENDLER, A. S., M.A. (Columbia) Asst. Prof., Rensselaer Poly. Inst., Troy, N.Y. *126 Winter St.*
HENDRICKSON, M. S., Ph.D. (Ohio State) Asso. Prof., Univ. of New Mexico, Albuquerque, N.M.
HENDRIX, GERTRUDE, A.M. (Illinois) Asst. Prof., Eastern Illinois State Teachers Coll., Charleston, Ill. *1425 Fourth St.*
HENNESSEY, J. J., REV., A.M. (Woodstock) Prof., Woodstock Coll., Woodstock, Md.
HENNING, RUTH E., M.A. (Illinois) Instr., Virginia Jr. Coll., Virginia, Minn. *303 LaSalle Apt.*
HENRICK, J. J., Student, Texas Christian Univ., Ft. Worth, Tex. *2220 Hemphill St. (4)*
HENRIKSEN, MELVIN, Ph.D. (Wisconsin) Asst. Prof., Univ. of Alabama, University, Ala.
HENRIQUES, ANNA S. (Mrs. D. E.), Ph.D. (Chicago) Asso. Prof., Univ. of Utah, Salt Lake City, Utah
HERBST, A. F., M.A. (Maryland) Asst. Prof., LaVerne Coll., LaVerne, Calif.
HEREN, MABEL M., M.S. (Northwestern) Emeritus Prof., Knox Coll., Galesburg, Ill. *Hanover, Ind.*
HERLIHY, F. W., B.S. (Chicago) Vice-Pres., Herlihy Mid-Continent Co., Chicago, Ill. *Box 81, Comstock, Mich.*
HERPEL, COLEMAN, A.M. (Harvard) Asso. Prof., Pennsylvania State Coll., Altoona Undergrad. Center, Altoona, Pa. *212 Coleridge Ave.*
HERR, GERTRUDE A., M.S. (Iowa S.C.) Asso. Prof., Iowa State Coll., Ames, Iowa
HERRERA, R. B., A.M. (U.C.L.A.) Instr., Los Angeles City Coll., Los Angeles, Calif. *6011 Fair Ave., North Hollywood, Calif.*
HERRICK, H. L., M.S. (Iowa) Mathematician, International Business Machines Corp., New York, N.Y. *63 West Hudson Ave., Apt. B-16, Englewood, N.J.*
HERRIOT, J. G., Ph.D. (Brown) Asso. Prof., Stanford Univ., Stanford, Calif.
HERSH, N. K., M.A. (Kansas) Instr., Tulane Univ., New Orleans, La. *1821 Milan (15)*
HERSTEIN, I. N., Ph.D. (Indiana) Cowles Commission, Univ. of Chicago, Chicago 37, Ill.

HERTZ, H. G., Ph.D. (Yale) Asso. Astronomer, U. S. Naval Observatory, Washington, D.C.
HERTZIG, MORRIS, M.S. (C.C.N.Y.) *6620 108 St., Forest Hills, N.Y.*
HERWITZ, P. S., B.A. (Cincinnati) Grad. Fellow, Univ. of Cincinnati, Cincinnati 20, Ohio. *109 Plant Rd., Chapel Hill, N.C.*
HERZOG, FRITZ, Ph.D. (Columbia) Asso. Prof., Michigan State Coll., East Lansing, Mich.
HESS, G. W., Ph.D. (Michigan) Prof., Howard Coll., Birmingham, Ala. *8009 Fourth Ave. S.*
HESTENES, M. R., Ph.D. (Chicago) Prof., Univ. of California at Los Angeles, Los Angeles, Calif.
HESTER, D. M., B.A. (Southern Methodist) Asst. Prof., Baker Univ., Baldwin City, Kan.
HETZELT, LUCILLE F., A.M. (Syracuse) Instr., Syracuse Univ., Syracuse, N.Y. *416 Bellevue Ave.*
HEWITT, EDWIN, Ph.D. (Harvard) Asst. Prof., Univ. of Washington, Seattle 5, Wash. *On leave:* Visiting Lecturer, Royal Univ. of Upsala, Upsala, Sweden
HEYDA, J. F., Ph.D. (Illinois) Mathematician, Research Dept., U. S. Naval Ordnance Plant, Indianapolis, Ind.
HEYMAN, R. J., Student, Illinois Inst. of Tech., Chicago, Ill. *4039 W. Armitage Ave. (39)*
HIBBARD, WILBUR, M. A. (Columbia) Instr., City Coll. of New York, New York, N.Y. *18 South Dr., Lawrence Brook Manor, New Brunswick, N.J.*
HICKERSON, T. F., A.M. (North Carolina) Prof., Univ. of North Carolina, Chapel Hill, N.C. *108 Battle Lane*
HICKS, J. E., M.S. (DePaul) Teacher, Chicago Board of Education, Chicago, Ill. *6411 Evans Ave. (37)*
HICKSON, A. O., Ph.D. (Chicago) Asst. Prof., Duke Univ., Durham, N.C. *2712 Legion Ave.*
HIGGINS, F. V., M.S. (Michigan) Asst. Prof., Fenn Coll., Cleveland, Ohio. *2769 Lancashire, Apt. 3 (6)*
HIGGINS, SMITH, JR., M.S. in Ed. (Notre Dame) Instr., Indiana Univ., South Bend Extension, South Bend, Ind. *1734 Southeast Dr.*
HIGGINS, T. J., Ph.D. (Purdue) Prof., Electrical Engg., Univ. of Wisconsin, Madison, Wis. *227 Princeton Ave.*
HIGHTOWER, RUBY U., Ph.D. (Missouri) Emeritus Prof., Shorter Coll., Rome, Ga. *309 E. Fourth Ave.*
HILDEBRANDT, E. H. C., Ph.D. (Michigan) Asso. Prof., Northwestern Univ., Evanston, Ill.
HILDEBRANDT, MARTHA, M.S. (Chicago) Head of Dept., Proviso Township High School, Maywood, Ill. *808 S. Second Ave.*
HILDEBRANDT, T. H., Ph.D. (Chicago) Prof., Univ. of Michigan, Ann Arbor, Mich. *1930 Cambridge Rd.*
HILDNER, R. C., Ph.D. (Ohio State) Asso. Prof., Univ. of New Mexico, Albuquerque, N.M.
HILFERTY, J. C., Rev., B.A. (St. Charles) Instr., St. Thomas More High School, Philadelphia, Pa. *401 S. Chester Rd., Swarthmore, Pa.*
HILL, A. G., M.S. (Wisconsin) Prof., North Dakota Agr. Coll., State College, N.D.
HILL, E. Y., Student, Univ. of British Columbia, Vancouver, B.C., Canada. *2505 W. Broadway*
HILL, H. A., B.S. (Washington) *R.F.D. 1, Box 41, Freewater, Ore.*
HILL, J. D., Ph.D. (Brown) Prof., Michigan State Coll., East Lansing, Mich. *443 Grove St.*
HILL, J. STANLEY, Fellow (Soc. of Actuaries) Asso. Actuary, Minnesota Mutual Life Insurance Co., 156 E. 6th St., St. Paul, Minn.
HILL, L. C., B.S. in Ed. (Miami U.) Grad. Student, Miami Univ., Oxford, Ohio. *705 Rhea Ave., Hamilton, Ohio*
HILL, L. S., Ph.D. (Yale) Asso. Prof., Hunter Coll., New York, N.Y. *22 Sagamore Rd., Bronxville 8, N.Y.*
HILL, M. A., JR., A.M. (North Carolina) Asso. Dean, Univ. of North Carolina, Chapel Hill, N.C. *333 Tenney Circle*
HILL, P. R., M.S. (Georgia) Asso. Prof., Univ. of Georgia, Athens, Ga. *190 Morton Ave.*
HILL, R. G., B.A. (Buffalo) Grad. Student, Univ. of Chicago, Chicago 37, Ill. *Room 52, Snell House, 5709 Ellis Ave.*
HILLE, EINAR, Ph.D. (Stockholm) Prof., Yale Univ., New Haven, Conn. *72 Edgehill Rd. (11)*
HILTON, H. K., M.A. (Northwestern) Asst. Manager, Weisz Delcalcomania, Chicago 6, Ill. *3822 N. Damen Ave. (18)*
HIND, A. T., JR., M.A. (Emory U.) *170 W. View Dr., Athens, Ga.*
HINMAN, H. H., A.M. (Columbia) Asst. Prof., City Coll. of New York, New York 31, N.Y.
HINRICHS, P. A., Student, Hofstra Coll., Hempstead, N.Y. *298 Bedford Ave., Uniondale, Hempstead, N.Y.*
HINRICHSEN, J. J. L., Ph.D. (Harvard) Asso. Prof., Iowa State Coll., Ames, Iowa
HIRSCH, W. M., M.S. (N.Y.U.) Inst. for Mathematics, New York Univ., New York, N.Y. *2791 University Ave. (63)*
HIRSCHMAN, I. I., JR., Ph.D. (Harvard) Asst. Prof., Washington Univ., St. Louis, Mo. *7396 Kingsbury Blvd. (5)*
HITCHCOCK, E. L., B.S. (Middle Tennessee) Compounder, Goodyear Tire & Rubber Co., Akron, Ohio. *284 Wheeler St. (4)*

HLAVATY, J. H., Ph.D. (Columbia) Head of Dept., Bronx High School of Science, 184 St. and Creston Ave., New York, N.Y. *250 Coligni Ave., New Rochelle, N.Y.*
HNATH, M. W., B.Ed. (Duquesne) Grad. Asst., Duquesne Univ., Pittsburgh 19, Pa.
HOAG, JESSIE M., M.A. (Alabama) Asst. Prof., Southwestern Louisiana Inst., Lafayette, La. *140 Clark Ct.*
HOAGLAND, D. B., M.A. (Missouri) Instr., Univ. of Missouri, Columbia, Mo. *1621 Hinkson Ave.*
HOARE, A. J., D.Sc. (Fairmount C.) Emeritus Prof., Univ. of Wichita, Wichita, Kan. *1717 N. Holyoke Ave.*
HOBBS, R. P., A.B. (Dartmouth) Vice-President, Rinehart and Co., Inc., 232 Madison Ave., New York 16, N.Y.
HOBBS, W. C., M.S. (Howard) Head of Dept., Morris Brown Coll., Atlanta, Ga.
HODGE, P. G., JR., Ph.D. (Brown) Asst. Prof., Univ. of California at Los Angeles, Los Angeles 24, Calif.
HODGES, J. R., B.S. (Tulane) Instr., Little Rock Junior Coll., Little Rock, Ark. *722 Battery, Apt. 6.*
HOEL, P. G., Ph.D. (Minnesota) Prof., Univ. of California at Los Angeles, Los Angeles 24, Calif.
HOERSCH, V. A., Ph.D. (Iowa) Asst. Prof., Univ. of Illinois, Urbana, Ill. *909 S. First St., Apt. 8, Champaign, Ill.*
HOFFMAN, A. J., Ph.D. (Columbia) Member, Applied Math. Labs., National Bureau of Standards, Washington 25, D.C.
HOFFMAN, H. E., B.A. (Buffalo) Grad. Student, Univ. of Oklahoma, Norman, Okla. *318½ W. Daws*
HOFFMAN, HELEN P., M.S. (Illinois) *Address Unknown*
HOFFMAN, JANICE M., B.S.Ed. (Bowling Green) Teacher, Seacrest High School, Delray Beach, Fla. *306 N. Swinton St.*
HOFFMAN, MARJORIE L. (Mrs.), M.A. (Stanford) Teacher, San Mateo Junior Coll., San Mateo, Calif. *56 Tilton Terr.*
HOFFMAN, RUTH I., A.M. (Colorado) Dean, Byers Junior High School, Denver, Colo. *338 Fox St. (9)*
HOGG, R. V., Ph.D. (Iowa) Asst. Prof., State Univ. of Iowa, Iowa City, Iowa
HOGGATT, V. E., JR., M.A. (Oregon S.C.) Teaching Fellow, Oregon State Coll., Corvallis, Ore.
HOHN, F. E., Ph.D. (Illinois) Asst. Prof., Univ. of Illinois, Urbana, Ill.
HOKANSON, J. S., A.M. (Michigan) Acting Instr., Ohio Univ., Athens, Ohio
HOKE, O. H., M.A. (North Carolina) Grad. Student, Univ. of Georgia, Athens, Ga.
HOLL, D. L., Ph.D. (Chicago) Prof., Iowa State Coll., Ames, Iowa
HOLLAND, L. R., M.S. (Oklahoma A. and M.) Prof., Eastern Oklahoma A. and M. Coll., Wilburton, Okla.
HOLLCROFT, T. R., Ph.D. (Cornell) Prof., Wells Coll., Aurora, N.Y.
HOLMES, C. T., Ph.D. (Harvard) Prof., Bowdoin Coll., Brunswick, Maine. *60 Spring St.*
HOLMES, C. V., M.A. (Mississippi) Asst. Prof., Murray State Coll., Murray, Ky. *702 W. Park St., Urbana, Ill.*
HOLT, E. W., M.S. (Chicago) Muhlenberg Coll., Allentown, Pa. *108 S. Chelsea Lane, Parkway Manor, R. 3, Allentown, Pa.*
HOLT, H. K., M.A. (Ohio State) Asst. Prof., Union Coll., Schenectady, N.Y. *Apt. 68, Dutchmen's Village*
HOLTOM, CARL, Ph.D. (Chicago) Asso. Prof., U. S. Air Force Inst. of Tech., Wright-Patterson Air Force Base, Dayton, Ohio
HOLTON, C. H., A.M. (Harvard) Asso. Prof., Georgia Inst. of Tech., Atlanta, Ga. *167 Fourth St., N.W.*
HOLUBAR, LEROY, B.S.E.E. (Colorado) Instr., Univ. of Colorado, Boulder, Colo. *1215 Grandview Ave.*
HOLZHAUER, F. X., B.S. (Detroit) Teaching Fellow, Univ. of Detroit, Detroit, Mich. *4768 Baldwin (13)*
HOLZINGER, J. R., M.S. (Cornell) Asst. Prof., Franklin and Marshall Coll., Lancaster, Pa. *203 Ruby St.*
HONIG, JULIUS, M.S. (Illinois) Research Analyst, Dept. of Defense, Washington 25, D.C. *3432 Lowell St., N.W. (16)*
HOOD, R. T., Ph.D. (Wisconsin) Pastor, Lincoln Baptist Church, R.D. 2, Macedon, N.Y.
HOOK, C. W., A.M. (North Carolina) Asso. Prof., Georgia Inst. of Tech., Atlanta, Ga. *616 Williams St., N.W.*
HOOKE, ROBERT, Ph.D. (Princeton) M.I.T. Dept. of Industrial Cooperation, Operations Evaluation Group, Washington, D.C. *2730 S. Cleveland St., Arlington, Va.*
HOOPER, DOROTHY S. (Mrs. W. J.), M.A. (California) Prof., The Principia Coll., Elsah, Ill.
HOOPS, R. R., County Engr., Perry County, New Lexington, Ohio
HOOVER, B. P., Ph.D. (Illinois) Asso. Prof., Carnegie Inst. of Tech., Pittsburgh, Pa. *2040 Fairlawn Ave. (21)*

HOOVER, N. C., M.A. (Kansas) Instr., Univ. of South Dakota, Vermillion, S.D. *18 Forrest*
HOPKINS, FANNIE, A.M. (Wisconsin) Teacher, High School, Waukesha, Wis. *138 S. East St.*
HOPKINS, L. A., Ph.D. (Chicago) Emeritus Asso. Prof., Univ. of Michigan, Ann Arbor, Mich.
HOPKINS, R. H., A.M. (Denver) Instr., Mississippi State Coll., State College, Miss.
HORENI, HILDEGARDE, M.A. (Minnesota) Instr., State Teachers Coll., Mankato, Minn. *330 Center St.*
HORNACEK, ROSE L., M.A. (Loyola U.) Instr., Univ. of Illinois, Navy Pier, Chicago, Ill. *3932 W. Fillmore St. (24)*
HORNE, B. C., JR., A.M. (North Carolina) Asst. Prof., Virginia Poly. Inst., Blacksburg, Va.
HORNE, J. G., JR., M.S. (Tulane) Instr., Tulane Univ., New Orleans, La. *2508 Napoleon Ave.*
HORSFALL, I. O., Ph.D. (Cornell) Emeritus Director, Extension Div., Univ. of Utah, Salt Lake City, Utah. *82 Virginia St. (3)*
HORTON, R. E., M.A. (U.C.L.A.) Lt. Col., U. S. Air Force
HORTON, T. R., M.S. (Florida) Asst. Commandant, Bolles School, Jacksonville, Fla.
HORWITZ, DAVID, Student, Illinois Inst. of Tech., Chicago, Ill. *4538 S. Woodlawn (15)*
HOSKINS, R. H., A.B. (Harvard) Actuarial Asso., Group Underwriting Dept., John Hancock Mutual Life Insurance Co., Boston, Mass.
HOSTETTER, I. M., Ph.D. (Washington) Asso. Prof., Oregon State Coll., Corvallis, Ore. *1398 Alta Vista Dr.*
HOSTINSKY, L. AILEEN, Ph.D. (Illinois) Asst. Prof., Temple Univ., Philadelphia 22, Pa.
HOUGHTON, D. B., A.M. (Michigan) Acting Chief, Analysis Section, Franklin Inst., Philadelphia 3, Pa. *294 Western Way, Princeton, N.J.*
HOUSEHOLDER, A. S., Ph.D. (Chicago) Principal Physicist, Oak Ridge National Lab., Oak Ridge, Tenn.
HOUSER, D. M., M.S. (Kansas S.C.) Mathematician, Electrical Staff Dept., Boeing Airplane Co., Wichita, Kan. *2720 E. 9th St., Apt. A*
HOUSTON, J. D., JR., B.S. (Clark C.) Teacher, Westside Senior High School, Talladega, Ala. *1337 Douglas St., S.W., Atlanta, Ga.*
HOVE, E. MARIE, M.S. (Iowa) Asst. Prof., Hofstra Coll., Hempstead, N.Y.
HOVEY, B. K., Ph.D. (Göttingen) Asso. Prof., Elec. Engg., Univ. of Pittsburgh, Pittsburgh 13, Pa.
HOWARD, AUGHTUM S., (Mrs. N. J.), Ph.D. (Kentucky) Prof., Kentucky Wesleyan Coll., Owensboro, Ky.
HOWARD, C. M., E.Mines (Alabama Poly.) Prof., North Texas Agr. Coll., Arlington, Tex. *605 S. Center St.*
HOWARD, L. N., Student, Graduate Coll., Princeton Univ., Princeton, N.J.
HOWCROFT, J. G., M.S. (West Virginia) Instr., Waynesburg Coll., Waynesburg, Pa. *659 Madison Ave., Morgantown 1, W. Va.*
HOWE, ANNA M., Ph.D. (Cornell) Prof., Cazenovia Junior Coll., Cazenovia, N.Y.
HOWE, G. K., B.S. (Worcester Poly.) Retired. *359 Fifth St., N.W., Atlanta, Ga.*
HOWE, T. D., JR., B.S. (Harvard) President, T. D. Howe Construction Co., Houston, Tex. *107 H. Memorial Dr. (7)*
HOWELL, J. M., M.A. (U.C.L.A.) Instr., Los Angeles City Coll., Los Angeles, Calif. *4140 W. 63rd St. (43)*
HOWELL, J. V., A.M. (North Carolina) Head of Dept., Mars Hill Coll., Mars Hill, N.C.
HOWELL, S. W., A.M. (South Dakota) Instr., Yankton Coll., Yankton, S.D.
HOWERTON, R. J., M.S. (Northwestern) Acting Head of Dept., Regis Coll., Denver, Colo. *1227 Milwaukee St.*
HOWIE, J. M., A.M. (Nebraska) Emeritus Prof., Nebraska Wesleyan Univ., Lincoln, Nebr. *1350 Ogden St., Denver 3, Colo.*
HOWLAND, L. A., Ph.D. (Munich) Emeritus Dean, Wesleyan Univ., Middletown, Conn. *29 Gordon Pl.*
HOYLE, B. O., M.S. (Southern Illinois U.) *29 E. 14th Ave., Columbus, Ohio*
HOYLE, H. B., JR., A.M. (North Carolina) Prof., Queens Coll., Charlotte, N.C. *1907 Vail Ave., Apt 3 (7)*
HOYLE, V. A., Ph.D. (Princeton) Prof., Univ. of North Carolina, Chapel Hill, N.C.
HRATZ, J. A., Rev., B.S. (Iowa) Asst. Prof., St. Ambrose Coll., Davenport, Iowa
HSIUNG, C. C., Ph.D. (Michigan) Lecturer, Northwestern Univ., Evanston, Ill.
HSU, K. C., M.S. (Kansas) Grad. Asst., Univ. of Kansas, Lawrence, Kan. *1334 Ohio St.*
HU, S. T., Ph.D. (Manchester) Asso. Prof., Tulane Univ., New Orleans, La. *Inst. for Advanced Study, Princeton, N.J.*
HUBBARD, J. F., Ed.M. (Boston Teachers) J. H. Goddard & Co., Boston, Mass. *346 Commonwealth Ave.*
HUBBS, H. N., Ph.D. (Cornell) Prof., Hobart and William Smith Colls., Geneva, N.Y. *538 S. Main St.*

HUBERT, W. G., Sc.D. (N.Y.U.) Asso. Prof., City Coll. of New York, New York, N.Y. *125 Lee Ave., Yonkers 5, N.Y.*
HUCK, RAYMOND, M.S. (Illinois) *1211 Winston Ave., Baltimore 12, Md.*
HUDSON, J. F., B.S. (Tennessee) Statistical Asst., Carbide and Carbon Chem. Corp., P.O. Box P, Oak Ridge, Tenn. *3926 Greenleaf Ave., Knoxville, Tenn.*
HUESTON, R. A., JR., M.A. (Brown) Instr., Chauncy Hall School, Boston, Mass. *329 Commonwealth Ave. (15)*
HUFF, C. W., M.S. (South Carolina) Teaching Fellow, Tulane Univ., New Orleans 18, La.
HUFF, G. B., Ph.D. (Illinois) Prof., Univ. of Georgia, Athens, Ga. *160 Greenwood Ct.*
HUFF, W. N., Ph.D. (Pennsylvania) Asst. Prof., Univ. of Oklahoma, Norman, Okla.
HUFFER, R. C., Ph.D. (Chicago) Prof., Beloit Coll., Beloit, Wis. *729 Hobart Pl.*
HUGGINS, MARY T. (Mrs. R. A.), A.B. (Vassar) Acting Instr., Stanford Univ., Stanford, Calif. *152 Willow Rd., Palo Alto, Calif.*
HUGGINS, MIRIAM O., A.B. (North Carolina) Teacher Grainger High School, Kinston, N.C. *112 E. Capitola Ave.*
HUGHES, H. K., Ph.D. (Michigan) Prof., Purdue Univ., Lafayette, Ind.
HUGHES, H. M., Ph.D. (California) Instr., Univ. of California, Berkeley, Calif. *1454 Bancroft Way (2)*
HUGHES, N. R., A.B. (Wabash) Instr., Wabash Coll., Crawfordsville, Ind. *R.R. 7*
HULL, RALPH, Ph.D. (Chicago) Prof., Purdue Univ., Lafayette, Ind.
HULL, S. L., M.S. (Iowa) Research Participant, Oak Ridge National Lab., Oak Ridge, Tenn. *107 Principia Lane, Apt. 102*
HULLINGHORST, D. W., B.S. (Tulane) Hydr. Engr., Civil Works Investigation Project, Army Base, Oakland, Calif. *2428 Dowling Pl., Berkeley 5, Calif.*
HUMMEL, J. A., B.S. (C.I.T.) U. S. Army. *50 Leroy Pl., Red Bank, N.J.*
HUMMEL, P. M., Ph.D. (Ohio State) Prof., Univ. of Alabama, University, Ala.
HUMPHREY, H. K., M.S. in E.E. (Union) President, Winnetka Trust and Savings Bank, Winnetka, Ill.
HUMPHREYS, M. GWENETH, Ph.D. (Chicago) Prof., Randolph-Macon Woman's Coll., Lynchburg, Va.
HUMPHREYS, T. R., M.A. (Oregon) Asst. Prof., New Jersey State Teachers Coll., Upper Montclair, N.J. *9 Bellaire Dr., Montclair, N.J.*
HUNDERTMARK, ELAINE, A.M. (Illinois) Grad. Student, Univ. of Illinois, Urbana, Ill. *714 W. Iowa St.*
HUNEKE, H. V., A.M. (Oklahoma) Instr., Univ. of Oklahoma, Norman, Okla. *604 Stinson, Apt. C*
HUNSAKER, N. C., Ph.D. (Rice) Asso. Prof., Utah State Agr. Coll., Logan, Utah
HUNT, BURROWES, Ph.D. (Colorado) Grad. Student, Univ. of Pennsylvania, Philadelphia, Pa. *746 Concord Ave., Drexel Hill, Pa.*
HUNT, G. H., C. E. (Cornell) Emeritus Asst. Prof., Univ. of California at Los Angeles, Los Angeles 24, Calif. *237 Tavistock Ave. (49)*
HUNT, MILDRED, Ph.D. (Chicago) Prof., Illinois Wesleyan Univ., Bloomington, Ill. *406 E. Walnut St.*
HUNTER, LOUISE S. (Mrs.), Ed.M. (Harvard) Asso. Prof., Virginia State Coll., Petersburg, Va.
HUNTINGTON, E. V., Ph.D. (Strasburg) Emeritus Prof., Mech., Harvard Univ., Cambridge, Mass. *48 Highland St.*
HURD, C. C., Ph.D. (Illinois) Theoretical Research Engr., International Business Machines Corp., 590 Madison Ave., New York 22, N.Y.
HURRY, J. A., A.M. (California) Dir., Fundamental Research and Development, Gates Rubber Co., Denver, Colo. *630 S. Canosa Ct.*
HURST, J. W., Ph.D. (Illinois) Prof., Montana State Coll., Bozeman, Mont. *522 S. Sixth St.*
HURT, J. M., A.M. (Texas) Instr., Univ. of Texas, Austin, Tex. *902 Stark St.*
HURWITZ, SOLOMON, Ph.D. (Columbia) Asst. Prof., City Coll. of New York, New York, N.Y. *4014 Ave. I., Brooklyn 10, N.Y.*
HURWITZ, W. A., Ph.D. (Göttingen) Prof., Cornell Univ., Ithaca, N.Y.
HUSKEY, H. D., Ph.D. (Ohio State) Asst. Dir., Inst. for Numerical Analysis, National Bureau of Standards, Los Angeles 24, Calif. *2237 Manning Ave. (64)*
HUSTON, R. E., Ph.D. (Chicago) Prof., Rensselaer Poly. Inst., Troy, N.Y. *6 Broadview Terr.*
HUTCHERSON, W. R., Ph.D. (Cornell) Prof., Univ. of Florida, Gainesville, Fla. *1120 N.E. Third St.*
HUTCHINSON, C. A., A.M. (Wittenberg) Prof., Univ. of Colorado, Boulder, Colo.
HUTCHINSON, L. C., Ph.D. (M.I.T.) Asso. Prof., Poly. Inst. of Brooklyn, Brooklyn 2, N.Y.
HUTCHINSON, L. P., Ph.D. (Kentucky) Asst. Prof., The Citadel, Charleston, S.C.
HUTCHINSON, T. C., M.S. (Penn. State) Project Engr., Sperry Gyroscope Co., Great Neck, N.Y. *1014 Denton Ave., New Hyde Park, L.I., N.Y.*

HYDE, A. R., M.A. (Trinity) Mathematician, Emhart Mfg. Co., Hartford, Conn. *156 Woodrow St., West Hartford 7, Conn.*
HYDE, EMMA, A.M. (Chicago) Emeritus Asso. Prof., Kansas State Coll., Manhattan, Kan.
HYDEMAN, W. R., A.M. (Syracuse) Mathematician, Navy Dept. *Navy 100, (NA-7) F.P.O., P.M., New York, N.Y.*
HYDEN, J. A., Ph.D. (Cornell) Prof., Vanderbilt Univ., Nashville 4, Tenn.
HYDER, GRACE M., B.S. (Massachusetts) Senior Actuarial Clerk, Commonwealth of Massachusetts, Dept. of Banking & Insurance, Boston, Mass. *651 Prospect St., Methuen, Mass.*
HYERS, D. H., Ph.D. (C.I.T.) Prof., Univ. of Southern California, Los Angeles 7, Calif.
HYMAN, M. A., M.A. (Maryland) Fulbright Scholar, Technical Univ., Delft, The Netherlands. *4021 Ninth St., N.W., Washington 11, D.C.*

IKENBERRY, ERNEST, Ph.D. (Louisiana) Asst. Prof., Alabama Poly. Inst., Auburn, Ala.
IKENBERRY, J. E., Ph.D. (Cornell) Prof., Madison Coll., Harrisonburg, Va.
INGALLS, E. E., Ph.D. (Michigan) Prof., Albion Coll., Albion, Mich. *614 Michigan Ave.*
INGERSOLL, JANE C., M.A. (Michigan) Research Asst., Los Alamos Scientific Lab., Los Alamos, N.M.
INGLE, R. G., A.B. (Bethany-Peniel C.) Vibration Engr., Chance Vought Aircraft, Dallas, Tex. *4322 Pensacola St. (11)*
INGRAHAM, M. H., Ph.D. (Chicago) Dean, Coll. of Letters and Sci., Univ. of Wisconsin, Madison, Wis.
INGRAM, W. H., M.S. (Cantab.) *150 Claremont Ave., New York 27, N.Y.*
INNIS, MARY E., A.M. (Smith) Teacher, The Helen Bush School, Seattle, Wash. *1711 36th Ave. (22)*
IRWIN, H. H., A.M. (Washington S.C.) Prof., State Coll. of Washington, Pullman, Wash.
ISAACS, RUFUS, Ph.D. (Columbia) *214 S. Medio Dr., Los Angeles 49, Calif.*
ISAACSON, EUGENE, Ph.D. (N.Y.U.) Asst. Prof., Inst. for Mathematics and Mechanics, New York Univ., New York, N.Y. *35-36 172nd St., Flushing 58, N.Y.*
ISHIHARA, JIRO, M.S. (Northwestern) Grad. Student, Northwestern Univ., Evanston, Ill. *2029 N. Sheffield Ave., Chicago 14, Ill.*
ITKEN, KARL, M.S. (George Washington) Administrative Officer, Rural Electrification Administration, Washington, D.C. *908 Newhall St., Silver Spring, Md.*
ITO, W. H., B.S. (Illinois Tech.) Instr., Univ. of Minnesota, Minneapolis 14, Minn.
ITURRALDE, VERBA M. (Mrs.), B.S. (Roanoke C.) Teacher, El Paso Public Schools, El Paso, Tex. *519 N. Santa Fe St.*
IVANOFF, J. M., B.E. (Gonzaga) *c/o Mr. Ben Ivanoff, Box 522, Wapato, Wash.*
IVANOFF, V. F., A.M. (California) *2070 Fell St., Apt. 6, San Francisco 17, Calif.*
IVERSON, P. F., A.B. (Hastings) Head of Dept., Potomac State School of West Virginia Univ., Keyser, W.Va.
IWANCHUK, R. Y., M.A. (Columbia) Asst. Prof., Kent State Univ., Kent, Ohio. *150 S. Mantua St.*

JABLONOWER, JOSEPH, Pd.M. (N.Y.U.) Member, Board of Examiners of the City of New York' New York, N.Y. *110 Livingston St., Brooklyn 2, N.Y.*
JACKSON, J. B., A.M. (Columbia) Dean of Men, Univ. of South Carolina, Columbia, S.C. *227 S. Waccamaw Ave. (5)*
JACKSON, ROSA L., Ph.D. (Chicago) Prof., Alabama Coll., Montevallo, Ala. *211 Moody St.*
JACKSON, S. B., Ph.D. (Harvard) Prof., Univ. of Maryland, College Park, Md.
JACOBSEN, R. S., M.S. (Iowa State) Asso. Prof., Luther Coll., Decorah, Iowa. *602 North St.*
JACOBSON, N. L., M.S. (Iowa) Instr., Graceland Coll., Lamoni, Iowa
JAEGER, C. G., Ph.D. (Missouri) Prof., Pomona Coll., Claremont, Calif. *1045 Yale Ave.*
JAEGER, G. H., M.A. (Minnesota) Asst. Prof., Macalester Coll., St. Paul, Minn.
JAEGER, R. J., JR., Student, Hofstra Coll., Hempstead, N.Y. *336 Garfield Ave., West Hempstead, N.Y.*
JAFFE, W. J., M.S. in I.E. (Columbia) Asso. Prof., Newark Coll. of Engineering, Newark 2, N.J. *, 1030 Anderson Ave., Palisade, N.J.*
JAGOE, EVA, B.A. (Alberta) Teacher, Calgary School Board, Calgary, Alta., Canada. *1431 26th St., W.*
JAMES, GLENN, Ph.D. (Columbia) Retired. *14068 Van Nuys Blvd., Pacoima, Calif.*
JAMES, H. A., M.S. (DePaul) Instr., Wilson Junior Coll., Chicago, Ill. *1709 N. Richmond St. (47)*
JAMES, R. D., Ph.D. (Chicago) Prof., Univ. of British Columbia, Vancouver, B.C., Canada
JAMESON, W. J., Student, Montana State Univ., Missoula, Mont. *67 Park Hill Rd., Billings, Mont.*
JAMISON, FREE, Ph.D. (Pittsburgh) Asso. Prof., San Jose State Coll., San Jose, Calif. *310 S. 15th St.*
JAMISON, G. H., M.S. (Chicago) Prof., Northeast Missouri State Teachers Coll., Kirksville, Mo. *807 S. Halliburton*

JAMISON, S. L., Ph.D. (California) Asst. Prof., Florida State Univ., Tallahassee, Fla.
JANES, W. C., A.M. (Nebraska) Asso. Prof., Kansas State Coll., Manhattan, Kan.
JANSSEN, R. E., JR., M.A. (Illinois) *1604 Avenue A, Gothenburg, Nebr.*
JASPER, S. J., Ph.D. (Kentucky) Asso. Prof., East Tennessee State Coll., Johnson City, Tenn.
JASPER, T. M., M.S. (Texas Tech.) Asst. Prof., Arlington State Coll., Arlington, Tex.
JAUTZ, M. L., Rev., M.S. (Marquette) Instr., St. Francis Seminary, Milwaukee 7, Wis. *3600 S. Kinnickinnic Ave.*
JEEVES, T. A., A.B. (California) Lecturer, Univ. of California, Berkeley, Calif. *3244 California St. (3)*
JEFFERY, R. L., Ph.D. (Cornell) Prof., Queen's Univ., Kingston, Ont., Canada
JEHN, L. A., M.S. (Michigan) Asst. Prof., Univ. of Dayton, Dayton 9, Ohio
JENKE, C. W., B.S. (St. Mary's U.) Chemist, San Antonio Brewing Association, San Antonio, Tex. *230 E. Carson St. (8)*
JENKINS, E. D., Ph.D. (Ohio State) Asso. Prof., Kent State Univ., Kent, Ohio
JENKINS, ELEANOR L., M.A. (North Carolina) Asso. Prof., Queens Coll., Charlotte, N.C. *1521 E. Seventh St.*
JENNINGS, S. A., Ph.D. (Toronto) Prof., Univ. of British Columbia, Vancouver, B.C., Canada
JENNINGS, WALTER, A.M. (Ohio State) Asst. Prof., Naval Postgraduate School, Annapolis, Md.
JENSEN, C. M., Ph.D. (Minnesota) Asst. Prof., Augustana Coll., Rock Island, Ill.
JERBERT, A. R., Ph.D. (Washington) Asso. Prof., Univ. of Washington, Seattle, Wash.
JOFFE, S. A., M.S. (N.Y.U.) Retired Actuary, Mutual Life Insurance Co., New York, N.Y. *515 W. 110th St. (25)*
JOHANSON, R. N., Ph.D. (Chicago) Asso. Prof., Boston Univ., Boston, Mass.
JOHN, FRITZ, Ph.D. (Göttingen) Prof., New York Univ., New York 3, N.Y. *30 Summit Ave., New Rochelle, N.Y.*
JOHN, F. W., M.A. (Columbia) Prof., Washington Square Coll., New York Univ., New York, N.Y. *100 Washington Sq., E.*
JOHN, R. T., Rev., M.S. (Notre Dame) Instr., Crosier Seminary, Onamia, Minn.
JOHN, P. W. M., M.A. (Oxon) Instr., Univ. of Oklahoma, Norman, Okla.
JOHNSON, A. L., JR., A.B. (Nebraska Wesleyan) Secretary, Crete Mills, Crete, Nebr.
JOHNSON, BARBARA (Mrs. D. S.), A.B. (Brown) Research Asst., Johns Hopkins Inst. for Cooperative Research, Baltimore 18, Md. *3301 St. Paul St.*
JOHNSON, C. A., A.M. (Northwestern) Asst. Prof., Univ. of Missouri, School of Mines and Met., Rolla, Mo.
JOHNSON, D. A., A.M. (Minnesota) Instr., Education, Univ. of Minnesota, Minneapolis 14, Minn.
JOHNSON, EVAN, JR., Ph.D. (Chicago) Prof., Pennsylvania State Coll., State College, Pa.
JOHNSON, FAY H. (Mrs.), A.B. (Howard Payne) Instr., Howard Payne Coll., Brownwood, Tex.
JOHNSON, G. P., M.A. (Minnesota) Grad. Student, Univ. of Minnesota, Minneapolis 14, Minn. *621 Mount Curve Blvd., St. Paul 5, Minn.*
JOHNSON, L. G., A.M. (Michigan) Mathematician, General Motors Research Div., Detroit 2, Mich. *668 Spencer, Ferndale 20, Mich.*
JOHNSON, L. W., Ph.D. (Princeton) Prof., Oklahoma A. and M. Coll., Stillwater, Okla. *1816 Arrowhead Dr.*
JOHNSON, M. L., A.M. (Western Reserve) Asst. Prof., Kent State Univ., Kent, Ohio. *425 E. Summit St.*
JOHNSON, N. W., JR., Student, Carleton Coll., Northfield, Minn.
JOHNSON, O. B., JR., B.S. (Chicago Tech.) Grad. Student, Roosevelt Coll., Chicago 5, Ill. *5852 S. Michigan Ave. (37)*
JOHNSON, P. B., Ph.D. (C.I.T.) Asso. Prof., Occidental Coll., Los Angeles 41, Calif. *1799 Campus Rd.*
JOHNSON, PEGGY, (Mrs. G. P.), B.A. (Carleton) *Apt. 4, 2411 S. 26th St., Arlington, Va.*
JOHNSON, R. A., Ph.D. (Harvard) Prof., Brooklyn Coll., Brooklyn, N.Y.
JOHNSON, R. E., Ph.D. (Wisconsin) Asso. Prof., Smith Coll., Northampton, Mass.
JOHNSON, ROBERTA F., Ph.D. (Cornell) Asso. Prof., Wilson Coll., Chambersburg, Pa.
JOHNSON, R. P., Ph.D. (Pittsburgh) Asso. Prof., Carnegie Inst. of Tech., Pittsburgh, Pa.
JOHNSON, W. G., M.A. (Michigan) Grad. Student, Syracuse Univ., Syracuse 10, N.Y. *15 Gibbons St.*
JOHNSON, W. H., M.S. (N.Y.U.) Mathematician, International Business Machines, 590 Madison Ave., New York, N.Y. *Hulda Hill Rd., Wilton, Conn.*
JOHNSON, W. W., Instr., John Huntington Poly. Inst., Cleveland, Ohio. *14058 Superior Rd. (18)*
JOHNSTON, ERNEST, M.S. (Illinois) Asst. Prof., Univ. of Minnesota, Minneapolis 14, Minn. *3444 Grand Ave., S. (8)*
JOHNSTON, F. E., Ph.D. (Illinois) Prof., George Washington Univ., Washington 6, D.C.
JOHNSTON, L. S., A.M. (Missouri) Prof., Univ. of Detroit, Detroit 21, Mich.
JOHNSTON, R. H., B.S. in Ed. (Miami U.) Instr., Miami Univ., Oxford, Ohio. *130 W. Church St.*
JOHNSTON, R. M., M.A. (Tulsa) Teacher, Tulsa Public Schools, Tulsa, Okla. *3544 E. King Pl. (12)*

JOHNSTON, S. A., Ph.D. (Stanford) Prof., Western Washington Coll. of Education, Bellingham, Wash. *819 Indian St.*
JONAS, PHILIP, B.S. (Brooklyn) Instr., R.C.A. Institutes, New York, N.Y. *1120 Bergen St., Brooklyn 16, N.Y.*
JONES, AYRLENE M. (Mrs.), M.A. (Texas) Asst. Prof., Univ. of Alabama, University, Ala.
JONES, A. W., Asso. Prof., Rensselaer Poly. Inst., Troy, N.Y.
JONES, B. W., Ph.D. (Chicago) Prof., Univ. of Colorado, Boulder, Colo.
JONES, C. H., Student, *1178 King St., Charleston, S.C.*
JONES, E. F. W., C. E. (Manhattan C.) Prof., Rollins Coll., Winter Park, Fla.
JONES, F. B., Ph.D. (Texas) Prof., Univ. of North Carolina, Chapel Hill, N.C. *208 McCauley St.*
JONES, FRANCES W. (Mrs. A. P.), M.A. (North Carolina) Asst. Prof., Milwaukee-Downer Coll., Milwaukee, Wis. *918 E. Knapp, Apt. 1*
JONES, HARRIS, B.S. (M.I.T.) Brig. General and Dean, U. S. Military Acad., West Point, N.Y.
JONES, H. J., M.A. (Texas) Instr., El Camino Coll., El Camino College, Calif.
JONES, I. L., M.A. (Missouri) Retired Asso. Prof., Armstrong Coll., Berkeley 4, Calif. *2127 Derby St. (5)*
JONES, JOHN, JR., M.A. (Peabody) Asst. Prof., Mississippi Southern Coll., Hattiesburg, Miss. *148 Daniels Rd., Chapel Hill, N.C.*
JONES, K. R., M.S. (Michigan) *1041 Tennessee St., Lawrence, Kan.*
JONES, L. G., M.A. (Oregon) Mathematician, Physical Research Unit, Boeing Airplane Co., Seattle, Wash. *3906 14th Ave. S. (8)*
JONES, L. O., A.M. (Peabody) Prof., William Jewell Coll., Liberty, Mo. *478 E. Mill St.*
JONES, MARGARET, E., A.M. (Ohio State) Asst. Prof., Ohio State Univ., Columbus 10, Ohio. *164 13th Ave. (1)*
JONES, M. W., Ph.D. (Colorado) Asso. Prof., Adams State Coll., Alamosa, Colo.
JONES, P. C., B.S. (M.I.T.) Science Editor, Bell Telephone Labs., 463 West St., New York 14, N.Y.
JONES, P. S., Ph.D. (Michigan) Asst. Prof., Univ. of Michigan, Ann Arbor, Mich.
JONES, R. W., A.M. (Pennsylvania) Asst. Prof., Univ. of Delaware, Newark, Del. *121 Townsend Rd.*
JONES, W. P., M.S. (Iowa) Grad. Student, State Univ. of Iowa, Iowa City, Iowa. *P.O. Box 215, Mocksville, N.C.*
JORDAN, C. W., JR., B.A. (Williams) Asst. Prof., Williams Coll., Williamstown, Mass. *Stetson Ct.*
JORDAN, D. B., M.S. (Brooklyn Poly.) Junior Engr., Sylvania Electric Products, Inc., Bayside, L.I., N.Y. *109-15 211th St., Bellaire 9, N.Y.*
JORDAN, H. E., Ph.D. (Chicago) Emeritus Asso. Prof., Univ. of Kansas, Lawrence, Kan. *1600 Kentucky St.*
JORDAN, T. L., JR., M.A (Vanderbilt) Asso. Prof., Wofford Coll., Spartanburg, S.C. *112 Howard Aden Apts.*
JUELICH, O. C., B.A. (Hofstra) Asst., Ohio State Univ., Columbus 10, Ohio
JUSTICE, H. K., Ph.D. (Cincinnati) Asst. Dean, Engg., Univ. of Cincinnati, Cincinnati 21, Ohio. *927 Kreis Lane*
JUSTIS, F. E., M.S. (West Virginia) Asst. Prof., Geneva Coll., Beaver Falls, Pa. *1506–16th Ave.*

KAC, MARK, Ph.D. (John Casimir U., Lwow) Prof., Cornell Univ., Ithaca, N.Y.
KAELIN, G. R., A.M. (California) Instr., Los Angeles City Coll., Los Angeles 27, Calif.
KAISER, JOHN, M.A. (Ohio State) Asso. Prof., Kent State Univ., Kent, Ohio
KALDENBERG, G. E., M.S. (Iowa S.C.) Asst. Prof., Iowa State Coll., Ames, Iowa. *648 Pammel Ct.*
KALINOWSKI, W. C., Rev., Ph.D. (St. Louis) Asst. Prof., St. John's Univ., Collegeville, Minn.
KALISCH, G. K., Ph.D. (Chicago) Asso. Prof., Univ. of Minnesota, Minneapolis 14, Minn.
KALISH, AIDA, A.M. (Columbia) Instr., Poly. Inst. of Brooklyn, Brooklyn, N.Y. *420 Crown St. (25)*
KALLETT, M. W., A.B. (Buffalo) Teacher, North Park Junior High School, Lockport, N.Y. *200 Jewett Ave., Buffalo 14, N.Y.*
KALTENBORN, H. S., Ph.D. (Michigan) Prof., Memphis State Coll., Memphis 11, Tenn.
KAMEL, HYMAN, M.S. (N.Y.U.) Instr., Univ. of Pennsylvania, Philadelphia, Pa. *4109 Girard Ave. (4)*
KANARIK, ROSELLA (Mrs.), Ph.D. (Pittsburgh) Lecturer, Univ. of Southern California, Los Angeles, Calif. *238 S. Mansfield Ave. (36)*
KANE, ROSEMARY, Student, Seton Hill Coll., Greensburg, Pa. *25 Warren St., South Waverly, N.Y.*
KANIA, R. P., M.A. (New Zealand U.) Lecturer, Teachers' Training Coll. *25 Norwood Rd., Bayswater, Auckland N. 3, New Zealand*
KANTER, L. H., Ph.D. (Stanford) Asso. Prof., Mississippi State Coll., State College, Miss.
KAPLAN, SIDNEY, A.M. (Brooklyn C.) Res. Mathematician, Office of the Comptroller of the Army, Washington, D.C. *1201 Valley Ave., S.E., Apt. 102 (20)*

KAPLAN, WILFRED, Ph.D. (Harvard) Asso. Prof., Univ. of Michigan, Ann Arbor, Mich. *1308 Olivia Ave.*
KAPLANSKY, IRVING, Ph.D. (Harvard) Asst. Prof., Univ. of Chicago, Chicago 37, Ill.
KARLIN, SAMUEL, Ph.D. (Princeton) Asst. Prof., Princeton Univ., Princeton, N.J.
KARNES, H. T., Ph.D. (Peabody) Asso. Prof., Louisiana State Univ., Baton Rouge 3, La.
KARNS, C. W., M.A. (Northwestern) Asst., Northwestern Univ., Evanston, Ill. *1638 Roberts Lane, Falls Church, Va.*
KARPINSKI, L. C., Ph.D. (Strassburg) Emeritus Prof., Univ. of Michigan, Ann Arbor, Mich. *1315 Cambridge Rd.*
KARR, LOIS, A.M. (Wisconsin) Asso. Prof., Lindenwood Coll., St. Charles, Mo.
KARRASS, ABRAHAM, Student, Brooklyn Coll., Brooklyn, N.Y. *590 Lafayette Ave. (16)*
KASKEY, GILBERT, M.A. (Delaware) Instr., Univ. of Delaware, Newark, Del. *17 New St.*
KASNER, EDWARD, Ph.D. (Columbia) Emeritus Prof., Columbia Univ., New York 27, N.Y. *430 W. 116th St. (27)*
KASRIEL, R. H., B.S. (Tampa) Part-time Instr., Univ. of Tampa, Tampa, Fla. *105 Observatory Ave., Charlottesville, Va.*
KASSLER, RAYMOND, B.A. (Brooklyn C.) Mathematician, Evans Signal Lab., Belmar, N.J. *P.O. Box 657, Asbury Park, N.J.*
KATO, CHOSABURO, Ph.D. (Ohio State) Asso. Prof., Denison Univ., Granville, Ohio
KATZ, MAX, C.P.A., Katz, Zuckerman and Co., 50 Broad St., New York 4, N.Y.
KAUFMAN, C. J., Student, Columbia Univ., New York 27, N.Y. *220 Cabrini Blvd. (23)*
KAUFMAN, HYMAN, Ph.D. (McGill) Research Geophysicist, Laboratory for Electronics, Inc., 11 Leon St., Boston 15, Mass.
KAUFMANIS, KARLIS, M.S. (Latvia) Asst. Prof., Gustavus Adolphus Coll., St. Peter, Minn.
KAWATA, T. K., B.A. (Southwestern C.) Grad. Student, DePaul Univ., Chicago, Ill. *611 W. Barry (14)*
KAYS, G. W., M.A. (Montclair) Instr., State Teachers Coll., Montclair, N.J.
KEARNEY, DORA E., A.M. (Minnesota) *1105 W. 23rd St., Cedar Falls, Iowa*
KECK, WINFIELD, Ph.D. (Brown) Asst. Prof., Physics, Lafayette Coll., Easton, Pa. *325 Hamilton St.*
KEEFER, HELEN S. (Mrs. Norman), *570 Western Ave., Albany, N.Y.*
KEEGAN, EDWARD, B.S. (Kansas S.T.C.) Grad. Student, Kansas State Teachers Coll., Pittsburg, Kan. *713 W. 3rd St.*
KEELER, B. C., A.M. (Columbia) Prof., Webb Inst. of Naval Architecture, New York, N.Y. *48 Sagamore Rd., Bronxville, N.Y.*
KEELER, C. A., M.A. (California) Head of Dept., Multnomah Junior Coll., Portland 4, Ore.
KEFFER, RALPH, A.M. (Wisconsin) Actuary, Aetna Life Insurance Co., Hartford 15, Conn.
KEHL, W. B., A.M. (Harvard) Instr., Massachusetts Inst. of Tech., Cambridge, Mass. *15 Comeau St.*
KELLER, E. R., A.M. (Tennessee) Teacher, Central High School, Fountain City, Tenn. *Route 8, Maryville, Tenn.*
KELLER, M. W., Ph.D. (Indiana) Asso. Prof., Purdue Univ., West Lafayette, Ind. *518 Carrolton Blvd.*
KELLEY, C. E., M.A. (Missouri) Special Agent, New York Life Insurance Co., 51 Madison, New York, N.Y. *218 W. 26th St., Texarkana, Tex.*
KELLEY, J. L., Ph.D. (Virginia) Asso. Prof., Tulane Univ., New Orleans, La.
KELLS, L. M., Ph.D. (Columbia) Asso. Prof., U. S. Naval Acad., Annapolis, Md. *23 Thompson St.*
KELLY, J. B., Ph.D. (M.I.T.) Member, Inst. for Advanced Study, Princeton, N.J. *80-83 Grenfell Ave., Kew Gardens, N.Y.*
KELLY, K. D., M.S. (Chicago) Prof., Fenn Coll., Cleveland 15, Ohio. *14719 Clifton Blvd., Lakewood 7, Ohio*
KELLY, L. M., Ph.D. (Missouri) Asst. Prof., Michigan State Coll., East Lansing, Mich.
KELLY, P. J., Ph.D. (Wisconsin) Asst. Prof., Santa Barbara Coll., Santa Barbara, Calif. *2706 Mesa School Lane*
KEMPNER, A. J., Ph.D. (Göttingen) Emeritus Prof., Univ. of Colorado, Boulder, Colo. *956 Thirteenth St.*
KENDALL, CLARIBEL, Ph.D. (Chicago) Prof., Univ of Colorado, Boulder, Colo. *1305 Euclid Ave.*
KENNEDY, E. C., Ph.D. (Rice) Research Engr., Consolidated Vultee Aircraft Corp., Daingerfield, Tex. *Box 295, Hughes Springs, Tex.*
KENNEDY, E. S., Ph.D. (Lehigh) Asso. Prof., American Univ. of Beirut, Beirut, Lebanon
KENNEDY, EVELYN M., A.M. (Cincinnati) Training Specialist, Naval Gun Factory, Washington 25, D.C. *1857 Mintwood Pl., N.W. (9)*
KENNEDY, R. A., Student, Illinois Inst. of Tech., Chicago, Ill. *7033 N. Monon Ave. (30)*
KENNEDY, RUTH E., M.A. (Texas) Teacher, Neville High School, Monroe, La. *708 Stubbs Ave.*
KENNER, M. R., B.A. (Rochester) Grad. Student, Univ. of Minnesota, Minneapolis 14, Minn. *710 13th Ave., S.E.*

KENNEY, J. F., A.M. (Michigan) Asst. Prof., Univ. of Wisconsin at Milwaukee, Milwaukee 3, Wis.
KENNISON, L. S., Ph.D. (C.I.T.) Asst. Prof., Brooklyn Coll., Brooklyn, N.Y. *193-19 109th Rd., Hollis 7, N.Y.*
KENNY, B. C., B.S. (Bethany) Grad. Asst., Lehigh Univ., Bethlehem, Pa.
KENT, J. R. F., Ph.D. (Illinois) Asso. Prof., Harpur Coll., Endicott, N.Y.
KEPPLER, KATHARINE B. (Mrs. Kurt), A.M. (Bryn Mawr) Head of Dept., Foxcroft School, Middleburg, Va.
KERALLA, J. A., B.S. (William & Mary) Engr., Electric Storage Battery Co., Philadelphia, Pa. *6463 Ditman St. (35)*
KERNS, GENE, A.B. (Chico) *P.O. Box 535, Susanville, Calif.*
KERR, C. E., B.A. (La Salle C.) Grad. Asst., Univ. of Delaware, Newark, Del.
KERR, F. J., *Merrick St., Pleasantville, Pa.*
KETCHUM, P. W., Ph.D. (Illinois) Prof., Univ. of Illinois, Urbana, Ill. *803 W. Illinois St.*
KEYES, G. A, M.S. (Syracuse) Asso. Instr., U. S. Merchant Marine Acad., Kings Point, N.Y.
KEYFITZ, NATHAN, B.S. (McGill) Senior Research Statistician, Dominion Bureau of Statistics, Ottawa, Ont., Canada. *5 Bristol Ave.*
KIBBEY, D. E., Ph.D. (Illinois) Asso. Prof., Syracuse Univ., Syracuse 10, N.Y.
KIEFER, E. C., M.S. (Michigan) Prof., James Millikin Univ., Decatur 24, Ill.
KIELY, E. R., Ph.D. (Columbia) Prof., Iona Coll., New Rochelle, N.Y.
KIERNAN, P. J., A.M. (Columbia) Head of Dept., Lawrenceville School, Lawrenceville, N.J. *25-49 38th St., Long Island City, N.Y.*
KIEVAL, H. S., Ph.D. (Cincinnati) Instr., Brooklyn Coll., Brooklyn 10, N.Y.
KILLEBREW, A. J., M.S. (Auburn) Asso. Prof., State Teachers Coll., Livingston, Ala.
KILLEN, C. G., M.S. (Louisiana) Prof., Northwestern State Coll., Natchitoches, La. *429 Henry Ave.*
KIMBALL, S. H., Ph.D. (Harvard) Prof., Univ. of Maine, Orono, Maine
KIMBALL, T. C., A.M. (Princeton) Master, Lawrenceville School, Lawrenceville, N.J.
KIMBER, J. E., JR., Student, Massachusetts Inst. of Tech., Cambridge, Mass. *294 Westgate W. (39)*
KIMME, E. G., Student, Univ. of Redlands, Redlands, Calif. *3766 Gaviota Ave., Long Beach 7, Calif.*
KINGSBURY, R. S., M.S. (Wisconsin) Mathematician, Operations Research Staff, Arthur D. Little, Inc., Cambridge, Mass. *MacIntosh Lane, Lincoln, Mass.*
KINGSLEY, E. H., M.S. (Northwestern) Mathematician, Aerial Measurements Lab., Northwestern Univ. Tech. Inst. *10 W. Elm St., Chicago 10, Ill.*
KINGSTON, H. R., Ph.D. (Chicago) Dean, Arts and Sci., Univ. of Western Ontario, London, Ont., Canada
KINGSTON, J. M., Ph.D. (Toronto) Asst. Prof., Univ. of Washington, Seattle 5, Wash.
KINNEY, A. E., Ed.D. (Columbia) Asst. Prof., N.Y. State Maritime Acad., Fort Schuyler, New York 61, N.Y.
KINNEY, EVELYN K. (Mrs.), M.A. (Illinois) Asst. Prof., Illinois Wesleyan Univ., Bloomington, Ill.
KINNEY, J. M., Ph.D. (Chicago) Emeritus Prof., Wilson Junior Coll., Chicago, Ill. *8058 Bennett Ave. (17)*
KINSELLA, J. J., Ed.D. (Columbia) Asso. Prof., Education, New York Univ., New York, N.Y.
KINSMAN, BLAIR, B.S. (Chicago) Mathematician, Chesapeake Bay Inst., Johns Hopkins Univ., St. Margarets, Md. *42 East St., Annapolis, Md.*
KINSOLVING, H. L., A.M. (Harvard) Asst. Prof., U. S. Naval Acad., Annapolis, Md.
KIOKEMEISTER, F. L., Ph.D. (Wisconsin) Asso. Prof., Mt. Holyoke Coll., South Hadley, Mass.
KIRBY, A. R., A.M. (Columbia) Asst. Prof., Education, Fordham Univ., New York, N.Y. *2921 Briggs Ave. (58)*
KIRBY, C. A., A.B. (California) Chief Computer, Western Geophysical Co., Sacramento, Calif. *5200 Del Norte Blvd. (17)*
KIRCHNER, W. H., B.S. (Worcester Poly.) Emeritus Prof., Univ. of Minnesota, Minneapolis 14, Minn. *722 Tenth Ave., S.E.*
KIRKHAM, DON, Ph.D. (Columbia) Prof., Soil Physics, Iowa State Coll., Ames, Iowa
KIRKWOOD, C. E., JR., M.S. (Georgia) Asso. Prof., Clemson Coll., Clemson, S.C.
KIRMSER, P. G., M.S. (Minnesota) Instr., Univ. of Minnesota, Minneapolis 14, Minn. *62 W. Congress St., Apt. 2, St. Paul 7, Minn.*
KISSEL, RUTH (Mrs. H. F.), M.A. (Houston) Asst. Prof., Univ. of Houston, Houston, Tex. *3874 Wentworth (4)*
KISSIN, SIDNEY, B.S. (Brooklyn C.) Teaching Fellow, Univ. of Massachusetts, Amherst, Mass.
KITCHENS, NELLIE M., M.A. (Missouri) Head of Dept., Hickman High School, Columbia, Mo. *1507 Rosemary Lane*
KLAFTER, LEONARD, A.B. (George Washington) Mathematician, Army Map Service, Washington, D.C. *416 Franklin St., N.E. (17)*
KLAMKIN, M. S., M.S. (Brooklyn Poly.) Instr., Poly. Inst. of Brooklyn, Brooklyn, N.Y.

KLAUBER, L. M., LL.D. (U.C.L.A.) Chmn. of the Board, San Diego Gas and Electric Co., San Diego, Calif. *233 W. Juniper St. (1)*
KLEE, V. L., JR., Ph.D. (Virginia) National Research Fellow, Inst. for Advanced Study, Princeton, N.J.
KLEENE, S. C., Ph.D. (Princeton) Prof., Univ. of Wisconsin, Madison, Wis.
KLEIN, J. S., M.S. (M.I.T.) Instr., Williams Coll., Williamstown, Mass. *231 Bedford Rd., Pleasantville, N.Y.*
KLEIN, JOSEPH, B.S. (Rutgers) Grad. Student, Rutgers Univ., New Brunswick, N.J. *169 Meade Ave., Passaic, N.J.*
KLIMCZAK, W. J., Ph.D. (Yale) Asst. Prof., Trinity Coll., Hartford 6, Conn.
KLINE, J. R., Ph.D. (Pennsylvania) Prof., Univ. of Pennsylvania, Philadelphia, Pa. *529 Riverview Rd., Swarthmore, Pa.*
KLINE, MORRIS, Ph.D. (N.Y.U.) Asso. Prof., New York Univ., New York, N.Y. *1188 E. Eighth St., Brooklyn 30, N.Y.*
KLINGER, E. L., A.M. (Illinois) Asst. Prof., Purdue Univ., Lafayette, Ind. *126 North St.*
KLIPHARDT, R. A., B.S. (Armour Inst. of Tech.) Instr., Civil Engg., Northwestern Univ., Evanston, Ill.
KLIPPLE, E. C., Ph.D. (Texas) Asso. Prof., Texas A. and M. Coll., College Station, Tex.
KLOPP, PATIENCE B., B.A. (Seton Hill) Fellow, American Council of Learned Societies, Yale Univ., New Haven, Conn. *17 Hillhouse Ave.*
KLOTZ, CELIA E., M.A. (Nebraska) Instr., State Coll. of Washington, Pullman, Wash.
KNAEBEL, BONAVENTURE, Rev., M.S. (Catholic U.) Dean, St. Meinrad's Coll., St. Meinrad, Ind.
KNEALE, S. G., M.A. (Kansas) Student, Harvard Univ., Cambridge, Mass. *27 Dana St.*
KNEBELMAN, M. S., Ph.D. (Princeton) Prof., State Coll. of Washington, Pullman, Wash. *2008 Indiana St.*
KNEDLER, P. A., A.M. (Pennsylvania) Prof., State Teachers Coll., Kutztown, Pa. *East Texas, Pa.*
KNEER, A. R., Student, Gonzaga Univ., Spokane, Wash. *803 W. 12th Ave. (9)*
KNIGHT, L. C., JR., A.M. (Kent) Asst. Prof., Muskingum Coll., New Concord, Ohio. *156 Lakeside Dr.*
KNIGHT, W. S., B.S. (Georgia) Second Lt., U. S. Army. *Box 131, Warner Robins, Ga.*
KNIPP, J. C., Ph.D. (Pittsburgh) Asso. Prof., Univ. of Pittsburgh, Pittsburgh, Pa.
KNOWLER, L. A., Ph.D. (Iowa) Head of Dept., State Univ. of Iowa, Iowa City, Iowa
KNOX, J. J., M.S. (Chicago) Prof., Dakota Wesleyan Univ., Mitchell, S.D. *620 S. Sanborn St.*
KNOX, R. H., JR., A.M. (Michigan) Asso. Prof., Virginia Military Inst., Lexington, Va. *313 Letcher Ave.*
KNOX, S. R., M.A. (Mississippi) Asst. Prof., Millsaps Coll., Jackson, Miss.
KOCH, E. H., JR., B.S. in E.E. (Pennsylvania) Retired. *Ogontz Manor, Apt. F 61, Ogontz and Olney Aves., Philadelphia 41, Pa.*
KOCH, R. J., B.S. (Chicago) Grad. Fellow, Tulane Univ., New Orleans, La.
KOCHER, F. T., JR., M.A. (Penn. State) Instr., Pennsylvania State Coll., State College, Pa.
KOEHLER, FULTON, Ph.D. (Minnesota) Asst. Prof., Univ. of Minnesota, Minneapolis 14, Minn.
KOEHLER, T. L., A.M. (Pennsylvania) Asst. Prof., Muhlenberg Coll., Allentown, Pa. *625 N. 24th St.*
KOELLNER, W. G., B.A. (Montclair S.T.C.) Teacher, Hillside High School, Hillside, N.J. *192 Galloping Hill Rd., Roselle Park, N.J.*
KOENEN, W. B., Student, Macalester Coll., St. Paul, Minn. *1853 Princeton Ave. (5)*
KOGER, W. M., Student, Southern Methodist Univ., Dallas, Tex. *Route 1, Box 35-A, Lancaster, Tex.*
KOHLMEYER, R. J., M.A. (Washington U.) Instr., Lafayette Coll., Easton, Pa.
KOKEN, J. C., M.A. (Missouri) Grad. Student, Univ. of Illinois, Urbana, Ill. *1403 W. Park Ave., Champaign, Ill.*
KOKOMOOR, F. W., Ph.D. (Michigan) Prof., Univ. of Florida, Gainesville, Fla.
KOLODNER, I. I., Ph.D. (N.Y.U.) Research Asso., Inst. for Mathematics, New York Univ., New York 3, N.Y. *230 E. 7th St. (9)*
KONHAUSER, J. D. E., B.S. (Penn. State) Instr., Pennsylvania State Coll., State College, Pa.
KOREN, CHARLES, M.A. (Columbia) Head of Dept., Bayonne Junior Coll., Bayonne, N.J. *222 Chancellor Ave., Newark 8, N.J.*
KOREVAAR, JACOB, Ph.D. (Leiden, Netherlands) Prof., Inst. of Tech., Delft, Netherlands. *Gebouwen voor Wiskunde, Jaffalaan, Delft, Netherlands*
KORGEN, R. L., Ph.D. (Harvard) Asso. Prof., Bowdoin Coll., Brunswick, Maine. *Prince's Point Rd.*
KOTLER, DAVID, M.A. (Columbia) Asst. Prof., Champlain Coll., Plattsburg, N.Y. *111 Hurley Ave., Kingston, N.Y.*
KOVARIK, A. F., Sc.D. (Manchester) Emeritus Prof. Physics, Yale Univ., New Haven, Conn. *145 Cottage St. (1)*

KOVITZ, A. T., B.S. (Purdue) Instr., Poly. Inst. of Brooklyn, Brooklyn, N.Y. *804 Ave. C, Bayonne, N.J.*
KOWALEWSKI, F. P., JR., M.A. (Buffalo) *69 Marne Rd., Buffalo 15, N.Y.*
KRABILL, D. M., Ph.D. (Ohio State) Prof., Bowling Green State Univ., Bowling Green, Ohio. *437 N. Prospect St.*
KRAFT, D. H., M.A. (Vanderbilt) Staff Mathematician, Wright-Patterson Air Force Base, Dayton, Ohio. *920 Seventh St., Huntington 1, W.Va.*
KRAL, I. H., B.S.R.E. (Indiana Tech.) Junior Engr., Bendix Aviation Research Labs., Detroit, Mich. *Route 2, Hobson Rd., Ft. Wayne, Ind.*
KRALL, H. L., Ph.D. (Brown) Prof., Pennsylvania State Coll., State College, Pa.
KRAMER, MAX, Ph.D. (Columbia) Asst. Prof., New Mexico Coll. of A. and M.A., State College, N.M.
KRAMER, S. B., B.S. (Brooklyn Poly.) Research Asst., Aeronautical Engg. and Applied Mech., Poly. Inst. of Brooklyn, Brooklyn 2, N.Y. *170 Bay 31 St. (14)*
KRAMER-LASSAR, EDNA E., Ph.D. (Columbia) Instr., Poly. Inst. of Brooklyn, Brooklyn, N.Y. *32 Lenox Rd. (26)*
KRANZER, H. C., Student, New York Univ., New York, N.Y. *2101 Creston Ave. (53)*
KRATHWOHL, W. C., Ph.D. (Chicago) Emeritus Prof., Inst. for Psychological Services, Illinois Inst. of Tech., Chicago 16, Ill.
KRATTIGER, J. T., A.M. (Southern Methodist) Asso. Prof., Southeastern State Coll., Durant, Okla.
KRAUS, G. R., M.S. (Carnegie) Head of Dept., Gannon Coll., Erie, Pa. *619 W. Ninth St.*
KRAVETZ, SAUL, Student, Harvard Univ., Cambridge 38, Mass. *3072 Brighton 13 St., Brooklyn 24, N.Y.*
KRAVITZ, SIDNEY, M.Aero E. (N.Y.U.) Electro-Mechanical Engr., Glenn L. Martin Co., Middle River, Md. *21 Oak Grove Dr., Baltimore 20, Md.*
KREIDER, O. C., Ph.D. (Iowa State) Asst. Prof., Iowa State Coll., Ames, Iowa
KRENTEL, W. D., M.S. (Oklahoma A.&M.) *805 N. Main, Apt. 8, San Antonio, Tex.*
KRETSCHMAR, E. L., JR., M.A.Ed. (Florida) Sub Postal Clerk, Bradenton Post Office, Bradenton, Fla. *1217 18th St., W.*
KRIEGSMAN, HELEN F., M.S. (Kansas S.T.C.) Instr., Kansas State Teachers Coll., Pittsburg, Kan. *210 E. Williams*
KROEGER, R. C., B.A. (Buffalo) Bell Aircraft Corp., Niagara Falls, N.Y. *165 University Ave., Buffalo 14, N.Y.*
KRONSBEIN, JOHN, Ph.D. (Leipzig) Head, Engg. Dept., Evansville Coll., Evansville, Ind.
KROS, F. A., M.S. (New Mexico A.&M.A.) Student, Univ. of Colorado, Boulder, Colo.
KRUEGER, R. L., Ph.D. (Marquette) Prof., Wittenberg Coll., Springfield, Ohio
KRUGER, OBERT, B.A. (Nebraska) Prof., St. John's Coll., Winfield, Kan.
KRUSE, A. H., A.B. (Kansas) Student, Univ. of Kansas, Lawrence, Kan. *314 W. 14th*
KRUSE, MERRY M. (Mrs.), B.S. (Iowa S.C.) Mathematician, Inst. for Numerical Analysis, Los Angeles 24, Calif. *3216 Sawtelle, Apt. 3 (34)*
KRUSE, W. E., A.M. (Columbia) Instr., St. Peter's Coll., Jersey City, N.J. *320 Webster Ave. (7)*
KRUSKAL, M. D., M.S. (N.Y.U.) Research Asso., Princeton Univ., Princeton, N.J. *15 W. 75th St., New York 23, N.Y.*
KRUSKAL, W. H., S.M. (Harvard) Committee on Statistics, Univ. of Chicago, Chicago 37, Ill.
KUBIS, J. F., Ph.D. (Fordham) Asso. Prof., Fordham Univ., New York, N.Y.
KUEBLER, R. R., JR., M.A. (Pennsylvania) Asso. Prof., Dickinson Coll., Carlisle, Pa. *212 N. Hanover St.*
KUHN, B. J., Rev., A.M. (St. Bonaventure) Prof., Siena Coll., Loudonville, N.Y.
KUHN, H. W., Ph.D. (Cornell) Emeritus Prof., Ohio State Univ., Columbus, Ohio. *1179 Fairview Ave.*
KULLBACK, SOLOMON, Ph.D. (George Washington) Lecturer, George Washington Univ., Washington, D.C. *1259 Van Buren St., N.W. (12)*
KUNKEL, P. V., Ph.D. (Columbia) Prof., Cedar Crest Coll., Allentown, Pa. *Box 104, Trexlertown, Pa.*
KUPPERMAN, MORTON, M.A. (George Washington) Statistician, Office of the Surgeon General, Dept. of the Army, Washington 25, D.C. *3229 Buena Vista Terr., S.E. (20)*
KURLAND, CHARLES, B.A. (Buffalo) Research Engr., Bell Aircraft Corp., Niagara Falls, N.Y. *130-B Kenville Rd., Buffalo 15, N.Y.*
KURZIN, W. H., M.S. (Chicago) Instr., Chicago City Coll., Herzl Branch, Chicago, Ill.
KUSHNER, JOSEPH, Student, New York Univ., New York, N.Y. *283 E. 168th St. (56)*
KUSICK, HELENE G., A.B. (California) Teacher, Placer Union High School, Auburn, Calif.
KUTMAN, HELEN L., (Mrs.), A.M. (Columbia) Instr., Hunter Coll., New York 21, N.Y. *2825 Grand Concourse, Box 58*
KUZNETS, G. M., Ph.D. (California) Asso. Prof., Univ. of California, Berkeley 4, Calif.

KYLE, G. D., Ph.D. (Minnesota) Prof., Agricultural, Mechanical and Normal Coll., Pine Bluff, Ark.

LACY, E. PAULINE, Ed.M. (Harvard) Prof., William Woods Coll., Fulton, Mo.
LADNER, A. C., A.M. (Brown) Asst. Prof., Denison Univ., Granville, Ohio
LAFFERTY, W. A., A.M. (Ohio State) Instr., Northwest Missouri State Coll., Maryville, Mo. *609 S. Buchanan St.*
LAFON, J. E., A.M. (Oklahoma) Asst. Prof., Univ. of Oklahoma, Norman, Okla. *806 Monnett St.*
LAGERSTROM, P. A., Ph.D. (Princeton) Asso. Prof., Aeronautics, California Inst. of Tech., Pasadena 4, Calif.
LAGRONE, J. W., A.M. (Vanderbilt) Asso. Prof., Clemson Coll., Clemson, S.C.
LAIDLAW, W. M., Student, Willamette Univ., Salem, Ore. *1570 Jefferson St.*
LAIRD, L. E., M.S. (Kansas S.T.C.) Instr., State Teachers Coll., Emporia, Kan. *18 Sunflower Apts., 11th and Missouri, Lawrence, Kan.*
LAKE, K. E., M.A. (Kansas) *1614 Kentucky, Lawrence, Kan.*
LALONDE, R. P., Lab. Asst., R.C.A. Victor Co., Montreal, P.Q., Canada. *3910 Mentana St.*
LAMB, R. C., A.M. (Virginia) Prof., U. S. Naval Acad., Annapolis, Md. *149 Monticello Ave.*
LAMBERT, R. J., M.S. (Iowa) Instr., Iowa State Coll., Ames, Iowa. *1901 55th St., Des Moines, Iowa*
LAMBERT, W. D., A.M. (Harvard) Retired, U. S. Coast and Geodetic Survey. *Box 687, Canaan, Conn.*
LAMPEN, A. E., A.M. (Michigan) Prof., Hope Coll., Holland, Mich. *552 College Ave.*
LAMPLAND, C. O., A.M. (Indiana) Lowell Observatory, Flagstaff, Ariz.
LANCASTER, E. R., M.A. (Wayne) Instr., Univ. of Detroit, Detroit, Mich. *12131 Woodward Ave., Highland Park, Mich.*
LANCASTER, O. E., Ph.D. (Harvard) Head, Applied Math. Branch, Bureau of Aeronautics, Navy Dept., Washington, D.C. *4219 Woodberry St., University Park, Md.*
LANCKTON, A. L., A.M. (Duke) *Indian Spring Rd., Rowayton, Conn.*
LANCZOS, CORNELIUS, Ph.D. (Szeged) Inst. for Numerical Analysis, Los Angeles 24, Calif. *2570 Amherst Ave. (34)*
LANDAU, SEYMOUR, B.A. (N.Y.U.) Instr., Univ. of Connecticut, Storrs, Conn. *511 Cortelyou Rd., Brooklyn, N.Y.*
LANDERS, A. W., Ph.D. (Chicago) Asso. Prof., Brooklyn Coll., Brooklyn 10, N.Y. *108-48 67th Dr., Forest Hills 75, N.Y.*
LANDERS, MARY K. (Mrs. A. W.), Ph.D. (Chicago) Asst. Prof., Hunter Coll., New York 21, N.Y. *108-48 67th Dr., Forest Hills 75, N.Y.*
LANDES, G. N., B.A. (Wooster) Math. Analyst, Sandia Corp., Albuquerque, N.M. *2235 Wilbur Ave.*
LANDIN, JOSEPH, Ph.D. (Notre Dame) Asst. Prof., Univ. of Illinois, Urbana, Ill.
LANDRY, A. E., Ph.D. (Johns Hopkins) Prof., Catholic Univ. of America, Washington, D.C. *3624 13th St., N.E. (17)*
LANE, E. P., Ph.D. (Chicago) Prof., Univ. of Chicago, Chicago 37, Ill.
LANE, F. O., JR., B.S. (New Mexico) Grad. Student, Univ. of New Mexico, Albuquerque, N.M. *P.O. Box 361, Mountainair, N.M.*
LANE, F. W., M.S. (St. Bonaventure) Asst. Prof., Champlain Coll., Plattsburgh, N.Y.
LANE, H. I., Ph.D. (Cornell) Prof., Hendrix Coll., Conway, Ark.
LANE, N. D., Ph.D. (Toronto) Asst. Prof., Carleton Coll., Ottawa, Ont., Canada. *71 Ossington Ave.*
LANE, R. A. C., A.B. (Indiana) Grad. Asst., Lehigh Univ., Bethlehem, Pa.
LANE, RUTH O., Ph.D. (Iowa) Asso. Prof., Sam Houston State Teachers Coll., Hunstville, Tex.
LANG, G. B., Ph.D. (Illinois) Asso. Prof., Univ. of Florida, Gainesville, Fla. *1617 N.W. 10th Ave.*
LANGE, LUISE, Ph.D. (Göttingen) Wilson Junior Coll., Chicago, Ill. *5851 Blackstone Ave.*
LANGENHOP, C. E., Ph.D. (Iowa S.C.) Acting Dir., Analytical Research Group, Princeton Univ., Princeton, N.J. *227-C Marshall St.*
LANGER, R. E., Ph.D. (Harvard) Prof., Univ. of Wisconsin, Madison 6, Wis. *822 Miami Pass (5)*
LANKALIS, M. T., B.S.E. (S.T.C., Bridgewater, Mass.) Instr., New Hampshire Tech. Inst., Manchester, N.H.
LANKTON, ROBERT, M.A. (Wayne) Asst. Prof., Iowa State Teachers Coll., Cedar Falls, Iowa
LANZ, J. C., Sc.M. (Brown) Instr., Hershey Junior Coll., Hershey, Pa.
LANZANO, PAOLO, Ph.D. (Rome) Instr., St. Louis Univ., St. Louis, Mo.
LAPAZ, LINCOLN, Ph.D. (Chicago) Prof., Univ. of New Mexico, Albuquerque, N.M.
LAPIDUS, LEO, A.M. (Boston U.) Instr., Michigan State Coll., East Lansing, Mich.
LAPOSKY, B. F., Draftsman, Laposky Drafting Service. *301 S. 6th St., Cherokee, Iowa*
LAREW, GILLIE A., Ph.D. (Chicago) Prof., Randolph-Macon Woman's Coll., Lynchburg, Va.
LARGUIER, E. H., Ph.D. (Michigan) Prof., Spring Hill College, Spring Hill Station, Mobile, Ala.

LARIVIERE, ROSE, M.A. (Illinois) Instr., Univ. of Illinois, Navy Pier, Chicago 11, Ill.
LARNEY, VIOLET H. (Mrs. N. H.), Ph.D. (Wisconsin) Asst. Prof., Kansas State Coll., Manhattan, Kan.
LAROE, RACHAEL A., A.M. (Tennessee) Instr., Grand Canyon Coll., Prescott, Ariz.
LARRIVEE, J. A., Ph.D. (Catholic U.) Asso. Prof., Univ. of Vermont, Burlington, Vt.
LARSEN, H. D., Ph.D. (Wisconsin) Prof., Albion Coll., Albion, Mich. *310 Burr Oak St.*
LARSEN, L. M., M.A. (Nebraska) Prof., Kearney State Coll., Kearney, Neb. *3217 1st Ave.*
LARSON, D. B., A.B. (Houghton) Grad. Student, Univ. of Buffalo, Buffalo 14, N.Y. *74 Pfohl Pl., Williamsville, N.Y.*
LARSON, OLGA, A.M. (Missouri) Asso. Prof., State Coll. for Women, Tallahassee, Fla.
LARSSON, R.D., A.B. (Maine) Asst. Prof., Clarkson Coll. of Tech., Potsdam, N.Y.
LARUE, J. A., M.S. (West Virginia) Instr., Morris Harvey Coll., Charleston, W.Va. *2456 Lexington Ave., Columbus, Ohio*
LASALLE, J. P., Ph.D. (C.I.T.) U. S. Navy
LASLEY, J. W., JR., Ph.D. (Chicago) Prof., Univ. of North Carolina, Chapel Hill, N.C. *523 E. Rosemary Lane*
LATIMER, C. G., Ph.D. (Chicago) Prof., Emory Univ., Emory University, Ga.
LATSHAW, ELMER, Tech. Engr., ACF-Brill Motors Co., Philadelphia, Pa. *1839 N. 60th St.*
LATSHAW, V. V., Ph.D. (Indiana) Asst. Prof., Lehigh Univ., Bethlehem, Pa. *708 Eighth Ave.*
LAUDANTE, M. R., Student, Poly. Inst. of Brooklyn, Brooklyn, N.Y. *182 Bay 13th St. (24)*
LAUSH, GEORGE, Ph.D. (Cornell) Asst. Prof., Univ. of Pittsburgh, Pittsburgh 13, Pa.
LAVOIE, E. S., A.B. (Brooklyn C.) *135 Sheridan Ave., Brooklyn, N.Y.*
LAWRENCE, H. R., A.M. (Michigan) Asst. Head, Aerodynamic Research Dept., Cornell Aeronautical Lab., Buffalo, N.Y. *4846 N. Harlem Rd., Snyder 21, N.Y.*
LAWRENCE, J. H., M.Ed. (Oklahoma) Teacher, Okemah High School, Okemah, Okla. *601 N. Fourth St.*
LAWRENCE, J. N. P., A.B. (Johns Hopkins) Fellow, Physics, Vanderbilt Univ., Nashville, Tenn. *1800 Primrose Ave.*
LAWS, L. S., A.M. (Stanford) Asst. Prof., Univ. of Minnesota, Minneapolis 14, Minn.
LAY, L. C., A.M. (Southern California) Teacher, John Muir Junior Coll., Pasadena, Calif. *593 W. Montana (3)*
LAYTON, L. L., M.S. (Illinois) Mathematician, Naval Ordnance Lab., Silver Spring 19, Md. *8619 Piney Branch Rd., Apt. 102*
LAYTON, W. I., Ph.D. (Geoge Peabody) Prof., Stephen F. Austin State Coll., Nacogdoches, Tex.
LAZAR, NATHAN, Ph.D. (Columbia) Asso. Prof., Ohio State Univ., Columbus, Ohio. *146 W. North Broadway (14)*
LEAVENS, D. H., A.M. (Yale) Retired. *1632 Wood Ave., Colorado Springs, Colo.*
LEAVITT, W. G., Ph.D. (Wisconsin) Asst. Prof., Univ. of Nebraska, Lincoln, Nebr. *2127 Park Ave.*
LEDFORD, CURTIS, A.M. (Georgia) Asst. Prof., Univ. of Bridgeport, Bridgeport 5, Conn.
LEE, A. S., JR., M.S. (Southern Methodist) Mathematician, Magnolia Petroleum Co., Dallas, Tex. *3620 Greenbrier Dr. (5)*
LEE, E. H., Ph.D. (Stanford) Prof., Brown Univ., Providence, R.I.
LEE, H. L., Ph.D. (Duke) Asst. Prof., Univ. of Tennessee, Knoxville, Tenn.
LEE, J. R., Ph.D. (Yale) Instr., Univ. of Michigan, Ann Arbor, Mich.
LEE, L. H. N., Ph.D. (Minnesota) Asst. Prof., Univ. of Notre Dame, Notre Dame, Ind.
LEE, MARY A., Ph.D. (Cornell) Asso. Prof., Sweet Briar Coll., Sweet Briar, Va.
LEE, R. E., M.S. (Missouri Schl. Mines & Met.) Asst. Prof., Missouri School of Mines and Met., Rolla, Mo.
LEE, T. H., M.A. (North Carolina) Adj. Prof., Univ. of South Carolina, Columbia, S.C. *On leave, 1951–52:* Univ. of North Carolina, Chapel Hill, N.C.
LEE, T. J., Student, Poly. Inst. of Brooklyn, Brooklyn, N.Y. *89-12 205th St., Hollis 7, N.Y.*
LEECH, J. S., Ph.D. (Yale) Prof., Colorado Coll., Colorado Springs, Colo.
LEEDS, B. R., A.B. (Brooklyn) Teacher, Franklin K. Lane High School, Brooklyn, N.Y. *69-22 136th St., Flushing, N.Y.*
LEES, MILTON, Student, Univ. of California, Berkeley, Calif. *375 Paseo Largavista, San Lorenzo, Calif.*
LEFSCHETZ, SOLOMON, Ph.D. (Clark) Prof., Princeton Univ., Princeton, N.J.
LEGRANGE, CELIA (Mrs.), M.A. (N.Y.U.) Teacher, Board of Education, New York, N.Y. *73-55 210th St., Bayside, L.I., N.Y.*
LEHMAN, MARGARET B. (Mrs. R. M.), A.M. (U.C.L.A.) *2315 24th St., Santa Monica, Calif.*
LEHMAN, MARIE P. (Mrs.), M.S. (Stanford) *3332 Emerson St., Palo Alto, Calif.*
LEHMANN, C. H., A.M. (Columbia) Asst. Prof., Engg., Cooper Union, New York 3, N.Y. *144-17 29th Ave., Flushing, N.Y.*
LEHMER, D. H., Ph.D. (Brown) National Bureau of Standards, Los Angeles 24, Calif. *16556 Chattanooga Pl., Pacific Palisades, Calif.*

LEHNER, G. R., B.S. (Loyola, Chicago) Grad. Asst., Univ. of Wisconsin, Madison, Wis. *1434 W. Argyle (40)*
LEHNER, JOSEPH, Ph.D. (Pennsylvania) Asso. Prof., Univ. of Pennsylvania, Philadelphia 4, Pa.
LEHR, MARGUERITE, Ph.D. (Bryn Mawr) Asso. Prof., Bryn Mawr Coll., Bryn Mawr, Pa.
LEIFER, H. R., Litt.M. (Pittsburgh) Chief, Registration & Research Section, Veterans Administration, Pittsburgh, Pa. *1059 N. Negley Ave. (6)*
LEIGHTON, WALTER Ph.D. (Harvard) Prof., Washington Univ., St. Louis 5, Mo.
LEIMANIS, EUGENE, Dr.Rer.Nat. (Hamburg) Asso. Prof., Univ. of British Columbia, Vancouver, B.C., Canada
LEISENRING, K. B., Ph.D. (Michigan) Instr., Univ. of Michigan, Ann Arbor, Mich. *517 E. Washington St.*
LEITHOLD, L. C., JR., M.A. (California) Instr., Phoenix Junior Coll., Phoenix, Ariz.
LELEIKO, MAX, B.S. (N.Y.U.) *1212 Ocean Ave., Brooklyn 30, N.Y.*
LEMAY, FERNAND, Student, Laval Univ., Quebec, P.Q., Canada. *144 Aberdeen*
LEMKE, C. E., B.A. (Buffalo) Grad. Student, Carnegie Inst. of Tech., Pittsburgh, Pa. *522 Ivy St. (32)*
LENNAHAN, C. M., M.S. (M.I.T.) Supervising Analyst, U. S. Weather Bureau, Washington 25, D.C.
LENSER, W. T., Sc.M. (Brown) *P.O. Box 1421, Lincoln, Nebr.*
LEONARD, H. B., Ph.D. (Colorado) Prof., Univ. of Arizona, Tucson, Ariz.
LEONARD, P. J., B.S. (Boston C.) Mathematician, Applied Mechanics Branch, Watertown Arsenal, Watertown, Mass. *71 Hampstead Rd., Jamaica Plain, Boston 30, Mass.*
LEONE, F. C., Ph.D. (Purdue) Instr., Case Inst. of Tech., Cleveland 6, Ohio
LEPORI, A. A., M.A. (Columbia) Instr., High School, East Orange, N.J. *915 Rahway Ave., Westfield, N.J.*
LERCHER, B. L., B.S. (Wisconsin) Asst., Univ. of Wisconsin, Madison, Wis. *1726 Regent St.*
LESER, TADEUSZ, Ph.D. (London) Asst. Prof., Univ. of Kentucky, Lexington, Ky. *105 James Ct., R.R. 4*
LESKOW, OLIVE, M.A. (Minnesota) Teacher, Tolleston School, Gary, Ind. *Graden Apts.,1806 W. 5th Ave., Apt. 6*
LESSARD, ROGER, B.A.Sc. (Montreal U.) Asst. Prof., Ecole Polytechnic, 1430 St. Denis, Montreal 18, Canada
LESTER, CAROLINE A., Ph.D. (Wisconsin) Asst. Prof., New York State Coll. for Teachers, Albany 3, N.Y. *105 S. Lake Ave. (3)*
LESTOURGEON, F. ELIZABETH, Ph.D. (Chicago) *208 N.E. 5th Ct., Delray Beach, Fla.*
LEUTERT, W. W., Dr.Sc.Math. (Zurich) Mathematician, Ballistics Research Labs., Aberdeen Proving Ground, Md.
LEVENSON, M. E., M.S. (N.Y.U.) Instr., Brooklyn Coll., Brooklyn, N.Y.
LEVEQUE, W. J., Ph.D. (Cornell) Asst. Prof., Univ. of Michigan, Ann Arbor, Mich.
LEVIN, H. S., S.B. (Chicago) Asst., Illinois Inst. of Tech., Chicago, Ill. *532 Addison St. (13)*
LEVINE, B. S., M.A. (Washington S.C.) Grad. Student, Columbia Univ., New York, N.Y. *131 Park Ave., Englewood, N.J.*
LEVINE, JACK, Ph.D. (Princeton) Prof., North Carolina State Coll., Raleigh, N.C.
LEVINE, L. D., A.B. (Brooklyn) *454 Central Ave., Brooklyn 21, N.Y.*
LEVIT, R. J., Ph.D. (California) Asso. Prof., Univ. of Georgia, Athens, Ga.
LEVITT, JOSEPH, M.M.E. (Brooklyn Poly.) Process Engr., M. W. Kellogg Co., Jersey City, N.J. *592 Stone Ave., Brooklyn 12, N.Y.*
LEVY, GENE, M.A. (Oklahoma) Instr., Univ. of Oklahoma, Norman, Okla.
LEVY, HARRY, Ph.D. (Princeton) Asso. Prof., Univ. of Illinois, Urbana, Ill.
LEVY, HERMAN, A.B. (N.Y.U.) Engr., Naval Air Exp. Station, Philadelphia Naval Base, Philadelphia, Pa. *2533 Irvington Rd., Drexel Hill, Pa.*
LEVY, S. L., M.S. (Illinois Tech.) Research Asso., Brown Univ., Providence 12, R.I. *344 Benefit St. (6)*
LEWIS, A. J., Ph.D. (Colorado) Emeritus Prof., Univ. of Denver, Denver 10, Colo. *2136 S. Josephine St.*
LEWIS, ANNE L., Ph.D. (Chicago) Asst. Prof., Woman's Coll., Univ. of North Carolina, Greensboro, N.C.
LEWIS, C. F., M.S. (Kansas S.C.) Asso. Prof., Kansas State Coll., Manhattan, Kan.
LEWIS, C. J., Rev., M.S. (Georgetown) Grad. Student, Brown Univ., Providence, R.I. *262 Blackstone Blvd.*
LEWIS, D. C., JR., Ph.D. (Harvard) Prof., Johns Hopkins Univ., Baltimore 18, Md.
LEWIS, E. V., Ph.D. (M.I.T.) Asst. Prof., Univ. of Delaware, Newark, Del. *109 North Rd., Lindamere, Wilmington, Del.*
LEWIS, EUNICE, A.M. (Oklahoma) Teacher, Univ. of Oklahoma Laboratory High School, Norman, Okla.

LEWIS, F. A., Ph.D. (Ohio State) Prof., Univ. of Alabama, University, Ala.
LEWIS, FLORENCE P., Ph.D. (Johns Hopkins) Emeritus Prof., Goucher Coll., Baltimore, Md. *2435 N. Charles St. (18)*
LEWIS, J. A., Ph.D. (Brown) Member, Tech. Staff, Bell Telephone Labs., Inc., Murray Hill, N.J. *20 Spring Hill Gardens, 767 Springfield Ave., Summit, N.J.*
LEWIS, J. H., A.M. (Washington & Jefferson) Dir., Cinemath Animation Studio, New York, N.Y. *32 Franklin Rd., Teaneck, N.J.*
LEWIS, KENNETH, E.E. (Syracuse) Asst. Prof., Univ. of Florida, Gainesville, Fla. *312 N.W. 10th St.*
LEWIS, P. E., Ph.D. (Illinois) Asso. Prof., North Carolina State Coll., Raleigh, N.C.
LI, J. C. R., Ph.D. (Iowa S.C.) Asso. Prof., Oregon State Coll., Corvallis, Ore.
LI, TA CHUNG-HENG, Ph.D. (Munich) Asst. Prof., Drake Univ., Des Moines 11, Iowa. *6019 Rollins Ave. (12)*
LIEBERKNECHT, MARY B., M.A. (Ohio State) Instr., Iowa State Coll., Ames, Iowa
LIEBLEIN, JULIUS, M.A. (Brooklyn) Mathematician, Statistical Engg. Lab., National Bureau of Standards, Washington 25, D.C.
LIFSHITZ, JAIME, M.S. (Mexico) *Address Unknown*
LIGHT, F. W., JR., M.D. (Johns Hopkins) Asst. Prof., Johns Hopkins Univ., Baltimore 18, Md. *719 E. Cold Spring Lane (12)*
LILLIE, ELLA L., A.M. (Michigan) Instr., Glendale Coll., Glendale 8, Calif. *1142 Paloma Dr., Arcadia, Calif.*
LIMPERT, J. V., A.M. (Syracuse) Instr., St. Lawrence Univ., Canton, N.Y. *4½ Pine St.*
LINARES, ENRIQUE, JR., B.S. in E.E. (U. of Santa Clara) Gen. Agt., Cerveceria Nacional, David, Republic of Panama. *P.O. Box 540, Panama City, Republic of Panama*
LINDAHL, C. H., M.S. (Colorado) Asst. Prof., Iowa State Coll., Ames, Iowa
LINDEMANN, NORMA L. (Mrs. R. A.), A.M. (Indiana) Research Asst., Univ. of Notre Dame, Notre Dame, Ind.
LINDSAY, J. W., M.A. (Texas) *605 Corwin, Apt. 1, Glendale 6, Calif.*
LINDSTROM, W. D., B.A. (Iowa) Grad. Student, Univ. of Iowa, Iowa City, Iowa
LINDQUIST, THEODORE, Ph.D. (Chicago) Emeritus Prof., Michigan State Normal Coll., Ypsilanti, Mich. *103 Elm St.*
LINDSTRUM, A. O., JR., Ph.D. (Illinois) Asst. Prof., Knox Coll., Galesburg, Ill.
LINEHAN, P. H., Ph.D. (Columbia) Emeritus Prof., City Coll. of New York, New York, N.Y. *924 West End Ave. (25)*
LINFIELD, B. Z., Ph.D. (Harvard) Asso. Prof., Univ. of Virginia, Charlottesville, Va. *1324 Hilltop Rd.*
LING, T. D., M.S. (Texas) Grad. Student, Univ. of Texas, Austin, Tex. *200 E. 22nd St. (12)*
LINNEY, MARGARET M., A.B. (Regis, Mass.) Engg. Aide, General Electric Co., Schenectady, N.Y. *1283 Glenwood Blvd.*
LINSCHEID, H. W., A.M. (Oklahoma) Asso. Prof., Southwestern Inst. of Tech., Weatherford, Okla. *116 E. Proctor St.*
LINTVEDT, D. R., M.S. (Wisconsin) Asst. Prof., Upsala Coll., East Orange, N.J. *76 Gillespie Rd., Bloomfield, N.J.*
LIOLIOS, LEO, *4933 Hutchinson, Chicago 41, Ill.*
LIPSEY, SALLY I. (Mrs. Robert), M.A. (Wisconsin) Lecturer, Hunter Coll., New York 21, N.Y. *160 Bush St. (53)*
LIPSICH, H. D., Ph.D. (Cincinnati) Asst. Prof., Univ. of Cincinnati, Cincinnati, Ohio
LITTAUER, S. B., Sc.D. (M.I.T.) Asso. Prof., Industrial Engg., Columbia Univ., New York 27, N.Y.
LITTERICK, W. S., Ed.D. (Rutgers) Dir. of Research Service, Stephens Coll., Columbia, Mo. *116 Park Hill Ave.*
LITTLE, T. M., Ph.D. (Maryland) Asst. Prof., Biology, Univ. of Nevada, Reno. Nev.
LITTLEJOHN, T. C., B.S. (Memphis S.C.) Grad. Student, Northwestern Univ., Evanston, Ill.
LITZINGER, MARIE, Ph.D. (Chicago) Prof., Mount Holyoke Coll., South Hadley, Mass.
LIVERS, J. J., JR., Ph.D. (Michigan) Prof., Montana State Coll., Bozeman, Mont.
LIVINGSTON, A. E., M.A. (Oregon) Grad. Asst., Univ. of Oregon, Eugene, Ore. *2202-1 Patterson Dr.*
LIVINGSTON, G. R., A.M. (California) Retired Prof., San Diego State Coll., San Diego, Calif. *4641 55th St. (5)*
LLOYD, D. B., Ph.D. (Catholic U.) Instr., James Ormond Wilson Teachers Coll., Washington 9, D.C.
LLOYD, F. H., B.S. (Westminster) Asst. Instr., Univ. of Missouri, Columbia, Mo.
LLOYD, OLWEN (Mrs. George) M.A. (Cambridge) Headmistress, Mount Vernon Seminary, Washington 7, D.C. *2100 Foxhall Rd.*

LOCH, JOSEPH, B.S. (Illinois Tech.) Project Engr., U. S. Air Force, AMC, Wright Field, Dayton, Ohio. *5204 Beaver Ave. (9)*
LOCKE, J. F., Ph.D. (Illinois) Prof., Birmingham-Southern Coll., Birmingham 4, Ala.
LOCKER, J. L., Student, Alabama Poly. Inst., Auburn, Ala. *226 W. Magnolia*
LOCKHART, B. J., Ph.D. (Illinois) Asso. Prof., U. S. Naval Postgraduate School, Annapolis, Md.
LOCKWOOD, E. C., A.M. (Brown) Head of Dept., Jaffna Coll., Vaddukoddai, Ceylon
LOEV, DAVID, Student, Univ. of Pennsylvania, Philadelphia, Pa. *2422 N. 33rd St. (32)*
LOEWEN, K. L., M.S. (Kansas S.C.) Administrative Asst., Mennonite Central Committee, Akron, Pa.
LOEWNER, CHARLES, Ph.D. (Prague) Prof., Stanford Univ., Stanford, Calif.
LOFLIN, Z. L., Ph.D. (Columbia) Prof., Southwestern Louisiana Inst., Lafayette, La.
LOGSDON, MAYME I., Ph.D. (Chicago) Emeritus Prof., Univ. of Chicago, Chicago, Ill. *Visiting Prof., Univ. of Miami, Coral Gables 34, Fla.*
LOH, ZUNG-NYI, M.A. (Cornell) Grad. Student, Univ. of Michigan, Ann Arbor, Mich. *1001 E. Huron*
LOHWATER, A. J., Ph.D. (Rochester) Instr., Univ. of Michigan, Ann Arbor, Mich.
LOKENSGARD, R. L., Ed.D. (Columbia) Head of Dept., State Teachers Coll., Winona, Minn.
LONDON, LIONEL, B.S.C.E. (Illinois) Stress Analyst, Barnes and Reinecke, Chicago, Ill. *1823 Pleasant Ave., Highland Park, Ill.*
LONG, FLORENCE, M.S. (Illinois) Asso. Prof., Earlham Coll., Earlham, Ind.
LONG, ROY, B.S. (Georgia) Job Analyst, Civilian Personnel Office, Camp Gordon, Ga. *3045 Bransford Rd., Augusta, Ga.*
LONG, R. W., M.A. (Washington & Jefferson) Asst. Prof., Washington and Jefferson Coll., Washington, Pa.
LONGENECKER, J. V., M.S. (Iowa) Actuary, Farmers and Bankers Life Insurance Co., Wichita, Kan.
LONGLEY, R. K., B.A. (Buffalo) Head of Dept., Canisteo Central School, Canisteo, N.Y. *5½ Russell St.*
LONGLEY, W. R., Ph.D. (Chicago) Emeritus Prof., Yale Univ., New Haven 11, Conn. *305 Lawrence St.*
LONNER, MAX, Rabbi, B.S. (C.C.N.Y.) Registrar, Mesifta Talmudical Seminary, Brooklyn 11, N.Y. *141 Penn St. (11)*
LONSETH, A. T., Ph.D. (California) Prof., Oregon State Coll., Corvallis, Ore.
LOOMER, A. P., M.A. (Acadia) Prof., Milton Coll., Milton, Wis.
LOONEY, DOROTHY K., Student, Regis Coll., Weston, Mass. *11 Farm Rd., Belmont 78, Mass.*
LORCH, E. R., Ph.D. (Columbia) Prof., Columbia Univ., New York 27, N.Y.
LORCH, LEE, Ph.D. (Cincinnati) Asso. Prof., Fisk Univ., Nashville 8, Tenn.
LORD, P. H., A.B. (Princeton) Graduate School, Princeton Univ., Princeton, N.J. *182 Hillside Ave., Englewood, N.J.*
LORING, S. J., B.S. (M.I.T.) Consulting Engr. *18 Chapman Rd., West Hartford 7, Conn.*
LOSEY, G. O., Student, Univ. of Michigan, Ann Arbor, Mich. *7817 Huber House, South Quad.*
LOTKIN, M. M., Ph.D. (Kiel) Chief, Machines Section, Ballistics Research Lab., Aberdeen Proving Ground, Md.
LOTT, F. W., JR., M.A. (Michigan) Asst. Prof., Iowa State Teachers Coll., Cedar Falls, Iowa
LOTZ, S. S., Lieutenant and Student, U. S. Navy, Tufts Coll., Medford 55, Mass. *P.O. Box 34, Lexington, Mass.*
LOUD, W. S., Ph.D. (M.I.T.) Asst. Prof., Univ. of Minnesota, Minneapolis 14, Minn.
LOVE, C. E., Ph.D. (Michigan) Emeritus Prof., Univ. of Michigan, Ann Arbor, Mich. *1915 Scottwood Ave.*
LOVETT, E. O., Ph.D. (Virginia; Leipzig) Emeritus President, Rice Inst., Houston 1, Tex.
LOWDENSLAGER, D. B., B.A. (St. Johns C.) Grad. Student, Univ. of Virginia, Charlottesville, Va.
LOWE, I. J., B.E.E. (Cooper Union) Grad. Student, Washington Univ., St. Louis 5, Mo. *1925 Quentin Rd., Brooklyn 29, N.Y.*
LOWEKE, G. P., Ph.D. (Berlin) Asso. Prof., Engg. Mech., Wayne Univ., Detroit 3, Mich.
LOWELL, L. I., B.S. in M.E. (Michigan S.C.) Grad. Student, Michigan State Coll., East Lansing, Mich., *R.R. 1, DeWitt, Mich.*
LOWENSTEIN, L. L., Ph.D. (Cornell) Prof., Kent State Univ., Kent, Ohio. *410 Stow St.*
LOWERY, M. J., A.B. (Northeastern S.C.) Asst. Instr., Univ. of Missouri, Columbia, Mo. *R.F.D. 1, Kinta, Okla.*
LOWNEY, R. E., M.A. (Michigan) Asso. Prof., Montana State Coll., Bozeman, Mont.
LOWRY, E. J., M.Sc. (Nebraska) Prof., Hastings Coll., Hastings, Nebr. *734 N. Bellevue*
LOWRY, W. C., M.Ed. (Ohio U.) Asst. Instr., Ohio State Univ., Columbus, Ohio. *1350 Highland St. (1)*
LUBBEN, R. G., Ph.D. (Texas) Asso. Prof., Univ. of Texas, Austin, Tex.

LUBIN, C. I., Ph.D. (Harvard) Asso. Prof., Univ. of Cincinnati, Cincinnati, Ohio. *3612 Washington Ave. (29)*
LUCAS, HELEN M. (Mrs. J. W.), M.S. (Iowa) Statistician, Bureau of the Census, Washington, D.C. *5101 V St., S.E. (19)*
LUCAS, W. A., M.E. (Stevens Tech.) Instr., Stevens Inst. of Tech., Hoboken, N.J.
LUFBURROW, R. A., M.S. (Purdue) Grad. Student, Univ. of Wisconsin, Madison, Wis. *177 Monroe Park*
LUIPPOLD, R. C., A.M. (Buffalo) Instr., New York State Coll. for Teachers, Albany, N.Y.
LUKACS, ELIZABETH C. (Mrs. Eugene), Teachers Certif. (Vienna) *3140 Wisconsin Ave., N.W., Apt. 704, Washington 16, D.C.*
LUKE, CYPRIAN, Brother, Ph.D. (Catholic U.) Head of Dept., St. Michael's Coll., Cerillos Rd., Santa Fe, N.M.
LUKENS, JAYWOOD, LL.B. (Fordham) Asso. Actuary, Geo. B. Buck, Consulting Actuary, New York, N.Y. *Midbrook Lane, Darien, Conn.*
LUNDBERG, G. H., A.M. (Vanderbilt) Instr., Vanderbilt Univ., Nashville, Tenn. *2310 Highland Ave.*
LUTHER, C. F., Ph.D. (Stanford) Dean, Coll. of Liberal Arts, Willamette Univ., Salem, Ore.
LUTHER, H. A., Ph.D. (Iowa) Prof., Texas A. and M. Coll., College Station, Tex.
LUYKX, JAN, Teacher, Tech. Coll. of Antwerp. *Provinciestraat, 97, Antwerpen, Belgium*
LYCHE, WALTER, M.A. (Minnesota) Asst. Prof., Augustana Coll., Sioux Falls, S.D.
LYLE, G. A., M.S. (Lehigh) Asso. Prof., U. S. Naval Acad., Annapolis, Md. *227 Wardour Dr.*
LYNCH, R. V., A.M. (Harvard) Instr., Phillips Exeter Acad., Exeter, N.H.
LYNDON, R. C., Ph.D. (Harvard) Asst. Prof., Princeton Univ., Princeton, N.J.
LYTLE, R. A., A.M. (Virginia) Adj. Prof., Univ. of South Carolina, Columbia, S.C. *2849 Sheffield Rd.*

MACCOLL, L. A., Ph.D. (Columbia) Member, Tech. Staff, Bell Telephone Labs., 463 West St., New York 14, N.Y.
MACCULLOUGH, G. H., Sc.D. (Michigan) Prof., Worcester Poly. Inst., Worcester 2, Mass.
MACCULLOUGH, R. H., M.S. (Lafayette) Retired Prof., Defiance Coll., Defiance, Ohio
MACDONALD, S. L., A.M. (Columbia) Emeritus Prof., Colorado A. and M. Coll., Fort Collins, Colo. *122 N. Thompson St., Springdale, Ark.*
MACDOUGAL, H. B., M.S. (Iowa) Prof., South Dakota State Coll., Brookings, S.D.
MACDUFFEE, C. C., Ph.D. (Chicago) Prof., Univ. of Wisconsin, Madison 6, Wis.
MACEWEN, D. M., Ph.D. (N.Y.U.) Asst. Prof., City Coll. of New York, New York, N.Y. *55 E. 65th St.*
MACKIE, E. L., Ph.D. (Chicago) Dean of Student Awards and Prof., Univ. of North Carolina, Chapel Hill, N.C. *702 Gimghoul Rd.*
MACLANE, G. R., Ph.D. (Rice) Prof., Rice Inst., Houston, Tex.
MACLANE, SAUNDERS, Ph.D. (Göttingen) Prof., Univ. of Chicago, Chicago 37, Ill.
MACNEIL, M. T., Student, Univ. of Detroit, Detroit, Mich. *16010 Ellsworth (27)*
MACNEILLE, H. M., Ph.D. (Harvard) Exec. Dir., American Mathematical Society, 80 Waterman St., Providence 6, R.I.
MACNEISH, H. F., Ph.D. (Chicago) Prof., Univ. of Miami, Coral Gables, Fla. *2705 Anderson Rd. (34)*
MACON, NATHANIEL, Ph.D. (North Carolina) Fulbright Fellow, Univ. of Amsterdam, Amsterdam, The Netherlands. *27 Myrtle Apts., Charlotte 3, N.C.*
MACPHAIL, M. S., Ph.D. (Oxford) Asso. Prof., Carleton Coll., Ottawa, Ont., Canada. *142 Brighton Ave.*
MACY, J. R., Student, Univ. of Chicago, Chicago 37, Ill. *8555 S. Maryland Ave. (19)*
MADDAUS, INGO, JR., Ph.D. (Michigan) Asst. Prof., Union Coll., Schenectady, N.Y.
MADDEN, J. E., M.S. (Brown) Instr., Portsmouth Priory School, Portsmouth, R.I. *2 Remsen Ave., Valley Stream, L.I., N.Y.*
MADDOX, A. C., A.M. (Columbia) Prof., Northwestern State Coll., Natchitoches, La. *325 Second St.*
MADDOX, J. L., JR., M.S. (Alabama Poly.) Engr., Philco Corp., Philadelphia, Pa. *Room 309, 5722 Greene St. (44)*
MADISON, M. L., M.S. (Colorado) Asso. Prof., Colorado A. and M. Coll., Fort Collins, Colo. *618 W. Myrtle*
MADOR, R. L., A.M. (Trinity C., Conn.) Mathematician, U. S. Navy Electronics Lab., San Diego, Calif. *3330 45th St. (5)*
MAGDA, R. A., Student, Univ. of Detroit, Detroit, Mich. *R.R. 1, River Canard, Ont., Canada*
MAGUIRE, J. M., Supervisor, Designer Tractor Engg. Dept., Ford Motor Co., Dearborn, Mich. *4951 Williamson*
MAHACHEK, JOY, Ph.D. (Pittsburgh) Head of Dept., State Teachers Coll., Indiana, Pa.

MAHURON, L. E., B.A. (Hardin-Simmons) Supervisor, I.B.M. Section, Sandia Corp., Albuquerque, N.M. *3226-U 34th St.*
MAILE, C. B., JR., A.B. (Hofstra) Metallurgist, Sperry Gyroscope Co., Great Neck, N.Y. *80-22 236th St., Bellerose 6, L.I., N.Y.*
MAINARDI, POMPEY, M.A. (Montclair) Asso. Prof., Newark Coll. of Engg., Newark, N.J. *120 E. 21st St., Paterson, N.J.*
MAKAROV, A. G., A.M. (Pennsylvania) Asst. Prof., Rutgers Univ., New Brunswick, N.J.
MALAK, F. J., B.S. (Ohio State) Instr., Youngstown Coll., Youngstown 2, Ohio
MALE, C. T., M.C.E. (Union) Prof., Union Coll., Schenectady, N.Y.
MALL, A. W., A.B. (Clark) Mathematician, Frankford Arsenal, Philadelphia, Pa. *27-C Parkway Apts., Haddonfield, N.J.*
MALLEY, R. W., B.S. (Southeastern Louisiana C.) Math. Analyst, Sandia Corp., Albuquerque, N.M. *Box 25, 3216 "A" St., Sandia Base*
MALLORY, V. S., Ph.D. (Columbia) Prof., New Jersey State Teachers Coll., Upper Montclair, N.J.
MALONEY, C. J., Ph.D. (Iowa S.C.) Chief, Statistics Branch, Camp Detrick, Frederick, Md.
MALTENFORT, MARTIN, A.M. (New Jersey S.T.C.) Lt., U. S. Army. *4000 Lee Blvd., Arlington 8, Va.*
MALY, DIS, M.A. (Harvard) Asso. Prof., Rensselaer Poly. Inst., Troy, N.Y.
MANCHESTER, R. E., A.M. (Michigan) Dean of Men, Kent State Univ., Kent, Ohio. *208 N. Lincoln St.*
MANCILL, J. D., Ph.D. (Chicago) Prof., Univ. of Alabama, University, Ala. *118 Kent Rd., Huntsville, Ala.*
MANDELBAUM, D. M., Student, Hillside High School, Hillside, N.J. *1600 Compton Pl. (5)*
MANDELBAUM, HUGO, Ph.D. (Hamburg) Instr., Wayne Univ., Detroit, Mich. *2538 Elmhurst (6)*
MANHEIM, J. H., M.S. (Illinois) Instr., Wilson Junior Coll., Chicago, Ill.
MANN, G. J., M.A. (Loyola, Chicago) Asst. Prof., U. S. Naval Acad., Annapolis, Md.
MANN, HENRY, B.A. (Brooklyn) *141-11 78th Rd., Flushing 67, N.Y.*
MANNING, F. L., Ph.D. (Cornell) Prof., Ursinus Coll., Collegeville, Pa. *68 Sixth Ave.*
MANNING, H. P., Ph.D. (Johns Hopkins) Emeritus Asso. Prof., Brown Univ., Providence 12, R.I. *148 Governor St. (6)*
MANOGUE, J. F., Student, Catholic Univ. of America, Washington, D.C. *3207 P St., N.W., (7)*
MANSFIELD, RALPH, M.S. (Chicago) Chief Engr., Jos. Weidenhoff, Inc., Chicago 24, Ill. *2211 E. 67th St. (49)*
MANSOUR, J. A., B.S. (Detroit) Teaching Fellow, Univ. of Detroit, Detroit, Mich. *2695 Carter (6)*
MANY, ANNA E., A.M. (Tulane) Counselor to Women, Newcomb Coll., New Orleans 18, La.
MAPLE, C. G., D.Sc. (Carnegie) U. S. Navy. *5030 Bradley Blvd., Chevy Chase 15, Md.*
MARBURGER, CLIFFORD, A.M. (Franklin and Marshall) *Denver, Pa.*
MARCH, H. W., Ph.D. (Munich) Emeritus Prof., Univ. of Wisconsin, Madison Wis.; Mathematician, U. S. Forest Products Lab., Madison, Wis. *1825 Summit Ave. (5)*
MARCHAND, E. W., M.S. (Washington U.) Physicist, Eastman Kodak Co., Rochester, N.Y. *192 Seville Dr. (17)*
MARCOU, R. J., Ph.D. (M.I.T.) Prof., Boston Coll., Chestnut Hill, Mass. *930 Beacon St., Newton Centre 59, Mass.*
MARDEN, MORRIS, Ph.D. (Harvard) Prof., Univ. of Wisconsin in Milwaukee, Milwaukee, Wis. *403 E. Carlisle Ave. (11)*
MARER, FRED, M.A. (Southern California) Instr., Los Angeles City Coll., Los Angeles, Calif. *855 N. Vermont Ave. (27)*
MARIA, D. MAY HICKEY (Mrs. A. J.), Ph.D. (Rice) Instr., Brooklyn Coll., Brooklyn, N.Y.
MARK, A. M., Ph.D. (Cornell) Asst. Prof., Southern Illinois Univ., Carbondale, Ill.
MARKLE, G. E., M.Sc. (Detroit) Asst. Prof., Engg., Univ. of Detroit, Detroit 6, Mich. *1993 Glendale (6)*
MARKOVIC, ZELJKO, Sc.D. (Zagreb) Prof., Univ. of Zagreb, Zagreb, Yugoslavia. *Gregorijanceva ul. 38*
MARKS, B. W., B.S. (Texas Western C.) Pvt., U. S. Army
MARKS, H. B., Student, Southern Methodist Univ., Dallas, Tex. *5538 Merrimac (6)*
MARLOW, W. H., Ph.D. (Iowa) Research Asso., Logistics Research Project, George Washington Univ., Washington 6, D.C. *7316 Forest Rd., Apt. 101, Hyattsville, Md.*
MARM, ANNA, A.M. (Kansas) Prof., Bethany Coll., Lindsborg, Kan. *741 N. Second St.*
MARQUARDT, MARY O., M.A. (Illinois) Grad. Student, Univ. of Illinois, Urbana, Ill. *102 S. Busey Ave.*
MARQUIS, R. H., Ph.D. (Chicago) Prof., Ohio Univ., Athens, Ohio
MARR, J. M., M.A. (Missouri) Asst. Instr., Univ. of Tennessee, Knoxville, Tenn.
MARRIAN, D. M., M.S. (Michigan) Master, Gilman Country School, Roland Park, Baltimore 10, Md. *1506 Shadyside Rd. (18)*

MARRIOTT, R. W., Ph.D. (Pennsylvania) Prof., Swarthmore Coll., Swarthmore, Pa. *213 Lafayette Ave.*
MARSH, D. C. B., JR., M.S. (Arizona) Asst., Univ. of Colorado, Boulder, Colo.
MARSH, R. W., A.B. (American U.) Grad. Student, George Washington Univ., Washington, D.C. *431 Randolph St., N.W. (11)*
MARSHALL, P. L., Student, Hofstra Coll., Hempstead, N.Y. *8 Leonard Ave., Freeport, N.Y.*
MARTENS, MILDRED M., M.A. (Michigan) Instr., Woodruff Senior High School, Peoria, Ill. *912 Fifth Ave. (6)*
MARTH, ELLA, Ph.D. (St. Louis U.) Dean of Women, Harris Teachers Coll., St. Louis, Mo. *4521a Clarence Ave.*
MARTIN, C. F., B.S. (U. S. Naval Acad.) Asst. Prof., Univ. of South Carolina, Columbia, S.C. *3219 Heyward St. (61)*
MARTIN, JEROME, Ph.D. (California) Research Dir., Commercial Solvents Corp., Terre Haute, Ind. *1334 S. Center St.*
MARTIN, J. A., Senior Partner, National Oil Co., Lisbon, Ohio. *234 W. Washington St.*
MARTIN, J. E., M.S. (Vanderbilt) Asst. Prof., Davidson Coll., Davidson, N.C.
MARTIN, M. H., Ph.D. (Johns Hopkins) Prof., Univ. of Maryland, College Park, Md.
MARTIN, MARGARET P., Ph.D. (Minnesota) Asst. Prof., School of Medicine, Vanderbilt Univ., Nashville 4, Tenn.
MARTIN, NORBERT, Rev., B.A. (Catholic U.) Instr., St. Francis Preparatory School, Spring Grove, Pa.
MARTIN, S. T., B.A. (Hampden-Sydney) Acting Head of Dept., Newberry Coll., Newberry, S.C.
MARTIN, W. A., A.M. (Alabama) Asst. Prof., Georgia Inst. of Tech., Atlanta, Ga. *2372 Glenwood Dr., N.E.*
MARTIN, W. T., Ph.D. (Illinois) Prof., Massachusetts Inst. of Tech., Cambridge 39, Mass. *On leave, 1951–52:* Inst. for Advanced Study, Princeton, N.J.
MARTINSON, MARGARET E., A.B. (Washburn) Instr., Washburn Municipal Univ., Topeka, Kan. *1600 MacVicar*
MARTYN, W. J., M.A. (New Zealand) Master, Otago Boys High School, Dunedin, New Zealand. *16 Sheen St., Roslyn, Dunedin, N.W. 1, New Zealand*
MASON, HAZEL L., A.M. (New Mexico) Grand Prairie Schools, Grand Prairie, Tex. *4946 Columbia Ave., Dallas 14, Tex.*
MASON, R. H., A.M. (Syracuse) Instr., Univ. of Florida, Gainesville, Fla.
MASON, W. E., M.E. (Michigan) Asso. Prof., Engg., Univ. of California at Los Angeles, Los Angeles 24, Calif.
MASSEY, W. L., A.M. (Duke) Asso. Prof., Univ. of Chattanooga, Chattanooga, Tenn.
MATHANY, H. V., A.M. (Washington S.C.) 445 N. Solano Ave., Albuquerque, N.M.
MATHESON, J. D., A.M. (George Washington) Col. and Asst. Prof., U. S. Military Acad., West Point, N.Y.
MATHEWS, C. W., JR., Ph.D. (Illinois) *421 N. Grace St., Marissa, Ill.*
MATHEWS, JON, Student, Pomona Coll., Claremont, Calif. *2591 Dearborn Dr., Hollywood 28, Calif.*
MATHEWSON, L. C., Ph.D. (Illinois) Prof., Dartmouth Coll., Hanover, N.H.
MATHIAS, H. R., A.M. (Indiana) Asso. Prof., Bowling Green State Univ., Bowling Green, Ohio. *123 E. Evers Ave.*
MATISON, HARRY, Ph.D. (Princeton) Teacher, Los Angeles City Coll., Los Angeles, Calif. *11941 Albers St., North Hollywood, Calif.*
MATLACK, D. W., B.A. (Grinnell) Mathematician, North American Aviation, Inc., Los Angeles 45, Calif. *827 Second St., Apt. F, Santa Monica, Calif.*
MATTHEWS, A. R., Student, Alabama Poly. Inst., Auburn, Ala. *c/o A. M. Matthews, Route 2, Decatur, Ala.*
MATTSON, H. F., JR., B.A. (Oberlin) *158 Hillsdale St., Hillsdale, Mich.*
MATTUCK, A. P., Student, Princeton Univ., Princeton, N.J. *890 E. 21st St., Brooklyn 10, N.Y.*
MAUCH, MARGARET, Ph.D. (Chicago) Asso. Prof., Univ. of Akron, Akron 4, Ohio. *Hetland, S.D.*
MAXFIELD, J. E., Ph.D. (Oregon) Mathematician, Naval Ordnance Test Station, China Lake, Calif. *214-A Ellis St.*
MAY, A. E., Ph.D. (Wisconsin) Asst. Dir., Univ. of Wisconsin, Racine, Wis. *224 12th St.*
MAY, KENNETH, Ph.D. (California) Asso. Prof., Carleton Coll., Northfield, Minn.
MAY, LIDA B., A.M. (Texas) Asst. Prof., Texas Tech. Coll., Lubbock, Tex. *2409 14th St.*
MAY, R. L., B.A. (Webb Inst.) Tech. Asst., Socony-Vacuum Oil Co., 26 Broadway, New York, N.Y. *157 Horton St., City Island, New York 64, N.Y.*
MAYER, E. S., M.A. (Colorado) Retired Prof., U. S. Naval Acad., Annapolis, Md. *115 Spa View Ave.*
MAYER, H. C., JR., M.S. (Iowa) Mathematician, General Electric Co., Richland, Wash.
MAYERSON, A. L., M.A. (Michigan) Principal Actuary, New York State Insurance Dept., New York, N.Y. *96 Sterling St., Brooklyn 25, N.Y.*
MAYOR, J. R., Ph.D. (Wisconsin) Prof., Univ. of Wisconsin, Madison, Wis. *441 Virginia Terr. (5)*

MAYS, W. J., A.M. (Vanderbilt) Asst. Actuary, Liberty Life Insurance Co., Greenville, S.C.
MCARTNEY, JUNE M., M.A. (Buffalo) Instr., Univ. of Buffalo, Buffalo 14, N.Y. *18 Dupont St. (8)*
MCBRIDE, ELNA B. (Mrs. J. B.), M.S. (Tennessee) Asst. Prof., Memphis State Coll., Memphis, Tenn.
MCBRIDE, W. H., M.S. (North Dakota) Instr., Univ. of North Dakota, Grand Forks, N.D. *A-11 Park Village*
MCBRIEN, V. O., Ph.D. (Catholic U.) Asso. Prof., Coll. of the Holy Cross, Worcester 3, Mass.
MCCALL, C. H., JR., A.B. (George Washington) Grad. Student, George Washington Univ., Washington, D.C. *6701–44th St., Chevy Chase 15, Md.*
MCCALLION, W. J., M.A. (McMaster) Lecturer, McMaster Univ., Hamilton, Ont., Canada. *24 Franklin Ave.*
MCCAMMAN, CAROL V., A.M. (California) Teacher, Calvin Coolidge High School, Washington, D.C. *3500 14th St., N.W., Apt. 514 (10)*
MCCARTHY, ANNE S., A.M. (Boston C.) *32 Auburn St., Brookline, Mass.*
MCCARTHY, E. D., A.M. (Penna. State) Asst. Prof., Engg., Univ. of Detroit, Detroit, Mich.
MCCARTHY, J. J., M.S. (N.Y.U.) Project Engr., Sperry Gyroscope Co., Great Neck, N.Y. *27 Carlton Terr., Stewart Manor, Garden City, N.Y.*
MCCARTHY, P. J., B.S. in E.E. (Notre Dame) Teaching Fellow, Univ. of Notre Dame, Notre Dame, Ind.
MCCARTY, M. E., M.A. (Michigan) Asst. Prof., Univ. of Florida, Gainesville, Fla.
MCCLELLAN, ADA A., B.S. (Chicago) Teacher, High School, Long Beach, Calif. *313 N. New Hampshire Ave., Los Angeles, Calif.*
MCCLELLAND, J. N., A.M. (Drake) Research Asst., Univ. of California Medical School, San Francisco, Calif. *818 N. Nicholas Ave., Fullerton 16, Calif.*
MCCLENON, R. B., Ph.D. (Yale) Prof., Grinnell Coll., Grinnell, Iowa. *1105 Park St.*
MCCLIMANS, J. W., Ph.D. (Peabody) Prof., Southeastern Louisiana Coll., Hammond, La.
MCCLINTOCK, W. J., M.A. (Michigan) Supt. of Schools, Trenary, Mich.
MCCLUNG, W. C., Student, Coll. of Wooster, Wooster, Ohio
MCCLURE, RUTH H. (Mrs.), M.S. (Illinois) *103 E. Vanderbilt Dr., Oak Ridge, Tenn.*
MCCOLGIN, GLADYS B. (Mrs.), A.M. (Radcliffe) *1556 Brookside Ave., Indianapolis, Ind.*
MCCONNELL, H. J., Engr., U. S. Engineer Office, Los Angeles, Calif. *R. 9, Box 2817, Carmichael, Calif.*
MCCONNELL, R. K., JR., A.M. (Columbia) Instr., New York Univ., New York, N.Y. *144 E. 74th St. (21)*
MCCORMICK, C. T., Ph.D. (Indiana) Prof., Illinois State Normal Univ., Normal, Ill.
MCCOY, DOROTHY, Ph.D. (Iowa) Prof., Wayland Coll., Plainview, Tex.
MCCOY, N. H., Ph.D. (Iowa) Prof., Smith Coll., Northampton, Mass. *53 Ridgewood Terr.*
MCCREA, W. H., Ph.D. (Cambridge) Prof., Royal Holloway Coll., Univ. of London, Englefield Green, Surrey, England
MCCREADY, G. T., B.S. (U. S. Naval Acad.) Capt., U. S. Navy
MCCROSSEN, GARNER, M.A. (Wyoming) Part-time Instr., Univ. of Colorado, Boulder, Colo. *Route 1, Box 363*
MCCUAN, M. G., M.A. (West Texas S.T.C.) Head of Dept., Amarillo Coll., Amarillo, Tex.
MCCULLEY, W. S., M.S. (Texas A. & M.) Asst. Prof., Texas A. and M. Coll., College Station, Tex.
MCCUNE, D. C., M.S. (Purdue) Instr., Purdue Univ., Lafayette, Ind.
MCCUSKEY, S. W., Ph.D. (Harvard) Prof., Case Inst. of Tech., Cleveland 6, Ohio
MCCUTCHEON, G. J., M.A. (Minnesota) Instr., University High School, Minneapolis, Minn. *90 Malcolm Ave., S.E. (14)*
MCDANIEL, W. C., Ph.D. (Wisconsin) Prof., Southern Illinois Univ., Carbondale, Ill.
MCDERMOT, R. G., B.S. (Pittsburgh) Instr., Western Pennsylvania Inst. of Tech., Pittsburgh, Pa. *2655 Amman St. (26)*
MCDOLE, R. D., M.S. (Oklahoma A. & M.) Asst. Prof., Oklahoma A. and M. Coll., Stillwater, Okla. *916 Connell Ave.*
MCDONALD, JANET, Ph.D. (Chicago) Asst. Prof., Vassar Coll., Poughkeepsie, N.Y.
MCDONALD, SOPHIA L., Ph.D. (California) Prof., Univ. of California, Berkeley, Calif. *2818 Webster St. (5)*
MCDONOUGH, D. L., Ph.D. (Pennsylvania) Head of Dept., South Philadelphia High School for Boys, Philadelphia, Pa. *1205 Belfield Ave., Drexel Hill, Pa.*
MCDOUGLE, EDITH A., A.B. (Delaware) Instr., Women's Coll., Univ. of Delaware, Newark, Del.
MCDOWELL, E. L., M.S. (Illinois Tech.) Instr., Illinois Inst. of Tech., Chicago 16, Ill. *819 Wenonah St., Oak Park, Ill.*
MCEWEN, G. F., Ph.D. (Stanford) Prof., Scripps Institution, Univ. of California, La Jolla, Calif.
MCEWEN, W. H., Ph.D. (Minnesota) Dean, Faculty of Grad. Studies and Research, Univ. of Manitoba, Winnipeg, Man., Canada. *29 Rothesay Apts., Preston Ave.*
MCEWEN, W. R., Ph.D. (Minnesota) Prof., Univ. of Minnesota, Duluth Branch, Duluth, Minn.
MCFADDEN, LEONARD, Ph.D. (Brown) Asso. Prof., Virginia Poly. Inst., Blacksburg, Va.

McFARLAN, L. H., Ph.D. (Missouri) Prof., Univ. of Washington, Seattle 5, Wash.
McFARLAND, DORA, Ph.D. (Chicago) Prof., Univ. of Oklahoma, Norman, Okla.
McFARLAND, M. M., A.A. (Santa Monica City C.) Surveyor, Engineering Service Corp., Los Angeles, Calif. *8942 DeHaviland Ave.* (*45*)
McGAR, F. H., JR., M.S. (Case) Instr., Sweet Briar Coll., Sweet Briar, Va.
McGAUGHEY, A. W., Ph.D. (Cincinnati) Asso. Prof., Bradley Univ., Peoria 5, Ill.
McGAUGHY, EDWARD, B.S. (Chicago) Lecturer, Columbia Univ., New York 27, N.Y.
McGAVOCK, W. G., Ph.D. (Duke) Prof., Davidson Coll., Davidson, N.C.
McGILLICUDDY, EUGENE, B.S. (Worcester Poly.) Instr., Coll. of the Holy Cross, Worcester, Mass.
McGLASSON, A. G., B.S. (Eastern Kentucky S.C.) Instr., Eastern Kentucky State Coll., Richmond, Ky.
McINNIS, S. W., A.M. (Florida) Asst. Prof., Univ. of Florida, Gainesville, Fla. *1417 W. McCormick St.*
McINTOSH, C. R., B.S. (Holy Cross) *73 Reservoir St., Lawrence, Mass.*
McINTOSH, D. L., M.A. (Denver) Teacher, South Denver High School, Denver 10, Colo.
McKEEHAN, J. E., M.A. (Oklahoma) Head of Dept., Skagit Valley Junior Coll., Mount Vernon, Wash. *916 S. 16th St.*
McKELVEY, J. V., Ph.D. (Cornell) Prof., Iowa State Coll., Ames, Iowa. *2117 Graeber St.*
McKELVEY, MARTHA M. (Mrs. J. V.), M.S. (Iowa) *2117 Graeber St., Ames, Iowa*
McKENNA, MARY, A.M. (Columbia) Retired. *110 Remsen St., Brooklyn 2, N.Y.*
McKENNEY, J. L., Rev., A.M. (Manhattan) Head of Dept., Providence Coll., Providence, R.I
McKENZIE, H. C., M.A. (Wisconsin) Instr., Univ. of Colorado, Boulder, Colo.
McKENZIE, W. H., M.A. (California) Instr., City Coll. of San Francisco, San Francisco 21, Calif. *224 Clement St.* (*18*)
McKEON, M. J. F., M.A. (New York State Coll. for Teachers) Instr., Rensselaer Poly. Inst., Troy, N.Y. *35 Grove Ave., Albany 3, N.Y.*
McKINSEY, J. C. C., Ph.D. (California) Visiting Prof., Philosophy, Stanford Univ., Stanford, Calif.
McKNELLY, R. D., M.A. (Oklahoma) Instr., Univ. of Oklahoma, Norman, Okla. *809 Chautauqua*
McKNIGHT, BETTY, M.A. (Southern Methodist) Asst. Prof., Centenary Coll., Shreveport, La.
McLACHLAN, E. K., M.A. (Texas) Asst. Prof., Baylor Univ., Waco, Tex.
McLAUGHLIN, J. E., Ph.D. (C.I.T.) Instr., Univ. of Michigan, Ann Arbor, Mich.
McLAUGHLIN, J. J., M.A. (Michigan) Head of Dept., State Teachers Coll., River Falls, Wis. *218 S. 8th St.*
McLAUGHLIN, K. F., Ph.D. (Iowa) Asst. Prof., U. S. Naval Acad., Annapolis, Md.
McLAUGHLIN, MARY L., A.B. (Regis C.) Grad. Student, Boston Univ., Boston, Mass. *9 Ellis St., Quincy 69, Mass.*
McLEAN, JESSIE, M.S. (Washington U.) Teacher, Harris Teachers and Junior Coll., St. Louis, Mo. *3676a Dover Pl.* (*16*)
McLEOD, S. A., M.A. (North Carolina) Dean, Lander Coll., Greenwood, S.C. *7 Sunset Dr.*
McLYNN, J. M., Mathematician, Engineering Research Associates. *239 Hawaii Ave., N.E., Washington, D.C.*
McMAHON, F. A., Ed.D. (Columbia) Sperry Gyroscope Co., Great Neck, N.Y. *84-12 35th Ave., Apt. 3N, Jackson Heights, N.Y.*
McMURDO, R. B., B.S.E.E. (U. S. Naval Acad.) Instr., Montana State Coll., Bozeman, Mont. *32 Central Ave., Whitefish, Mont.*
McMURTRIE, K. A., M.S. (Syracuse) *General Delivery, West Point Pleasant, N.J.*
McNABB, W. K., A.M. (Michigan) Instr., Hockaday Junior Coll., Dallas 6, Tex.
McNAIR, J. S. M.S. (Chicago) Instr., Canal Zone Junior Coll., Balboa Heights, Canal Zone
McNEAL, R. L., B.S. in E.E. (Purdue) Statistical Engr., General Motors Proving Ground, Milford, Mich.
McNEARY, S. S., A.M. (Pennsylvania) Asso. Prof., Drexel Inst. of Tech., Philadelphia 4, Pa.
McPHERSON, J. C., B.A. (Alabama Poly.) Asst., Duke Univ., Durham, N.C.
McPHERSON, J. R., B.S. (North Dakota) Teacher, Benson County Agr. School, Maddock, N.D. *Leeds, N.D.*
McSHANE, E. J., Ph.D. (Chicago) Prof., Univ. of Virginia, Charlottesville, Va. *209 Maury Ave.*
McVOY, K. W., JR., Student, New College, Oxford, England
MEAD, SALLIE P. (Mrs.), A.M. (Columbia) Member, Tech. Staff, Bell Telephone Labs., 463 West St., New York 24, N.Y.
MEADOR, G. E., A.M. (Oklahoma) Prof., Oklahoma City Univ., Oklahoma City, Okla. *1701 N.W. 30th St.*
MEADOWS, P. E., M.A. (Ohio State) Asso. Prof., Carroll Coll., Waukesha, Wis. *100 Racine Ave.*
MEANS, E. A., Student, Friends Univ., Wichita, Kan. *801 Maple* (*12*)
MEARS, FLORENCE M., Ph.D. (Cornell) Prof., George Washington Univ., Washington 6, D.C.
MEBANE, W. N., JR., A.M. (Cornell) Prof., Davidson Coll., Davidson, N.C.

MEDER, A. E., JR., A.M. (Columbia) Dean, Rutgers Univ., New Brunswick, N.J.
MEDLIN, G. W., M.A. (North Carolina) Part-time Instr., Univ. of North Carolina, Chapel Hill, N.C. *32 Davie Circle*
MEHLENBACHER, L. E., Ph.D. (Michigan) Prof., Univ. of Detroit, Detroit 21, Mich. *1207 Longfellow Ave., Royal Oak, Mich.*
MEIER, J. A., Student, Franklin and Marshall Coll., Lancaster, Pa. *139 Montrose Ave., Bausman, Pa.*
MEIER, PAUL, M.A. (Princeton) Forrestal Research Center, Princeton, N.J. *39 Vandeventer Ave.*
MEJIA, J. E., E.E. (Rensselaer) Chief, Bridge Div., Highway Dept., San Salvador, El Salvador. *2a. C. P. y 21 Av. SUR*
MELA, D. F., M.A. (Michigan) *4209 Four Mile Run Dr., S., Arlington 4, Va.*
MELGAR, A. V., Secretary. *P.O. Box 45, Banco Internacional del Peru, Arequipa, Peru*
MELVILLE, C. E., A.B. (Northwestern) Emeritus Prof., Clark Univ., Worcester, Mass. *950 Main St. (3)*
MELVIN, D. S. C., B.S. (Maryland) Grad. Asst., Univ. of Maryland, College Park, Md. *3927 Cranston Ave., Baltimore 29, Md.*
MENDELSOHN, N. S., Ph.D. (Toronto) Asso. Prof., Univ. of Manitoba, Winnipeg, Man., Canada
MENDELSON, BERT, B.A. (Columbia) Grad. Asst., Univ. of Nebraska, Lincoln, Nebr.
MENDENHALL, WILLIAM, III, M.S. (Bucknell) Lt. (j.g.), U. S. Navy. *213 N. Arch St., Montoursville, Pa.*
MENGER, KARL, Ph.D. (Vienna) Prof., Illinois Inst. of Tech., Chicago, Ill. *5506 N. Wayne Ave. (40)*
MENKE, H. E., A.M. (Ohio State) Asso. Prof., Heidelberg Coll., Tiffin 4, Ohio. *267 E. Market St. (3)*
MERGENDAHL, T. E., M.S. (Tufts) Prof., Tufts Coll., Medford 55, Mass. *128 Professor's Row*
MERKEL, R. B., A.B. (Sacramento) Instr., Sacramento Unified School District, Sacramento, Calif. *4915 Lawrence Dr. (17)*
MERKES, E. P., M.S. (DePaul) Lecturer, DePaul Univ., Chicago, Ill. *6405 S. Rockwell (29)*
MERRILEES, D. S., B.A. (Syracuse) Grad. Student, Indiana Univ., Bloomington, Ind. *Box 343, Apalachin, N.Y.*
MERRILL, A. S., Ph.D. (Chicago) Prof., Montana State Univ., Missoula, Mont.
MERRILL, L. L., Ph.D. (Rensselaer) Clarkson Coll., Potsdam, N.Y. *56 Leroy St.*
MERRILL, R. B., B.S. (Purdue) Statistician, Quality Control Dept., Wisconsin Steel Works, Chicago, Ill. *261 Vine St., Hammond, Ind.*
MERRIMAN, G. M., Ph.D. (Cincinnati) Prof., Univ. of Cincinnati, Cincinnati 21, Ohio
MERTIE, J. B., JR., Ph.D. (Johns Hopkins) Senior Geologist, U. S. Geological Survey, Room 12, Bldg. 213, Naval Gun Factory, Washington 25, D.C.
MESERVE, B. E., Ph.D. (Duke) Asst. Prof., Univ. of Illinois, Urbana, Ill. *121 W. Franklin*
MESNER, D. M., M.S. (Northwestern) Grad. Student, Michigan State Coll., East Lansing, Mich. *700 Tisdale Ave., Lansing 10, Mich.*
MESSICK, C. A., Ph.D. (Chicago) Retired. *1100 Tyler St., Topeka, Kan.*
MESSICK, J. F., Ph.D. (Johns Hopkins) Emeritus Prof., Emory Univ., Emory University, Ga. *1096 Clifton Rd., N.E., Atlanta, Ga.*
METTLER, J. W., M.A. (Bucknell) Supervisor, Educational Testing Service, 20 Nassau St., Princeton, N.J.
MEWALDT, N. H., M.A. (Wisconsin) Prof., Northern State Teachers Coll., Aberdeen, S.D.
MEWBORN, A. B., Ph.D. (C.I.T.) *P.O. Box 1748, Monterey, Calif.*
MEYER, B. C., Ph.D. (Stanford) Asst. Prof., Univ. of Arizona, Tucson, Ariz.
MEYER, GENEVIEVE T. (Mrs.), M.A. (Marquette) Instr., Univ. of Wisconsin in Milwaukee, Milwaukee, Wis. *535 E. Homer St.*
MEYER, H. A., Ph.D. (Iowa) Asso. Prof., Univ. of Florida, Gainesville, Fla. *1929 N.W. 7th Lane*
MEYER, H. L., JR., Ph.D. (Chicago) Asso. Prof., Univ. of Chicago, Chicago 37, Ill.
MEYER, HERMAN, Ph.D. (Chicago) Head of Dept., Univ. of Miami, Coral Gables, Fla. *2470 S.W. 24th St., Miami, Fla.*
MEYER, RUDOLF, B.A. (Buffalo) Bell Aircraft Corp., Niagara Falls, N.Y. *318 Paramount Pkwy., Kenmore 17, N.Y.*
MICHAL, A. D., Ph.D. (Rice) Prof., California Inst. of Tech., Pasadena 4, Calif.
MICHALIK, E. R., M.A. (Pittsburgh) Asst. Prof., Univ. of Pittsburgh, Pittsburgh, Pa.
MICHALUP, ERIC, Actuary, Compania Nac. Anonima de Seguros, La Previsora, Apartado 848, Caracas, Venezuela
MICHEL, R. J., Ph.D. (Missouri) Head of Dept., Southeast Missouri State Teachers Coll., Cape Girardeau, Mo. *225 N. Ellis St.*
MICHIE, J. N., A.M. (Michigan) Emeritus Prof., Texas Tech. Coll., Lubbock, Tex. *2110 17th St.*
MICKLE, E. J., Ph.D. (Ohio State) Prof., Ohio State Univ., Columbus 10, Ohio
MICULKA, V. A., M.S. (Texas Tech.) Instr., Frank Phillips Coll., Borger, Tex. *1212 Cooley Dr.*
MIDDLEMISS, R. R., M.S. (Colorado) Prof., Washington Univ., St. Louis 5, Mo.

MIELKE, P. T., Ph.D. (Purdue) Asst. Prof., Wabash Coll., Crawfordsville, Ind.
MIHALEK, R. J., Student, Illinois Inst. of Tech., Chicago, Ill. *3122 N. Spaulding Ave. (18)*
MIKSA, F. L., Switchman, Illinois Bell Telephone Co., Aurora, Ill. *613 Spring St.*
MILAM, ANITA B. (Mrs.), M.A. (Maryland) Instr., Washington Missionary Coll., Takoma Park, Washington 12, D.C.
MILES, E. J., Ph.D. (Chicago) Asso. Prof., Yale Univ., New Haven, Conn. *87 Marvel Rd.*
MILES, E. P., JR., Ph.D. (Duke) Asso. Prof., Alabama Poly. Inst., Auburn, Ala.
MILES, H. J., Ph.D. (California) Prof., Univ. of Illinois, Urbana, Ill.
MILGRAM, MARTIN, *1364 E. 53rd St., Brooklyn 34, N.Y.*
MILKMAN, JOSEPH, Ph.D. (N.Y.U.) Asst. Prof., U. S. Naval Acad., Annapolis, Md. *71 Shipwright St.*
MILLER, AKELEY, *717 E. Main St., Vermillion, S.D.*
MILLER, A. E., B.A. (Pennsylvania) Asst. Research Engr., Burroughs Adding Machine Co., Philadelphia, Pa. *7316 N. 21st St. (38)*
MILLER, A. L., Ph.D. (Harvard) Treasurer and General Manager, C. N. Miller Co., Boston, Mass. *28 Clark Rd., Babson Park 57, Mass.*
MILLER, C. E., Ph.D. (Toronto) Prof., Univ. of Saskatchewan, Saskatoon, Sask., Canada
MILLER, C. F., Student, George Pepperdine Coll., Los Angeles, Calif. *232 N. Belmont (26)*
MILLER, D. D., Ph.D. (Michigan) Lt. Comdr., U. S. Navy. *3040 Idaho Ave., N.W., Apt. 116, Washington 16, D.C.*
MILLER, E. B., A.M. (Chicago) Prof., Illinois Coll., Jacksonville, Ill. *1252 W. College Ave.*
MILLER, E. D., A.B. (Stanford) Head of Dept., Yuba Junior Coll., Marysville, Calif.
MILLER, F. D., A.M. (Michigan) Mathematician, Bell Aircraft Corp., Niagara Falls, N.Y. *116 Parkwood Ave., Kenmore 17, N.Y.*
MILLER, F. H., Ph.D. (Columbia) Prof., Cooper Union, Cooper Square, New York 3, N.Y.
MILLER, G. T., Ph.D. (Purdue) Asst. Prof., Purdue Univ., Lafayette, Ind. *601 Dodge St., West Lafayette, Ind.*
MILLER, HARLAN C., Ph.D. (Texas) Prof., Texas State Coll. for Women, Denton, Tex.
MILLER, IRWIN, Student, Alfred Univ., Alfred, N.Y. *3710 Ave. S, Brooklyn, N.Y.*
MILLER, J. J., B.S. in M.E. (New Mexico) Supervisor, App. Math. Section, Sandia Corp., Albuquerque, N.M. *220 N. Solano*
MILLER, K. W., M.S. in E.E. (Union) Asst. Dir. of Research, Armour Research Foundation, Chicago 16, Ill. *10431 S. Oakley (43)*
MILLER, L. H., Ph.D. (Ohio State) Asst. Prof., Ohio State Univ., Columbus, Ohio. *1671 Rhoda Ave. (12)*
MILLER, LEONARD, A.M. (Brooklyn) Chief, Research and Statistics, New York City Veteran Service Centers, New York, N.Y. *77 Linden Blvd., Brooklyn, N.Y.*
MILLER, NORMAN, Ph.D. (Harvard) Prof., Queen's Univ., Kingston, Ont., Canada
MILLER, R. A., M.A. (Mississippi) Asst. Prof., Univ. of Mississippi, University, Miss.
MILLER, W. I., Ph.D. (Pittsburgh) Asso. Prof., Bucknell Univ., Lewisburg, Pa. *220 S. Third St.*
MILLER, W. M., Ph.D. (Illinois) Retired. *78 Woodside Ave., Amherst, Mass.*
MILLMAN, GEORGE, M.A. (Columbia) Mathematician, Army Map Service, Washington 16, D.C. *1287 Brentwood Rd., N.E. (18)*
MILLS, C. N., Ph.D. (Wisconsin) Prof., Illinois State Normal Univ., Normal, Ill. *304 Virginia Ave.*
MILLS, W. H., Ph.D. (Princeton) Instr., Yale Univ., New Haven, Conn. *219 Cooper Pl. (15)*
MILLSAPS, KNOX, Ph.D. (C.I.T.) Prof., MCRR, Wright-Patterson Air Force Base, Dayton, Ohio
MILNE, W. E., Ph.D. (Harvard) Prof., Oregon State Coll., Corvallis, Ore. *Route 4, Box 55*
MILOS, J. F., A.M. (Columbia) Asso. Prof., U. S. Naval Acad., Annapolis, Md.
MILOSEVICH, KOVINA, Student, Univ. of Skoplje, Skoplje, Yugoslavia. *ul. Marksova Br. 44*
MINAS, J. S., Student, Wayne Univ., Detroit 1, Mich. *3756 W. Grand Ave. (6)*
MINO, P. M., Rev., M.A. (Columbia) Teacher, St. Francis Coll., Loretto, Pa.
MINOIS, SERGE, Agregation des Sc.Math. (Paris) Asst. Prof., Lycee Lakanol e Sceaux., 7 Rue Candelot, Bourg-la-Reine (Seine), France
MINTON, P. D., M.S. (Southern Methodist) Grad. Student, Univ. of North Carolina, Chapel Hill, N.C. *127 Jackson Circle*
MIRICK, G. R., A.M. (Columbia) Asst. Dir., Div. of Teachers College Schools, Teachers College, New York, N.Y. *525 W. 120th St. (27)*
MIRSKY, LEONID, M.Sc. (London) Lecturer, Univ. of Sheffield, Sheffield, England
MISER, H. J., Ph.D. (Ohio State) Operations Analyst, Hq. U. S. Air Force, Washington 25, D.C. *2713 Blaine Dr., Chevy Chase 15, Md.*
MISER, NELLIE P. (Mrs. W. L.), A.B. (Huron C.) McKendree Coll., Lebanon, Ill.
MISER, W. L., Ph.D. (Chicago) Retired Prof., Vanderbilt Univ., Nashville 4, Tenn. Member of Faculty: McKendree Coll., Lebanon, Ill.
MISRA, B. B., M.S. (Michigan) Asst. Prof., Ravenshaw Coll., Cuttack, India

MITCHELL, A. K., Ph.D. (Johns Hopkins) Aerodynamicist, Applied Physics Lab., Johns Hopkins Univ., Silver Spring, Md. *6306 Maple Ave., Chevy Chase 15, Md.*
MITCHELL, BENJAMIN ERNEST, Ph.D. (Columbia) Prof., Univ. of Mississippi, University, Miss.
MITCHELL, BENJAMIN EVANS, Ph.D. (Wisconsin) Asst. Prof., Alabama Poly. Inst., Auburn, Ala.
MITCHELL, JOSEPHINE M., Ph.D. (Bryn Mawr) Asst. Prof., Univ. of Illinois, Urbana, Ill.
MITCHELL, M. L., A.B. (Hofstra) Actuarial Student, New York Life Insurance Co., New York, N.Y. *200 W. Merrick Rd., Baldwin, L.I., N.Y.*
MITCHELL, W. W., JR., M.S. (Colorado) Instr., Phoenix Coll., Phoenix, Ariz. *1333 W. Mulberry Dr.*
MITTMAN, BENJAMIN, B.S. (Illinois Tech.) Teaching Asst., Univ. of California at Los Angeles, Los Angeles 24, Calif. *1815½ S. Colby Ave. (25)*
MODE, E. B., A.M. (Harvard) Prof., Boston Univ., Boston 15, Mass. *33 Longmeadow Rd., Wellesley 81, Mass.*
MOELLER, A. C., M.S. (Michigan) Instr., Marquette Univ., Milwaukee 3, Wis.
MOLINA, E. C., Special Lecturer, Newark Coll. of Engineering, Newark 2, N.J. *141 Dodd St., East Orange, N.J.*
MOLIVER, MARTIN, Ed.M. (Temple) Teacher, Benjamin Franklin High School, Philadelphia, Pa. *1620 E. Washington Lane (38)*
MOLLER, R. W., Ph.D. (Catholic U.) Asst. Prof., Catholic Univ. of America, Washington, D.C. *604 Girard St., N.E. (17)*
MOLLOY, C. T., Ph.D. (N.Y.U.) Head of Math. Section, Kellex Corp., New York 7, N.Y. *5 Corey Rd., Denville, N.J.*
MONTAGUE, HARRIET F., Ph.D. (Cornell) Prof., Univ. of Buffalo, Buffalo 14, N.Y.
MONTGOMERY, A. G., M.A. (Minnesota) Asst. Prof., Coll. of St. Thomas, St. Paul, Minn.
MONTGOMERY, DEANE, Ph.D. (Iowa) Prof., Inst. for Advanced Study, Princeton, N.J.
MONTGOMERY, J. T., B.S. (Notre Dame) Teaching Fellow, Univ. of Notre Dame, Notre Dame, Ind.
MONTGOMERY, MABEL D., M.A. (Buffalo) Instr., Univ. of Buffalo, Buffalo, N.Y. *81 Princeton Blvd., Kenmore 17, N.Y.*
MONTROLL, E. W., Ph.D. (Pittsburgh) Head of Physics Branch, Office of Naval Research, Navy Bldg., Code 421, Washington, D.C.
MOOMAW, D. W., Student, Univ. of Nebraska, Lincoln, Neb. *1233 S. 24th (2)*
MOON, PARRY, M.S. (M.I.T.) Asso. Prof., Massachusetts Inst. of Tech., Cambridge, Mass.
MOONEY, MOTHER MARY GERARD, M.A. (Creighton) Teacher, Ursuline Coll., New Orleans 18, La. *2635 State St.*
MOORE, A. H., B.M.E. (Pratt Inst.) Instr., Pratt Inst., Brooklyn, N.Y. *1585 Unionport Rd., New York 62, N.Y.*
MOORE, B. C., A.M. (Princeton) Asso. Prof., Texas A. and M. Coll., College Station, Tex.
MOORE, C. N., Ph.D. (Harvard) Prof., Univ. of Cincinnati, Cincinnati, Ohio. *219 Woolper Ave.*
MOORE, DEWEY, A.B. (Berea Coll.) Aeronautical Research Scientist, National Advisory Committee for Aeronautics, Langley Field, Va. *1005 Indiana, Newport News, Va.*
MOORE, D. C., A.M. (Emory) Asst. Prof., Emory Junior Coll., Valdosta, Ga.
MOORE, E. F., Ph.D. (Missouri) Prof., School of Business, Baylor Univ., Waco, Tex.
MOORE, G. E., Ph.D. (Illinois) Asso. Prof., Univ. of Illinois, Urbana, Ill. *1113 W. Union St., Champaign, Ill.*
MOORE, LILLIAN, Ed.D. (N.Y.U.) Teacher, Far Rockaway High School, Far Rockaway, New York, N.Y. *885 N. 28th St., Philadelphia 30, Pa.*
MOORE, L. T., Ph.D. (Johns Hopkins) Asso. Prof., Brooklyn Coll., Brooklyn, N.Y. *205 Hicks St.*
MOORE, M. G., Ph.D. (Illinois) Prof., Bradley Univ., Peoria 5, Ill. *503 W. Melbourne St.*
MOORE, MARIE A., D.Ed. (Colorado) Prof., Harris Teachers Coll., St. Louis, Mo. *6209 Murdoch (9)*
MOORE, T. W., Ph.D. (Yale) Asso. Prof., U. S. Naval Acad., Annapolis, Md.
MOORE, W. A., A.M. (Chicago) Emeritus Prof., Birmingham-Southern Coll., Birmingham 4, Ala.
MOORE, W. L., Ph.D. (Illinois) Prof., Univ. of Louisville, Louisville, Ky. *R.F.D. 2, Box 59, Coral Ridge, Ky.*
MOORMAN, R. H., Ph.D. (Peabody) Prof., Tennessee Poly. Inst., Cookeville 10, Tenn.
MOOTS, E. E., Ph.D. (Iowa) Prof., Cornell Coll., Mt. Vernon, Iowa. *710 Eighth Ave. N.*
MORALES, J. A., M.A. (Florida) Instr., Univ. of Florida, Gainesville, Fla. *1211 N.W. Tenth Ave.*
MORAN, C. W., Ph.D. (Illinois) Instr., Wright Junior Coll., Chicago 45, Ill. *2950 Jarlath St. (45)*
MORDELL, L. J., M.A. (Cambridge) Prof., St. John's Coll., Cambridge Univ., Cambridge, England
MORDUCHOW, MORRIS, D.Ae.E. (Brooklyn Poly.) Asst. Prof., Poly. Inst. of Brooklyn, Brooklyn 2, N.Y. *1470 Sterling Pl. (13)*
MOREDOCK, H. S., JR., Ph.D. (California) Asst. Prof., Sacramento State Coll., Sacramento 18, Calif.

MORELLI, MICHAEL, S.B. (M.I.T.) *Civilian Housing, Room 720, Hq. Feamcom, A.P.O. 323, San Francisco, Calif.*
MORELOCK, J. C., M.A. (Missouri) Teaching Asst., Univ. of Florida, Gainesville, Fla.
MORENO, JENARO, Grad. (Inst. Pedag.) Prof., Inst. Pedagogico, Santiago, Chile
MORENUS, EUGENIE M., Ph.D. (Columbia) Emeritus Prof., Sweet Briar Coll., Sweet Briar, Va. *Cleveland, N.Y.*
MORGAN, EDITH L. (Mrs. Joseph), M.A. (Texas Christian) Instr., Texas Christian Univ., Fort Worth 9, Tex.
MORGAN, F. M., Ph.D. (Cornell) Headmaster, Clark School, Hanover, N.H.
MORGAN, J. E., Student, St. Mary's Univ. of Texas, San Antonio, Tex. *819 Victoria St.*
MORGAN, K. B., Ed.D. (Columbia) Instr., Mt. Kisco High School, Mt. Kisco, N.Y. *20 Carpenter Ave.*
MORGAN, W. D., Traffic Analyst, D. L. Piazza Co., Minneapolis 3, Minn. *1764 St. Anthony Ave., St. Paul 4, Minn.*
MORISON, J. S., M.S. (Illinois) Mathematician, Douglas Aircraft Co., Santa Monica, Calif. *1445-C Stanford St.*
MORLAN, SABRINA, M.S. (Michigan) Instr., Univ. of Wichita, Wichita, Kan.
MORLEY, R. K., Ph.D. (Clark) Prof., Worcester Poly. Inst., Worcester, Mass. *43 Laconia Rd. (5)*
MORREL, J. S., Ph.D. (Illinois) Principal Research Engr., Bendix Radio, Baltimore 4, Md. *Circle Rd., Ruxton 4, Md.*
MORREY, C. B., JR., Ph.D. (Harvard) Prof., Univ. of California, Berkeley, Calif. *210 Yale Ave. (8)*
MORRILL, D. E., B.A. (Mississippi) Lt., U. S. Army
MORRILL, W. K., Ph.D. (Johns Hopkins) Asso. Prof., Johns Hopkins Univ., Baltimore 18, Md.
MORRIS, C. C., A.M. (Harvard) Emeritus Prof., Ohio State Univ., Columbus, Ohio. *1100 Urlin Ave. (12)*
MORRIS, C. R., M.A. (Kentucky) Mathematician, U. S. Navy Hydrographic Office, Suitland, Md. *3118 Parkway Terr. Dr., Apt. 5*
MORRIS, F. R., Ph.D. (California) Prof., Fresno State Coll., Fresno 4, Calif.
MORRIS, MAX, Ph.D. (Chicago) Prof., Case Inst. of Tech., Cleveland, Ohio
MORRISON, D. R., Ph.D. (Wisconsin) Asst. Prof., Tulane Univ., New Orleans, La. *Apt. 69-H, Stadium Pl.*
MORRISON, MILTON, M.A. (Columbia) Instr., Stevens Inst. of Tech., Hoboken, N.J.
MORRISON, NATHAN, A.B. (Brooklyn C.) Principal Actuary, N.Y. State Div. of Placement and Unemployment Insurance, New York 17, N.Y. *196 Elm St., New Rochelle, N.Y.*
MORRISON, R. D., M.S. (Oklahoma A.&M.) Asst. Prof., Oklahoma A. and M. Coll., Stillwater, Okla.
MORROW, D. C., Ph.D. (Chicago) Asso. Prof., Wayne Univ., Detroit 1, Mich.
MORROW, H. W., JR., M.A. (South Dakota) Instr., Industrial Engg., Univ. of Florida, Gainesville, Fla.
MORSE, D. S., Ph.D. (Cornell) Prof., Union Coll., Schenectady 8, N.Y.
MORSE, J. T., B.S. (Case) Asst., Massachusetts Inst. of Tech., Cambridge 39, Mass.
MORSE, MARSTON, Ph.D. (Harvard) Prof., Inst. for Advanced Study, Princeton, N.J.
MORTELL, H. DOROTHY, M.A. (Catholic U.) Instr., Catholic Univ. of America, Washington, D.C. *1040 Otis St., N.E. (17)*
MORTOLA, A. J., M.S. (C.C.N.Y.) St. Peter's Coll., Jersey City 6, N.J. *884 Riverside Dr., New York 32, N.Y.*
MOSER, LEO, Ph.D. (North Carolina) Lecturer, Univ. of Alberta, Edmonton, Alta., Canada
MOSESSON, Z. I., Ph.D. (Harvard) Chief Actuarial Asst., Prudential Insurance Co. of America, Newark 1, N.J. *104 N. Munn Ave., East Orange, N.J.*
MOSEY, ABIGAIL M., M.A. (Syracuse) Instr., Hobart and William Smith Colls., Geneva, N.Y.
MOSKOVITZ, DAVID, Ph.D. (Brown) Prof., Carnegie Inst. of Tech., Pittsburgh 13, Pa.
MOSKOWITZ, P. M., B.S. (Columbia) Physicist, New York Naval Shipyard, Brooklyn, N.Y. *626 E. 51st St. (3)*
MOSS, J. H., JR., M.S. (N.Y.U.) Asst. Prof., Alabama Poly. Inst., Auburn, Ala.
MOSSMAN, THIRZA A., A.M. (Chicago) Asso. Prof., Kansas State Coll., Manhattan, Kan.
MOSTELLER, F. C., Ph.D. (Princeton) Prof., Harvard Univ., Cambridge 38, Mass. *28 Pierce Rd., Belmont, Mass.*
MOSTON, L. T., Ph.D. (Harvard) Dean, Waynesburg Coll., Waynesburg, Pa. *498 Second Ave.*
MOTT, G. R., Student, Hofstra Coll., Hempstead, N.Y. *11 Fourth St., Oceanside, N.Y.*
MOULTON, C. E., B.A. (Miami U.) Grad. Student, Univ. of Buffalo, Buffalo 14, N.Y. *West Main St., Box 336, Arcade, N.Y.*
MOULTON, E. J., Ph.D. (Chicago) Prof., Northwestern Univ., Evanston, Ill. *840 Apple Tree Lane, Glenview, Ill.*
MOULTON, F. R., Ph.D. (Chicago) Administrative Secretary, A.A.A.S., 1515 Massachusetts Ave., N.W., Washington 5, D.C.
MOUNT, B. H., JR., M.S. (Princeton) Asst. Prof., Univ. of Pittsburgh, Pittsburgh 21, Pa.

MOURSUND, A. F., Ph.D. (Brown) Prof., Univ. of Oregon, Eugene, Ore.
MOUZON, E. D., JR., Ph.D. (Illinois) Prof., Southern Methodist Univ., Dallas 5, Tex.
MOYE, W. B., M.S. (Mercer U.) Asst. Prof., Georgia Teachers Coll., Collegeboro, Ga.
MOYLS, B. N., Ph.D. (Harvard) Asst. Prof., Univ. of British Columbia, Vancouver, B.C., Canada
MUGUERCIA, R. V., B.S.L. (Inst. de Stgo. de Cuba) Prof., Colegio Bautista El Salvador, Calle 3a, No. 455, Rpto. Sueno, Santiago de Cuba, Cuba. *Calle A #206*
MUHLY, H. T., Ph.D. (Johns Hopkins) Asso. Prof., State Univ. of Iowa, Iowa City, Iowa
MULCRONE, T. F., M.S. (Catholic U.) Asst. Prof., Spring Hill Coll., Spring Hill Station, Mobile, Ala.
MULLAN, C. E., Ph.D. (Pittsburgh) Servomechanisms Engr., Bell Aircraft Corp., Niagara Falls, N.Y. *340 Westmoreland Rd., Snyder, N.Y.*
MULLER, ELSIE C., M.A. (Michigan) Instr., LaSalle-Peru Twp. High School and Junior Coll., LaSalle, Ill.
MULLER, G. M., M.S. (Yale) Grad. Student, Harvard Univ., Cambridge 38, Mass.
MULLIGAN, J. H., JR., Ph.D. (Columbia) Asst. Prof., Dept. of Elec. Engg., New York Univ., New York 53, N.Y.
MULLINGS, M. E., Ph.D. (Cincinnati) Prof., Abilene Christian Coll., Abilene, Tex.
MULLINS, G. W., Ph.D. (Columbia) Emeritus Prof., Barnard Coll., New York 27, N.Y.
MUNDHJELD, SIGURD, Ph.D. (Nebraska) Prof., Concordia Coll., Moorhead, Minn.
MUNRO, W. D., Ph.D. (Minnesota) Asst. Prof., Univ. of Minnesota, Minneapolis 14, Minn. *39 Clarence Ave., S.E. (14)*
MUNSHOWER, C. W., Ph.D. (N.Y.U.) Prof., Colgate Univ., Hamilton, N.Y.
MURDOCH, D. C., Ph.D. (Toronto) Asso. Prof., Univ. of British Columbia, Vancouver, B.C., Canada
MURDOCK, W. L., A.B. (Cornell) Mathematician, General Electric Electronics Lab., Syracuse, N.Y. *61 Oswego St., Baldwinsville, N.Y.*
MURNAGHAN, F. D., Ph.D. (Johns Hopkins) Prof., Instituto Tecnologico de Aeronautica, Sao Jose dos Campos, Estado de Sao Paulo, Brazil
MURPHY, C. H., JR., M.A. (Johns Hopkins) Mathematician, Ballistic Research Lab., Aberdeen Proving Ground, Md.
MURPHY, MARY L., M.S. (Boston C.) Secretary, Math. Dept., Boston Coll., Chestnut Hill 67, Mass. *70 Lowell Ave., Watertown 72, Mass.*
MURPHY, R. R., M.A. (Oklahoma) Prof., Panhandle A. and M. Coll., Goodwell, Okla.
MURRAY, C. A., A.M. (Texas) Prof., West Texas State Coll., Canyon, Tex.
MURRAY, F. J., Ph.D. (Columbia) Prof., Columbia Univ., New York, N.Y. *2 Arden St. (34)*
MURRAY, J. G., M.A. (Catholic U.) Actuarial Clerk, Metropolitan Life Insurance Co., 1 Madison Ave., New York, N.Y. *150 Rutland Rd., Brooklyn 25, N.Y.*
MURRAY, J. J., Rev., M.A. (Gonzaga) Instr., Gonzaga Univ., Spokane, Wash.
MURRAY, J. P., Rev., M.A. (Boston C.) Asso. Prof., Fairfield Univ., Fairfield, Conn.
MURRAY, S. B., M.S. (Chicago) Asso. Prof., Mississippi State Coll., State College, Miss.
MURRAY, V. F., B.S. (St. Andrew's, Scotland) *Box 247, Hoboken, N.J.*
MURRAY, W. R., M.S. (Cornell) Mathematician, Armed Forces Security Agency, Dept. of Defense, Washington 25, D.C. *4001 N. 5th St., Arlington, Va.*
MUSCH, E. J., M.A. (Kent) Instr., Univ. of Louisville, Louisville, Ky. *145 E. Kingston Ave. (8)*
MUSSELMAN, J. R., Ph.D. (Johns Hopkins) Prof., Western Reserve Univ., Cleveland 6, Ohio
MYATT, D. J., M.M.E. (Louisiana) Instr., Antioch Coll., Yellow Springs, Ohio
MYERS, E. F., A.B. (Geneva) Instr., Univ. of Pittsburgh, Pittsburgh, Pa. *322 13th Ave., New Brighton, Pa.*
MYERS, N. G., JR., M.A. (Penn. State) Statistician, Lord Manufacturing Co., Erie, Pa. *3409 Peach St.*
MYERS, S. B., Ph.D. (Harvard) Prof., Univ. of Michigan, Ann Arbor, Mich.
MYERS, S. S., M.Ed. (Cincinnati) Instr., Ohio State Univ., Columbus, Ohio. *186 W. Woodruff, Apt. 1 (1)*
MYERS, W. H., Ph.D. (Stanford) Prof., San Jose State Coll., San Jose, Calif.

NACE, H. W., M.A. (Cornell) Prof., Lawrence Inst. of Tech., Detroit, Mich. *4223 Robina Ave., Berkeley, Mich.*
NAHIKIAN, H. M., Ph.D. (North Carolina) Asso. Prof., North Carolina State Coll., Raleigh, N.C.
NAIDITCH, SAM, Ph.D. (C.I.T.) Asst. Prof., Ohio State Univ., Columbus, Ohio
NANAGARA, PRASERT, B.S. (Philippines) Secretary-General, Univ. of Agriculture, Bangkhen, Bangkok, Thailand. *On leave:* Grad. Student, Cornell Univ., Ithaca, N.Y. *107 Harvard Pl.*
NASH, E. E., M.S. (Rensselaer) Asst. Prof., Rensselaer Poly., Troy, N.Y. *Averill Park, N.Y.*
NASH, F. P., A.M. (Columbia) Teacher, Groton School, Groton, Mass.
NASIR, A. R., M.A. (Panjab U.) Asst. Prof., Panjab Coll. of Engg. & Tech., Lahore, Pakistan, *38, Kucha Chughatian, Mochi*

NASSAU, J. J., Ph.D. (Syracuse) Dir. of Observatory, Case Inst. of Tech., Cleveland 6, Ohio
NASTUCOFF, P. M., Diploma (U. of Moscow) Instr., Univ. of Notre Dame, Notre Dame, Ind.
NATHAN, D. S., Ph.D. (Cincinnati) Asst. Prof., City Coll. of New York, New York, N.Y. *3525 Perry Ave. (67)*
NAUMAN, GERTRUDE C., *International House, 1414 E. 59th St., Chicago 37, Ill.*
NEALE, A. B., B.S. in M.E. (Detroit Tech.) Mechanical Engr., Naval Aircraft Factory, Philadelphia, Pa. *1524 Clairmount St., Apt. 4-A (6)*
NEELLEY, J. H., Ph.D. (Yale) Prof., Carnegie Inst. of Tech., Pittsburgh, Pa. *300 Broadmoor Ave. (16)*
NEELY, W. H., M.S. (Southern California) Asst. Prof., Texas Christian Univ., Fort Worth, Tex.
NEFF, I. F., M.S. (Chicago) Emeritus Prof., Drake Univ., Des Moines, Iowa. *2801 Brattleboro Ave. (11)*
NEFF, J. D., M.S. (Kansas S.C.) Grad. Asst., Purdue Univ., Lafayette, Ind.
NEHARI, ZEEV, Ph.D. (Hebrew U., Jerusalem) Prof., Washington Univ., St. Louis 5, Mo.
NEHRBAS, C. J., A.B. (C.C.N.Y.) Moses, Nehrbas and Tyler, 20 Pine St., New York 5, N.Y.
NEILSON, J. A. S., B.A. (British Columbia) *4448 W. Ninth Ave., Vancouver, B.C., Canada*
NEILSON, J. O., A.B. (Augustana C.) Cpl., U. S. Air Force
NEISIUS, W. V., M.S. (Emory U.) Asst. Prof., Georgia Inst. of Tech., Atlanta, Ga. *561 Page Ave., N.E.*
NELSON, A. L., Ph.D. (Chicago) Prof., Wayne Univ., Detroit 1, Mich.
NELSON, C. A., Ph.D. (Chicago) Prof., New Jersey Coll. for Women, Rutgers Univ., New Brunswick, N.J.
NELSON, H. E., Ph.D. (Iowa) Prof., Augustana Coll., Rock Island, Ill.
NELSON, J. B., A.M. (Southern California) Instr., Los Angeles City Coll., Los Angeles, Calif. *1667 Rotary Dr. (26)*
NELSON, MARY, M.S. (Iowa) Asst. Prof., Utah State Agr. Coll., Logan, Utah
NELSON, R. E., M.A. (Dartmouth) Asso. Prof., Dickinson Coll., Carlisle, Pa. *716 W. North St.*
NELSON, SARA L., Ph.D. (Cornell) Prof., Georgia State Coll. for Women, Milledgeville, Ga. *512 N. Columbia St.*
NELSON, T. A., M.Ed. (Maryland) *225 Jefferson St., N.W., Washington 11, D.C.*
NELSON, THEODORA S., M.S. (Illinois) Asst. Prof., Nebraska State Teachers Coll., Kearney, Neb.
NELSON, W. K., M.S. (Colorado) Asso. Prof., Univ. of Colorado, Boulder, Colo. *925 Grandview Ave.*
NEMECEK, VIVIAN, M.A. (Oklahoma) 514 N. Crawford, Norman, Okla.
NEMENYI, P. F., D.Sc. (Berlin) Physicist, Research Dept., Naval Ordnance Lab., Whiteoak, Md. *3015 Legation St., N.W., Washington, D.C.*
NEMMERS, F. E., M.S. (Iowa) Grad. Student, State Univ. of Iowa, Iowa City, Iowa. *2936 N. Hackett Ave., Milwaukee 11, Wis.*
NESBEDA, PAUL, Ph.D. (Pisa) Instr., Catholic Univ. of America, Washington, D.C. *3423 Oakwood Terr., N.W., Washington 10, D.C.*
NEUBAUER, GRETA, A.M. (Wyoming) Asst. Prof., Univ. of Wyoming, Laramie, Wyo.
NEUMANN, R. K., B.A. (George Washington) Mathematician, Applied Physics Lab., Johns Hopkins Univ., Silver Spring, Md. *930 Wayne Ave.*
NEUSTADTER, S. F., Ph.D. (California) Instr., Harvard Univ., Cambridge 38, Mass.
NEUWIRTH, S. I., M.S. (N.Y.U.) Bio-Statistician, Biological Research Labs., Schering Corp., 86 Orange St., Bloomfield, N.J.
NEVINS, W. V., III, A.M. (Columbia) Asst. Prof., Alfred Univ., Alfred, N.Y.
NEWBURG, E. A., Student, Purdue Univ., Lafayette, Ind. *1846 Belleview Pl., Indianapolis 22, Ind.*
NEWELL, C. R., B.A. (Toronto) Instr., Niagara Univ., Niagara University, N.Y. *1761 Peer St., Niagara Falls, Ont., Canada*
NEWELL, R. F., B.A. (South Carolina) Instr., N. Y. State Inst., 1685 Elmwood Ave., Buffalo 7, N.Y.
NEWHOUSE, ALBERT, Ph.D. (Chicago) Prof., Univ. of Houston, Houston, Tex. *5505 Grand Lake, Bellaire, Tex.*
NEWMAN, J. J., Student, Harvard Univ., Cambridge 38, Mass. *220 W. 107th St., New York 25, N.Y.*
NEWMAN, MORRIS, M.A. (Columbia) Radio Bldg., National Bureau of Standards, Washington 25, D.C.
NEWMAN, W. J., B.A. (De Paul) Grad. Student, De Paul Univ., Chicago, Ill. *7035 S. Parnell Ave. (21)*
NEWSOM, C. V., Ph.D. (Michigan) Asso. Commissioner for Higher Education, Univ. of the State of New York, Albany 1, N.Y. *50 Euclid Ave. (3)*
NEWSON, MARY W., Ph.D. (Göttingen) Emeritus Prof., Eureka Coll., Eureka, Ill. *Lake Dalecarlia, Lowell, Ind.*
NEWTON, ABBA V., Ph.D. (Chicago) Asso. Prof., Vassar Coll., Poughkeepsie, N.Y. *12 Collegeview Ave.*

NEWTON, G. A., A.M. (Trinity) Retired Prof., Trinity Univ., San Antonio, Tex. *613 Highland Ave., Waxahachie, Tex.*
NEWTON, T. A., B.S. (Colorado A.&M.) Grad. Student, Univ. of Georgia, Athens, Ga.
NIBARGER, AGNES E., M.S. (Wichita) Instr., Univ. of Wichita, Wichita, Kan. *1611 N. Hillside (14)*
NICHOLAS, C. P., B.S. (U. S. Military Acad.) Col. and Prof., U. S. Military Acad., West Point, N.Y.
NICHOLLS, R. S., B.Sc. (London) Miles Labs., Inc., Elkhart, Ind. *13 St. Joseph Manor*
NICHOLS, I. C., Ph.D. (Michigan) Emeritus Prof., Louisiana State Univ., Baton Rouge, La. *915 N. Sixth St.*
NICHOLS, LEONARD, Student, New Jersey State Teachers Coll., Upper Montclair, N.J. *14 Liberty St., Newton, N.J.*
NICHOLSON, W. A., JR., Transmission Man, American Tel. & Tel. Co., Charlotte, N.C. *911 S. Tryon St., Apt. 20*
NICKEL, J. A., M.S. (Oregon S.C.) Research Asst., Indiana Univ., Bloomington, Ind.
NICKERSON, HELEN K., (Mrs. W. J.), Ph.D. (Radcliffe) *Washington Rd., R.D. 3, Princeton, N.J.*
NICKL, A. F., A.B. (Queens C.) Instr., U. S. Merchant Marine Acad., Kings Point, N.Y.
NICKOL, J. P., Ph.D. (Fribourg) Prof., Physics, Rensselaer Poly. Inst., Troy, N.Y. *Box 837, Maxwell Rd., Newtonville, N.Y.*
NICOLET, JUSTIN, M.A. (Wisconsin) Structural Engr., Dept. of Subways and Highways, Chicago, Ill. *4148 N. Paulina St. (13)*
NIEDRAUER, R. V., Student, Univ. of Buffalo, Buffalo 14, N.Y. *134 Warren Ave., Kenmore 17, N.Y.*
NIELSEN, W. A., M.Ed. (U.C.L.A.) Bakersfield Junior Coll., Bakersfield, Calif. *1818 Terrace Way*
NIERSBACH, P. M., A.M. (Southern California) Head of Dept., High School, Bell, Calif. *6925 Passaic St., Huntington Park, Calif.*
NIKODYM, O. M., Ph.D. (Warsaw) Prof., Kenyon Coll., Gambier, Ohio
NILES, N. O., M.S. (St. Lawrence) Asst. Prof., U. S. Naval Acad., Annapolis, Md.
NILSON, E. N., Ph.D. (Harvard) Asst. Prof., Trinity Coll., Hartford, Conn.
NIVEN, IVAN, Ph.D. (Chicago) Prof., Univ. of Oregon, Eugene, Ore.
NOBLE, ANDREWA R., Ph.D. (California) *4489 Moraga Ave., Oakland 11, Calif.*
NOBLE, C. A., Ph.D. (Göttingen) Emeritus Prof., Univ. of California, Berkeley 4, Calif. *2224 Piedmont Ave.*
NOBLE, K. L., A.M. (Colorado S.C.) Asst. Prof., Univ. of Denver, Denver, Colo. *2925 W. Douglas Pl. (11)*
NODVIK, J. S., Student, Carnegie Inst. of Tech., Pittsburgh, Pa. *713 Second St., Canonsburg, Pa.*
NOLAN, GRACE M., M.A. (Loyola U.) Instr., Univ. of Illinois, Navy Pier, Chicago, Ill. *2308 W. 35th Pl. (9)*
NOLLER, RUTH B. (Mrs.), Ed.M. (Buffalo) Instr., Univ. of Buffalo, Buffalo 14, N.Y. *145 E. Morris Ave. (14)*
NOLSTAD, A. R., D.Ed. (Pittsburgh) Asst. Prof., North Carolina State Coll., Raleigh, N.C.
NORDEN, M. L., B.S. (M.I.T.) Mathematician, Johns Hopkins Univ., Operations Research Office, 6410 Connecticut Ave., Chevy Chase, Md.
NORDGAARD, M. A., Ph.D. (Columbia) Prof., Upsala Coll., East Orange, N.J.
NORDHAUS, E. A., Ph.D. (Chicago) Asso. Prof., Michigan State Coll., East Lansing, Mich.
NORDSTROM, C. H., A.M. (Lehigh) Scientific Research Div., Military Security Board, A.P.O. 742, c/o Postmaster, New York, N.Y.
NORMAN, P. B., Ph.D. (Columbia) Project Engr., Curtiss-Wright Corp., Carlstadt, N.J. *P.O. Box 302, Union, N.J*
NORRIS, M. J., Ph.D. (Harvard) Asst. Prof., Coll. of St. Thomas, St. Paul, Minn. *2111 Lincoln Ave. (5)*
NORRIS, R. E., A.M. (Illinois) Dean, State Teachers Coll., Milwaukee 11, Wis.
NORRIS, W. H., A.M. (Pennsylvania) Head of Dept., Maury High School, Norfolk, Va. *707 Redgate Ave. (7)*
NORSKOG, EDNA M., M.A. (Columbia) Instr., Illinois State Normal Univ., Normal, Ill.
NORTHROP, E. P., Ph.D. (Yale) Asso. Dean of the Coll., Univ. of Chicago, Chicago, Ill. *5464 Cornell Ave. (15)*
NORTON, FRANCES A., A.B. (Birmingham-Southern) Teaching Fellow, Alabama Poly. Inst., Auburn, Ala.
NORTON, W. G., B.S. in Ed. (Vermont) Research Engr., Carbide & Carbon C. C., K-25, Oak Ridge, Tenn. *106 Thornton Rd.*
NORWOOD, L. R., M.S. (Yale) Mathematician, U. S. Army Signal Corps Labs., Ft. Monmouth, N.J. *170 Manor Dr., Red Bank, N.J.*
NOTLEY, LLEWELLYN, A.M. (Texas) Supt. of Schools, Teague, Tex.
NOVAK, J. D., M.S. (Chicago) Asso. Prof., Univ. of South Carolina, Columbia, S.C. *802 Poinsettia Rd.*

NOVOA, L. G., D.P.M.S. (Havana) Asso. Prof., Havana Univ., Escuela de Ciencias, Havana, Cuba
NOWLAN, F. S., Ph.D. (Chicago) Prof., Univ. of British Columbia, Vancouver, B.C., Canada. *Visiting Prof.*, Univ. of Illinois, Navy Pier, Chicago 11, Ill.
NOWLAN, MABEL I., M.S. (Michigan) Clerk, Dept. of the Army, Washington, D.C. *5810 N.W. First Pl., Miami 38, Fla.*
NYHOFF, V. D., M.S. (Kansas S.C.) *Box 12, Atwood, Kan.*
NYSWANDER, J. A., Ph.D. (Chicago) Asso. Prof., Univ. of Michigan, Ann Arbor, Mich.

OAKLEY, C. O., Ph.D. (Illinois) Prof., Haverford Coll., Haverford, Pa.
O'BEIRNE, T. H., M.A. (Glasgow) Chief Mathematician, Messrs. Barr & Stroud, Ltd., Anniesland, Glasgow, W. 3, Scotland. *14 Princess Gardens, W. 2*
OBERG, E. N., Ph.D. (Minnesota) Asso. Prof., Univ. of Iowa, Iowa City, Iowa. *On Leave 1951–52:* North American Aviation, Downey, Calif.
OBERIST, C. J., M.A. (Columbia) Chief, Mathematics Section, U. S. Merchant Marine Acad., Kings Point, N.Y.
O'CONNOR, D. G., B.E.E. (Manhattan C.) Design Engr., I.B.M. Corp., Endicott, N.Y. *12 Liberty Ave., Union, N.Y.*
O'CONOR, J. S., Rev., M.S. (M.I.T.) Head of Dept., Physics, St. Joseph's Coll., Philadelphia 31, Pa.
O'DAY, J. L., Student, Gonzaga Univ., Spokane, Wash. *511 E. Boone Ave. (11)*
ODIN, EUGENE, M.E. (Cornell) Senior Project Engr., Arma Corp., 254 36th St., Brooklyn, N.Y. *229-17 141st Ave., Laurelton 10, N.Y.*
ODLAND, LEOLA P., B.S. (Oregon) Grad. Asst., Univ. of Kansas, Lawrence, Kan. *1236 Oread Ave.*
O'DONNELL, G. A., Rev., Ph.D. (St. Louis U.) Dean, Grad. School, Boston Coll., Chestnut Hill 67, Mass.
O'DONNELL, RUTH E., Ph.D. (Pennsylvania) Asso. Prof., Duquesne Univ., Pittsburgh 19, Pa. *3 Emerson St., Grafton (5)*
OEDER, R. D., M.S. (Oregon S.C.) Research Asst., Los Alamos Scientific Lab., Los Alamos, N.M. *3412-B Urban St.*
OEHMKE, R. H., M.A. (Detroit) Student, Univ. of Chicago, Chicago 37, Ill. *5658 Drexel (37)*
OEHMKE, THERESA M. (Mrs. R. H.), M.A. (Detroit) Computer, Air Weapons Research, Univ. of Chicago, Chicago, Ill. *5658 Drexel (37)*
OERGEL, C. T., B.S. in M.E. (Penna. State) Devel. Engr., General Electric Co., Bloomfield, N.J. *Middle Valley, N.J.*
OESTERLE, R. A., M.A. (Colorado S.C.) Instr., Oregon Coll., La Grande, Ore. *No. 18 Eocene Court Apts.*
OGAWA, GEORGE, A.M. (Washington S.C.) Lt., U. S. Army
OGDEN, E. B., Ph.D. (Boston U.) Dean, Union Coll., Lincoln, Neb. *4626 Bancroft St.*
OGILVY, C. S., A.M. (Columbia) Part-time Instr., Syracuse Univ., Syracuse 10, N.Y. *605 University Ave. (10)*
OGLESBY, E. J., A.M. (Virginia) Prof., Univ. of Virginia, Charlottesville, Va. *Box 1887, University, Va.*
OHMER, M. M., M.S. (Tulane) Asst. Prof., Southwestern Louisiana Inst., Lafayette, La. *215 Buena Vista Blvd.*
OHNSORG, F. R., B.A. (St. Thomas) Grad. Student, Univ. of Minnesota, Minneapolis 14, Minn. *1230 Edmund Ave., St. Paul 4, Minn.*
OLBRICH, L. P., Student, St. John's Univ., Brooklyn, N.Y. *69-14 Forest Ave. (27)*
OLDENBURGER, RUFUS, Ph.D. (Chicago) Mathematician, Woodward Governor Co., Rockford, Ill.
OLDS, C. D., Ph.D. (Stanford) Prof., San Jose State Coll., San Jose, Calif. *P.O. Box 462, Los Altos, Calif.*
OLDS, E. G., Ph.D. (Pittsburgh) Prof., Carnegie Inst. of Tech., Pittsburgh, Pa. *222 Gladstone Rd. (17)*
O'LEARY, A. J., M.S. (Catholic U.) Prof., St. Anselm's Coll., Manchester, N.H.
OLIPHANT, M. W., M.A. (Johns Hopkins) Instr., Georgetown Univ., Washington 7, D.C. *104 S. Park Dr., Arlington, Va.*
OLIVE, GLORIA, M.A. (Wisconsin) *Address Unknown*
OLLIVIER, ARTHUR, Ph.D. (Iowa) Prof., Mississippi State Coll., State College, Miss.
OLLMANN, L. F., Ph.D. (Michigan) Prof., Hofstra Coll., Hempstead, N.Y.
OLMSTED, J. M. H., Ph.D. (Princeton) Asso. Prof., Univ. of Minnesota, Minneapolis 14, Minn. *100 Malcolm Ave., S.E.*
OLMSTED, MARGARET, A.M. (Illinois) Asso. Prof., Augustana Coll., Rock Island, Ill. *330 19th St.*
OLNEY, HELEN, M.S. (Oregon) Prof., Hiram Coll., Hiram, Ohio. *Garfield, Rd.*
OLPIN, J. L., M.S. (Colorado) Instr., Eastern Arizona Junior Coll., Thatcher, Ariz.

OLSEN, C. E., B.S. (U. S. Naval Acad.) Instr., Univ. of Illinois, Navy Pier, Chicago 11, Ill. *238 Ridgeland Ave., Waukegan, Ill.*
OLSON, C. L., Student, George Pepperdine Coll., Los Angeles, Calif. *550 W. 90th St., (44)*
OLSON, EMMA J., Ph.D. (Chicago) Prof., Kent State Univ., Kent, Ohio. *1075 Crain Ave.*
OLSON, F. R., M.A. (Kent) Part-time Instr., Duke Univ., Durham, N.C.
OLSON, H. L., Ph.D. (Chicago) Prof., Indiana Tech. Coll., Fort Wayne, Ind.
OLSON, VIRGINIA L. (Mrs. F. R.), M.A. (Kent) Duke Univ., Durham, N.C.
ONDRAK, T. B., B.A. (St. Procopius) Instr., Univ. of Illinois, Navy Pier, Chicago, Ill. *2422 S. 61st Ave., Cicero 50, Ill.*
ONEIL, MARGHRITA L., B.A. (Bryn Mawr) Calculator, Towers, Perrin, Forster & Crosby, Philadelphia, Pa. *206 E. Willow Grove Ave., Chestnut Hill (18)*
O'NEILL, ANNE F., Ph.D. (Radcliffe) Asst. Prof., Smith Coll., Northampton, Mass. *115 Elm St.*
OPATOWSKI, IZAAK, D.Math. (R. Univ., Turin) Asst. Prof., Univ. of Chicago, Chicago 37, Ill. *1047 W. North Shore Ave. (26)*
OPPENHEIM, J. H., Ph.B. (Chicago) *409 E. Green St., Connellsville, Pa.*
O'QUINN, R. L., Ph.D. (Peabody) Asso. Prof., Louisiana State Univ., Baton Rouge, La.
ORAVEC, R. J., M.S. (Case) Instr., Fenn Coll., Cleveland, Ohio. *29821 Lake Shore Blvd., Willowick Village, Ohio*
ORE, OYSTEIN, Ph.D. (Oslo) Prof., Yale Univ., New Haven, Conn.
ORLIN, HYMAN, A.B. (George Washington) Mathematician, Coast and Geodetic Survey, Washington, D.C. *12811 Holdridge Rd., Glenmont, Silver Spring, Md.*
ORLOFF, DANIEL, Ph.D. (Chicago) *27 Linda Dr., Buffalo, N.Y.*
ORMSBY, E. F., M.S. (Syracuse) Instr., Pennsylvania State Coll., State College, Pa.
ORR, MARTIN, M.A. (Columbia) Mathematician, Signal Corps Center, Ft. Monmouth, N.J. *3871 Sedgwick Ave., New York 63, N.Y.*
ORR, S. R., B.A. (Hiram) Asst. Research Physicist, Mound Laboratories, Miamisburg, Ohio. *1008 Ellen Dr., Middletown, Ohio*
ORSHANSKY, BERNICE, B.A. (Hunter) Student, Teachers Coll., Columbia Univ., New York, N.Y. *1374 Bronx River Ave. (59)*
OSBORN, H. A., A.B. (Princeton) *420-A Butler Ave., Princeton, N.J.*
OSBORN, R. C., A.M. (Texas) Instr., Univ. of Texas, Austin, Tex. *3404 West Ave.*
O'SHAUGHNESSY, LOUIS, Ph.D. (Pennsylvania) Head of Dept., Applied Mech., Virginia Poly. Inst., Blacksburg, Va.
O'SHEA, E. F., Rev., A.B. (Woodstock) Student, Weston Coll., Weston 93, Mass.
OSNER, H. J., M.A. (California) Instr., Modesto Junior Coll., Modesto, Calif. *416 Oak St.*
OSTERBERG, ALICE, A.M. (Columbia) Sub. Instr., Brooklyn Coll., Brooklyn, N.Y. *215 E. 23rd St. (26)*
OSTROFSKY, MORRIS, Ph.D. (Wisconsin) Prof., Duquesne Univ., Pittsburgh 19, Pa.
OSTROM, T. G., Ph.D. (Minnesota) Asso. Prof., Montana State Univ., Missoula, Mont.
OSTROWSKI, A. M., Ph.D. (Göttingen) Prof., Univ. of Basle, Basle, Switzerland. *Mathem. Seminar, 21 Rheinsprung*
OTIS, F. F., M.S. (Wisconsin) Asst. Prof., Michigan Coll. of Mining and Tech., Sault Sainte Marie, Mich.
OTT, E. R., Ph.D. (Illinois) Prof., Rutgers Univ., New Brunswick, N.J. *199 Sterling Dr., Orange, N.J.*
OTTER, R. R., Ph.D. (Indiana) Asst. Prof., Univ. of Notre Dame, Notre Dame, Ind.
OTTESON, ELLI, A.M. (Wisconsin) Head of Dept., High School, Eau Claire, Wis. *705 Whipple St.*
OTTO, MARIANNE S. (Mrs. W. R.), M.S. (Marquette) Lecturer, Marquette Univ., Milwaukee 3, Wis. *2118-B N. 33rd St. (8)*
OURSLER, C. C., M.S. (Chicago) Instr., Indiana Univ., Gary Center, Gary, Ind. *4832 Van Buren St.*
OVERMAN, J. R., Ph.D. (Michigan) Dean of Faculties, Bowling Green State Univ., Bowling Green, Ohio
OVERMAN, P. W., M.S. (Indiana) Asst. Prof., Purdue Univ., Lafayette, Ind. *422 Harrison St., West Lafayette, Ind.*
OVERN, O. E., Ph.D. (Columbia) State Teachers Coll., Milwaukee 11, Wis.
OWCHAR, MARGARET, Ph.D. (Minnesota) Asso. Prof., Southwest Missouri State Coll., Springfield, Mo.
OWENS, A. J., Ph.D. (Florida) *211 Bellevue, Cape Girardeau, Mo.*
OWENS, F. W., Ph.D. (Chicago) Emeritus Prof., Pennsylvania State Coll., State College, Pa. *462 E. Foster Ave.*
OWENS, HELEN B. (Mrs. F. W.), Ph.D. (Cornell) Emeritus Asst. Prof., Pennsylvania State Coll., State College, Pa. *462 E. Foster Ave.*
OXLEY, J. E., B.E.E. (Cooper Union) Actuarial Student, New York Life Insurance Co., 51 Madison Ave., New York 10, N.Y. *640 79th St., Brooklyn 9, N.Y.*

OXLEY, T. D., JR., B.A. (Texas Christian) Grad. Student, Purdue Univ., Lafayette, Ind. *Box 667, Cary Hall East, West Lafayette, Ind.*
OXTOBY, J. C., A.M. (California) Asso. Prof., Bryn Mawr Coll., Bryn Mawr, Pa.

PACHUCKI, CHESTER, M.S. (De Paul) Instr., De Paul Univ., Chicago, Ill. *2907 W. 24th Blvd.*
PAGANO, S. J., M.A. (Washington U.) Instr., Missouri School of Mines and Met., Rolla, Mo.
PAIGE, L. J., Ph.D. (Wisconsin) Asst. Prof., Univ. of California at Los Angeles, Los Angeles 24, Calif. *2728 Veteran Ave. (34)*
PALL, GORDON, Ph.D. (Chicago) Prof., Illinois Inst. of Tech., Chicago 16, Ill. *3034 W. Eastwood Ave. (25)*
PALLADINO, JAMES, M.S. (N.Y.U.) Grad. Student, New York Univ., New York, N.Y. *498 Graham Ave., Brooklyn 22, N.Y.*
PALMBACH, E. J., B.S. (Lawrence C.) *108 E. Carroll St., Portage, Wis.*
PALMER, HASELL, M.S. (Tennessee) Instr., Univ. of Alabama, University, Ala.
PALMER, H. A., Ph.D. (Oklahoma) Chemist, United Gas Corp., Shreveport, La. *4121 Overton Dr.*
PALMER, T. P., A.M. (Harvard) Asso. Prof., Rose Poly. Inst., Terre Haute, Ind.
PALMER, W. N., M.A. (New York S.C. for Teachers) *1111 Elm St., Utica 3, N.Y.*
PANCOE, ARTHUR, B.S. (Wisconsin) *Address Unknown*
PARIS, J. F., Student, George Pepperdine Coll., Los Angeles, Calif. *1121 W. 79th St. (44)*
PARK, BART, M.S. (Michigan C. of Mining & Tech.) Prof., Michigan Coll. of Mining and Tech., Houghton, Mich.
PARK, D. S., A.B. (Tufts) Mathematician, National Bureau of Standards, Washington 25, D.C. *311 Briland St., Apt. 203, Alexandria, Va.*
PARK, EUGENE, A.M. (Lehigh) Asst. Prof., Clemson Coll., Clemson, S.C. *111 Park Way*
PARK, R. S., Ph.D. (Kentucky) Prof., Eastern Kentucky State Teachers Coll., Richmond, Ky. *213 Burnam Ct.*
PARKE, N. G., III, Ph.D. (M.I.T.) President, Parke Mathematical Labs., Concord, Mass.
PARKER, BOB, A.M. (Texas Tech.) Asst. Prof., Texas Tech. Coll., Lubbock, Tex.
PARKER, C. D., B.S. (Michigan S.C.) Grad. Student, Michigan State Coll., East Lansing, Mich.
PARKER, F. D., Ph.D. (Case) Asst. Prof., St. Lawrence Univ., Canton, N.Y. *R.F.D. 2*
PARKER, H. G., M.S. in Ed. (Alabama Poly.) Grad. Student, Univ. of Florida, Gainesville, Fla. *3808 11th Ave., Tampa, Fla.*
PARKER, J. E., A.M. (Fisk) Dir., Audio-Visual Aids Center, North Carolina Coll., Durham, N.C.
PARKER, S. T., Ph.D. (Cincinnati) Prof., Kansas State Coll., Manhattan, Kan.
PARKER, W. V., Ph.D. (Brown) Prof., Alabama Poly. Inst., Auburn, Ala.
PARKINSON, G. A., Ph.D. (Wisconsin) Asso. Prof., Univ. of Wisconsin in Milwaukee, Milwaukee, Wis.
PARKS, D. K., M.A. (Denver) Instr., Lower Columbia Junior Coll., Longview, Wash. *223 Freeport Villa*
PARRISH, H. C., M.S. (North Texas S.T.C.) Asst. Prof., North Texas State Coll., Denton, Tex.
PARRISH, LOIS K. (Mrs.), B.S. (Eastern Kentucky S.T.C.) *Arch St., Vevay, Ind.*
PARTER, S. V., M.S. (Illinois Tech.) Research Asst., Los Alamos Scientific Lab., Los Alamos, N.M. *528 35th St., Box 44*
PARTINGTON, C. R., M.S. (Purdue) Instr., Emory Univ., Emory University, Ga.
PATE, R. S., Ph.D. (Illinois) Prof., Michigan State Normal Coll., Ypsilanti, Mich. *224 N. Summit St.*
PATIN, H. A., S.M. (Chicago) Instr., Wilson Junior Coll., Chicago, Ill. *6437 University Ave. (37)*
PATTEN, W. E., C.E. (Cornell) Hydrologist, U. S. Dept. of Agriculture, Soil Conservation Service, Spartanburg, S.C. *1304 Main St., South Boston, Va.*
PATTERSON, B. C., Ph.D. (Johns Hopkins) President, Washington & Jefferson Coll., Washington, Pa. *238 E. Wheeling St.*
PATTERSON, D. O., Ph.D. (Minnesota) Prof., Colorado State Coll. of Education, Greeley, Colo. *809 19th St.*
PATTERSON, G. W., A.M. (Columbia) Asst. Prof., Elec. Engr., Univ. of Pennsylvania, Philadelphia, Pa. *312 Dartmouth Ave., Swarthmore, Pa.*
PATTERSON, J. M., M.A. (Columbia) Mathematician, Bendix Aviation Corp., Detroit, Mich. *18637 Cherrylawn, (21)*
PATTERSON, K. B., A.M. (Princeton) Asst. Prof., Duke Univ., Durham, N.C. *1024 Monmouth Ave.*
PATTERSON, P. B., M.A.E. (Florida) Instr., Univ. of Florida, Gainesville, Fla.
PAXTON, E. K., A.M. (Columbia) Retired. *P.O. Box 754, Lexington, Va.*
PAXTON, W. D., JR., B.A. (Pomona) Insurance Inspector, Retail Credit Co., San Bernardino, Calif. *P.O. Box 322, Bryn Mawr, Calif.*
PAYDON, J. F., Ph.D. (Northwestern) Asst. Prof., U. S. Naval Acad., Annapolis, Md.
PAYNE, A. H., B.S. (Appalachian S.T.C.) Grad. Student, Univ. of North Carolina, Chapel Hill, N.C. *179-A Jackson Circle*

PAYNE, C. K., Ph.D. (N.Y.U.) Emeritus Asso. Prof., New York Univ., New York, N.Y. *24 Valley Rd., Butler, N.J.*
PAYNE, G. W., B.S. (St. Mary's U.) Grad. Student, St. Mary's Univ., San Antonio, Tex. *714 Hoover*
PAYNE, MARY H. (Mrs. W. T.), Ph.D. (Brown) Asst. Prof., Michigan State Coll., East Lansing, Mich. *133 Woodmere*
PAYTON, A. F., Student, Rutgers Univ., New Brunswick, N.J. *141 Reynolds Pl., South Orange, N.J.*
PEACH, M. O., M.S. (Carnegie) Asso. Prof., Engg. Mech., Univ. of Notre Dame, South Bend, Ind.
PEAK, PHILIP, M.S. (Iowa) Asst. to Dean, Indiana Univ., Bloomington, Ind. *R.R. 6*
PEARSON, ERNA H. (Mrs.), M.A. (Texas) *1331 Emory Circle, N.E., Atlanta, Ga.*
PEARSON, PATRICIA M., M.A. (Oregon S.C.) Teaching Fellow, Oregon State Coll., Corvallis, Ore. *239 N. 8th St., Apt. 8*
PEASE, D. K., M.S. (Connecticut) Instr., Univ. of Connecticut, Hartford, Conn. *65 Cedar Ridge Rd., Newington 11, Conn.*
PECKHAM, C. G., M.S. (Illinois) Asso. Prof., Univ. of Dayton, Dayton 9, Ohio
PEDRICK, G. B., B.S. (Oklahoma A.&M.) AEC Predoctoral Fellow, Univ. of Kansas, Lawrence, Kan. *1317 Ohio St.*
PEEPLES, W. D., JR., Ph.D. (Georgia) Asst. Prof., Howard Coll., Birmingham, Ala. *913 N. 51st St. (6)*
PEGRAM, ANNIE M., A.M. (Duke) *308 Buchanan Blvd., Durham, N.C.*
PEHRSON, E. W., A.M. (California) Emeritus Prof., Univ. of Utah, Salt Lake City 1, Utah. *1456 Kensington Ave. (5)*
PEISER, A. M., Ph.D. (Cornell) Senior Mathematician, Hydrocarbon Research, Inc., 115 Broadway, New York 6, N.Y.
PEJSA, A. J., M. S. (Marquette U.) Asst. Prof., U. S. Naval Acad., Anapolis, Md. *Admiral Heights*
PENCE, SALLIE E., Ph.D. (Illinois) Asso. Prof., Univ. of Kentucky, Lexington 29, Ky.
PENNELL, W. O., B.S. (M.I.T.) Retired Chief Engr., Southwestern Bell Tel. Co., St. Louis, Mo. *69 Court St., Exeter, N.H.*
PENNEY, WALTER, Analyst, Navy Dept., Washington, D.C. *600 20th St., N.W.*
PENNINGTON, J. V., Ph.D. (Rice) Dir., Drilling Research, Inc., Houston 2, Tex. *1937 Portsmouth St. (6)*
PENNINGTON, WILLIAM, JR., *427 W. College St., Florence, Ala.*
PEPPER, P. M., Ph.D. (Cincinnati) Asso. Prof., Ohio State Univ. Research Foundation, Columbus 10, Ohio
PEREZ, FRANCISCO, Ph.D. (Chicago) Univ. of the Philippines, Coll. of Liberal Arts, Manila, Philippines
PERISHO, C. R., A.M. (Haverford) Asso. Prof., Nebraska Wesleyan Univ., Lincoln 4, Nebr.
PERKINS, F. W., Ph.D. (Harvard) Prof., Dartmouth Coll., Hanover, N.H. *8 Prospect St.*
PERKINS, LILLIAN G., M.A. (South Carolina) Asst. Prof., Univ. of South Carolina, Columbia, S.C. *2923 Heyward St. (5)*
PERKINS, O. L., M.A. (Atlanta) Instr., Campbell Street High School, Daytona Beach, Fla.
PERLIN, I. E., Ph.D. (Chicago) Asso. Prof., Georgia Inst. of Tech., Atlanta, Ga.
PERLSTEIN, SHIRLEY A. (Mrs.), M.A. (Oregon) Invoice Clerk, Lever Bros., 80 Varick St., New York, N.Y. *337 E. 24th St. (10)*
PERRY, C. L., JR., Ph.D. (Michigan) Senior Mathematician, Oak Ridge National Lab., Oak Ridge, Tenn. *121 W. Malta Rd.*
PERRY, D. B., A.M. (Stanford) International Business Machines Corp., Los Angeles 14, Calif. *14321 Valerio St., Van Nuys, Calif.*
PETERHANS, W. A., M.A. (Academy Weimar) Prof., Visual Training, Illinois Inst. of Tech., Chicago 16, Ill. *3140 S. Michigan Ave. (16)*
PETERS, A. S., Ph.D. (N.Y.U.) Asso. Prof., New York Univ., New York, N.Y. *54 Maplewood St., Larchmont, N.Y.*
PETERS, ANN C., Ed.D. (Columbia) Asso. Prof., Kenne Teachers Coll., Keene, N.H.
PETERS, I. D., M.S. (West Virginia) Asst. Prof., West Virginia Univ., Morgantown, W. Va.
PETERS, J. W., Ph.D. (Johns Hopkins) Asso. Prof., Univ. of Illinois, Urbana, Ill.
PETERS, MARY A., M.S. (Iowa) Teacher, Elgin High School and Junior Coll., Elgin, Ill. *376 E. Chicago St.*
PETERS, P. I., M.A. (Ohio State) Asso. Prof., Union Coll., Barbourville, Ky. *124 School St.*
PETERS, RUTH M., Ph.D. (Radcliffe) Asso. Prof., St. Lawrence Univ., Canton, N.Y.
PETERSEN, G. M., Ph.D. (Toronto) Lecturer, Brandon Coll., Brandon, Man., Canada
PETERSON, D. J., B.A. (Occidental C.) Grad. Student, Univ. of California at Los Angeles, Los Angeles 24, Calif. *2227 Patricia Ave. (64)*
PETERSON, H. C., M.A. (Denver) Instr., Univ. of Denver, Denver, Colo. *1015 E. Cedar Ave.*
PETERSON, J. C., M.S. in Ed. (North Dakota) Asst. Prof., Univ. of North Dakota, Grand Forks, N.D.

PETERSON, O. J., Ph.D. (Michigan) Prof., State Teachers Coll., Emporia, Kan. *1417 West St.*
PETERSON, R. P., JR., Ph.D. (U.C.L.A.) Instr., Univ. of Washington, Seattle 5, Wash.
PETERSON, S. R., M.A. (Columbia) Instr., Univ. of New Hampshire, Durham, N.H.
PETERSON, T. S., Ph.D. (Ohio State) Asso. Prof., Univ. of Oregon, Eugene, Ore. *1364 E. 18th Ave.*
PETRIE, G. W., III, Ph.D. (Lehigh) Govt. Representative, I.B.M. Corp., Washington 6, D.C. *6524 First St., N.W. (12)*
PETSCHEK, ELISABETH J., A.B. (Vassar) *1000 Park Ave., New York 28, N.Y.*
PETTIS, B. J., Ph.D. (Virginia) Prof., Tulane Univ., New Orleans 18, La.
PETTIT, H. P., Ph.D. (Illinois) Prof., Marquette Univ., Milwaukee, Wis. *Waterford, Wis.*
PETTUS, MARY, M.A. (Chicago) Instr., Sue Bennett Coll., London, Ky.
PEYTON, P. B., JR., B.E. (Virginia) Asst. Prof., Davidson Coll., Davidson, N.C.
PFLAUM, C. W., A.M. (Pennsylvania) Master, Darrow School, New Lebanon, N.Y.
PHALEN, H. R., Ph.D. (Chicago) Prof., Coll. of William and Mary, Williamsburg, Va. *130 Chandler Ct.*
PHELAN, K. S., M.A. (Columbia) Asst. Prof., St. John's Univ., Brooklyn, N.Y. *15 Alfred Rd., W., Merrick, N.Y.*
PHELPS, C. R., Ph.D. (Harvard) Asst. Prof., Rutgers Univ., New Brunswick, N.J.
PHILLIPS, E. C., REV., Ph.D. (Johns Hopkins) Spiritual Dir., Novitiate of St. Andrew-on-Hudson, Poughkeepsie, N.Y.
PHILLIPS, O. L., M.A. (North Texas S.C.) Prof., East Carolina Teachers Coll., Greenville, N.C.
PHIPPS, C. G., Ph.D. (Minnesota) Prof., Univ. of Florida, Gainesville, Fla.
PICKARD, W. L., Ph.D. (Peabody) Asso. Prof., Northern Illinois State Teachers Coll., DeKalb, Ill.
PIECHAN, R. G., Student, Central Michigan Coll. of Education, Mount Pleasant, Mich. *751 N. Third St., Rogers City, Mich.*
PIERCE, JESSE, Ph.D. (Michigan) Research Engr., Northrop Aircraft, Hawthorne, Calif. *26331 Alta Vista, Harbor City, Calif.*
PIERSON, A. D., A.M. (Missouri) Head of Dept., Junior Coll. of Kansas City, Kansas City, Mo. *7217 Summit Ave. (5)*
PIGNANI, T. J., M.S. (Bucknell) Part-time Instr., Univ. of North Carolina, Chapel Hill, N.C. *402 Hillsboro St.*
PIHLBLAD, MARGARET M., A.B. (Kansas) Grad. Student, Univ. of Kansas, Lawrence, Kan. *1201 W. Campus*
PIKE, J. M., B.S. (Kansas S.T.C.) Grad. Student, Kansas State Teachers Coll., Pittsburg, Kan. *P.O. Box 138, Frontenac, Kan.*
PINEIRO, M. T., B.S. (Havana) *Blanquizar 112, Luyano, Havana, Cuba*
PINGRY, R. E., Ph.D. (Cornell) Asst. Prof., Univ. of Illinois, Urbana, Ill. *1509 W. Church St., Champaign, Ill.*
PINKERTON, R. M., B.S. (Bradley) Aeronautical Research Scientist, National Advisory Committee for Aeronautics, Langley Field, Va. *131 LaSalle Ave., Hampton, Va.*
PINSON, R. B., M.A. (Texas) Asso. Prof., Stephen F. Austin State Coll., Nacogdoches, Tex.
PINZKA, C. F., B.S. (Rutgers) Instr., Xavier Univ., Cincinnati, Ohio
PIPES, C. J., Ph.D. (Oklahoma) Asst. Prof., Southern Methodist Univ., Dallas 5, Tex. *4628 March Ave. (9)*
PIRANIAN, GEORGE, Ph.D. (Rice) Asst. Prof., Univ. of Michigan, Ann Arbor, Mich.
PIRENIAN, Z. M., M.S. (Florida) Asso. Prof., Univ. of Florida, Gainesville, Fla.
PIRRONG, C. M., JR., Ed.M. (Oklahoma City U.) Asst. Prof., Oklahoma City Univ., Oklahoma City, Okla. *912 N.W. 22nd St. (6)*
PITCHER, A. E., Ph.D. (Harvard) Prof., Lehigh Univ., Bethlehem, Pa. *422 W. Broad St.*
PITNER, C. G., Prof., Harding Coll., Searcy, Ark.
PITT, W. B., Student, Univ. of Rochester, Rochester 3, N.Y.
PITTS, R. J., A.M. (Michigan) Prof., Fort Valley State Coll., Fort Valley, Ga.
PIXLEY, H. H., Ph.D. (Chicago) Asso. Prof., Wayne Univ., Detroit 1, Mich.
PIZA, P. A., *P.O. Box 627, San Juan, Puerto Rico*
PLAIT, O. A., B.S. (Illinois Tech.) U. S. Navy. *1315 S. Kolin, Chicago 23, Ill.*
PLANT, L. C., M.S. (Chicago) Emeritus Prof., Michigan State Coll., East Lansing, Mich. *231 Oakhill Ave.*
PLITHIDES, COSTAS, Degree of Math. (Univ. of Athens, Greece) *Address Unknown*
PLOENGES, E. W., A.M. (Michigan) Asst. Dean, James Millikin Univ., Decatur, Ill. *317 Linden Pl.*
PLUMMER, LOUISA G., M.S. (Minnesota) Asst. Prof., West Liberty State Coll., West Liberty, W. Va.
PLYMALE, R. B., M.A. (Columbia) Asso. Prof., Mercer Univ., Macon, Ga. *1329 Adams St., Apt. B.*
PODMELE, THERESA L., A.M. (Buffalo) Teacher, Bennett High School, Buffalo, N.Y. *356 Lisbon Ave. (15)*

POLAK, E. J., Asst. Instr., Bucknell Univ., Lewisburg, Pa. *Apt. 3A, Bucknell Village, R.D. 1*
POLAN, L. R., M.S. (West Virginia) Asso. Prof., Alfred Univ., Alfred, N.Y.
POLLACK, M. F., Master (Berlin) Teacher, Union Linden High School, Linden, Calif. *504 E. Fremont, Stockton 3, Calif.*
POLLAK, BARTH, Student, Illinois Inst. of Tech., Chicago 16, Ill. *7437 S. Shore Dr. (49)*
POLLARD, HARRY, Ph.D. (Harvard) Asso. Prof., Cornell Univ., Ithaca, N.Y. *112 Auburn St.*
POLLARD, H. S., Ph.D. (Wisconsin) Prof., Miami Univ., Oxford, Ohio. *350 Patterson Ave.*
POLLEY, J. C., Ph.D. (Cornell) Prof., Wabash Coll., Crawfordsville, Ind.
POLYA, GEORGE, Ph.D. (Budapest) Prof., Stanford Univ., Stanford, Calif. *2260 Dartmouth St., Palo Alto, Calif.*
POMILLA, F. R., M.S. (Fordham) Instr., St. John's Univ., Brooklyn, N.Y. *48 Pulaski St. (6)*
POND, H. L., M.S. (George Washington) Physicist, David Taylor Model Basin, Navy Dept., Washington 7, D.C. *3415 38th St., N.W. (16)*
PONDS, J. W., M.S. (Howard) Mathematician, U.S. Air Force, Washington 25, D.C. *3984 Ames St., N.E. (19)*
POOL, FLORENCE E., M.A. (Nebraska) Instr., Extension Div., Univ. of Nebraska, Lincoln, Nebr.
POOLE, A. R., Ph.D. (C.I.T.) Asst. Prof., Oregon State Coll., Corvallis, Ore.
POPOW, J. W., M.A. (Maryland) Asst. Prof., U. S. Naval Acad., Annapolis, Md.
POPPEN, H. A., Ph.D. (George Peabody) Consulting Psychologist, Rohrer, Hibler & Replogle, 135 S. LaSalle St., Chicago, Ill. *4240 Linden St., Western Springs, Ill.*
PORGES, ARTHUR, M.S. (Illinois Tech.) *2743 San Marino St., Los Angeles 6, Calif.*
PORITSKY, HILLEL, Ph.D. (Cornell) Consulting Mathematician, General Electric Co., 1 River Rd., Schenectady, N.Y.
PORTER, D. H., Ph.D. (Indiana) Instr., Indiana Univ., Bloomington, Ind. *Hoosier Courts 17-3*
PORTER, M. B., M.A. (Texas) Asst. Prof., Southwest Texas State Teachers Coll., San Marcos, Tex. *1120 W. Hopkins St.*
PORTER, RUTH E., M.S. (Oregon S.C.) Asst. Prof., John Brown Univ., Siloam Springs, Ark. *114 N. Madison St.*
PORTMANN, W. O., Student, Kent State Univ., Kent, Ohio. *915 Camden Ave., S.W., Canton, Ohio*
POST, E. L., Ph.D. (Columbia) Prof., City Coll. of New York, New York 31, N.Y. *610 W. 173rd St. (32)*
POSTON, P. L., M.B.S. (Michigan) Actuary, Great Lakes Mutual Life Insurance Co., Detroit, Mich. *2952 W. Philadelphia (6)*
POTTS, L. D., B.M.E. (Ohio State) Research Engr., Linde Air Products Co., Buffalo, N.Y. *328 Hendricks Blvd., Eggertsville 21, N.Y.*
POULIOT, ADRIEN, M.A. (Laval) Dean of Faculty of Sciences, Laval Univ., Quebec, P.Q., Canada
POUND, V. E., Ph.D. (Toronto) Prof., Univ. of Buffalo, Buffalo 14, N.Y. *190 Capen Blvd.*
POUNDER, D. W., M.S. (Illinois Tech.) Aerodynamicist, A. V. Roe Canada, Ltd., Toronto, Ont., Canada. *19 Glen Gordon Rd. (9)*
POUNDER, I. R., Ph.D. (Chicago) Prof., Univ. of Toronto, Toronto, Ont., Canada
POWELL, J. E., Ph.D. (Chicago) Prof., Michigan State Coll., East Lansing, Mich. *137 Bogue St.*
POWELL, K. F., B.S. (Pittsburgh) Customer Engr., International Business Machines Corp., New York, N.Y. *201 5th St., Ellwood City, Pa.*
POWELL, LILY B., M.A. (Colorado) Instr., Univ. of Colorado, Boulder, Colo. *605 Thirteenth St.*
PRATT, GERTRUDE V., A.M. (Michigan) Asst. Prof., Central Michigan Coll. of Education, Mt. Pleasant, Mich.
PREBLE, W. G., B.S. (Tulane) Computer, U. S. Army Corps of Engineers, New Orleans, La. *549 Mehle Ave., Arabi, La.*
PRENOWITZ, WALTER, Ph.D. (Columbia) Asso. Prof., Brooklyn Coll., Brooklyn 10, N.Y.
PRENTICE, W. N., A.M. (Brown) Grad. Student, Syracuse Univ., Syracuse, N.Y. *102 Walnut Pl.*
PRESNELL, ROBERTA E. (Mrs.), M.A. (Beloit) Instr., Rockford Coll., Rockford, Ill. *803 N. Chicago Ave.*
PRETZ, P. S., A.M. (St. John's U.) Head of Dept., St. Benedict's Coll., Atchison, Kan.
PRICE, D. J., Ph.D. (London) *4, Brittany Ct., New Church Rd., Hove 3, Sussex, England*
PRICE, G. B., Ph.D. (Harvard) Prof., Univ. of Kansas, Lawrence, Kan.
PRICE, H. F., Ph.D. (Pennsylvania) Prof., Pacific Univ., Forest Grove, Ore. *303 Second St., S.*
PRICE, H. V., Ph.D. (Iowa) Head of Dept., Univ. High School, Iowa City, Iowa
PRICE, IRENE, Ph.D. (Indiana) Acting Chief, Tech. Staff, Air Research & Development Command, Holloman Air Force Base, N.M.
PRICE, J. F., Ph.D. (Oregon S.C.) Asst. Prof., Oregon State Coll., Corvallis, Ore.
PRIEST, R. E., M.S. (Illinois) Asst., Univ. of Illinois, Urbana, Ill.
PRIESTER, G. C., Ph.D. (Michigan) Head of Dept., Univ. of Minnesota, Minneapolis 14, Minn.
PROCTOR, S. W., S.M. (Chicago) Instr., Vashon High School, St. Louis, Mo. *2846 Pine Blvd. (3)*
PROTTER, M. H., Ph.D. (Brown) Member, Inst. for Advanced Study, Princeton, N.J.

PRYOR, J. E., Ph.D. (Louisiana) Prof., Chemistry and Physics, Harding Coll., Searcy, Ark.
PRYZIE, J. B., B.S. (C.C.N.Y.) Supt. Asst., Consolidated Edison Co. of New York, Inc., New York, N.Y. *31-15 21st Ave., Long Island City 5, N.Y.*
PUCKETT, W. T., JR., Ph.D. (Virginia) Asso. Prof., Univ. of California at Los Angeles, Los Angeles 24, Calif.
PUGSLEY, D. W., M.S. (Michigan) Prof., Berea Coll., Berea, Ky.
PULLIAM, F. M., Ph.D. (Illinois) Asst. Prof., U. S. Naval Postgraduate School, Annapolis, Md.
PURCELL, E. J., Ph.D. (Cornell) Prof., Univ. of Arizona, Tucson, Ariz. *1702 Lind Rd.*
PURDIE, K. S., B.S. (V.M.I.) Prof., Virginia Military Inst., Lexington, Va. *313 Letcher Ave.*
PURSEL, C. A., M.A. (Southern California) Instr., California State Poly. Coll., San Luis Obispo, Calif. *1890 San Luis Dr.*
PUTNAM, A. L., Ph.D. (Harvard) Asso. Prof., Univ. of Chicago, Chicago 37, Ill.
PUTNAM, R. G., Ph.D. (Chicago) Prof., New York Univ., New York, N.Y. *115 Riverview Ave., Tarrytown, N.Y.*
PYLE, H. R., Ph.D. (California) Prof., Whittier Coll., Whittier, Calif. *530 N. Bright Ave.*

QUAID, L. J., B.S. (Illinois) Asst. Prof., Univ. of Minnesota, Minneapolis 14, Minn. *236 Melbourne Ave., S.E.*
QUALLEY, A. O., A.M. (Iowa) Superintendent, Farnhamville Indep. School, Farnhamville, Iowa
QUALLS, V. Elise, A.B. (Berea) Instr., Tennessee Poly. Inst., Cookeville, Tenn.
QUERRY, J. W., Ph.D. (Iowa) Prof., Sam Houston State Teachers Coll., Huntsville, Tex.
QUINN, GRACE S. (Mrs. R. B.) Ph.D. (Ohio State) *723 S. Adams St., Arlington 4, Va.*
QUINN, J. J., M.S. (N.Y.U.) Instr., Bayonne Junior Coll., Bayonne, N.J. *75 Garretson Ave.*
QUIRMBACH, A. H., M.S. (V.P.I.) Asst. Prof., Univ. of Alabama, Gadsden, Ala. *522 W. 8th Ave., Attalla, Ala.*

RADER, C. B., M.S. (Houston) Asst. Prof., Univ. of Houston, Houston, Tex. *3719 Merrick*
RADER, M. A., A.M. (Lehigh) Prof., Moravian Coll., Bethlehem, Pa.
RADOR, TIBOR, Ph.D. (Szeged) Prof., Ohio State Univ., Columbus, Ohio. *92 Walhalla Rd. (2)*
RAHN, EDRIS P., A.M. (California) Teacher, Union High School, Hayward, Calif. *1456 Glen Dr., San Leandro, Calif.*
RAINBOW, HENRY, B.A. (Cambridge) Shell Research Lab., Bellaire Blvd., Houston 5, Tex.
RAINE, P. W. A., A.M. (Virginia) Teacher, Newport News High School, Newport News, Va.
RAINICH, G. Y., Master in Pure Math. (Kazan) Prof., Univ. of Michigan, Ann Arbor, Mich. *602 Oswego St.*
RAINVILLE, E. D., Ph.D. (Michigan) Asso. Prof., Univ. of Michigan, Ann Arbor, Mich. *2864 Bellewood Ave., East Ann Arbor, Mich.*
RAISBECK, GORDON, Ph.D. (M.I.T.) Member, Tech. Staff, Bell Telephone Labs., Murray Hill, N.J. *Apt. H-29, Franklin Village, Morristown, N.J.*
RAKER, P. H., A.M. (Michigan) Instr., General Motors Inst., Flint, Mich. *2629 Colby St. (4)*
RALL, L. B., B.S. (Puget Sound) *1710 S.W. 13th St., Portland 1, Ore.*
RAMBO, SUSAN M., Ph.D. (Michigan) Emeritus Prof., Smith Coll., Northampton, Mass. *71 Ridgewood Terr.*
RAMLER, O. J., Ph.D. (Catholic U.) Prof., Catholic Univ. of America, Washington 17, D.C. *12 Girard St., N.E. (2)*
RAMSDELL, B. D., M.S. (Michigan) Instr., Orlando Junior Coll., Orlando, Fla.
RAMSDELL, G. E., A.M. (Harvard) Emeritus Prof., Bates Coll., Lewiston, Maine. *40 Mountain Ave.*
RAMSEY, L. W., M.S. (Texas A. & M.) Asst. Prof., Texas Christian Univ., Fort Worth 9, Tex.
RAMSEY, MARGARET, A.M. (Oregon) Asso. Prof., Linfield Coll., McMinnville, Ore. *Route 3, Box 37*
RAND, R. C., Ph.D. (Maryland) Senior Mathematician, Johns Hopkins Applied Physics Lab., Silver Spring, Md. *9413 Thornhill Rd.*
RANDICH, E. A., M.S. (Pittsburgh) Development Engr., United Engineering and Foundry Co., Pittsburgh, Pa. *120 Winchester Dr., R.F.D. 1, Allison Park, Pa.*
RANDOLPH, H. L., B.A. (California) Bookkeeper, Bank of America, Long Beach, Calif. *2337 Stearnlee (4)*
RANDOLPH, J. F., Ph.D. (Cornell) Prof., Univ. of Rochester, Rochester 3, N.Y. *171 Highland Parkway*
RANKIN, F. MOZELLE, M.A. (Ohio State) Physicist, Carbide & Carbon Chemicals Corp., Y-12, Oak Ridge, Tenn. *Eff. Apt. A-1, #140*
RANKIN, J. M., A.M. (California) Prof., Coll. of Idaho, Caldwell, Idaho. *1810 Ash St.*
RANKIN, R. M., M.A. (Chicago) Prof., Missouri School of Mines, Rolla, Mo. *1604 N. Pine St.*
RANKIN, W. W., A.M. (North Carolina) Prof., Duke Univ., Durham, N.C.
RANSOM, W. R., A.M. (Harvard) Prof., Tufts Coll., Medford, Mass. *13 Barrows Rd., Reading, Mass.*

RAO, B. S. M. Prof., Central Coll., Bangalore, India

RAPOPORT, ANATOL, Ph.D. (Chicago) Asst. Prof., Univ. of Chicago, Committee on Mathematical Biology, 5741 Drexel Ave., Chicago, Ill.

RAPP, MARY K., A.M. (Illinois) Instr., Illinois Inst. of Tech., Chicago, Ill. *217 Washington Blvd., Oak Park, Ill.*

RAPP, P. C., B.A. (Buffalo) Engr., Bell Aircraft Corp., Niagara Falls, N.Y. *1029 West Ave., Buffalo 13, N.Y.*

RASMUSEN, RUTH B., Ph.D. (Chicago) Instr., Wilson Junior Coll., Chicago, Ill. *6105 S. Woodlawn Ave.*

RASMUSSEN, O. M., M.S. (Kansas S.T.C.) Instr., Univ. of Kansas, Lawrence, Kans. *814 E. 13th St.*

RASOR, E. A., M.S. (Ohio State) Actuarial Mathematician, Social Security Admin., Washington 25, D.C. *1085 N. Manchester St., Arlington, Va.*

RASOR, ELLEN F., A.M. (Duke) Adjunct Prof., Univ. of South Carolina, Columbia, S.C. *Cross Hill, S.C.*

RATHBUN, V. J., M.S. (Catholic U.) Instr., Mount St. John Normal, Dayton 10, Ohio

RATNER, L. T., Ph.D. (U.C.L.A.) Asst. Prof., Vanderbilt Univ., Nashville, Tenn.

RAUCH, L. L., Ph.D. (Princeton) Asso. Prof., Univ. of Michigan, Ann Arbor, Mich.

RAUCH, S. E., Ph.D. (Stanford) Asso. Prof., Santa Barbara Coll., Santa Barbara, Calif.

RAUDENBUSH, H. W., Ph.D. (Columbia) Asso. Prof., Queens Coll., Flushing, N.Y.

RAWLINS, C. H., JR., Ph.D. (Johns Hopkins) Prof., U. S. Naval Postgraduate School, Annapolis, Md. *13 Franklin St.*

RAYBOULD, ETHEL H., M.A. (Queensland) Senior Lecturer, Univ. of Queensland, Brisbane, Australia

RAYHER, EDWARD, A.M. (Columbia) Head of Dept., Bergen Junior Coll., Teaneck, N.J. *198 Elm Ave.*

RAYL, ADRIENNE S., Ph.D. (Chicago) Asso. Prof., Univ. of Alabama, Birmingham, Ala. *Apt. 506, Sheraton, 2909 Highland Ave.*

RAYNOR, G. E., Ph.D. (Princeton) Prof., Lehigh Univ., Bethlehem, Pa. *349 Eighth Ave.*

READ, C. B., Ph.D. (Colorado S.C.) Prof., Univ. of Wichita, Wichita 6, Kan.

READE, M. O., Ph.D. (Rice) Asst. Prof., Univ. of Michigan, Ann Arbor, Mich.

REAGAN, C. A., A.M. (Kansas) Emeritus Prof., Friends Univ., Wichita 12, Kan. *1201 S. Martinson (12)*

REAGAN, L. M., A.M. (Kansas) Asso. Prof., Univ. of Wichita, Wichita, Kan. *1056 N. Market St. (5)*

REAVES, CAROLINE M., A.M. (Oklahoma) Emeritus Prof., Coker Coll., Hartsville, S.C. *1006 Fifth St.*

RECHARD, O. H., Ph.D. (Wisconsin) Dean, Liberal Arts, Univ. of Wyoming, Laramie, Wyo.

RECHARD, O. W., Ph.D. (Wisconsin) Staff Member, Los Alamos Scientific Lab., Los Alamos, N.M. *2345-B 48th St.*

RECHT, A. W., Ph.D. (Chicago) Prof., Univ. of Denver, Denver 10, Colo. *2233 S. St. Paul St.*

RECKZEH, J. K., A.M. (Kentucky) Asst. Prof., State Teachers Coll., Jersey City 5, N.J.

RECTOR, R. W., A.M. (Stanford) Asst. Prof., U. S. Naval Acad., Annapolis, Md. *7 Steele Ave.*

REDDEN, J. E., M.S. (Iowa State) Asso. Prof., Tarleton State Coll., Stephenville, Tex.

REDDICK, H. W., Ph.D. (Columbia) Emeritus Prof., New York Univ., New York, N.Y. *Mexico, N.Y.*

REDHEFFER, R. M., Ph.D. (M.I.T.) Instr., Univ. of California at Los Angeles, Los Angeles 24, Calif. *264 Conway Ave. (24)*

REDMOND, P. J., B.E.E. (Cooper Union) Grad. Student, Univ. of Birmingham, Birmingham, England. *c/o U. S. Educ. Commission in the United Kingdom, 55 Upper Brook St., London W. 1, England*

REECE, R. H., A.M. (Colorado) President Emeritus, New Mexico School of Mines, Socorro, N.M.

REED, F. W., Ph.D. (Virginia) Prof., Ohio Univ., Athens, Ohio. *61 Columbia Ave.*

REED, J. W., B.S. (New Mexico) Staff Member, Meterological Section, Sandia Corp., Albuquerque, N.M. *1211 S. Van Buren Ave.*

REED, L. J., Ph.D. (Pennsylvania) Vice-President, Johns Hopkins Univ., Baltimore 5, Md. *615 N. Wolfe St. (5)*

REEKS, M. R., M.S. (Stevens Inst.) Asso. Prof., Stevens Inst. of Tech., Hoboken, N.J. *32 W. Fairmount Ave., Maywood, N.J.*

REES, C. J., Ph.D. (Pennsylvania) Dean, Graduate School, Univ. of Delaware, Newark, Del.

REES, DOROTHY L., A.M. (Texas) Asst. Prof., Trinity Univ., San Antonio 1, Tex. *220 W. Park Ave.*

REES, MINA S., Ph.D. (Chicago) Dir., Mathematical Science, Office of Naval Research, Bldg. T-3, Rm. 2719, Navy Dept., Washington 25, D.C.

REES, P. K., Ph.D. (Rice) Asso. Prof., Louisiana State Univ., Baton Rouge, La.
REEVE, W. D., Ph.D. (Minnesota) Emeritus Prof., Columbia Univ., New York, N.Y. *525 W. 120th St. (27)*
REEVES, R. F., B.S.E.E. (Colorado) Instr., Iowa State Coll., Ames, Iowa. *2117 Graeber*
REGAN, FRANCIS, Ph.D. (Michigan) Prof., St. Louis Univ., St. Louis, Mo.
REHBERG, C. F., A.M. (Columbia) Asso. Prof., New York Univ., New York, N.Y. *25-28 84th St., Jackson Heights, N.Y.*
REIBER, F. A., M.S. (Florida) *Y.M.C.A., 2020 Witherell St., Detroit 26, Mich.*
REICHELDERFER, P. V., Ph.D. (Ohio State) Prof., Ohio State Univ., Columbus 10, Ohio. *328 E. Beechwold Blvd.*
REID, ANN T., Student, Duke Univ., Durham, N.C. *42 Carrera St., St. Augustine, Fla.*
REID, C. H. JR., M.Ed. (Colorado) Instr., Ft. Lewis A. and M. Coll., Hesperus, Colo.
REID, W. P., Ph.D. (Pittsburgh) Mathematician, U. S. Naval Ordnance Test Station, China Lake, Calif.
REID, W. T., Ph.D. (Texas) Prof., Northwestern Univ., Evanston, Ill.
REILLY, G. A., B.S. (St. Mary's U.) Clerk, American Hospital & Life Insurance Co., San Antonio, Tex. *519 W. Mistletoe Ave. (1)*
REINER, IRMA M., Ph.D. (Cornell) Instr., Danville Community Coll., Danville, Ill. *707 W. Church St., Champaign, Ill.*
REINER, IRVING, Ph.D. (Cornell) Asst. Prof., Univ. of Illinois, Urbana, Ill. *707 W. Church St., Champaign, Ill.*
REINGOLD, HAIM, Ph.D. (Cincinnati) Asso. Prof., Illinois Inst. of Tech., Chicago 16, Ill.
REINHARDT, H. E., M.A. (Washington S.C.) Teaching Asst., State Coll. of Washington, Pullman, Wash. *1427 Ninth Ave., Lewiston, Idaho*
REINSCH, B. P., Ph.D. (Illinois) Prof., Florida Southern Coll., Lakeland, Fla. *191 Lake Morton Dr.*
REISEL, R. B., B.S. (DePaul) Grad. Student, Univ. of Chicago, Chicago 37, Ill. *2425 Dakin St. (18)*
REISSNER, ERIC, Ph.D. (M.I.T.) Prof., Massachusetts Inst. of Tech., Cambridge 39, Mass.
REITWIESNER, G. W., S.M. (Harvard) Mathematician, Ballistic Research Lab., Aberdeen Proving Ground, Md. *Box 141, Aberdeen, Md.*
REKLIS, VIRGINIA M. (Mrs. E. P.), Ph.D. (Illinois) *Sherwood Lane, R.D. 1, Havre de Grace, Md.*
RENNEMANN, CONRAD, JR., M.A. (Nebraska) Aero. Research Scientist, National Advisory Committee for Aeronautics, Langley Field, Va. *410 Mallory Ave., Hampton, Va.*
RENNO, J. G., JR., M.S. (Michigan State) Grad. Asst., Univ. of Wisconsin, Madison 6, Wis. *803 State St. (5)*
RENSIN, HYMEN, M.A. (Columbia) Lecturer, City Coll. of New York, New York, N.Y. *75 W. Mosholu Parkway, Bronx 67, N.Y.*
RENTON, P. H., M.S. (Pittsburgh) Dean of Instruction, West Virginia Inst. of Tech., Montgomery, W. Va. *309 Fayette Pike*
RENZEMA, DOROTHY G., M.S. (Michigan) Instr., Coll. of Wooster, Wooster, Ohio. *141 Garland St., S.E., Grand Rapids 8, Mich.*
REUBER, J. M., M.S. (Chicago) Asst. Prof., Chemistry, Coll. of St. Thomas, St. Paul, Minn.
REVES, G. E., Ph.D. (Cincinnati) Asso. Prof., The Citadel, Charleston, S.C.
REX, E. C., M.S. (Washington) Asst. Prof., George Pepperdine Coll., Los Angeles 44, Calif. *1021 W. 81st St. (44)*
REYNOLDS, C. N., Ph.D. (Harvard) Prof., West Virginia Univ., Morgantown, W. Va. *217 McLane Ave.*
REYNOLDS, J. B., Ph.D. (Moravian) Emeritus Prof., Lehigh Univ., Bethlehem, Pa. *Sugar Run, Bradford Co., Pa.*
REYNOLDS, J. O., Ph.D. (North Carolina) Prof., East Carolina Coll., Greenville, N.C.
REYNOLDS, LENA E., A.M. (California) Instr., Fullerton Junior Coll., Fullerton, Calif.
REYNOLDS, T. L., Ph.D. (North Carolina) Prof., Millsaps Coll., Jackson, Miss.
RHODES, A. P., A.M. (Stanford) Civil Engr., U. S. Naval Training Center, San Diego 33, Calif. *1885 Sheridan Ave. (3)*
RHODES, C. E., Ph.D. (Cincinnati) Prof., Alfred Univ., Alfred, N.Y.
RHODES, M. C., Ph.D. (Peabody) Dean, Univ. of Tampa, Tampa 6, Fla.
RIBEIRO, H. B., Dr.Sc. (Zurich) Asso. Prof., Univ. of Nebraska, Lincoln 8, Nebr.
RICE, E. C., M.A. (George Peabody) Grad. Student, George Peabody Coll. for Teachers, Nashville 4, Tenn.
RICE, HARRIS, A.M. (Harvard) Prof., Worcester Poly. Inst., Worcester 2, Mass.
RICE, HARRY L., M.S. (Iowa) Prof., Univ. of Omaha, Omaha, Nebr. *130 Fifth Ave., Council Bluffs, Iowa*

RICE, HERBERT L., Student, St. Mary's Univ., San Antonio, Tex. *1400 W. Mistletoe Ave.*
RICE, J. N., Ph.D. (Catholic U.) Asso. Prof., Catholic Univ. of America, Washington, D.C. *3326 13th St. N.E.* (*17*)
RICE, R. B., A.B. (Wooster) Senior Physicist, Research Dept., Phillips Petroleum Co., Bartlesville, Okla.
RICE, W. D., M.A. (Illinois) Head of Dept., Olivet Nazarene Coll., Kankakee, Ill. *Bourbonnais, Ill.*
RICH, C. L., Ph.M. (Wisconsin) Asso. Prof., Stout Inst., Menomonie, Wis. *Route 6*
RICHARDS, AUDREY I., M.A. (Columbia) Instr., Utica Coll., Utica, N.Y. *40 Jewett Pl.*
RICHARDS, T. R., M.S. (Bucknell) Asst. Prof., Wilkes Coll. Wilkes-Barre, Pa., *173 Gaylord Ave., Plymouth, Pa.*
RICHARDSON, C. H., Ph.D. (Michigan) Prof., Bucknell Univ., Lewisburg, Pa. *401 S. Sixth St.*
RICHARDSON, D. P., Ph.D. (Chicago) Prof., Univ. of Arkansas, Fayetteville, Ark. *524 Leverett St.*
RICHARDSON, J. W., Student, South Dakota School of Mines & Tech., Rapid City, S.D.
RICHARDSON, MOSES, Ph.D. (Columbia) Asst. Prof., Brooklyn Coll., Brooklyn, N.Y.
RICHERT, D. H., A.M. (Colorado) Emeritus Prof., Bethel Coll., North Newton, Kan.
RICHMAN, JUDITH A., Student, Univ. of Pennsylvania, Philadelphia, Pa. *4639 Walnut St.* (*39*)
RICHMOND, A. E., Instr., Multnomah Coll., Portland, Ore. *P.O. Box 441* (*7*)
RICHMOND, C. A., B.S. (Pomona) *Tyngsboro, Mass.*
RICHTMEYER, C. C., Ph.D. (Colorado S.C.) Prof., Central Michigan Coll. of Education, Mt. Pleasant, Mich.
RICKARD, HORTENSE, A.M. (Ohio State) Asst. Prof., Ohio State Univ., Columbus, Ohio. *79 Beaumont Rd.* (*2*)
RICKEY, F. A., Ph.D. (Louisiana) Prof., Louisiana State Univ., Baton Rouge, La.
RICUPITO, J. R., Student, Univ. of Florida, Gainesville, Fla.
RIDDLE, D. F., A.B. (Illinois) Grad. Student, Univ. of Michigan, Ann Arbor, Mich. *2636 Euclid Ave., Berwyn, Ill.*
RIDER, P. R., Ph.D. (Yale) Prof., Washington Univ., St. Louis 5, Mo. *On leave 1951–52:* Flight Research Lab. (DCRRS-11), Wright-Patterson Air Force Base, Dayton, Ohio
RIDLEY, J. W., JR., Junior Chemist, Barrett Div., Allied Chemical & Dye Corp., Margaret & Bermuda Sts., Philadelphia, Pa. *3111 Bucknell Terr.* (*45*)
RIEBE, N. W., B.S. (New Mexico) Research Asst., Los Alamos Scientific Lab., Los Alamos, N.M. *P.O. Box 1138, Santa Fe, N.M.*
RIES, H. F., A.B. (Michigan) Principal Actuary, State of Colorado Insurance Dept., 100 State Office Bldg., Denver, Colo.
RIES, J. F., M.A. (Delaware) Instr., Kings Coll., Kings College, Del.
RIESS, J. K., Ph.D. (Brown) Asso. Prof., Tulane Univ., New Orleans 15, La. *17 Audubon Blvd.* (*18*)
RIGGS, C. L., Ph.D. (Kentucky) Asst. Prof., East Texas State Teachers Coll., Commerce, Tex.
RIGGS, L. G., Ph.D. (Northwestern) Asst. Prof., Ohio Univ., Athens, Ohio
RILEY, J. D., M.A. (Kansas) Grad. Student, Univ. of Kansas, Lawrence, Kan. *1135 Maine St.*
RINE, T. E., M.S. (Iowa) Asst. Prof., Illinois State Normal Univ., Normal, Ill.
RINEHART, R. F., Ph.D. (Ohio State) Prof., Case Inst. of Tech., Cleveland, Ohio
RINGENBERG, L. A., Ph.D. (Ohio) Prof., Eastern Illinois State Coll., Charleston, Ill.
RIORDAN, JOHN, B.S. (Yale) Member, Tech. Staff, Bell Telephone Labs., Inc., 463 West St., New York 14, N.Y.
RIPPE, D. D., M.A. (Nebraska) Teaching Fellow, Univ. of Michigan, Ann Arbor, Mich. *1049 Woburn Ct., Willow Run, Mich.*
RIPY, SARA L., M.A. (Kentucky) Instr., Univ. of Kentucky, Lexington, Ky.
RISSLER, W. J., A.M. (New Jersey S.T.C.) Instr., State Teachers Coll., Frostburg, Md. *61 Park Ave.*
RITCHIE, A. A., M.S. (Oklahoma A. & M.) Teacher, Ponca City Senior High School, Ponca City, Okla.
RITCHIE, GORDON, B.A. (McMaster) Teacher, Sharbot Lake High School, Sharbot Lake, Ont., Canada. *30 Rosslyn Ave., N., Hamilton, Ont., Canada*
RITGER, P. D., A.M. (Pennsylvania) Grad. Student, New York Univ., New York, N.Y. *69 Dover St., Newark 6, N.J.*
RITTER, E. K., Ph.D. (Virginia) Head, Simulation & Computation Dept., Willow Run Research Center, Willow Run Airport, Ypsilanti, Mich.
RITTERMAN, M. B., M.S. (N.Y.U.) Instr., Long Island Univ., Brooklyn, N.Y. *7201 Ridge Blvd.* (*9*)
RIVLIN, T. J., M.A. (Harvard) Grad. Student, Harvard Univ., Cambridge, Mass.
ROBB, D. L., Ph.D. (Catholic U.) Asso. Prof., Baldwin-Wallace Coll., Berea, Ohio
ROBB, J. M., A.M. (Michigan) Retired, Univ. of Southern California, Los Angeles, Calif. *1198 W. 29th St.* (*7*)

ROBBINS, C. K., A.M. (Harvard) Asso. Prof., Purdue Univ., West Lafayette, Ind. *418 Vine St.*
ROBBINS, E. S., M.A. (Wichita) Teaching Asst., Univ. of California, Berkeley, Calif.
ROBBINS, H. E., Ph.D. (Harvard) Asso. Prof., Univ. of North Carolina, Chapel Hill, N.C.
ROBERTS, B. D., Ph.D. (Iowa) Dean, New Mexico Highlands Univ., Las Vegas, N.M.
ROBERTS, G. G., A.M. (Kentucky) Asso. Prof., Berea Coll., Berea, Ky. *6 Estill St.*
ROBERTS, J. H., Ph.D. (Texas) Prof., Duke Univ., Durham, N.C.
ROBERTS, VIRGINIA E. (Mrs. R. T.), M.A. (Texas Tech.) Instr., Texas Tech. Coll., Lubbock, Tex. *2616 37th St.*
ROBERTSON, FRED, A.M. (Indiana) Asst. Prof., Iowa State Coll., Ames, Iowa
ROBERTSON, J. M., B.S. (Clemson) Part-time Instr., Junior Coll. of Augusta, Augusta, Ga.
ROBINSON, A. J., M.S. (Emory) Asso. Prof., Alabama Poly. Inst., Auburn, Ala.
ROBINSON, G. DEB., Ph.D. (Cambridge) Asso. Prof., Univ. of Toronto, Toronto, Ont., Canada
ROBINSON, G. N., A.B. (Boston U.) Oiler, Pumping Station, New Bedford, Mass. *R.F.D. 2, Acushnet Station*
ROBINSON, H. A., Ph.D. (Johns Hopkins) Prof., Agnes Scott Coll., Decatur, Ga.
ROBINSON, L. V., Ph.D. (Harvard) Mathematician, Aberdeen Proving Ground, Md. *P.O. Box 124, Aberdeen, Md.*
ROBINSON, RALPH M., M.S. (Drake) Instr., Iowa State Coll., Ames, Iowa, *1222 Northwestern Ave.*
ROBINSON, RAPHAEL M., Ph.D. (California) Prof., Univ. of California, Berkeley 4, Calif.
ROBINSON, ROBIN, Ph.D. (Harvard) Prof., Dartmouth Coll., Hanover, N.H. *16 Allen St.*
ROBINSON, SELBY, Ph.D. (Iowa) Asst. Prof., City Coll. of New York, New York, N.Y. *100 Carlyle Pl., Roslyn Heights, N.Y.*
ROBINSON, V. N., Ph.D. (Chicago) Lt. Comdr. and Asso. Prof., U. S. Naval Acad., Annapolis, Md.
ROBINSON, W. J., Ph.D. (Ohio State) Prof., Centre Coll. of Kentucky, Danville, Ky.
ROBISON, G. B., M.A. (Columbia) Teaching Fellow, Cornell Univ., Ithaca, N.Y.
ROCHE, E. J., REV., M.Sc. (Notre Dame) Prof., St. Dunstan's Univ., Charlottetown, P.E.I., Canada
ROCK, SIBYL M., B.A. (U.C.L.A.) Tech. Consultant, Consolidated Engineering Corp., Pasadena, Calif. *275 N. Craig Ave. (8)*
RODABAUGH, L. D., Ph.D. (Ohio State) Asso. Prof., Southern Illinois Univ., Carbondale, Ill.
RODGERS, T. G., A.M. (Wisconsin) Emeritus Dean, New Mexico Highlands Univ., Las Vegas, N.M. *1018 Fourth St.*
RODRIGUEZ, MARGARITA, Dr. C.F.M. (Havana) Teacher, Vedado High School, Vedado, Havana, Cuba. *Calzada de Columbia 16, Alturas de Miramar, Marianao, Cuba*
ROESSLER, E. B., Ph.D. (California) Prof., Coll. of Agr., Univ. of California, Davis, Calif.
ROGERS, C. A., M.S. (North Texas S.C.) Asst. Prof., Univ. of Houston, Houston, Tex. *4606 Spruce St., Bellaire, Tex.*
ROGERS, H. P., A.M. (Illinois) Asst. Prof., Univ. of New Mexico, Albuquerque, N.M. *238 N. Mesilla Ave.*
ROGERS, J. C., Hon.D.Ed. (Piedmont) Retired President, Univ. of Georgia, Athens, Ga. *Dahlonega, Ga.*
ROGERS, P. C., M.A. (Boston U.) Grad. Asst., Univ. of Maryland, College Park, Md.
ROGNLIE, P. A., M.S. (North Dakota) Asst. Prof., Univ. of North Dakota, Grand Forks, N.D.
ROHDE, F. VIRGINIA, Ph.D. (Kentucky) Instr., Univ. of Florida, Gainesville, Fla.
ROLAND, F. M., Student, St. Mary's Univ., San Antonio, Tex. *1810 W. Hildebrand (1)*
ROLFE, KATHRYN B. (Mrs. R. W.), M.S. (U. of Washington) *1416 Albany Ave., Modesto, Calif.*
ROLFS, H. Q., M.A. (Kansas) Instr., Washburn Univ., Topeka, Kan. *Apt. 11, University Pl.*
ROLL, ROSE, A.M. (Columbia) Asst., Washington Irving High School, New York, N.Y. *205 E. 78th St.*
ROMAN, IRWIN, Ph.D. (Chicago) Geophysicist, U. S. Geological Survey, Baltimore 2, Md. *722 Hunting Pl. (29)*
ROORDA, ETHEL, M.S. (Iowa) Asso. Prof., Buena Vista Coll., Storm Lake, Iowa
ROOT, R. E., Ph.D. (Chicago) Retired Prof., U. S. Naval Postgraduate School, Annapolis, Md. *7 Franklin St.*
RÓSE, G. F., M.A. (Wisconsin) Instr., Univ. of Wisconsin, Madison, Wis. *111 N. Randall Ave. (5)*
ROSE, I. H., A.M. (Brooklyn) Asst. Prof., Univ. of Massachusetts, Amherst, Mass.
ROSE, N. J., M.E. (Stevens Tech.) Instr., Stevens Inst. of Tech., Hoboken, N.J.
ROSEN, J. S., Ph.D. (Washington U.) Prof., Univ. of Kansas City, Kansas City 4, Mo.
ROSEN, SAUL, Ph.D. (Pennsylvania) Asst. Prof., Drexel Inst. of Tech., Philadelphia 4, Pa.
ROSENBACH, J. B., M.S. (Illinois) Prof., Carnegie Inst. of Tech., Pittsburgh, Pa. *2550 Beechwood Blvd., Squirrel Hill (17)*
ROSENBAUM, IRA, M.A. (Harvard) Asst. Prof., Univ. of Miami, Coral Gables, Fla. *3049 S.W. 21st Terr.*

ROSENBAUM, JOSEPH, Ph.D. (Cornell) Teacher, Milford School, Milford, Conn. *148 N. Whitney St., Hartford, Conn.*
ROSENBAUM, LOUISE J. (Mrs. R. A.), Ph.D. (Colorado) Asst. Prof., Reed Coll., Portland 2, Ore.
ROSENBAUM, R. A., Ph.D. (Yale) Prof., Reed Coll., Portland 2, Ore.
ROSENBERG, ALEX, M.A. (Toronto) Univ. of Michigan, Ann Arbor, Mich.
ROSENBERG, G. S., Student, Illinois Inst. of Tech., Chicago, Ill. *417 Hoodridge Dr., Pittsburgh 34, Pa.*
ROSENBLOOM, P. C., Ph.D. (Stanford) Asso. Prof., Univ. of Minnesota, Minneapolis 14, Minn.
ROSENBLUM, G. K., Student, Univ. of Pennsylvania, Philadelphia, Pa. *4932 N. Front St. (20)*
ROSENFELD, A. I., M.A. (Columbia) *475 W. 186th St., New York 33, N.Y.*
ROSENFELD, ABRAHAM, M.S. (M.I.T.) Prof., O.T.D. Ordnance School, Aberdeen Proving Ground, Md. *R.D. 2, Aberdeen, Md.*
ROSENTHAL, ARTHUR, Ph.D. (Munich) Prof., Purdue Univ., Lafayette, Ind.
ROSENTHALL, EDWARD, Ph.D. (C.I.T.) Asso. Prof., McGill Univ., Montreal, P.Q., Canada
ROSS, A. E., Ph.D. (Chicago) Prof., Univ. of Notre Dame, Notre Dame, Ind.
ROSS, D. T., A.B. (Oberlin) Teaching Asst., Massachusetts Inst. of Tech., Cambridge, Mass. *201 Prospect St. (39)*
ROSS, ELOISE M. (Mrs.), M.A. (Oregon) *722 N. Tejon, Colorado Springs, Colo.*
ROSS, F. E., A.B. (Kentucky) Auditor, Div. of Rates & Services, Dept. of Motor Transportation, Frankfort, Ky.
ROSS, J. W., M.A. (Texas) Asso. Prof., Texas Lutheran Coll., Seguin, Tex. *305 S. Randolph Ave., Bryan, Tex.*
ROSS, L. L., M.A. (Ohio State) Stress Analyst, Babcock and Wilcox Co., Barberton, Ohio. *524 E. Washington, Medina, Ohio*
ROSS, LOUIS, A.M. (Akron) Asst. Prof., Univ. of Akron, Akron 2, Ohio. *879 Whittier Ave.*
ROSS, R. M., M.A. (Indiana) Asst. Prof., Rose Poly. Inst., Terre Haute, Ind.
ROSS, S. L., A.M. (Boston U.) Grad. Student, Boston Univ., Boston, Mass. *North Shapleigh, Maine*
ROSSER, J. B., Ph.D. (Princeton) Prof., Cornell Univ., Ithaca, N.Y.
ROSSKOPF, M. F., Ph.D. (Brown) Asso. Prof., Syracuse Univ., Syracuse 10, N.Y.
ROTH, J. P., M.S. (Michigan) Research Asst., Univ. of Michigan, Ann Arbor, Mich. *216 S. Ingalls*
ROTH, S. G., A.M. (North Carolina) Asst. Prof., New York Univ., New York 3, N.Y.
ROTH, W. E., Ph.D. (Wisconsin) Prof., Univ. of Tulsa, Tulsa, Okla.
ROTHE, E. H., Ph.D. (Berlin) Asso. Prof., Univ. of Michigan, Ann Arbor, Mich. *413 S. Forest Ave.*
ROTHE, JANE S. (Mrs. E. H.), Ph.D. (Michigan) *413 S. Forest Ave., Ann Arbor, Mich.*
ROTHLISBERGER, HAZEL M., B.A. (Iowa S.T.C.) Asso. Prof., Univ. of Dubuque, Dubuque, Iowa
ROULEAU, W. G., M.A. (George Washington) Mathematician, Army Map Service, Washington 25, D.C. *2800 Quebec St., N.W., Apt. 1043 (8)*
ROUNDS, E. D., B.S. (Scranton) Instr., Univ. of Scranton, Scranton. Pa
ROUSE, L. J., Ph.D. (Michigan) Asst. Prof., Univ. of Michigan, Ann Arbor, Mich. *712 Onondaga St.*
ROWE, J. E., B.A. (Tennessee) Research Engr., Carbide and Carbon Chem. Corp., Oak Ridge, Tenn. *72 Outer Dr.*
ROWLAND, ANNIE N. (Mrs.), M.S. (Texas Tech.) Instr., Texas Tech. Coll., Lubbock, Tex. *Rushland Park, Rt. 5*
ROWLAND, J. J., A.M. (Oregon) Junior Engr., Boeing Airplane Co., Seattle, Wash. *3022 16th St., S. (44)*
ROWLAND, S. A., A.B. (Ouachita) Prof., Ohio Wesleyan Univ., Delaware, Ohio. *45 Oak Hill Ave.*
ROWLEY, R. P., M.S. (Syracuse) Head of Dept., Amherst Central High School, Buffalo, N.Y. *186 Hamilton Dr., Snyder, N.Y.*
ROYDEN, H. L., Ph.D. (Harvard) Asst. Prof., Stanford Univ., Stanford, Calif.
ROYSTER, W. C., M.A. (Kentucky) Instr., Univ. of Kentucky, Lexington, Ky.
RUBIN, HANAN, A.B. (N.Y.U.) Instr., New York Univ., New York, N.Y. *155 W. 162nd St. (52)*
RUDERMAN, H. D., A.M. (Columbia) Teacher, Manhattan High School of Aviation Trades, New York 21, N.Y. *2391 Webb Ave. (63)*
RUDIN, B. D., B.S. (C.I.T.) Grad. Asst., Univ. of Southern Calif., Los Angeles, Calif. *4055D West Blvd. (8)*
RULON, P. J., Ph.D. (Minnesota) Prof., Education, Harvard Univ., Cambridge 38, Mass. *Peabody House, 13 Kirkland St.*
RUMBLE, DOUGLAS, A.M. (Emory) *442 N. Julia Ave., DeLand, Fla.*
RUMNEY, ETHEL A., M.S. (Chicago) *309 N. Main St., Sandwich, Ill.*
RUNGE, LULU L., A.M. (Wisconsin) Asst. Prof., Univ. of Nebraska, Lincoln, Nebr.

RUNNING, T. R., Ph.D. (Wisconsin) Emeritus Prof., Univ. of Michigan, Ann Arbor, Mich. *1019 Michigan Ave.*
RUPP, C. A., Ph.D. (Chicago) Major, Signal Corps, Dept. of Defense, Washington, D.C. *2807-A S. Abingdon St., Fairlington, Arlington, Va.*
RUSCH, C. E., A.M. (Wisconsin) Prof., Mission House Coll., R.R. 3, Plymouth, Wis.
RUSH, JACOB, A.B. (Columbia) Teacher, Brooklyn Tech. High School, Brooklyn 1, N.Y.
RUSK, EVELYN C. (Mrs. W. S.), Ph.D. (Cornell) Prof., Wells Coll., Aurora, N.Y.
RUSK, W. J., M.A. (Toronto) Emeritus Prof., Grinnell Coll., Grinnell, Iowa. *1415 Park St.*
RUSS, ALEXANDER, A.M. (Duke) C.P.A., Naval Aircraft Factory, Philadelphia, Pa. *1546 Levick St. (24)*
RUSSELL, B. M., Rev., M.S. (Michigan) Asst. Prof., Gannon Coll., Erie, Pa.
RUSSELL, HELEN G., Ph.D. (Radcliffe) Prof., Wellesley Coll., Wellesley 81, Mass.
RUSSELL, J. P., Sc.M. (Brown) Asst. Prof., Poly. Inst. of Brooklyn, Brooklyn, N.Y. *27 E. 89th St.*
RUTLAND, L. W., JR., M.S. (East Texas S.T.C.) Instr., Univ. of Colorado, Boulder, Colo.
RUTLEDGE, J. D., B.A. (Swarthmore) Programmer, Trainee, Eckert-Mauchly Computer Corp., Philadelphia, Pa. *4621 Cedar Ave. (43)*
RUTLEDGE, W. A., Ph.D. (Tennessee) Asst. Prof., Alabama Poly. Inst., Auburn, Ala.
RUTT, N. E., Ph.D. (Pennsylvania) Prof., Louisiana State Univ., Baton Rouge, La.
RYAN, D. R., M.A. (Gonzaga) Head of Dept., Gonzaga Univ., Spokane 11, Wash.
RYAN, J. B., Rev. Br., B.S. (Seattle) Treasurer, Rice High School, New York 27, N.Y.
RYSER, H. J., Ph.D. (Wisconsin) Asst. Prof., Ohio State Univ., Columbus 10, Ohio

SAASTAD, ARTHUR, M.A. (Northwestern) Asst. Prof., De Paul Univ., Chicago, Ill. *1309 Maple Ave., Evanston, Ill.*
SACHS, J. M., Ph.D. (Chicago) Instr., Chicago Teachers Coll., Chicago, Ill.
SACKS, LOUIS, M.S. (Carnegie) Instr., Carnegie Inst. of Tech., Pittsburgh 13, Pa.
SADOWSKY, M. A., Ph.D. (Berlin) Asso. Prof., Illinois Inst. of Tech., Chicago 16, Ill.
SAIBEL, E. A., Ph.D. (M.I.T.) Prof., Carnegie Inst. of Tech., Pittsburgh 13, Pa.
SAILE, MARCIA C., A.M. (Michigan) Instr., Bay City Junior College, Bay City, Mich. *2300 Center Ave., Apt. 4*
SALEM, RAPHAEL, Sc.D. (Paris) Prof., Massachusetts Inst. of Tech., Cambridge, Mass.
SALERNO, JOHN, A.B. (Brooklyn) Mathematician, U. S. Coast and Geodetic Survey, New York, N.Y. *105-45 88th St., Ozone Park 17, N.Y.*
SALISBURY, E. L., M.S. (Idaho) Instr., State College of Washington, Pullman, Wash.
SALKIND, CHARLES, M.S. (C.C.N.Y.) Teacher, S. J. Tilden High School, Brooklyn 3, N.Y. *1304 New York Ave.*
SALTARELLI, G. X., M.S. (Notre Dame) Bell Aircraft Corp., Niagara Falls, N.Y. *156 Freund St., Buffalo 15, N.Y.*
SALTZER, CHARLES, M.S. (Brown) Instr., Case Inst. of Tech., Cleveland 6, Ohio
SAMELSON, HANS, Dr.Sc.Math. (Fed. Inst. of Tech., Zurich) Asso. Prof., Univ. of Michigan, Ann Arbor, Mich.
SAMMET, JEAN E., M.A. (Illinois) Computer, Actuarial Div., Metropolitan Life Insurance Co., 1 Madison Ave., New York 10, N.Y. *500 West End Ave. (24)*
SAMOLOFF, JACOB, B.A. (Toronto) Instr., Lehigh Univ., Bethlehem, Pa.
SAMUELS, ALTA H. (Mrs.), M.A. (L.S.U.) Asst. Prof., Univ. of Mississippi, University, Miss.
SANCHEZ-DIAZ, RAFAEL, Ph.D. (California) Prof., New Mexico School of Mines, Socorro, N.M.
SANDER, F. L., Student, Michigan State College, East Lansing, Mich. *1020 West Washtenaw St., Lansing 15, Mich.*
SANDERS, J. E., Retired Meteorologist, *109 E. Glenwood Ave., Knoxville 17, Tenn.*
SANDERS, S. T., M.S. (Chicago) Emeritus Prof., Louisiana State Univ., Baton Rouge, La. *1323 Old Shell Road, Mobile, Ala.*
SANDHAM, H. F., B.A. (Trinity, Dublin) *5 St. Helen's Rd., Blackrock, Dublin, Ireland*
SANDLER, BARNEY, B.A. (Brooklyn C.) Instr., New York State Univ., Inst. of Applied Arts & Science, Brooklyn, N.Y. *15 Argyle Rd. (18)*
SANDLIN, W. T., A.B. (Marshall C.) Manager and Advertising Director, C. E. Sandlin & Son, Richmond, Ky.
SANDT, J. E., A.M. (Lafayette) Asst. Prof., Marietta Coll., Marietta, Ohio
SANDWICK, C. M., SR., B.A. (Lehigh) Teacher, Easton High School, Easton, Pa. *819 Spring Garden St.*
SANDY, J. M., B.S. in Ed. (Missouri) Asst. Instr., Univ. of Missouri, Columbia, Mo. *Route 3, Box 243, Joplin, Mo.*
SANFORD, VERA, Ph.D. (Columbia) Prof., State Teachers Coll., Oneonta, N.Y.
SANGER, R. G., Ph.D. (Chicago) Prof., Kansas State Coll., Manhattan, Kan. *1400 Houston St.*
SANGREN, W. C., Ph.D. (Michigan) Sr. Mathematician, Oak Ridge National Lab., Oak Ridge, Tenn. *112 Meadow Rd.*

SARD, ARTHUR, Ph.D. (Harvard) Asso. Prof., Queens Coll., Flushing 55, N.Y. *146-19 Beech Ave.*
SARNO, A. H., M.S. (C.C.N.Y.) Asst. Prof., St. John's Univ., Brooklyn, N.Y. *104 Village Ave., Elmont, L.I., N.Y.*
SARTOR, R. A., Student, Univ. of Detroit, Detroit, Mich. *1121 Marlboro St., Sandusky, Ohio*
SASSER, ANNIE G. H. (Mrs.), M.S. (Prairie View) Asst. Prof., Tennessee A. and I. State Coll., Nashville, Tenn.
SAUNDERS, F. W., M.A. (North Carolina) Prof. Coker Coll., Hartsville, S.C.
SAUNDERS, R. B., Ph.D. (Minnesota) Asso. Prof., Oregon State Coll., Corvallis, Ore. *2605 Tyler St.*
SAUNDERS, W. G., B.S. (Randolph-Macon) Asst. Instr., Virginia Military Inst., Lexington, Va. *Milford, Va.*
SAUTE, GEORGE, A.M. (Brown) Prof., Rollins Coll., Winter Park, Fla.
SAUTER, W. L., Student, Univ. of Louisville, Louisville, Ky. *1103 Ellison Ave. (4)*
SAVARO, JOSEPHINE, Librarian, Univ. of Scranton, Scranton 3, Pa.
SAVIT, C. H., M.S. (C.I.T.) Mathematician, Western Geophysical Co., Los Angeles, Calif. *6033 Ventura Canyon Ave., Van Nuys, Calif.*
SAWYER, J. W., M.A. (Missouri) Asst. Prof., Atlanta Div., Univ. of Georgia, Atlanta, Ga.
SAYLES, H. A., Retired. *120 Elm St., Orange, N.J.*
SCARBOROUGH, J. B., Ph.D. (Johns Hopkins) Emeritus Prof., U. S. Naval Academy, Annapolis, Md. *Ferry Farms*
SCARF, HERBERT, Student, Temple Univ., Philadelphia, Pa. *516 McKean St. (48)*
SCHACH, ARTHUR, A.M. (Columbia) Instr., Philosophy, City Coll. of New York, New York, N.Y. *147-01 77th Rd., Kew Garden Hills, N.Y.*
SCHAEFER, L. E., M.A. (Michigan State) Instr., General Motors Inst., Flint, Mich. *605 East Dewey St. (5)*
SCHAEFER, P. T., B.A. (Rochester) Grad. Student, Univ. of Rochester, Rochester, N.Y. *340 Garson Ave. (9)*
SCHAEFFER, A. C., Ph.D. (M.I.T.) Prof., Univ. of Wisconsin, Madison, Wis.
SCHAER, L. R., Student, Univ. of Buffalo, Buffalo 14, N.Y. *50 Crestwood Ave. (16)*
SCHAFER, R. D., Ph.D. (Chicago) Asst. Prof., Univ. of Pennsylvania, Philadelphia 4, Pa.
SCHATTEN, ROBERT, Ph.D. (Columbia) Asso. Prof., Univ. of Kansas, Lawrence, Kan.
SCHATZ, J. A., B.S. in E.E. (Virginia Poly. Inst.) Student, Brown Univ., Providence 12, R.I. *103 South Angell St.*
SCHECTER, SAMUEL, A.B. (Brooklyn) Instr., Lehigh Univ., Bethlehem, Pa.
SCHEFFE, HENRY, Ph.D. (Wisconsin) Asso. Prof., Columbia Univ., New York 27, N.Y.
SCHEIER, M. A., Rev., M.S. (Catholic) Dean, Div. Natural Science, St. Bonaventure Coll., St. Bonaventure, N.Y.
SCHELKUNOFF, S. A., Ph.D. (Columbia) Research Mathematician, Bell Telephone Labs., New York, N.Y. *Murray Hill, N.J.*
SCHELL, E. D., A.M. (Western Reserve) Chief, Math. and Electronic Comp. Br., Air Force Hdqs., Washington 25, D.C. *General Delivery, Vienna, Va.*
SCHILLING, O. F. G., Ph.D. (Marburg) Asso. Prof., Univ. of Chicago, Chicago, Ill.
SCHILLO, P. J., B.A. (Buffalo) Grad. Student, Univ. of Buffalo, Buffalo 14, N.Y. *288 May St.*
SCHLAUCH, L. R., M.A. (Penn State) Analyst, Dept. of Defense, Armed Forces Security Agency, Washington 25, D.C. *6610 Wilson Blvd., Falls Church, Va.*
SCHLAUCH, W. S., A.M. (Columbia) Emeritus Prof., New York Univ., New York, N.Y. *14 Lohman Place, Dumont, N.J.*
SCHLESINGER, STEWART, M.S. (Illinois Tech.) Research Asst., Los Alamos Scientific Lab., Los Alamos, N.M. *3710 Gold St., Apt. #9*
SCHMID, ERWIN, M.A. (George Washington) Mathematician, U. S. Coast and Geodetic Survey, Washington 25, D.C. *7300 Newburn Dr. (16)*
SCHMIED, R. W., Ph.D. (Brown) *1013 West 23rd St., Austin 21, Texas*
SCHNECKENBURGER, EDITH R., Ph.D. (Michigan) Asst. Prof., Univ. of Buffalo, Buffalo 14, N.Y. *476 Shirley Ave. (15)*
SCHNEPP, R. F., Ph.D. (Fribourg) Prof., St. Mary's Univ. of San Antonio, San Antonio 1, Texas
SCHOCKEN, KLAUS, Ph.D. (Berlin) Biophysicist, Army Medical Department Field Research Lab., Fort Knox, Ky.
SCHODERBEK, J. J., M.S. (Carnegie) Electronics Engr., Glen L. Martin Co., Baltimore 3, Md. *2 Mars Ave., Apt. C (21)*
SCHOENBERG, I. J., Ph.D. (Jassy) Prof., Inst. for Numerical Analysis, National Bureau of Standards, Los Angeles 24, Calif. *3226 Glendon Ave. (34)*
SCHOLL, L. F., M.A. (Buffalo) Supervisor of Math., Board of Education, Buffalo 2, N.Y.
SCHOLOMITI, N. C., B.S. (N.Y.U.) Instr., Univ. of Illinois, Navy Pier, Chicago 11, Ill. *2200 W. Cullom Ave. (18)*
SCHOLTEN, RUTH, (Mrs.), M.A. (Drake) Instr., South Dakota State Coll., Brookings, S.D. *416½ 9th Ave.*

SCHOLZ, HERBERT, JR., M.A. (North Carolina) Asso. Prof., Oklahoma A. and M., Stillwater, Okla. *523 Blakley St.*
SCHOR, HARRY, M.S. in Ed. (C.C.N.Y.) Chm. of Dept., Far Rockaway High School, Far Rockaway, Queens, N.Y. *2348 Ocean Parkway, Brooklyn 23, N.Y.*
SCHRADER, DOROTHY V., M.A. (Boston C.) Instr., Coll. of Saint Teresa, Winona, Minn.
SCHRAUT, K. C., Ph.D. (Cincinnati) Prof., Univ. of Dayton, Dayton 9, Ohio
SCHREIBER, E. W., A.M. (Chicago) Prof., Western Illinois State Teachers Coll., Macomb, Ill. *719 W. Adams St.*
SCHRIRO, G. R., M.A. (N.Y.U.) Teacher, Farmingdale High School, Farmingdale, L.I., N.Y. *17-A 180 Lenox Rd., Huntington Station, L.I., N.Y.*
SCHROEDER, H. F., M.S. (Louisiana) Prof., Louisiana Poly. Inst., Ruston, La.
SCHUB, PINCUS, Ph.D. (Dropsie C.) Instr., Univ. of Pennsylvania, Philadelphia, Pa. *State Hospital, Trenton 8, N.J.*
SCHULT, VERYL G. (Miss), A.M. (George Washington) Head of Dept., Washington Public Schools, Washington, D.C. *Wardman Park Hotel, Apt. 410-A (8)*
SCHULTZ, O. T., M.S. (Chicago) Physicist, Naval Ordnance Lab., Silver Spring, Md. *561 Southampton Dr., Northwest Park*
SCHUMAKER, J. A., M.A. (Illinois) Asst. Prof., MacMurray Coll. for Women, Jacksonville, Ill. *903 West College Ave.*
SCHURRER, AUGUSTA L., A.M. (Wisconsin) Instr., Iowa State Teachers Coll., Cedar Falls, Iowa. *618 West 12th St.*
SCHUSTER, SEYMOUR, M.A. (Columbia) Instr., Pennsylvania State Coll., State College, Pa.
SCHUTZBERGER, HENRY, M.S. (Illinois) Research Mathematician, Sandia Laboratory, Albuquerque, N.M. *1611 Ridgecrest Circle*
SCHWALLER, R. L., B.S. (St. Norbert) QM3, *Sub. Gru. 1, Div. 4, U.S.S. Markab, Orange, Tex.*
SCHWARTZ, ABRAHAM, Ph.D. (M.I.T.) Asst. Prof., City College of New York, New York, N.Y. *321 Windsor Rd., Englewood, N.J.*
SCHWARTZ, A. A., M.S. in Ed. (C.C.N.Y.) Claims Examiner, U. S. Railroad Retirement Board, New York, N.Y. *55 Parade Place, Brooklyn 26, N.Y.*
SCHWARTZ, B. L., M.S. (Carnegie) Instr., Duquesne Univ., Pittsburgh 22, Pa. *815 Norwich Ave. (26)*
SCHWARTZ, CECIL, B.S. (Mississippi S.C.) Instr., Univ. of Illinois, Undergrad. Div., Navy Pier, Chicago, Ill.
SCHWARTZ, H. M., Ph.D. (Pennsylvania) Asso. Prof., Physics, Univ. of Arkansas, Fayetteville, Ark. *P.O. Box 1607*
SCHWARTZ, H. W., Sc.M. (N.Y.U.) Chanute Air Force Base, Ill. *320 West Pells St., Paxton, Ill.*
SCHWEITZER, A. R., Ph.D. (Chicago) *452 Oakdale Ave., Chicago, Ill.*
SCHWEIZER, BERTHOLD, B.S. (M.I.T.) Mathematician, Naval Ordnance Lab., White Oak, Md. *228 N. Manor Circle, Takoma Park, Md.*
SCHWEPPE, E. J., M.S. (Illinois) Asst., Univ. of Illinois, Urbana, Ill. *Apt. 26, 206 N. Race St.*
SCHWID, NATHAN, Ph.D. (Wisconsin) Prof., Univ. of Wyoming, Laramie, Wyo.
SCIBIORSKI, TRYPHENA HOWARD (Mrs. A. C.), A.M. (Michigan) Instr., Wayne Univ., Detroit 1, Mich.
SCOBERT, W. G., M.S. (Oregon) Asso. Prof., Idaho State Coll., Pocatello, Idaho
SCOTT, CAROL S. (Mrs. Wm. C.), M.A. (Western Reserve) Instr., St. Petersburg Jr. Coll., St. Petersburg 4, Fla. *1190 8th St. N.*
SCOTT, E. J., Ph.D. (Cornell) Asst. Prof., Univ. of Illinois, Urbana, Ill. *105 W. Michigan*
SCOTT, P. C., Ph.D. (Peabody) Head of Dept., East Carolina Teachers Coll., Greenville, N.C.
SCOTT, W. R., Ph.D. (Ohio State) Asst. Prof., Univ. of Kansas, Lawrence, Kan.
SCOTT, W. T., Ph.D. (Rice) Asso. Prof., Northwestern Univ., Evanston, Ill.
SEALANDER, B. E., B.S. (Seton Hall) Sr. Research Engr., Republic Aviation, Farmingdale, L.I., N.Y.
SEALANDER, C. E., Ph.D. (Iowa) Asst. Prof., Ohio State Univ., Columbus 10, Ohio. *162 Garden Road (2)*
SEALE, R. Q., Ph.D. (Stanford) Prof., Southern Methodist Univ., Dallas, Texas
SEAMONS, R. S., M.S. (Washington) Instr., Yakima Valley Jr. Coll., Yakima, Wash. *1012 South Second Ave.*
SEARS, HELEN WRIGHT, M.A. (Loyola) Instr., Univ. of Illinois, Navy Pier, Chicago, Ill. *2858 Dickens Ave. (47)*
SEBER, R. C., M.S. (Iowa) Asst. Prof., Wisconsin State Coll., Eau Claire, Wis.
SEBESTA, C. F., M.A. (Pittsburgh) Instr., Univ. of Pittsburgh, Ellsworth Center, Pittsburgh, Pa. *4843 Baptist Rd. (27)*
SECADA, C. F., S.M. (M.I.T.) Production Officer, Callao Naval Shipyard, Copacabana 565, Lima, Peru
SEDGEWICK, C. H. W., Ph.D. (Brown) Prof., Univ. of Connecticut, Storrs, Conn.

SEDNEY, RAYMOND, D.Sc. (Carnegie) Aerodynamicist, Douglas Aircraft Co., Santa Monica, Calif. *2358 S. Bentley Ave., Los Angeles 64, Calif.*
SEEBALD, H. A., M.A. (Lehigh) *20 Ely Pl., East Orange, N.J.*
SEEBECK, C. L., JR., Ph.D. (North Carolina) Asso. Prof., Univ. of Alabama, University, Ala. *P.O. Box 814*
SEEBER, R. R., JR., A.B. (Harvard) Senior Staff Member, International Business Machines Corp., New York, N.Y.
SEELY, N. T., JR., M.A. (Pennsylvania) Instr., Arkansas A. M. & N. Coll., Pine Bluff, Ark. *136 Edgewood Ave., New Haven, Conn.*
SEIBEL, D. W., B.S. (Denison) Grad. Asst., Kent State Univ., Kent, Ohio. *150 Benedict Ave., Norwalk, Ohio*
SEIDEL, WLADIMIR, Ph.D. (Munich) Prof., Univ. of Rochester, Rochester 3, N.Y.
SEIDEN, ESTHER, Ph.D. (California) Asst. Prof., Univ. of Buffalo, Buffalo 14, N.Y.
SEIDENBERG, ABRAHAM, Ph.D. (Johns Hopkins) Asst. Prof., Univ. of California, Berkeley 4, Calif. *1900-B Woolsey St. (3)*
SEIDLIN, JOSEPH, Ph.D. (Columbia) Dean of Grad. School, Alfred Univ., Alfred, N.Y.
SELBY, SAMUEL, Ph.D. (Chicago) Prof., Univ. of Akron, Akron 4, Ohio
SELLERS, W. H., M.S. (West Virginia) Head of Dept., Alderson-Broaddus Coll., Philippi, W. Va.
SEMPLE, R. J., M.A. (Toronto) Instr., Princeton Univ., Princeton, N.J.
SENDROW, M. J., *1705 P St., N.W., Apt. 33, Washington, D.C.*
SENSALE, C. J., Student, New Jersey State Teachers College, Upper Montclair, N.J. *201 North First St., Paterson, N.J.*
SENTELL, MAE DELL, M.S. (Alabama Poly.) Instr., Alabama Poly. Inst., Auburn, Ala.
SERBYN, W. D., B.A. (Chicago) Grad. Student, Univ. of Chicago, Chicago 37, Ill.
SEUBERT, G. A., B.S. (Pittsburgh) *Box 107, Latrobe, Pa.*
SEUGLING, W. R., M.A. (U.C.L.A.) Research Asst., Univ. of California at Los Angeles, Los Angeles, Calif.
SEWARD, D. M., Ph.D. (Duke) Prof., Ouachita Coll., Arkadelphia, Ark.
SEWELL, J. R., B.S. (Rollins) *1017 Aloma Ave., Winter Park, Fla.*
SEWELL, W. E., Col. and Prof., State Univ. of Iowa, Iowa City, Iowa
SEYBOLD, M. ANICE, Ph.D. (Illinois) Prof., North Central Coll., Naperville, Ill.
SHANKS, E. B., Ph.D. (Illinois) Asso. Prof., Vanderbilt Univ., Nashville, Tenn. *Taggart Ave.*
SHANKS, M. E., Ph.D. (Iowa) Asso. Prof., Purdue Univ., Lafayette, Ind.
SHAPIRO, AARON, M.S. (C.C.N.Y.) Lecturer, Brooklyn Coll., Brooklyn, N.Y. *120 Kenilworth Pl. (10)*
SHAPIRO, E. I., M.A. (Cornell) Instr., Brooklyn Coll., Brooklyn, N.Y. *2865 Ocean Ave. (29)*
SHAPIRO, PAUL, M.S. (George Washington) *1224 Emerson St., N.W., Washington, D.C.*
SHARMA, H. G. S., Prof., Lingaraj Coll., Belgaum, India
SHARP, W. T., M.A. (Toronto) Asst. Research Officer, National Research Council of Canada, Chalk River, Ont., Can. *Staff Hotel, Deep River, Ont., Can.*
SHARPE, B. B., M.A. (Buffalo) Major, U. S. Army. *48 Princeton Blvd., Kenmore, N.Y.*
SHARTLE, S. M., Surveyor and Computer, Office of County Surveyor, Danville, Ind.
SHAUB, H. C., Ph.D. (Cornell) Prof., Washington and Jefferson Coll., Washington, Pa.
SHEAR, R. E., B.A. (Syracuse) Asst., Harpur Coll., Endicott, N.Y. *4 Donald St., Johnson City, N.Y.*
SHEFFER, I. M., Ph.D. (Harvard) Prof., Pennsylvania State Coll., State College, Pa.
SHELDON, D. C., Ph.D. (California) Head of Dept., Clemson Coll., Clemson, S.C.
SHELTON, S. W., JR., M.A. (George Peabody) Asso. Prof., Northwestern State Coll., Natchitoches, La.
SHENTON, W. F., Ph.D. (Johns Hopkins) Prof., American Univ., Washington 16, D.C. *3605 Porter St., N.W. (16)*
SHEPHARD, R. W., Ph.D. (California) Asso. Social Scientist, Rand Corp., Santa Monica, Calif.
SHEPHERD, C. M., M.S.Ch.E. (Case) Research Electrochemist, Naval Research Lab., Washington, D.C. *3959 Nichols Ave. S.W. (20)*
SHEPHERD, W. L., M.S. (Oklahoma A. and M.) Instr., Univ. of Oregon, Eugene, Ore.
SHEPPARD, ELYSE G. (Mrs.), M.A. (Michigan) Asst. Prof., Univ. of Tampa, Tampa 6, Fla.
SHERER, C. R., A.M. (Nebraska) Prof., Texas Christian Univ., Fort Worth 9, Texas
SHERIDAN, L. W., Ph.D. (Catholic) Asso. Prof., Coll. of St. Thomas, St. Paul 1, Minn.
SHERWOOD, G. E. F., Ph.D. (Chicago) Retired Prof., Univ. of California at Los Angeles, Los Angeles, Calif. *1580 San Remo Drive, Pacific Palisades, Calif.*
SHERWOOD, JEANETTE H., M.A. (Michigan) Instr., Univ. of Detroit, Detroit, Mich. *743 Lakepointe, Grosse Pointe Park 30, Mich.*
SHEWHART, W. A., Ph.D. (California) Research Engr., Bell Telephone Labs., Murray Hill, N.J. *158 Lake Drive, Mountain Lakes, N.J.*
SHIVELY, L. S., Ph.D. (Chicago) Prof., Ball State Teachers Coll., Muncie, Ind. *2025 W. Jackson St.*

SHIVELY, R. L., M.A.(Michigan) Instr., Western Reserve Univ., Cleveland 6, Ohio
SHNIAD, HAROLD, Ph.D.(U.C.L.A.) Asst. Prof., Univ. of Arkansas, Fayetteville, Ark.
SHOEMAKER, R. W., M.S.(Toledo) Asst. Prof., Univ. of Toledo, Toledo, Ohio. *2647 Thoman Place (13)*
SHOOK, C. A., Ph.D.(Johns Hopkins) Prof., Lehigh Univ., Bethlehem, Pa. *R.D. 4*
SHORE, D. P., A.M.(Peabody) Teacher, Carter-Riverside High School, Fort Worth, Tex. *1511 N. Riverside Dr.*
SHOTWELL, C. E., M.A.(Missouri) Asst. Prof., Univ. of the South, Sewanee, Tenn.
SHREVE, D. R., Ph.D.(Illinois) Mathematician, Carter Oil Co., Research Lab., Tulsa 2, Okla. *2709 E. 10th St. (4)*
SHRINER, W. O., Ph.D.(Michigan) Prof., Indiana State Teachers Coll., Terre Haute, Ind. *2525 N. Ninth St.*
SHULER, C. EUCEBIA, Ph.D.(Peabody) Asso. Prof., Univ. of South Carolina, Columbia, S.C.
SHUMWAY, R. R., A.B.(Minnesota) Retired Prof., Univ. of Minnesota, Minneapolis 14, Minn. *4736 Washburn Ave. South (10)*
SHUSTER, C. N., Ph.D.(Columbia) Prof., State Teachers Coll., Trenton, N.J. *2393 Pennington Road*
SICELOFF, L. P., Ph.D.(Columbia) Prof., Columbia Univ., New York 27, N.Y.
SIDEBOTTOM, K. J., B.S.(Indiana Central C.) Instr., Indiana Central Coll., Indianapolis, Ind. *Route 1, Box 695 (44)*
SIEDBAND, J. H., M.S.(Chicago) Instr., Wilson Jr. Coll., Chicago, Ill. *4705 N. Central Park Ave. (25)*
SIGLER, L. G., B.S.(Oklahoma A. and M.) Grad. Student, Cambridge Univ., Cambridge, England. *79 Hurst Park Ave.*
SIGLEY, D. T., Ph.D.(Illinois) Asso. Prof., Johns Hopkins Univ., Silver Spring, Md.
SILBER, JACK, M.S.(Chicago) Asso. Prof., Roosevelt Coll., Chicago 5, Ill. *4908 N. Springfield Ave. (25)*
SILVA, J. A., Ph.D.(Duke) Research Associate, Inst. for Co-op. Research, Johns Hopkins Univ., Baltimore, Md. *1124A E. Belvedere Ave. (12)*
SILVERMAN, EDWARD, Ph.D.(California) Staff member, Sandia Corp., Albuquerque, N.M. *3216 "A" St., Rm. 32*
SILVERMAN, L. L., Ph.D.(Missouri) Prof., Dartmouth Coll., Hanover, N.H.
SILVERSTEN, I. W., B.A.(Brooklyn C.) Supervisor, Babcock & Wilcox Co., 19 Rector St., New York, N.Y. *153-13 77th Rd., Flushing, L.I., N.Y.*
SIMESTER, J. H., M.A.(Toronto) Prof., Univ. of Louisville, Louisville 8, Ky.
SIMMONS, H. A., Ph.D.(Chicago) Asso. Prof., Northwestern Univ., Evanston, Ill. *1125 Davis St.*
SIMON, W. G., Ph.D.(Chicago) Prof. and Vice-Pres., Western Reserve Univ., Cleveland 6, Ohio
SIMONEL, JEROME, B.A.(Paris) Justice, French Oceania High Court. *Papeete, Tahiti Island, French Oceania*
SIMONS, W. H., Ph.D.(California) Asst. Prof., Univ. of British Columbia, Vancouver, B.C., Can.
SIMONSON, S. C., M.A.(Minnesota) Instr., Univ. of North Dakota, Grand Forks, N.D. *1013 Belmont Rd.*
SIMPSON, T. MCN., JR., Ph.D.(Chicago) Prof., Randolph-Macon Coll., Ashland, Va.
SIMPSON, T. M., Ph.D.(Wisconsin) Emeritus Dean, Grad. School, Univ. of Florida, Gainesville, Fla. *423 S.W. 13th St.*
SINCLAIR, MARY EMILY, Ph.D.(Chicago) Emeritus Prof., Oberlin Coll., Oberlin, Ohio. *175 Morgan St.*
SINGER, DAVID, B.C.E.(N.Y.U.) Structural Design Engr., Corbett, Tinghir & Co., 39 Broadway, New York, N.Y. *1521 Ocean Ave., Brooklyn 30, N.Y.*
SINGER, JAMES, Ph.D.(Princeton) Asso. Prof., Brooklyn Coll., Brooklyn 10, N.Y. *3054 Bedford Ave.*
SISAM, C. H., Ph.D.(Cornell) Emeritus Prof., Colorado Coll., Colorado Springs, Colo. *3805 Benton St., N.W., Washington, D.C.*
SISK, AUGUSTUS, Ph.D.(Cornell) Prof., Maryville Coll., Maryville, Tenn. *117 Miller Ave.*
SISTER AGNES THERESE, A.M.(Columbia) Prof., Coll. of Mount St. Joseph, Mount St. Joseph, Ohio
SISTER ANN ELIZABETH, Ph.D.(Wisconsin) Prof., St. Mary Coll., Xavier, Kan.
SISTER AUGUSTINE MARIA, Ph.D.(Fordham) Principal, The Mary Louis Academy High School, 86-44 178th St., Jamaica 3, N.Y.
SISTER CATHERINE MARIE, Ph.D.(Boston C.) President, Trinity Coll., Washington 17, D.C.
SISTER CHARLES ANNE, M.A.(Ohio State) Dean of Studies, Coll. of St. Mary of the Springs, Columbus 3, Ohio
SISTER CONSTANTIA, B.S.(St. Joseph's) Instr., Seton High School, 2800 N. Charles St., Baltimore, Md.
SISTER ESTHER MARIA, A.M.(St. Elizabeth) Prof., Coll. of St. Elizabeth, Convent Station, N.J.

SISTER GABRIELLE MARIE, M.A. (Catholic) Instr., Dunbarton Coll., Washington, D.C. *2935 Upton St. (8)*
SISTER GERTRUDE MARIE, M.S. (St. Louis) Prof., Marian Coll., R.R. 17, Box 14, Indianapolis 44, Ind.
SISTER IRENE, M.A. (Catholic) Teacher, Annhurst Coll., Box 569, Putnam, Conn.
SISTER JEANETTE, Ph.D. (Catholic) Asso. Prof., Mount St. Scholastica Coll., Atchison, Kan.
SISTER LAURENTINE MARIE, A.M. (Trinity C., D.C.) Head of Dept., Emmanuel Coll., 400 The Fenway, Boston 15, Mass.
SISTER M. ANITA, A.M. (Seton Hall C.) Instr., Caldwell Coll., Caldwell, N.J.
SISTER M. CLAUDIA, Ph.D. (Michigan) Asst. Prof., Coll. of St. Francis, Joliet, Ill.
SISTER M. CLOTILDA, A.M. (Canisius) Instr., Mercyhurst Coll., Erie, Pa.
SISTER M. CONSTANTIA, M.A. (Notre Dame) Head of Dept., Central Catholic High School, Toledo, Ohio. *6832 Convent Blvd., Sylvania, Ohio*
SISTER M. DEPAZZI, Ph.D. (Notre Dame) Dean, Briar Cliff Coll., Sioux City 17, Iowa
SISTER M. DOROTHEA, M.A. (Cornell) Instr., Nazareth Coll., Rochester, N.Y. *4245 East Ave. (18)*
SISTER M. HELEN, Ph.D. (Catholic) Chm. of Dept., Mount St. Scholastica Coll., Atchison, Kan.
SISTER M. IRMA, M.A. (Michigan) Instr., Siena Heights Coll., Adrian, Mich.
SISTER M. JOANNE, Ph.D. (Washington) Prof., St. Benedict's Coll., St. Joseph, Mo.
SISTER M. LAURINE, A.M. (Catholic) Registrar, Brescia Coll., Owensboro, Ky.
SISTER M. LEONA, M.S. (Notre Dame) Instr., Mercy Coll., 8200 W. Outer Dr., Detroit 19, Mich.
SISTER M. LEONARDA, Ph.D. (Catholic) Prof., Regis Coll., Weston 93, Mass. *Wellesley St.*
SISTER M. MADELEINE ROSE, M.S. (St. Louis) Pres., Coll. of the Holy Names, Oakland 12, Calif.
SISTER M. MECHTILDIS, M.S. (Marquette) Head of Dept., Viterbo Coll., La Crosse, Wis.
SISTER M. MERCEDES, M.S. (Notre Dame) Instr., Mary Manse Coll., Toledo, Ohio. *2413 Collingwood Ave. (10)*
SISTER M. MERCEDES, A.B. (St. Scholastica) Registrar, Coll. of St. Scholastica, Duluth 2, Minn.
SISTER M. MICHAEL, M.S. (Catholic) Instr., Mt. Mercy Coll., 3333 Fifth Ave., Pittsburgh 13, Pa.
SISTER M. MIRABELLA, M.S. (Catholic) Instr., Alverno Teachers Coll., 1413 S. Layton Blvd., Milwaukee 4, Wis.
SISTER M. NICHOLAS, Ph.D. (Catholic) Head of Dept., Marymount Coll., Salina, Kan.
SISTER M. PACHOMIA, A.M. (Missouri) Coll. of St. Teresa, Kansas City, Mo.
SISTER M. PRUDENTIA, A.M. (Catholic) Coll. of St. Scholastica, Duluth 2, Minn.
SISTER M. ROSALIN, A.M. (Catholic) Head of Dept., Ursuline Coll., Louisville 6, Ky. *3115 Lexington Rd.*
SISTER M. ROSE AGNES, M.A. (Catholic) Instr., Salve Regina Coll., Ochre Point Ave., Newport, R.I.
SISTER M. TARCISIUS, M.A. (Duquesne) Teacher, St. Peter's Convent, 715 Market St., McKeesport, Pa.
SISTER M. THOMAS A KEMPIS, Ph.D. (Michigan) Principal, St. Joseph Central High School, Ironton, Ohio
SISTER M. VIRGILIA, M.S. (Notre Dame) Teacher, Madonna Coll., Plymouth, Mich.
SISTER MARIA CORONA, Ph.D. (Fordham) Dean, Coll. of Mount St. Joseph, Mount St. Joseph, Ohio
SISTER MARIA LOYOLA, A.M. (Fordham) Asso. Prof., Coll. of Mount St. Vincent, New York 63, N.Y.
SISTER MARIE GERTRUDE, M.S. (Notre Dame) Prof., Seton Hill Coll., Greensburg, Pa.
SISTER MARIE GRACE, M.S. (Catholic) Vice-Pres., Barry Coll., Miami 38, Fla.
SISTER MARIOLA, A.M. (Wisconsin) Prof., Rosary Coll., River Forest, Ill.
SISTER MARY AGNETA, B.A. (St. Teresa) Teacher, St. Clare Academy, Sylvania, Ohio
SISTER MARY BENEDICTA, A.B. (Good Counsel C.) Good Counsel Coll., White Plains, N.Y.
SISTER MARY CHARLES, M.A. (Catholic) Asst. Prof., Immaculate Heart Coll., Los Angeles, Calif. *2021 No. Western Ave. (27)*
SISTER MARY CHARLOTTE, A.M. (Catholic) Registrar, St. Xavier Coll., 4900 Cottage Grove Ave., Chicago 15, Ill.
SISTER MARY CLEOPHAS, Ph.D. (St. Louis) Head of Dept., Science, Notre Dame Coll., South Euclid 21, Ohio
SISTER MARY CORDIA, Ph.D. (Johns Hopkins) Head of Dept., Coll. of Notre Dame of Maryland, N. Charles St., Baltimore 10, Md.
SISTER MARY CORMAC, M.S. (Notre Dame) Asst. Prof., Marywood Coll., Scranton 9, Pa.
SISTER MARY DEBORAH, B.A. (Seton Hill C.) Instr., Seton Hill Coll., Greensburg, Pa.
SISTER MARY EDMUND, Ed.M. (Cincinnati) Business Manager & Instr., Our Lady of Cincinnati Coll., Edgecliff, Walnut Hills, Cincinnati 6, Ohio
SISTER MARY ESTHER, A.M. (California) Instr., Mundelein Coll., 6363 Sheridan Rd., Chicago, Ill.
SISTER MARY FELICE, Ph.D. (Catholic U.) Prof., Mount Mary Coll., Milwaukee 10, Wis.
SISTER MARY FERRER, Ph.D. (Notre Dame) Head of Dept., St. Xavier Coll., 4900 Cottage Grove Ave., Chicago 15, Ill.

SISTER MARY HENRIETTA, Ph.D. (Catholic U.) Head of Dept., Teachers Coll., Cincinnati 12, Ohio. *1409 Freeman Ave. (14)*
SISTER MARY LEONTIUS, Ph.D. (Michigan) Instr., Coll. of St. Teresa, Winona, Minn.
SISTER MARY OF MERCY, A.M. (Catholic U.) Instr., Incarnate Word Coll., San Antonio 9, Tex.
SISTER MARY MICHAEL, A.M. (Catholic U.) Head of Dept., D'Youville Coll., 320 Porter Ave., Buffalo 1, N.Y.
SISTER MARY MICHAIL, M.S. (Marquette) Head of Dept., Clarke Coll., Dubuque, Iowa
SISTER MARY OLIVE, M.S. (Marquette) Head of Dept., Cardinal Stritch Coll., 3195 S. Superior St., Milwaukee 7, Wis.
SISTER MARY PAULA, M.S. (Notre Dame) Prof., Marygrove Coll., 8425 W. McNichols Rd., Detroit 21, Mich.
SISTER MARY PETRONIA, Ph.D. (Notre Dame) Teacher, Mount Mary Coll., Milwaukee 13, Wis.
SISTER MARY RAPHAEL, A.B. (Chestnut Hill) Instr., Immaculata Coll., Immaculata, Pa.
SISTER MARY SERAPHIM, Ph.D. (Minnesota) Head of Dept., Coll. of St. Catherine, St. Paul 1, Minn.
SISTER MARY SYLVESTER, M.S. (St. Louis) Instr., Mundelein Coll., 6363 Sheridan Rd., Chicago 40, Ill.
SISTER MARY TERESINE, Ph.D. (Catholic U.) Instr., Fontbonne Coll., Wydown and Big Bend Blvd., St. Louis 5, Mo.
SISTER MIRIAM FRANCIS, Ph.D. (Catholic U.) Instr., Xavier Univ., Washington and Pine St., New Orleans, La.
SISTER MIRIAM PATRICK, B.S. (St. Mary's) Grad. Asst., St. Mary's Coll., Notre Dame, Ind.
SISTER NOEL MARIE, A.M. (Catholic U.) Head of Dept., Coll. of Saint Rose, Albany 3, N.Y.
SISTER PRESENTATION, A.M. (Catholic U.) Instr., Our Lady of the Lake Coll., San Antonio, Tex.
SISTER ROSE GERTRUDE, Ph.D. (Catholic U.) Prof., Mount St. Mary's Coll., 12001 Chalon Rd., Los Angeles 24, Calif.
SISTER ROSE MARGARET, M.S. (Notre Dame) Prof., Loretto Heights Coll., Loretto, Colo.
SISTER ST. THOMAS OF CANTERBURY, M.A. (Fordham) Teacher, Villa Maria Academy, Country Club Rd., Bronx 61, New York, N.Y.
SISTER TERESA MARIE, A.M. (Boston C.) Prof., Coll. of Our Lady of the Elms, Chicopee, Mass.
SISTER VINCENT DE PAUL, M.S. (Notre Dame) Head of Dept., Mary Manse Coll., Toledo 10, Ohio
SKELDING, A. Z., A.B. (C.C.N.Y.) Asst. Mgr., National Council on Compensation Insurance, New York, N.Y. *162 Hamilton Rd., Hempstead, L.I., N.Y.*
SKINNER, A. T., M.A. (Columbia) Asst. Prof., Champlain Coll., Plattsburg, N.Y.
SKLAR, ABE, M.S. (Chicago) Grad. Student, Univ. of Chicago, Chicago 37, Ill. *4834 N. Troy St. (25)*
SKOLNIK, SAMUEL, M.A. (Southern California) Instr., Los Angeles City Coll., Los Angeles, Calif.
SLECHTICKY, J. L., M.S. (Washington) Asst. Prof., New Mexico Highlands Univ., Las Vegas, N.M.
SLEIGHT, E. R., A.M. (Albion) Lecturer, Univ. of Richmond, Richmond, Va.
SLEPIN, BENJAMIN, LL.B. (South Jersey Law School) Dir. Mathematical Lab., 422 S. 57th St., Philadelphia 43, Pa.
SLOAN, A. R., A.M. (Vanderbilt) Prof., Carson-Newman Coll., Jefferson City, Tenn. *150 N. Eastview Ave.*
SLOAN, J. L., M.A. (Duke) Asso. Prof., Presbyterian Coll., Clinton, S.C.
SLOATMAN, J. K., JR., B.S. (Union C.) Cmdr., U. S. Navy. *725 North 17th St., Albuquerque, N.M.*
SLOTNICK, M. M., Ph.D. (Harvard) Chief Mathematician, Geophysical Interpretation, Humble Oil and Refining Co., Houston, Tex. *4421 Roseneath Dr. (4)*
SMAIL, L. L., Ph.D. (Columbia) Prof., Lehigh Univ., Bethlehem, Pa.
SMALL, W. A., A.B. (Rochester) Grad. Asst., Univ. of Rochester, Rochester, N.Y. *10 Parkside Court (9)*
SMART, R. E., B.S. (Ohio State) Engineer, Anchor Hocking Glass Corp., Lancaster, Ohio. *518 Frederick St. South*
SMILEY, C. H., Ph.D. (California) Prof. and Dir. of Ladd Observatory, Brown Univ., Providence 12, R.I.
SMILEY, M. F., Ph.D. (Chicago) Prof., State Univ. of Iowa, Iowa City, Iowa. *1716 East Court St.*
SMITH, ALTON H., Student, George Pepperdine Coll., Los Angeles, Calif. *1842 Maine St., Long Beach 6, Calif.*
SMITH, AUBREY H., Ph.D. (Brown) Asso. Prof., Purdue Univ., Lafayette, Ind.
SMITH, A. J., Ph.D. (Pennsylvania) Asst. Prof., Montana School of Mines, Butte, Mont.
SMITH, C. B., Ph.D. (Wisconsin) Asso. Prof., Univ. of Florida, Gainesville, Fla. *604 N.W. 8th Pl.*
SMITH, C. D., Ph.D. (Iowa) Asso. Prof., Univ. of Alabama, University, Ala. *Box 2686*
SMITH, C. V. L., Ph.D. (Harvard) Head, Computer Branch, Office of Naval Research, Washington, D.C. *Box 125, Lovettsville, Va.*

SMITH, C. W., A.M. (Wisconsin) Emeritus Pres., State Teachers Coll., Superior, Wis. *401 E. Third St.*
SMITH, D. J., M.S. (Notre Dame) Actuarial Trainee, Royal-Liverpool Group of Insurance Companies, 150 William St., New York 8, N.Y. *47 Oakland Ave., Bloomfield, N.J.*
SMITH, D. M., Ph.D. (Chicago) Prof., Georgia Inst. of Tech., Atlanta, Ga.
SMITH, E. C., JR., B.S. (Stanford) Grad. Student, Brown Univ., Providence, R.I. *310 Thayer St. (6)*
SMITH, EDWIN R., Ph.D. (Munich) Retired Prof., Iowa State Coll., Ames, Iowa. *1113 Clark Ave.*
SMITH, ELMER R., A.M. (Vanderbilt) Retired Prof., State Coll. for Women, Tallahassee, Fla. *648 W. Call St.*
SMITH, E. S., Ph.D. (Virginia) Prof., Univ. of Cincinnati, Cincinnati 21, Ohio
SMITH, F. C., Ph.D. (Michigan) Prof., Coll. of St. Thomas, St. Paul 1, Minn.
SMITH, F. E., Ph.D. (Catholic U.) Asso. Prof., Brooklyn Coll., Brooklyn, N.Y. *Box 49, Wantagh, N.Y.*
SMITH, G. W., Ph.D. (Illinois) Prof., Univ. of Kansas, Lawrence, Kan. *1730 Illinois St.*
SMITH, GENEVA M., A.B. (Maine) Head of Dept., Plymouth Normal School, Plymouth, N.H.
SMITH, H. W., Ph.D. (Texas) Prof., Oklahoma A. and M. Coll., Stillwater, Okla. *80 College Circle*
SMITH, I. W., A.M. (Illinois) Emeritus Prof., North Dakota Agr. Coll., State College, N.D. *1126 13th St. North*
SMITH, JAMES C., M.S. (St. Louis) Instr., Loyola Univ. of Los Angeles, Los Angeles 45, Calif.
SMITH, JEROME C., Ph.D. (Cornell) Asst. Prof., Lafayette Coll., Easton, Pa.
SMITH, J. P., Rev., A.M. (Woodstock) Georgetown Univ., Washington 7, D.C.
SMITH, LOIS B. (Mrs. R. B.), A.M. (Kansas) *8357 Allison Ave., La Mesa, Calif.*
SMITH, L. W., Ph.D. (Washington and Lee) Emeritus Prof., Washington and Lee Univ., Lexington, Va.
SMITH, MALCOLM, B.S. (Illinois Tech.) Lt., U. S. Air Force. *8705 S. Chicago Ave., Chicago 17, Ill.*
SMITH, O. D., M.A. (Oregon State) Teaching Asst., Univ. of Southern California, Los Angeles, Calif. *1131 W. 36th St.*
SMITH, P. K., Ph.D. (Illinois) Prof., Louisiana Poly. Inst., Ruston, La.
SMITH, R. E., Ph.D. (Pittsburgh) Asso. Prof., Duquesne Univ., Pittsburgh, Pa. *939 Sedalia Ave., Avalon, Pittsburgh 2, Pa.*
SMITH, R. G., Ph.D. (Kansas) Prof., State Teachers Coll., Pittsburg, Kan.
SMITH, S. J., A.M. (Pittsburgh) Chm. of Dept., State Teachers Coll., Lock Haven, Pa.
SMITH, S. R., Ph.D. (Indiana) Asso. Prof., Univ. of Wyoming, Laramie, Wyo.
SMITH, T. C., M.A. (N.Y.U.) Asst. Prof., Phillips Univ., Enid, Okla.
SMITH, T. F., M.A. (Cincinnati) Grad. Student, Univ. of Cincinnati, Cincinnati, Ohio. *3760 Ault Park Ave. (8)*
SMITH, WENDALL A., B.S. (Vermont) Physicist, Johns Hopkins Univ., Applied Physics Lab., Silver Spring, Md. *1725 17th St., N.W., Washington 9, D.C.*
SMITH, WILLIAM A., M.A. (Syracuse) Asst. Prof., St. Lawrence Univ., Canton, N.Y. *Forestport, N.Y.*
SMITH, W. F., A.M. (Kentucky) Asst. Prof., Univ. of Detroit, Detroit 21, Mich.
SMITH, W. N., M.A. (Northwestern) Asst. Prof., Univ. of Wyoming, Laramie, Wyo.
SMULIN, RUBIN, B.S. (Miami) Instr., Frame Refrigeration-Electrical Inst., Aviation Bldg., Miami, Fla. *1348 S.W. 2nd St. (35)*
SMURTHWAITE, R. J., B.A. (Buffalo) Computer, Bell Aircraft Co., Niagara Falls, N.Y. *537 Prospect St., Lockport, N.Y.*
SMYTH, RUTH B. (Mrs. B. J.), A.M. (Oberlin) Instr., College of Wooster, Wooster, Ohio. *273 Morgan St., Oberlin, Ohio*
SMYTH, S. GRACE, A.M. (Columbia) Dean of Women, Knox Coll., Galesburg, Ill.
SMYTHE, W. R., JR., Student, Rollins Coll., Winter Park, Fla. *634 E. Colonial Dr., Orlando, Fla.*
SNADER, D. W., Ed.D. (Columbia) Prof., Univ. of Illinois, Urbana, Ill. *Box 545, Tolono, Ill.*
SNAPPER, ERNST, Ph.D. (Princeton) Asso. Prof., Univ. of Southern California, Los Angeles 7, Calif.
SNIDER, R. L., M.A. (Missouri) Instr., Community High School, Blue Island, Ill.
SNIVELY, L. C., M.S.E.E. (Colorado) Asst. Prof., Univ. of Colorado, Boulder, Colo. *814 Pine St.*
SNOW, D. O., M.A. (Acadia) Asso. Prof., Acadia Univ., Wolfville, Nova Scotia, Can.
SNYDER, A. D., A.M. (Wisconsin) Prof., Union Coll., Schenectady, N.Y. *1592 Union St.*
SNYDER, A. K., M.A. (N.Y.U.) Prof., Valley Forge Military Jr. Coll., Wayne, Pa.
SNYDER, B. R., M.A. (Boston) Lawyer, 510 Tower Bldg., Baltimore, Md. *5711 Chinquapin Pkwy. (12)*
SNYDER, W. S., Ph.D. (Ohio State) Asso. Prof., Univ. of Tennessee, Knoxville, Tenn.
SOBCZYK, ANDREW, Ph.D. (Princeton) Staff Member, Los Alamos Scientific Lab., Los Alamos, N.M. *53 Pine Ridge Rd., Arlington 74, Mass.*
SOGLIN, ALBERT, M.S. (Illinois Tech.) Instr., Illinois Inst. of Tech., Chicago, Ill. *4523 Drexel Blvd. (15)*

SOHON, F. W., Rev., Ph.D. (Georgetown) Dean of Grad. School, Georgetown Univ., Washington, D.C.
SOKOLNIKOFF, ELIZABETH S., Ph.D. (Wisconsin) Asst. Prof., Univ. of Wisconsin, Madison 6, Wis.
SOKOLNIKOFF, I. S., Ph.D. (Wisconsin) Prof., Univ. of California at Los Angeles, Los Angeles 24, Calif.
SOLKY, C. G., Student, New York Univ., New York 53, N.Y. *1136 Teller Ave. (56)*
SOLLINS, A. D., M.A. (George Washington) Mathematician, U. S. Coast and Geodetic Survey, Washington, D.C. *1717 Keokee St., Hyattsville, Md.*
SOLLOWAY, C. B., B.S. (Indiana S.T.C.) *569 N. Rossmore Ave., Los Angeles 4, Calif.*
SOLOMON, GUSTAVE, Student, Yeshiva Coll., New York, N.Y. *485 Ashford St., Brooklyn 7, N.Y.*
SOLOMON, HERBERT, M.A. (Columbia) Mathematical Statistician, Office of Naval Research, Washington 25, D.C.
SOLOMON, J. L., B.A. (Oberlin) Solomon & Sons, 467 10th Ave., New York, N.Y. *115 Central Park West (23)*
SOLOMONT, FLORENCE N. (Mrs. Stanley), B.A. (Radcliffe) *49 Miller Rd., Newton Center 59, Mass.*
SOMERS, E. V., Ph.D. (Pittsburgh) Research Engr., Magnolia Petroleum Co., Field Research Labs., Dallas, Tex. *308 E. Beech St., Hazleton, Pa.*
SORENSON, J. C., M.S. (Oregon) Math. Statistician, Quality Control Sec., Hill A.F.B., Ogden, Utah. *526 22nd St.*
SORGENFREY, R. H., Ph.D. (Texas) Asst. Prof., Univ. of California at Los Angeles, Los Angeles 24, Calif.
SORRELLS, RUTH C. (Mrs. C. C.), A.M. (Columbia) *619 N. Montclair St., Dallas, Tex.*
SOUSLEY, C. P., Ph.D. (Johns Hopkins) Prof., Rose Poly. Inst., Terre Haute, Ind. *92 Potomac Ave.*
SOUTH, D. E., Ph.D. (Michigan) Prof., Univ. of Kentucky, Lexington, Ky.
SOUTHARD, T. H., Ph.D. (Ohio State) Asso. Prof., Inst. for Numerical Analysis, 405 Hilgard Ave., Los Angeles 24, Calif.
SOWUL, J. G., M.A. (Detroit) Grad. Student, Univ. of Detroit, Detroit, Mich. *11625 Mitchell (12)*
SPARKS, F. W., Ph.D. (Chicago) Prof., Texas Tech. Coll., Lubbock, Tex.
SPAULDING, S. P., M.A. (Boston) Ordnance Engr., U. S. Naval Torpedo Station, Newport, R.I. *118 Ruggles Ave., Apt. 2-E*
SPEAR, JOSEPH, A.M. (Boston) Prof., Northeastern Univ., Boston 15, Mass. *316 Huntington Ave.*
SPEARS, O. S., M.S. (Ala. Poly. Inst.) Instr., Univ. of Oklahoma, Norman, Okla.
SPEARS, W. K., A.B. (Dartmouth) Lt., U. S. Army. *P.O. Box 491, Red Bank, N.J.*
SPEAS, J. D., Student, Butler Univ., Indianapolis, Ind. *Whiteland, Ind.*
SPECHT, E. J., Ph.D. (Minnesota) Prof., Emmanuel Missionary Coll., Berrien Springs, Mich.
SPECHT, R. D., Ph.D. (Wisconsin) Asso. Mathematician, The Rand Corp., Santa Monica, Calif. *15015 McKendree Ave., Pacific Palisades, Calif.*
SPEECE, H. E., M.A. (Texas Christian) Instr., North Carolina State Coll., Raleigh, N.C.
SPEICHER, P. I., A.M. (Pennsylvania) Asso. Prof., Albright Coll., Reading, Pa.
SPENCELEY, G. W., A.M. (Harvard) Asso. Prof., Miami Univ., Oxford, Ohio. *402 E. Church St.*
SPENCER, H. E., Ph.D. (Cornell) Asso. Prof., Virginia Poly. Inst., Blacksburg, Va. *606 Progress St.*
SPENCER, MARY C., M.S. (Cornell) Emeritus Prof., Newcomb Coll., New Orleans, La. *1111 Lowerline St. (18)*
SPENCER, R. L., M.A. (Michigan) Instr., Dearborn Jr. Coll., Dearborn, Mich. *3081 Pardee*
SPENCER, VIVIAN E., Ph.D. (Pennsylvania) Chief, Minerals Section, U. S. Bureau of the Census, Washington 25, D.C. *5126 Chevy Chase Pkwy. N.W. (8)*
SPERLINE, M. E., B.S. (Kansas S.C.) Instr., Univ. of Colorado, Boulder, Colo.
SPERRY, PAULINE, Ph.D. (Chicago) Asso. Prof., Univ. of California, Berkeley, Calif.
SPICER, C. A., Ph.D. (Johns Hopkins) Prof., Western Maryland Coll., Westminster, Md. *17 Ridge Rd.*
SPIEGEL, M. R., Ph.D. (Cornell) Asst. Prof., Rensselaer Poly. Inst., Troy, N.Y.
SPIEGELTHAL, E. S., Student, Washington Square Coll., New York Univ., New York, N.Y. *1795 Riverside Dr. (34)*
SPINKS, M. J., Retired. *Wilmington, Ohio*
SPITZBART, ABRAHAM, Ph.D. (Harvard) Asst. Prof., Univ. of Wisconsin in Milwaukee, Milwaukee 3, Wis.
SPOHN, R. H., M.A. (Lehigh) Mathematician, The Kellex Corp., 233 Broadway, New York, N.Y. *754 Scotland Rd., Orange, N.J.*
SPONG, R. A., M.S. (Northwestern) Mathematician, U. S. Navy Underwater Sound Lab., Fort Trumbull, New London, Conn. *R.F.D. Niantic, Conn.*
SPOONER, G. A., B.S. (Teachers Coll. of Conn.) Asst. Instr., Teachers Coll. of Connecticut, New Britain, Conn.
SPRAGENS, W. H., JR., Ph.D. (Cincinnati) Asso. Prof., Univ. of Mississippi, University, Miss. *806 Van Buren Ave., Oxford, Miss.*

SPRINGER, C. E., Ph.D. (Chicago) Prof., Univ. of Oklahoma, Norman, Okla.
SPURGEON, VIVIAN, M.A. (Peabody) *328 South Pine, Nowata, Okla.*
SQUIRES, P. C., Ph.D. (Princeton) *P.O. Box 72, Ridgeway Station, Stamford, Conn.*
STABLER, E. R., Ed.D. (Harvard) Asso. Prof., Hofstra Coll., Hempstead, N.Y.
STAHL, K. H., Ph.D. (Pittsburgh) Asso. Prof., Univ. of Colorado, Boulder, Colo.
STAIR, R. H., A.B. (Lawrence C.) Instr., Clark Jr. Coll., Vancouver, Wash. *Rt. 7, Box 1082*
STALEY, J. McD., M.S. (Michigan) Instr., Colorado A. and M. Coll., Ft. Collins, Colo. *Rt. 4, Box 272*
STALEY, R. C., Ph.D. (Michigan) Prof., Univ. of North Dakota, Grand Forks, N.D. *321 Cambridge St., University Station*
STALLARD, F. W., M.S. (Brown) Instr., East Tennessee State Coll, Johnson City, Tenn. *On leave:* 310-B Dormitory, Chapel Hill, N.C.
STALLEY, R. D., M.A. (Oregon S.C.) Grad. Asst., Univ. of Oregon, Eugene, Ore.
STAMEY, W. L., M.A. (Missouri) Instr., Univ. of Missouri, Columbia, Mo.
STAMPER, A. L., Second Lt., U. S. Army
STANAITIS, O. E., Ph.D. (Wurzburg) Asst. Prof., St. Olaf Coll., Northfield, Minn. *301 N. Linden*
STANDERFER, J. A., B.A. (N. Texas S.C.) U. S. Navy. *606 W. Acheson St., Denison, Tex.*
STANLEY, E. L., M.S. (Tennessee) Asst. Prof., Clemson Coll., Clemson, S.C. *150 Riverside Dr.*
STANLEY, H. S., A.M. (Harvard) Asso. Prof., Univ. of Georgia, Athens, Ga.
STANWICK, C. A., B.S. in E.E. (Washington) Elec. Engr., *131 Rynda Rd., South Orange, N.J.*
STAPLEFORD, E. T., M.A. (Ohio State) Instr., Kent State Univ., Kent, Ohio
STAPP, M. C., M.A. (Peabody) Asst. Prof., Univ. of Alabama, University, Ala.
STARCH, G. W., B.S. (Oklahoma A. and M.) Jr. Mining Geologist, New Cornelia Mine, Phelps Dodge Corp., Ajo, Ariz.
STARCHER, G. W., Ph.D. (Illinois) Dean, Ohio Univ., Athens, Ohio. *Northwood Dr.*
STARK, L. W., M.A. (Duke) Mathematician, U. S. Government. Red Stone Arsenal. *Kildare Apts., Kildare Ave., Huntsville, Ala.*
STARK, MARION E., Ph.D. (Chicago) Prof., Wellesley Coll., Wellesley 81, Mass. *6 Waban St.*
STARK, R. H., Ph.D. (Northwestern) Mathematician, Los Alamos Scientific Lab., Los Alamos, N.M. *P.O. Box 1663*
STARKE, E. P., Ph.D. (Columbia) Prof., Rutgers Univ., New Brunswick, N.J.
STARR, D. W., Ph.D. (Illinois) Prof., Southern Methodist Univ., Dallas, Tex. *3937 Wentwood Dr. (5)*
STARRETT, A. L., A.M. (Harvard) Asso. Prof., Georgia Inst. of Tech., Atlanta, Ga.
STAUFFER, J. R. K., M.S. (Chicago) Instr., Rhode Island State Coll., Kingston, R.I.
STAVINOHA, E. A., M.S. (Oklahoma A. and M.) Teacher, Brownsville High School, Brownsville, Tex. *1744 Stanford Ave., Ebony Heights*
STAYER, J. C., A.M. (Pittsburgh) Dean of Men, Juniata Coll., Huntingdon, Pa. *1618 Moore St.*
STECHSCHULTE, V. C., Rev., Ph.D. (California) Prof., Xavier Univ., Cincinnati 7, Ohio
STEED, D. V., Ph.D. (California) Prof., Univ. of Southern California, Los Angeles, Calif. *1731 W. 80th St. (44)*
STEEN, F. H., Ph.D. (Harvard) Prof., Allegheny Coll., Meadville, Pa. *R.D. 1*
STEEN, P. O., M.A. (Wyoming) Asst. Prof., Univ. of Wyoming, Laramie, Wyo.
STEIN, F. M., M.S. (Iowa) Asso. Prof., Iowa Wesleyan Coll., Mount Pleasant, Iowa. *707 College Ave.*
STEIN, MARIE A., Student, Adelphi Coll., Garden City, N.Y. *282 N. Franklin St., Hempstead, N.Y.*
STEINBACH, E. M., A.B. (Northern S.T.C.) Instr., Univ. of Detroit, Detroit, Mich.
STEINBERG, H. L., A.B. (U.C.L.A.) Consulting Chemist, *P.O. Box 2035, Main Postoffice, Atlanta, Ga.*
STEINBERG, ROBERT, Ph.D. (Toronto) Instr., Univ. of California at Los Angeles, Los Angeles 24, Calif.
STEINHAUS, H. W., Ph.D. (Göttingen) Research Asst., Equitable Life Assurance Soc., 393 Seventh Ave., New York, N.Y.
STEINKAMP, W. I., Pvt., U. S. Army. *Route 1, Box 141, Brooks, Ore.*
STELFORD, NORMA K., A.M. (Northwestern) Asst. Prof., Northern Illinois State Teachers Coll. De Kalb, Ill.
STELLING, LOIS B., M.A. (Chicago) Instr., Univ. of Illinois, Navy Pier, Chicago, Ill. *5909 W. Lake St.*
STELSON, H. E., Ph.D. (Iowa) Prof., Michigan State Coll., East Lansing, Mich. *300 Beal St.*
STEMPNITZKY, ISAAY, S.M. (M.I.T.) *1498 Beacon St., Brookline 46, Mass.*
STENGLE, T. R., B.S. (Franklin & Marshall) Grad. Student, Univ. of Michigan, Ann Arbor, Mich. *116 College Ave., Lancaster, Pa.*
STENNES, FLORENCE SWALLOW (Mrs.), M.A. (Illinois) Instr., North Dakota Agr. Coll., State College, N.D. *Perley, Minn.*
STEPHENS, C. F., Ph.D. (Michigan) Prof., Morgan State Coll., Baltimore 12, Md. *1014 West 43rd St. (11)*

STEPHENS, EUGENE, M.S. (Washington) Emeritus Asst. Prof., Washington Univ., St. Louis 5, Mo. *6916 Millbrook Blvd., Apt. 102, University City 5, Mo.*
STEPHENS, R. C., Ph.D. (Iowa) Prof., Knox Coll., Galesburg, Ill.
STEPHENS, R. P., Ph.D. (Johns Hopkins) Emeritus Prof., Univ. of Georgia, Athens, Ga. *230 Woodlawn Ave.*
STERN, MARVIN, M.S. (N.Y.U.) Math. Consultant, Republic Aviation Corp., Farmingdale, L.I., N.Y.
STERRETT, ANDREW, JR., M.S. (Pittsburgh) Instr., Ohio Univ., Athens, Ohio. *8 Strathmore Blvd.*
STERRETT, J. K., M.A. (Kansas) Mathematician, Ballistics Research Lab., Aberdeen Proving Ground, Md. *1723-B Aberdeen Rd., Loch Raven Village, Towson 4, Md.*
STETSON, J. M., Ph.D. (Princeton) Prof., Coll. of William and Mary, Williamsburg, Va. *232 Jamestown Rd.*
STEUDLE, MARY R., M.A. (Canisius) Teacher, Kensington High School, Buffalo 15, N.Y. *390 Minnesota Ave.*
STEVENS, H. R., B.A. (Buffalo) Teaching Fellow, Univ. of Buffalo, Buffalo 14, N.Y. *258 Hoyt St. (13)*
STEVENS, W. R., A.B. (George Washington) Meteorologist, U. S. Weather Bureau, New Orleans, La. *1124 Second St. (13)*
STEVENSON, GUY, Ph.D. (Illinois) Prof., Univ. of Louisville, Louisville, Ky.
STEWARD, R. F., M.Sc. (Rutgers) Instr., Virginia Military Inst., Lexington, Va.
STEWART, B. M., Ph.D. (Wisconsin) Asso. Prof., Michigan State Coll., East Lansing, Mich. *1601 W. Ottawa St., Lansing, Mich.*
STEWART, C. E., B.S. (Illinois Tech.) Research Asst., Armour Research Foundation, Chicago 16, Ill. *3140 So. Michigan Ave., Apt. 211 (16)*
STEWART, F. M., Ph.D. (Harvard) Asst. Prof., Brown Univ., Providence 12, R.I.
STEWART, J. C., Ph.D. (Illinois) Asso. Prof., Lawrence Coll., Appleton, Wis.
STEWART, J. K., Ph.D. (West Virginia) Asso. Prof., West Virginia Univ., Morgantown, W.Va. *355 Kingwood St.*
STEWART, R. C., M.A. (Yale) Instr., Trinity Coll., Hartford 6, Conn.
STILES, R. B., A.B. (Middlebury) Instr., Vanderbilt Univ., Nashville, Tenn. *Trimble Rd. (5)*
STILWELL, M. F., A.M. (Syracuse) Asst. Prof., U. S. Naval Academy, Annapolis, Md. *Weems Creek, R.F.D. #4*
STINETORF, ROSCOE, Ph.D. (Pennsylvania) Prof., Wagner Coll., Staten Island, N.Y.
STIPE, C. G., Ph.D. (Michigan) Prof., Michigan Coll. of Mining and Tech., Houghton, Mich. *307 Agate St.*
STOCKMAN, J. F., B.A. (Willamette) Part-time Instr., Univ. of Colorado, Boulder, Colo. *Quonset 116, Vetsville*
STOKES, RUTH W., Ph.D. (Duke) Asst. Prof., Syracuse Univ., Syracuse 10, N.Y.
STOKES, W. G., M.A. (Sam Houston S.T.C.) Instr., Navarro Jr. Coll., Corsicana, Tex.
STOLL, R. R., Ph.D. (Yale) Asso. Prof., Lehigh Univ., Bethlehem, Pa.
STONE, M. H., Ph.D. (Harvard) Prof., Univ. of Chicago, Chicago 37, Ill.
STONE, R. B., A.M. (Harvard) Asso. Prof., Purdue Univ., Lafayette, Ind. *615 Russell St., West Lafayette, Ind.*
STONE, SANDERS, Student, Computer, Wilmington Coll. Undergrad. Center, Wright-Patterson A.F.B., Dayton, Ohio. *85 Bragg Pl. (8)*
STONE, S. N., M.S. (N.Y.U.) Mathematician, U. S. Coast and Geodetic Survey, Washington, D.C. *600 E. 21st St., Brooklyn 26, N.Y.*
STONE, W. M., Ph.D. (Iowa State) Engr., Boeing Airplane Co., Seattle, Wash. *2717 18th Ave. S. (44)*
STONEHAM, R. G., Sc.M. (Brown) Lecturer, Univ. of California, Berkeley 4, Calif. *2130 Essex St. (5)*
STONER, IRWIN, M.A. (North Carolina) Asst. Scientist, Rosemount Research Center, Univ. of Minnesota, Minneapolis, Minn. *3221 31st Ave. S. (6)*
STOPHER, E. C., Ph.D. (Iowa) Prof., Fort Hays Kansas State Coll., Hays, Kan. *500 West 6th St.*
STORY, HELEN F., A.M. (Wellesley) *2762 Burlington Ave., St. Petersburg, Fla.*
STOTZ, H. C., A.M. (Boston U.) Asso. Prof., U. S. Naval Academy, Annapolis, Md.
STOUFFER, E. B., Ph.D. (Illinois) Prof., Univ. of Kansas, Lawrence, Kan. *1019 Maine St.*
STOVALL, W. B., JR., M.A. (Florida) Lt., U.S.N.R. *Box 4526 Jacksonville 1, Fla.*
STOWELL, C. J., Ph.D. (Illinois) Dean, McKendree Coll., Lebanon, Ill. *810 Belleville St.*
STRATTON, W. T., Ph.D. (Washington) Prof., Kansas State Coll., Manhattan, Kan.
STRAUSS, W. J., M.S. (Chicago) Mathematician, Air Weapons Research Center, Univ. of Chicago, Chicago 37, Ill. *5227 Kimbark Ave. (15)*
STREBE, D. D., M.A. (Buffalo) Instr., Univ. of Buffalo, Buffalo 14, N.Y. *15 Seneca Parkside (20)*
STREET, A. T., M.A. (Northwestern) Asso. Prof., Roosevelt Coll., Chicago 5, Ill.
STREET, R. E., Ph.D. (Harvard) Asso. Prof., Univ. of Washington, Seattle 5, Wash.

STRIGHT, I. L., Ph.D. (Western Reserve) Prof., State Teachers Coll., Indiana, Pa. *1384 Philadelphia St.*
STROBEL, C. F., Ph.D. (Illinois) Asso. Prof., North Carolina State Coll., Raleigh, N.C. *3310 Pollock Pl.*
STROBEL, E. O., A.B. (Pennsylvania) Statistician, Dept. of Economic Affairs, United Nations, Lake Success, N.Y. *58-39 Bell Blvd., Bayside, L.I., N.Y.*
STROCK, E. E., A.M. (Yale) Asst. Actuary, Prudential Insurance Co., Newark, N.J. *55 Monroe Pl., Bloomfield, N.J.*
STRONG, CORA, A.M. (Michigan) Retired Prof., Woman's Coll., Univ. of North Carolina, Greensboro, N.C. *109 Adams St.*
STRONG, O. E., M.S. (Indiana S.T.C.) Asso. Prof., Indiana State Teachers Coll., Terre Haute, Ind.
STROW, F. D., M.A. (Bowling Green) Head, Dept. of Math., Perrysburg High School, Perrysburg, Ohio. *Custar, Ohio*
STRUBLE, R. A., Ph.D. (Notre Dame) Douglas Aircraft Co., Santa Monica, Calif. *18755 Runnymede, Reseda, Calif.*
STRUTTON, E. B., M.S. (Toledo) Lecturer, Univ. of California at Los Angeles, Los Angeles, Calif. *P.O. Box 84, Vista, Calif.*
STRUYK, ADRIAN, A.M. (Columbia) Teacher, High School, Clifton, N.J. *232 Jefferson St., Paterson 2, N.J.*
STUBBE, J. S., Ph.D. (Cincinnati) Insr., Clark Univ., Worcester 3, Mass.
STUCKEY, C. S., Ph.M. (Wisconsin) Supervisor, Standard Register Co., 595 Madison Ave., New York, N.Y. *67 Canterbury Dr., Ramsey, N.J.*
STULKEN, E. J., A.M. (Texas) Chief Seismologist, Geophysical Service, Inc., Dallas, Tex. *3709 Gillon Ave.*
STURGES, FALKA G. (Mrs.), A.M. (California) Teacher, Hassler High School, Redwood City, Calif. *1790 Hopkins Ave.*
STURLEY, E. A., M.A. (Yale) Asst. Prof., Allegheny Coll., Meadville, Pa.
STURM, R. G., Ph.D. (Illinois) Dir., Auburn Research Foundation, Alabama Poly. Inst., Auburn, Ala.
SUBRAMU, MR. S., M.Sc. (Mysore) Lecturer, Technical Inst., Tata Iron and Steel Co., Ltd., Jamshedpur-1, Bihar, India
SUDBOROUGH, D. R., M.S. (Michigan) Asst. Prof., Central Michigan Coll., Mt. Pleasant, Mich. *1016 S. Franklin St.*
SUFFA, MARY C., A.M. (Brown) Prof., Elmira Coll., Elmira, N.Y.
SULLIVAN, J. A., Ph.D. (Indiana) Instr., Univ. of Notre Dame, Notre Dame, Ind.
SULLIVAN, J. R., M.A. (Georgetown) Instr., Clemson Coll., Clemson, S.C. *On leave:* Grad. Student, Univ. of North Carolina. *18-C Glen Lennox, Chapel Hill, North Carolina*
SULLIVAN, MILDRED M., Ph.D. (Radcliffe) Asst. Prof., Queens Coll., 65-30 Kissena Blvd., Flushing, N.Y.
SUMNER, RUTH G. (Mrs.), Ed.M. (Stanford) Teacher, Oakland High School, Oakland, Calif. *Kilkare Rd., Sunol, Calif.*
SUNSERI, MARY V., M.A. (Stanford) Acting Asst. Prof., Stanford Univ., Stanford, Calif.
SUSSMAN, IRVING, M.A. (Johns Hopkins) Univ. of San Francisco, San Francisco, Calif. *36 Ackley Ct., Walnut Creek, Calif.*
SUTER, ANNA K., A.M. (Indiana) Instr., Purdue Univ., Extension Div., Indianapolis, Ind. *902 N. Meridian St.*
SUTER, M. FRANCES, A.M. (Illinois) Asst. Prof., Roanoke Coll., Salem, Va. *Ft. Defiance, Va.*
SUTHERLAND, ETHEL, Ph.D. (Columbia) Prof., Longwood Coll., Farmville, Va. *509 Beech St.*
SUTTER, R. R., Student, Illinois Inst. of Tech., Chicago, Ill. *3611 W. 62nd St. (29)*
SUTTON, C. S., M.S. (M.I.T.) Asst. Prof., The Citadel, Charleston, S.C.
SVOBODA, A. F., M.S. (De Paul) Asst. Prof., De Paul Univ., Chicago 14, Ill. *1815 S. Troy St. (23)*
SWAFFIELD, J. R., Student, Boston Univ., Boston, Mass. *Whipple Rd., Tewksbury, Mass.*
SWAFFORD, E. G., A.M. (Syracuse) Asst. Prof., U. S. Naval Academy, Annapolis, Md.
SWAIN, R. L., Ph.D. (Texas) Prof., State Teachers Coll., New Paltz, N.Y.
SWANSON, A. G., Ph.D. (Michigan) Prof., Gustavus Adolphus Coll., St. Peter, Minn.
SWANSON, C. A., B.A. (British Columbia) Grad. Student, Univ. of British Columbia, Vancouver, B.C., Can. *4678 W. 9th Ave.*
SWANSON, E. L., A.M. (Colorado C.) Asso. Prof., South Dakota State School of Mines, Rapid City, S.D.
SWANSON, L. W., Ph.D. (Minnesota) Mathematician, Applied Science Dept., I.B.M. Corp., Chicago, Ill. *1503 S. Fairview Ave., Park Ridge, Ill.*
SWANSON, RUTH KJERSTI, M.S. (Oklahoma) Instr., Friends Univ., Wichita 12, Kan.
SWEETLAND, P. C., M.S. (Fort Hays) Grad. Asst., Michigan State Coll., East Lansing, Mich. *409A Hickory Lane*
SWIFT, ELIJAH, Ph.D. (Göttingen) Dean, Univ. of Vermont, Burlington, Vt. *415 S. Willard St.*

SWIFT, J. D., Ph.D. (C.I.T.) Asst. Prof., Univ. of California at Los Angeles, Los Angeles 24, Calif.
SWINGLE, P. M., Ph.D. (Michigan) Prof., Univ. of Miami, Coral Gables, Fla. *4970 North Kendall. Dr., Box 126-D, Route 2, Miami 43, Fla.*
SWOKOWSKI, E. W., M.S. (Marquette) Teaching Asst., Univ. of Wisconsin, Madison, Wis.
SWORDS, Rev. R. J., M.A. (Harvard) Student, Rathfornham Castle, Dublin, Ireland. *42 Granville St., Springfield, Mass.*
SYDNOR, T. E., M.A. (Whittier) Instr., Pasadena City Coll., Pasadena, Calif. *1386 N. Mar Vista Ave. (6)*
SYER, H. W., Ed.D. (Harvard) Asso. Prof., School of Education, Boston Univ., Boston, Mass.
SYLWESTER, A. V., Instr., California Concordia Coll., Oakland 5, Calif. *6325 Camden St.*
SYNGE, J. L., Sc.D. (Dublin) Prof., Dublin Inst. for Advanced Studies, Dublin, Ireland. *64 Merrion Sq., Dublin, Ireland*
SZASZ, OTTO, Ph.D. (Budapest) Prof., Univ. of Cincinnati, Cincinnati, Ohio
SZEGÖ, GABOR, Ph.D. (Vienna) Prof., Stanford Univ., Stanford, Calif.

TAAM, CHOY-TAK, Ph.D. (Harvard) Asst. Prof., Univ. of Missouri, Columbia, Mo.
TABOR, EDWIN, A.B. (California) Retired. *2319½ Haste St., Berkeley 4, Calif.*
TALACKO, J. V., D.S. (Prague) Asst. Prof., Marquette Univ., Milwaukee 3, Wis. *2503 So. 10th St. (7)*
TALIAFERRO, CARRIE B., A.M. (Columbia) Emeritus Prof., State Teachers Coll., Farmville, Va. *Box 188, Orange, Va.*
TALIAFERRO, R. C., Ph.D. (Virginia) Teacher, Portsmouth Priory School, Portsmouth, R.I.
TALKINGTON, A. D., M.A. (Missouri) Instr., De Pauw Univ., Greencastle, Ind. *518 E. Washington*
TAN, KAIDY, Tutor, Anglo-Chinese Coll., Kulangsu, Amoy, China. *c/o Mr. Cheng Hsun-Han, School of Physiology, Univ. of Hong Kong, Hong Kong*
TANG, N. Y., M.S. (Michigan) Actuarial Dept., Manhattan Life Insurance Co., 120 W. 57th St., New York 19, N.Y.
TANI, P. Y., M.S. (Stanford) Research Engr., Minneapolis-Honeywell Regulator Co., Minneapolis, Minn. *2600 Ridgway Rd.*
TANIMOTO, T. T., Ph.D. (Pittsburgh) Asst. Prof., Allegheny Coll., Meadville, Pa.
TAPPAN, A. HELEN, Ph.D. (Cornell) Prof., Western Coll., Oxford, Ohio
TAPPER, ETHEL W., B.S. (Illinois) Librarian and Asst. Prof., Aurora Coll., Aurora, Ill.
TAPSCOTT, R. P., Student, Univ. of Kentucky, Lexington, Ky. *1924 Griffith Pl. W., Owensboro, Ky.*
TARTLER, ALEXANDER, Ph.D. (Pennsylvania) Prof., Lafayette Coll., Easton, Pa. *321 Porter St.*
TATE, JENNIE L., A.M. (Wisconsin) Head of Dept., McMurry Coll., Abilene, Tex. *1301 Orange St.*
TAUB, A. H., Ph.D. (Princeton) Prof., Univ. of Illinois, Urbana, Ill.
TAUSSIG, R. A., M.B.A. (California) Instr., Golden Gate Coll., San Francisco, Calif. *2741 Parker St., Berkeley 14, Calif.*
TAYLOR, A. E., Ph.D. (C.I.T.) Prof., Univ. of California at Los Angeles, Los Angeles 24, Calif.
TAYLOR, ANN A., B.S. (Georgia) *Bowdre Apt. E, Greene St., Augusta, Ga.*
TAYLOR, ANNA L., M.A. (Missouri) Mathematician, U. S. Naval Ordnance Test Station, Pasadena, Calif. *1119½ Pine St., South Pasadena, Calif.*
TAYLOR, C. F., B.S. (Northeastern U.) Electrical Engr., General Electric Co., Lynn, Mass. *50 Sachem St.*
TAYLOR, E. H., Ph.D. (Harvard) Emeritus Prof., Eastern Illinois State Teachers Coll., Charleston, Ill. *885 Seventh St.*
TAYLOR, F. H., B.S. in Ed. (Bucknell) *Box 43, East McKeesport, Pa.*
TAYLOR, F. J., A.B. (St. Thomas) Prof., Coll. of St. Thomas, St. Paul 1, Minn.
TAYLOR, H. E., Ph.D. (Rice) Asst. Prof., Florida State Univ., Tallahassee, Fla. *213 Crest St.*
TAYLOR, J. H., Ph.D. (Chicago) Prof., George Washington Univ., Washington 6, D.C.
TAYLOR, J. S., Ph.D. (California) Prof., Univ. of Pittsburgh, Pittsburgh 13, Pa.
TAYLOR, MARGARET O., M.S. (Pittsburgh) Mathematician, Gulf Research and Development Co., P.O. Drawer 2038, Pittsburgh, Pa.
TAYLOR, MILDRED E., Ph.D. (Illinois) Prof., Mary Baldwin Coll., Staunton, Va.
TAYLOR, WILLIAM CHARLES, Th.B. (Philadelphia Divinity School) Head of Dept. of Engg., Univ. of Tennessee Jr. Coll., Martin, Tenn.
TAYLOR, WILLIAM CLARE, Ph.D. (Wisconsin) Mathematician, Interior Ballistics Lab., Aberdeen Proving Ground, Md.
TAYLOR, W. W., M.S. (East Texas S.T.C.) Instr., East Texas State Teachers Coll., Commerce, Tex. *1707 Jackson St.*
TEAR, R. T., M.S. (Pittsburgh) Lt., U. S. Air Force. Wright-Patterson A.F.B., Dayton, Ohio
TEATS, JEAN E., M.A. (Pittsburgh) Asst. Prof., Univ. of Pittsburgh, Pittsburgh 13, Pa.
TEED, Rev. R. J., M.A. (Illinois) Instr., Fournier Inst. of Tech., Lemont, Ill.
TEGELS, C. J., B.A. (St. John's U.) *1100 Wayburn, Grosse Pointe Park 30, Mich.*

TEMPLE, M. M., M.S. (Mississippi) Asst. Prof., Mississippi State Coll., State College, Miss.
TEMPLE, V. B., A.M. (Texas) Prof., Louisiana Coll., Pineville, La.
TEMPLE, W. B., A.M. (Louisiana State) Asso. Prof., Louisiana Poly. Inst., Ruston, La.
TEMPLETON, MARY M., A.M. (North Carolina) *201 McLelland Ave., Mooresville, N.C.*
TEODORO, D. T., B.S. (Michigan) Instr., Univ. of Detroit, Detroit, Mich. *2048 Delaware (6)*
TERAMI, TAKASHI, Ph.D. (California) Asst. Prof., St. Thomas Coll., St. Paul, Minn.
TERRY, C. M., M.S. (Iowa S.C.) Analyst, Dept. of Defense, Washington, D.C. *800 S. Greenbrier St., Arlington, Va.*
TERRY, P. D., Student, McMaster Univ., Hamilton, Ont., Can. *76 Pacific Ave., Toronto, Ont., Can.*
TERRY, RUTH F. (Mrs.), M.A. (Duke) Asst. Prof., Florida Southern Coll., Lakeland, Fla. *127 East Belvedere*
TERZUOLI, A. J., M.S. (N.Y.U.) Instr., Poly. Inst. of Brooklyn, Brooklyn, N.Y.
THÉBAULT, VICTOR, Inspecteur honoraire d'Assurances, *"Le Paradis," Tennie, Sarthe, France*
THEILHEIMER, FEODOR, Ph.D. (Berlin) Mathematician, Naval Ordnance Lab., White Oak, Md. *2213 Washington Ave., Silver Spring, Md.*
THIELMAN, H. P., Ph.D. (Ohio State) Prof., Iowa State Coll., Ames, Iowa
THIGPEN, J. I., M.S. in C.E. (North Carolina S.C.) Chemical Engr., Hercules Powder Co., Wilmington, Del. *1807 Market St.*
THOMAS, A. W., JR., Student, George Pepperdine Coll., Los Angeles, Calif. *3445 West 76th St. (43)*
THOMAS, BETTY, A.M. (Alabama) Asst. Prof., Judson Coll., Marion, Ala.
THOMAS, BROTHER L., Ph.D. (St. Louis) Dean, Christian Brothers Coll., Memphis 10, Tenn.
THOMAS, C. F., M.S. (Case) Emeritus Prof., Case Inst. of Tech., Cleveland 6, Ohio. *3240 Scarborough Rd., Cleveland Heights, Ohio*
THOMAS, EARL, M.S. (Louisiana State) Supervisor, Geophysical Service, Inc., Dallas 1, Tex. *1705 West 30th St., Austin 3, Tex.*
THOMAS, G. B., JR., Ph.D. (Cornell) Asso. Prof., Massachusetts Inst. of Tech., Cambridge 39, Mass.
THOMAS, G. H. M., M.S. (Wisconsin) Teaching Asst., Univ. of Wisconsin, Madison 6, Wis. *130 East Gilman St. (3)*
THOMAS, J. M., Ph.D. (Pennsylvania) Prof., Duke Univ., Durham, N.C. *2215 Cranford Rd.*
THOMAS, O. M., A.M. (Colorado S.C.) Asst. Prof., U. S. Naval Academy, Annapolis, Md.
THOMAS, P. D., A.M. (Oklahoma) Mathematician, U. S. Coast & Geodetic Survey, Dept. of Commerce, Washington 25, D.C. *Chesapeake Gardens Apts., 304 Augusta St., Apt. H, Aberdeen, Md.*
THOMAS, R. B., JR., Student, Gonzaga Univ., Spokane 11, Wash. *N. 4717 Altamont St.*
THOMAS, R. W., Ph.D. (Pittsburgh) Dean, Washington and Jefferson Coll., Washington, Pa.
THOMAS, T. Y., Ph.D. (Princeton) Prof., Indiana Univ., Bloomington, Ind.
THOMPSON, F. B., Ph.D. (California) Mathematician, The Rand Corp., Santa Monica, Calif. *12173 Leven Lane, Los Angeles 49, Calif.*
THOMPSON, G. C., B.S. in E.E. (Union C.) Asso. Actuary, Security Mutual Life Insurance Co., Exchange St., Binghamton, N.Y.
THOMPSON, J. E., A.M. (Columbia) Prof., Pratt Inst., Brooklyn 5, N.Y. *183 Steuben St.*
THOMPSON, J. R., M.A. (Minnesota) Instr., Univ. of Wisconsin in Milwaukee, Milwaukee, Wis. *5693 No. 40th St. (9)*
THOMPSON, L. O., B.S. (W.Va. Inst. of Tech.) 17181 Birchcrest Dr., Detroit 21, Mich. *Clear Creek, W.Va.*
THOMPSON, MATILDA B., M.A. (Columbia) Asso. Prof., North Dakota Agr. Coll., State College, N.D.
THOMPSON, R. H., M.A. (Kansas) Prof., Sterling Coll., Sterling, Kan. *117½ West Cleveland*
THOMPSON, S. L., M.A. (Michigan) Asso. Prof., Alabama Poly. Inst., Auburn, Ala. *346 Payne St.*
THOMPSON, W. I., A.M. (California) Instr., Los Angeles City Coll., Los Angeles, Calif. *122 W. 50th St. (37)*
THOMSEN, D. L., Ph.D. (M.I.T.) Research Engr., Jet Propulsion Lab., Caltech, 4800 Oak Grove Dr., Pasadena 3, Calif.
THOMSON, J. F., Ph.D. (Michigan) Asst. Prof., Tulane Univ., New Orleans, La. *139 S. Genois St.*
THORNE, C. J., Ph.D. (Iowa State) Asso. Prof., Univ. of Utah, Salt Lake City, Utah.
THORNTON, H. B., A.M. (Cincinnati) Patent Examiner, U. S. Patent Office, Washington, D.C. *4609 Polk St. N.E.*
THORNTON, MARIAN W. (Mrs.) Ph.D. (Minnesota) *372 N. Cleveland Ave., St. Paul 4, Minn.*
THORNTON, W. G., M.S. (Florida State) Teacher, Madison High School, Madison, Fla.
THORO, D. E., Student, Univ. of Florida, Gainesville, Fla. *Box 2252, University Station*
THORP, ELLA, A.B. (Minnesota) Asst. Prof., Univ. of Minnesota, Minneapolis 14, Minn. *656 Jefferson St. N.E.*
THORP, LOLLIE BELLE (Mrs. John Gray) M.S. (Louisiana State) Teacher, Public School, Odessa, Tex.

THRALL, R. M., Ph.D. (Illinois) Asso. Prof., Univ. of Michigan, Ann Arbor, Mich.
THROCKMORTON, V. C., A.M. (Southern California) Instr., Los Angeles City Coll., Los Angeles, Calif. *13054 Otsego St., Sherman Oaks, Calif.*
THRON, W. J., Ph.D. (Rice) Asst. Prof., Washington Univ., St. Louis 5, Mo.
THULLEN, PETER, Ph.D. (Münster) *Apartado 1710, Panama, Republic of Panama*
THURMAN, J. C., M.A. (Vanderbilt) Asst. Prof., Vanderbilt Univ., Nashville 5, Tenn.
THURSTON, H. S., Ph.D. (Wisconsin) Prof., Univ. of Alabama, University, Ala.
TIDD, R. F., M.A. (Buffalo) Instr., Canisuis Coll., Buffalo, N.Y. *35 Albemarle St. (7)*
TIERNEY, J. A., Ph.D. (Maryland) Asst. Prof., U. S. Naval Academy, Annapolis, Md.
TIKSON, MICHAEL, M.A. (Lehigh) Office of Air Research, Air Development Command, Dayton, Ohio
TILLER, G. L., Ph.D. (Kentucky) Asso. Prof., Southwestern at Memphis, Memphis, Tenn.
TILLOTSON, D. B., A.M. (Boston U.) Asst. Prof., Northwest Nazarene Coll., Nampa, Idaho
TIMON, W. E., JR., B.S. (Northwestern S.C., Louisiana) Grad. Fellow, Tulane Univ., New Orleans 18, La.
TINNAPPEL, H. E., M.A. (Ohio State) Asst. Prof., Bowling Green State Univ., Bowling Green, Ohio
TINNER, J. C., M.S. (Chicago) Asso. Prof., Florida A. and M. Coll., Tallahassee, Fla.
TITTLE, M. E., M.A. (Texas) Instr., Texas A. and M. Coll., College Station, Tex.
TOALSON, WILMONT, M.A. (Kansas) Asst. Prof., Fort Hays Kansas State Coll., Hays, Kan. *517 W. 16th St.*
TODD, JOHN, B.S. (Belfast) Chief, Computation Lab., National Bureau of Standards, Washington 25, D.C.
TOLLE, L. F., A.M. (St. John's U.) Prof., Teachers Coll., St. John's Univ., Brooklyn, N.Y. *240-23 141st Ave., Rosedale 10, N.Y.*
TOM, D. Y. C., M.S. (Illinois) Engr., Howard, Needles, Tammen & Bengendoff, Consulting Engineers, 55 Liberty St., New York, N.Y. *2911 Koali Rd., Honolulu 36, Hawaii*
TOMPKINS, C. B., Ph.D. (Michigan) Mathematician, Engineering Research Associations, Inc., Washington, D.C. *4115 46th St. N.W.*
TONEY, H. S., A.M. (Ohio State) Prof. and Mathematician, Engg. Div., U.S.A.A.F., Wright Field, Dayton, Ohio. *R.R. #2, R.F.D., Cedarville, Ohio*
TOOPS, H. A., Ph.D. (Columbia) Prof., Psychology, Ohio State Univ., Columbus 10, Ohio. *1430 Cambridge Blvd. (12)*
TOPEL, B. J., Ph.D. (Notre Dame) Prof., Carroll Coll., Helena, Mont.
TOPP, C. W., Ph.D. (Case) Research Associate, Case Inst. of Tech., Cleveland, Ohio. *1524 Compton Rd., Cleveland Heights 18, Ohio*
TORGERSON, N. W., Student, George Pepperdine Coll., Los Angeles, Calif. *618 West 124th St.*
TORIAN, J. G., B.S. (St. Mary's Univ. of San Antonio) Student, St. Mary's Univ. of San Antonio, San Antonio, Tex. *1231 W. Russell Pl.*
TORNHEIM, LEONARD, Ph.D. (Chicago) Asst. Prof., Univ. of Michigan, Ann Arbor, Mich. *211 Pine Ridge*
TORRANCE, C. C., Ph.D. (Cornell) Asso. Prof., U. S. Naval Postgraduate School, Annapolis, Md. *Eu-De-So, R.F.D. 3*
TORREY, MARIAN M., Ph.D. (Cornell) Prof., Goucher Coll., Baltimore, Md. *Mary Fisher Hall, Towson, Md.*
TOVANI, E. P., E.E. (Colorado) Asst. Prof., Univ. of Colorado, Boulder, Colo.
TOWNSEND, B. B., M.S. (North Texas S.C.) Asst. Prof., Louisiana State Univ., Baton Rouge, La.
TOWNSEND, R. R., M.A. (Lehigh) Instr., Pennsylvania State Coll., Pottsville, Pa. *Route 1, Orefield, Pa.*
TRABKA, E. A., M.A. (Rochester) Mathematician, Cornell Aeronautical Lab., Buffalo, N.Y. *18 Norway Parkway (8)*
TRACEY, J. I., Ph.D. (Johns Hopkins) Asso. Prof., Yale Univ., New Haven, Conn. *84 McKinley Ave. (15)*
TRAMMELL, G. J., JR., M.S. (Tulane) Instr., Louisiana State Univ., Baton Rouge, La.
TRANSUE, W. R., Ph.D. (Lehigh) Prof. Kenyon Coll., Gambier, Ohio
TRENNT, EVELYN L., M.A. (Illinois) Instr., Springfield Jr. Coll., Springfield, Ill.
TREUENFELS, P. M., M.S. (N.Y.U.) Mathematician, Aberdeen Proving Ground, Md. *214 Ballistics Research Lab.*
TRIFAN, DEONISIE, Ph.D. (Brown) Asst. Prof., Univ. of Arizona, Tucson, Ariz. *3033 E. Hawthorne St.*
TRIGG, C. W., M.S. (Southern California) Instr. and Asst. to Dean, Los Angeles City Coll., Los Angeles 27, Calif.
TRIMBLE, H. C., Ph.D. (Wisconsin) Asso. Prof., Iowa State Teachers Coll., Cedar Falls, Iowa
TRIPP, M. O., Ph.D. (Columbia) Emeritus Prof., Wittenberg Coll., Springfield, Ohio. *218 W. Cecil St.*
TROSS, C. H., B.S. (Illinois) Grad. Asst., Univ. of Illinois, Urbana, Ill. *701 W. Ohio*

TROTZUK, GEORGE, Automobile Engr., Chrysler Corp., Detroit, Mich. *16796 Woodingham Dr. (21)*
TRUAN, HALSEY K. (Mrs. David) B.S. (Pittsburgh) Computer, Atomic Power Div., Westinghouse Bettis Field, Pittsburgh 30, Pa. *7462 Schoyer Ave.*
TRUEBLOOD, D. V., B.S. (Washington) Engr., Pacific Tel. and Tel. Co., Seattle, Wash. *Route 1, Box 164-C, Bellevue, Wash.*
TRUESDELL, C. A., Ph.D. (Princeton) Visiting Prof., Graduate Inst. for Applied Math., Indiana Univ., Bloomington, Ind.
TRUMP, P. L., Ph.D. (Wisconsin) Asso. Prof., Univ. of Wisconsin, Madison, Wis. *2228 Eton Ridge (5)*
TUCKER, A. W., Ph.D. (Princeton) Prof., Princeton Univ., Princeton, N.J.
TUCKER, B. A., A.M. (Millsaps C.) Prof., Southeastern Louisiana Coll., Hammond, La.
TUCKER, C. B., M.S. (Brown) Asso. Prof., State Teachers Coll., Emporia, Kan.
TUCKER, W. J., Asst. Projects Engr., Sperry-Radio Engineering Research. *34 Metropolitan Oval, Parkchester, Bronx 62, N.Y.*
TUCKERMAN, BRYANT, Ph.D. (Princeton) Asst. Prof., Oberlin Coll., Oberlin, Ohio
TUKEY, J. W., Ph.D. (Princeton) Prof., Princeton Univ., Princeton, N.J.
TULLER, ANNITA, Ph.D. (Bryn Mawr) Asst. Prof., Hunter Coll., New York, N.Y. *139-62 Pershing Crescent, Jamaica 2, N.Y.*
TULLIER, P. M., JR., M.S. (Louisiana State) Asst. Prof., Loyola Univ., New Orleans, La. *2900 Calhoun St. (18)*
TULLOCH, L. H., A.M. (Brown) Asso. Prof., Southwest Texas State Coll., San Marcos, Tex.
TUNELL, GEORGE, Ph.D. (Harvard) Asso. Prof., Univ. of California at Los Angeles, Los Angeles, Calif. *801 Toyopa Dr., Pacific Palisades, Calif.*
TURNER, BIRD M., Ph.D. (Bryn Mawr) Emeritus Prof., West Virginia Univ., Morgantown, W. Va. *1411 Fifth St., Moundsville, W. Va.*
TURNER, LONA L., M.A. (Michigan) Instr., Univ. of Illinois, Navy Pier, Chicago, Ill.
TURNER, NURA D., M.S. (Iowa) Instr., N.Y. State College for Teachers, Albany, N.Y.
TURRITTIN, H. L., Ph.D. (Wisconsin) Asso. Prof., Univ. of Minnesota, Minneapolis, Minn. *4046 Beard Ave. S.*
TYE, BRADFORD, M.Sc. (N.Y.U.) Prof. Bethany Coll., Bethany, W. Va.
TYLER, C. M., JR., Ph.D. (Pittsburgh) Asst. Prof., Carnegie Inst. of Tech., Pittsburgh, Pa.
TYLER, G. W., Ph.D. (V.P.I.) Operations Analyst, U. S. Air Force, Washington 25, D.C. *831 24th St. South, Arlington 2, Va.*
TYLER, JOHN, A.M. (William & Mary) Prof., U. S. Naval Academy, Annapolis, Md. *3 Southgate Ave.*
TYSVER, J. B., M.A. (Washington S.C.) Teaching Fellow, Univ. of Michigan, Ann Arbor, Mich. *1404 Erving Court, Willow Run Village, Mich.*

UHL, H. R., B.A. (Buffalo) Lt. Comdr., U. S. Navy
UHLER. H. S., Ph.D. (Johns Hopkins) Emeritus Prof., Yale Univ., New Haven, Conn. *296 Spring St., Meriden, Conn.*
UHRHAN, JANE A., M.A. (Indiana) Instr., Butler Univ., Indianapolis, Ind. *3005 Broadway*
UHRICH, G. E., Ph.D. (Washington) Asst. Prof., Montana State Coll., Bozeman, Mont.
ULLMAN, J. L., Ph.D. (Stanford) Instr., Univ. of Michigan, Ann Arbor, Mich. *On leave 1951–52:* Box 2795, Stanford, Calif.
ULLMAN, NELLY S., M.S. (Columbia) Instr., Poly. Inst. of Brooklyn, Brooklyn, N.Y. *2 Grace Court*
ULLSVIK, B. R., Ph.D. (Wisconsin) Prof., Illinois State Normal Univ., Normal, Ill.
ULMER, GILBERT, Ph.D. (Kansas) Asst. Dean, Univ. of Kansas, Lawrence, Kan. *1836 Vermont St.*
ULRICH, F. E., Ph.D. (Harvard) Asso. Prof., Rice Inst., Houston 1, Tex. *37 Ray St., Schenectady, N.Y.*
UNDERWOOD, R. S., Ph.D. (Chicago) Prof., Texas Tech. Coll., Lubbock, Tex. *2220 Broadway*
UPTON, C. B., A.M. (Columbia) Emeritus Prof., Teachers Coll., Columbia Univ., New York, N.Y.
URNER, S. E., Ph.D. (Harvard) Prof., Los Angeles State Coll., Los Angeles, Calif. *6217 Matilija Ave., Van Nuys, Calif.*
UTZ, W. R., JR., Ph.D. (Virginia) Asst. Prof., Univ. of Missouri, Columbia, Mo.

VALENTINE, F. A., Ph.D. (Chicago) Asso. Prof., Univ. of California at Los Angeles, Los Angeles, Calif.
VALLANDINGHAM, J. T., A.B. (Georgetown C.) Head of Dept., Cumberland Coll., Williamsburg, Ky.
VAN ALSTYNE, J. P., B.S. (Hamilton) Instr., Hamilton Coll., Clinton, N.Y.

VAN ARKEL, G. H., M.S. (Washington) Chm., Science Div., Everett Jr. Coll., Everett, Wash. *1010 Hoyt Ave.*
VAN ARNAM, R. N., M.S. (Cornell) Asst. Prof., Lehigh Univ., Bethlehem, Pa. *705 First Ave.*
VAN BERGEN, F. E., Sc.D. (Brussels) Teacher, Royal Athenaeum, St. Niklaas, Belgium. *Broedersstraat 37*
VAN BUSKIRK, H. C., Ph.D. (Cornell) Emeritus Prof., California Inst. of Tech., Pasadena, Calif. *390 S. Holliston Ave. (5)*
VANCE, E. P., Ph.D. (Michigan) Chm. of Dept., Oberlin Coll., Oberlin, Ohio. *174 Forest St.*
VANDE CASTLE, J. J., B.S. (St. Norbert) Instr., St. Norbert Coll., West De Pere, Wis. *209 Reid St.*
VANDERBURGH, E. L., M.A. (Wyoming) Instr., Pueblo Jr. Coll., Pueblo, Colo. *937 Veta Ave.*
VAN DEVENTER, L. R., M.S. (Illinois) Asst. Prof., Eastern Illinois State Coll., Charleston, Ill.
VANDIVER, H. S., Prof., Univ. of Texas, Austin 12, Tex.
VANDORT, H. J., REV., B.D. (Kenyon) Curate, Grace Episcopal Church, Grand Rapids 3, Mich. *130 Langdon Ave. N.E.*
VAN ENGEN, HENRY, Ph.D. (Michigan) Head of Dept., Iowa State Teachers Coll., Cedar Falls, Iowa
VAN HOOK, B. O., M.A. (Vanderbilt) Asso. Prof., Mississippi Southern Coll., Hattiesburg, Miss.
VAN HORN, C. E., Ph.D. (Chicago) Prof., Wartburg Coll., Waverly, Iowa. *302 Fourth Ave. N.W,*
VANN, M. H., B.S. (Berry C.) Dean of Men, Reinhardt Coll., Waleska, Ga.
VAN ORSTRAND, C. E., M.S. (Michigan) Retired Geophysicist, U. S. Geological Survey, Washington, D.C. *Route 2, Manito, Ill.*
VAN SANT, HELEN E., A.M. (Columbia) *R.D. 2, Bristol, Pa.*
VAN SCHAACK, G. B., Ph.D. (Harvard) Missouri Botanical Garden, 2315 Tower Ave., St. Louis, Mo.
VAN VOORHIS, W. R., Ph.D. (Pennsylvania State) Prof., Fenn Coll., Cleveland, Ohio. *838 Caledonia Rd., Cleveland Heights, Ohio*
VARINEAU, V. J., Ph.D. (Wisconsin) Asso. Prof., Univ. of Wyoming, Laramie, Wyo.
VARNER, W. W., B.S. (Colorado) Instr., Univ. of Colorado, Boulder, Colo. *1075 Eleventh St.*
VARNHORN, MARY C., Ph.D. (Catholic U.) Asso. Prof., Trinity Coll., Washington, D.C. *1120 Poplar Grove Ave., Baltimore 16, Md.*
VARNUM, E. C., M.S. (Michigan) Mathematician, Barber-Colman Co., Rockford, Ill. *2128 Fremont St.*
VASS, J. I., Ph.D. (Wisconsin) Retired Instr., Univ. of Wisconsin in Milwaukee, Milwaukee, Wis. *5141 N. Santa Monica Blvd.*
VATNSDAL, J. R., Ph.D. (Michigan) Prof., State Coll. of Washington, Pullman, Wash. *1916 B St.*
VAUGHAN, H. E., Ph.D. (Michigan) Asst. Prof., Univ. of Illinois, Urbana, Ill. *907 S. Vine St.*
VAUGHN, G. F., A.B. (Butler U.) Grad. Student, Butler Univ., Indianapolis, Ind. *4640 Kingsley Dr.*
VAUSE, R. Z., JR., M.A. (Duke) Asst. Prof., Clemson Coll., Clemson, S.C.
VEATCH, R. W., A.M. (Northwestern) Asso. Prof., Univ. of Tulsa, Tulsa 4, Okla. *332 N. Santa Fe St.*
VEBLEN, OSWALD, Ph.D. (Chicago) Emeritus Prof., Inst. for Advanced Study, Princeton, N.J. *58 Battle Rd.*
VEDOVA, G. C., Ph.D. (Maryland) Prof., Newark Coll. of Engineering, Newark 2, N.J.
VEHSE, C. H., Ph.D. (Brown) Asso. Prof., West Virginia Univ., Morgantown, W. Va. *505 Cambridge Ave.*
VELEK, ELEANOR R., B.A. (Carleton) Trainee, Trust Dept., First National Bank of Chicago, Chicago, Ill. *2115 So. 59th Ave. Cicero 50, Ill.*
VELESZ, REV. DUNSTAN, M.S. (De Paul) Instr., Quincy Coll., Quincy, Ill.
VEST, M. L., Ph.D. (Michigan) Asso. Prof., West Virginia Univ., Morgantown, W. Va. *417 Elm St.*
VICHICH, T. E., B.S. (Mich. Coll. of M. & T.) Michigan Coll. of Mining and Tech., Houghton, Mich.
VICK, C. M., Supervisor, Math. Analyst Section, Sandia Corp., Albuquerque, N.M. *525 N. Montclaire*
VICK, G. R., A.M. (Sam Houston S.T.C.) Asso. Prof., Sam Houston State Teachers Coll., Huntsville, Tex. *Box 21, Woodrow Wilson Center, Norman, Okla.*
VILKELIS, W. V., Student, Rutgers Univ., New Brunswick, N.J. *1916 West 7th St., Dunellen, N.J.*
VINCENT, KATHLEEN V., A.B. (Regis) *29 Manchester Rd., Newton Highlands, Mass.*
VINOGRADE, BERNARD, Ph.D. (Michigan) Asso. Prof., Iowa State Coll., Ames, Iowa
VIRTS, R. O., A.M. (Indiana) Chm. of Dept., Central High School, Fort Wayne, Ind. *2222 Hoagland Ave., Apt. A-1, (6)*
VITALE, R. L., M.E. (Poly. Inst. of Brooklyn) Asst. Elec. Engr., Board Transportation, City of New York, N.Y. *366 Avenue T, Brooklyn 23, N.Y.*

VITOUSEK, M. J., B.S. (Stanford) Research Asst., Stanford Univ., Stanford, Calif.
VLACHOS, N. D., A.M. (Pennsylvania) Mathematician, Kellett Aircraft Corp., Camden Airport, New Jersey. *4737 Browning Rd., Merchantville, N.J.*
VOLLMER, J. E., M.A. (Detroit) Instr., Univ. of Vermont, Burlington, Vt. *151 South Prospect St.*
von NEUMANN, JOHN, Ph.D. (Budapest) Prof., Inst. for Advanced Study, Princeton, N.J. *26 Wescott Rd.*
von SCHELLING, HERMANN, Ph.D. (Berlin) Biomathematician, Naval Medical Research Lab., U. S. Submarine Base, New London, Conn. *Eccleston's Apartments, R.F.D. 1, Groton, Conn.*
VOORHEES, E. A., JR., A.B. (Maryville C.) Fellow, Vanderbilt Univ., Nashville, Tenn. *906 19th Ave. S.*
VOPNI, SYLVIA, M.A. (Washington) Head of Dept. of Tech., Edison Technical School, Seattle 22, Wash. *2107-28 West, Seattle 99, Wash.*
VORONOVICH, W. E., Grad. (Polytechnic Inst. of Riga) The Lewis Machine Co., Cleveland, Ohio. *9820 Parkview Ave. (5)*
VROOMAN, S. I., M.A. (Columbia) Asst. Prof., Rensselaer Poly. Inst., Troy, N.Y. *On leave:* 58 E. Armory Ave., Champaign, Ill.

WADE, B. T., M.A. (K.S.U.) Asst. Prof., Clemson Coll., Clemson, S.C. *312 Meadow Lane*
WADE, T. L., JR., Ph.D. (Virginia) Head of Dept., Florida State Univ., Tallahassee, Fla. *1003 N. Washington St.*
WADKINS, O. L., JR., B.A. in E.E. (Florida) Research Engr., North American Aviation, Los Angeles, Calif. *Route 1, Box K-20, Simi, Calif.*
WAGNER, F. J., M.S. (St. Louis) Grad. Fellow, Univ. of Notre Dame, Notre Dame, Ind.
WAGNER, I. F., JR., M.S. (V.P.I.) Chm. of Dept., Bethesda-Chevy Chase Sr. High School, Bethesda, Md. *Box 303, College Park, Md.*
WAGNER, J. F., M.S. (Michigan) Asst. Prof., Univ. of Colorado, Boulder, Colo. *960 14th St.*
WAGNER, R. D., Ph.D. (Wisconsin) Asst. Prof., Univ. of Wisconsin, Madison, Wis. *2614 Stevens St.*
WAGNER, R. W., Ph.D. (Michigan) Prof., Univ. of Massachusetts, Amherst, Mass.
WAHAB, J. H., Ph.D. (North Carolina) Asst. Prof., Georgia Inst. of Tech., Atlanta, Ga.
WAHLERT, H. E., A.M. (Princeton) Asso. Prof., Washington Square Coll., New York Univ., New York, N.Y. *50 Strickland Pl., Manhasset, N.Y.*
WAIDER, K. J., A.M. (California) Instr., Physics, Univ. of San Francisco, San Francisco, Calif.
WAINWRIGHT, THOMAS, B.S. (Montana S.C.) Grad. Student, Univ. of Notre Dame, Notre Dame, Ind.
WAITE, ETTA A., A.M. (Columbia) Teacher, Prospect Heights High School, Brooklyn 25, N.Y. *491 New Milford Ave., Oradell, N.J.*
WALDEN, EARL, Ph.D. (Illinois) Asso. Prof., New Mexico Coll. of A. and M. A., State College, N.M.
WALDER, O. E., A.M. (Nebraska) Prof., South Dakota State Coll., Brookings, S.D.
WALDMAN, G. E., M.A. (Columbia) Teacher, Tottenville High School, Staten Island 7, N.Y. *70 Peters Pl., Red Bank, N.J.*
WALKER, C. R., M.E.E. (Poly. Inst. of Brooklyn) Engineer, American Tel. and Tel. Co., New York, N.Y.
WALKER, D. T., B.S. (Wofford C.) Grad. Student, Univ. of Georgia, Athens, Ga.
WALKER, G. L., Ph.D. (Cornell) Asst. Prof., Purdue Univ., Lafayette, Ind.
WALKER, G. W., B.D. (Auburn Theol. Sem.) Pastor, Walden Presbyterian Church, 2065 Bailey Ave., Buffalo 11, N.Y.
WALKER, L. A., A.M. (Stanford) Teacher, High School, San Mateo, Calif. *336 San Antonio St.*
WALKER, LILA P., M.A. (North Carolina) Grad. Student, Univ. of North Carolina, Chapel Hill, N.C. *26-B Glen Lennox*
WALKER, R. J., Ph.D. (Princeton) Prof., Cornell Univ., Ithaca, N.Y.
WALKLEY, S. E., M.A. (Illinois) Asst., Univ. of Illinois, Urbana, Ill.
WALL, B. M., A.M. (Sam Houston S.T.C.) Asso. Prof., Sam Houston State Teachers Coll., Huntsville, Tex.
WALL, D. D., Ph.D. (California) I.B.M. Corp., Los Angeles, Calif. *1931 Glendon (25)*
WALL, F. J., B.S. (Sul Ross S.T.C.) Research Asst., Los Alamos Scientific Lab., Los Alamos, N.M. *3534G Trinity Dr.*
WALL, H. S., Ph.D. (Wisconsin) Prof., Univ. of Texas, Austin, Tex.
WALL, J. G., M.A. (North Carolina) Grad. Asst., Univ. of Georgia, Athens, Ga.
WALLACE, A. D., Ph.D. (Virginia) Prof., Tulane Univ., New Orleans, La.
WALLACE, F. A., M.S. (Florida) Instr., Jacksonville Jr. Coll., Jacksonville, Fla. *4834 Attleboro St. (5)*
WALLACE, R. T., M.A. (British Columbia) Asso. Prof., Victoria Coll., Victoria, B.C., Can. *830 Richmond Rd.*

WALLACE, WILLIAM, A.M. (Boston U.) Asst. Prof., Northeastern Univ., Boston, Mass.
WALLACH, Israel, M.S. (N.Y.U.) Teacher, Thomas Jefferson High School, Brooklyn, N.Y. *816 St. John's Pl., (16)*
WALLICK, E. E., Ed.M. (Temple U.) Head of Dept., Senior High School, Lakewood, N.J. *140 East Seventh St.*
WALSH, FRANCES E., M.S. (St. Louis U.) Instr., Ursuline Coll., Paola, Kan. *321 North 12th St., Atchison, Kan.*
WALSH, J. L., Ph.D. (Harvard) Prof., Harvard Univ., Cambridge 38, Mass.
WALSH, MOTHER MARY ELIZABETH, M.A. (Boston) Instr., Newton Coll. of the Sacred Heart, Newton 59, Mass.
WALTER, R. M., A.M. (Columbia) Asso. Prof., New Jersey Coll. for Women, New Brunswick, N.J.
WALTERS, ELEANOR B., M.A. (Duke) Asso. Prof., Delta State Teachers Coll., Cleveland, Miss.
WALTERS, LILLIE C. (Mrs.) A.M. (Colorado) Instr., Univ. of Colorado, Boulder, Colo.
WALTERS, R. E., B.A. (Akron) Mathematician, Ballistics Research Lab., Aberdeen Proving Ground, Md. *224 North Stokes St., Havre de Grace, Md.*
WALTERS, S. S., Ph.D. (U.C.L.A.) *117 Lexington St., Belmont, Mass.*
WALTON, C. B., B.S. (Pennsylvania) Sr. Engr., Federal Power Commission, Washington, D.C. *3822 McKinley St., Chevy Chase, Md.*
WALTON, JEAN B., Ph.D. (Pennsylvania) Dean of Women, Pomona Coll., Claremont, Calif.
WALTON, T. O., Ph.D. (Michigan) Prof., Kalamazoo Coll., Kalamazoo 49, Mich.
WALTZ, A. K., Ph.D. (Cornell) Retired Asso. Prof., Clarkson Coll., Potsdam, N.Y. *Chewsville, Md.*
WAMPOLE, C. R., Student, St. John's Univ., Brooklyn, N.Y. *53-20 196th St., Flushing, N.Y.*
WANG, CHIH-YI, M.A. (Minnesota) Teaching Asst., Univ. of Minnesota, Minneapolis, Minn. *2829 16th Ave. S. (7)*
WANG, E. H., Dip. Eng., Techn. (Vienna) Chief, Special Engg. Studies, Air Materiel Comm., Dayton, Ohio. *601 Monroe Dr., Xenia, Ohio*
WAPPLE, A. R., A.M. (California) Instr., Texas A. and M. Coll., College Station, Tex.
WARD, J. A., Ph.D. (Wisconsin) Prof., Univ. of Kentucky, Lexington, Ky.
WARD, J. E., M.S. (Northwestern) Instr., Univ. of Wisconsin, Kenosha Ext. Center, Kenosha, Wis. *7406 8th Ave.*
WARD, L. E., Ph.D. (Harvard) Mathematician, U. S. Navy, China Lake, Calif. *206-B Ellis St.*
WARD, L. E., JR., M.S. (Tulane) Grad. Fellow, Tulane Univ., New Orleans, La. *7608 St. Charles Ave., Apt. H (18)*
WARD, MORGAN, Ph.D. (C.I.T.) Prof., California Inst. of Tech., Pasadena 4, Calif.
WARD, SUSIE L., M.A. (Alabama) Instr., Univ. of Alabama, University, Ala. *611 13th Ave., Tuscaloosa, Ala.*
WARDWELL, J. F., Ph.D. (Johns Hopkins) Prof., Colgate Univ., Hamilton, N.Y. *62 Broad St.*
WARNER, F. C., M.A. (Buffalo) Instr., Univ. of Buffalo, Buffalo 14, N.Y.
WARNER, W. H., M.S. (Carnegie) Research Asst., Carnegie Inst. of Tech., Pittsburgh, Pa. *650 Morewood Ave. (13)*
WARNOCK, W. G., Ph.D. (Illinois) Prof., Rensselaer Poly. Inst., Troy, N.Y. *Box 71, Melrose, N.Y.*
WARREN, K. L., Ph.D. (Michigan S.) Asso. Prof., Physics, Kent State Univ., Kent, Ohio
WARSCHAWSKI, S. E., Ph.D. (Basel) Prof., Univ. of Minnesota, Minneapolis 14, Minn.
WASHBURNE, A. C., Grad. (U. S. Military Acad.) Actuary Emeritus, Berkshire Life Insurance Co., Pittsfield, Mass. *134 E. Housatonic St.*
WASOW, W. R., Ph.D. (N.Y.U.) Mathematician, Inst. for Numerical Analysis, National Bureau of Standards, Los Angeles, Calif.
WASSON, H. P., A.M. (Columbia) Asso. Prof., Newark Coll. of Engineering, Newark, N.J. *282 William St., East Orange, N.J.*
WATERMOLEN, N. M., M.Sc. (Wisconsin) Instr., St. Norbert Coll., West De Pere, Wis.
WATKEYS, C. W., A.M. (Harvard) Prof., Univ. of Rochester, Rochester, N.Y. *287 Dartmouth St.*
WATSON, G. C., A.M. (Virginia) Asst. Prof., North Carolina State Coll., Raleigh, N.C.
WATT, MARTHA W., A.M. (Columbia) Emeritus Asso. Prof., Wheaton Coll., Norton, Mass. *35 Humboldt Ave., Providence 6, R.I.*
WATTS, C. B., A.B. (Indiana) Astronomer, U. S. Naval Observatory, Washington 25, D.C.
WAYNE, ALAN, M.S. in Ed. (C.C.N.Y.) Chm. of Dept., Williamsburgh Vocational High School, Brooklyn 6, N.Y. *141-21 78th Rd., Flushing, N.Y.*
WAYNE, W. J., Student, Butler Univ., Indianapolis, Ind. *933 Tecumpseh St. (1)*
WEAR, LAURA B., M.A. (Minnesota) Grad. Student, Univ. of Pennsylvania, Philadelphia, Pa. *Brooke Rd., Wayne, Pa.*
WEAR, L. E., Ph.D. (Johns Hopkins) Emeritus Asso. Prof., California Inst. of Tech., Pasadena, Calif. *2247 Lambert Dr.*

WEAVER, C. L., B.S. (Kent) Asst. Actuary, New England Mutual Life Insurance Co., 501 Boylston St., Boston, Mass.
WEAVER, WARREN, Ph.D. (Wisconsin) Dir. for the Natural Sciences, Rockefeller Foundation, New York, N.Y. *160 Brite Ave., Scarsdale, N.Y.*
WEBB, D.L., Ph.D. (C.I.T.) Asso. Prof., Univ. of Arizona, Tucson, Ariz.
WEBBER, G. C., Ph.D. (Chicago) Prof., Univ. of Delaware, Newark, Del.
WEBER, BETTY R., M.A. (Southern California) Asst. Prof., Univ. of South Carolina, Columbia, S.C. *1516 Columbia College Dr.*
WEBER, W. W., A.M. (Georgia) Asso. Prof., Univ. of South Carolina, Columbia, S.C. *1516 Columbia College Dr.*
WEBSTER, G. M., M.A. (Oklahoma) Research Physicist, Carter Oil Co., Tulsa, Okla. *1133 N. Lewis*
WEBSTER, M. S., Ph.D. (Pennsylvania) Asso. Prof., Purdue Univ., Lafayette, Ind.
WEDEL, A. M., Ph.D. (Iowa State) Asst. Prof., Bethel Coll., North Newton, Kan.
WEDEL, E. B., A.M. (Oklahoma) Asso. Prof., Univ. of Wichita, Wichita, Kan. *3600 E. 21st St., Route 3 (15)*
WEEBER, MARGARET C., A.M. (Chicago) Asso. Prof., Teachers Coll. of Connecticut, New Britain, Conn.
WEEG, G. P., M.S. (Oklahoma A. and M.) Instr., St. Ambrose Coll., Davenport, Iowa. *1623 Florence Lane*
WEGNER, K. W., Ph.D. (Wisconsin) Asso. Prof., Carleton Coll., Northfield, Minn.
WEHAUSEN, J. V., Ph.D. (Michigan) Executive Editor, Mathematical Reviews, American Mathematical Society, 80 Waterman St., Providence 6, R.I. *50 Lincoln Ave., West Barrington, R.I.*
WEIDA, F. M., Ph.D. (Iowa) Prof., George Washington Univ., Washington 6, D.C. *7130 Hampden Lane, Bethesda 14, Md.*
WEIDEMAN, W. G., B.E.E. (Marquette) Grad. Student, Marquette Univ., Milwaukee, Wis. 2170 N. 56th St. (8)
WEINER, L. M., S.M. (Chicago) Grad. Student, Univ. of Chicago, Chicago, Ill. *3312 West Eastwood Ave. (25)*
WEINSTOCK, ELIZABETH (Mrs. Robert), B.A. (Vassar) Grad. Student, Stanford Univ., Stanford, Calif. *894 San Jude Ave., Palo Alto, Calif.*
WEISNER, LOUIS, Ph.D. (Columbia) Asso. Prof., Hunter Coll., New York, N.Y.
WEISS, ANDREW, Student, Univ. of Detroit, Detroit, Mich. *4855 Beck Rd., Belleville, Mich.*
WEISS, E. H., Sc.M. (Brown) Statistician, Bureau of Census, Washington, D.C. *3305 Terrace Dr., Suitland, Md.*
WEISS, J. A., M.A. (Ohio State) *398 South Ogden Ave., Columbus 4, Ohio*
WEISS, MARIE J., Ph.D. (Stanford) Prof., Newcomb Coll., New Orleans 18, La.
WEISSLER, H. E., SR., B.Sc. (St. Mary's) Instr., St. Mary's Univ., San Antonio, Tex. *115 Vaughan Pl. (1)*
WEITZENHOFFER, A. M., M.A. (Brown) Research Associate, Univ. of Detroit, College of Dentistry, 630 Jefferson Ave., Detroit 26, Mich.
WELCH, MARGARET M., Student, Texas Christian Univ., Fort Worth, Tex. *2818 5th Ave. (10)*
WELCHONS, A. M., A.M. (Indiana) Head of Dept., Arsenal Tech. Schools, Indianapolis, Ind. *509 N. Drexel Ave. (1)*
WELLS, C. P., Ph.D. (Iowa State) Asso. Prof., Michigan State Coll., East Lansing, Mich.
WELLS, E. D., A.M. (Minnesota) Asso. Prof., Univ. of Pittsburgh, Pittsburgh, Pa. *735 S. Negley Ave. (32)*
WELLS, MARGUERITE F., M.A. (Alabama) Statistician, *34 N. Central Ave., Fairborn, Ohio*
WELLS, MARY EVELYN, Ph.D. (Chicago) Emeritus Prof., Vassar Coll., Poughkeepsie, N.Y. *Southport, Maine*
WELLS, N. W., A.M. (Rice) Asst. Prof., Sam Houston State Teachers Coll., Huntsville, Tex. *1928 Avenue N½, Galveston, Tex.*
WELLS, V. H., Ph.D. (Michigan) Prof., Williams Coll., Williamstown, Mass. *3 Chapin Court*
WELLS, W. J., M.S. (Michigan) Grad. Student, Univ. of Minnesota, Minneapolis 14, Minn.
WELMERS, E. T., Ph.D. (Michigan) Chief of Dynamics Group, Bell Aircraft Corp., Niagara Falls, N.Y. *165 Fayette Ave., Kenmore, N.Y.*
WELMERS, INA W. (Mrs. E. T.) A.M. (Michigan) *165 Fayette Ave., Kenmore 17, N.Y.*
WENDEL, J. G., Ph.D. (C.I.T.) Mathematician, Rand Corp., Santa Monica, Calif. *150 San Vicente Blvd.*
WENDROFF, BURTON, Student, New York Univ., New York, N.Y. *207 South First Ave., Mount Vernon, N.Y.*
WENNER, A. M., B.S. (Gustavus Adolphus) Grad. Asst., Oregon State Coll., Corvallis, Ore.
WENTE, IRENE L., M.S. (Chicago) Instr., South Dakota State Coll., Brookings, S.D.
WERDELIN, R. A., Student, Trinity Coll., Hartford, Conn. *172 George St.*

WERKMAN, MARY K., M.S. (Chicago) Teacher, Parker High School, Chicago, Ill. *7139 S. Normal Blvd. (21)*
WESCOTT, M. E., Ph.D. (Northwestern) Asst. Prof., Northwestern Univ., Evanston, Ill. *2011 Beechwood Dr., Wilmette, Ill.*
WESOLOWSKI, EDMUND, B.S. (Elmhurst) Spectrographer, Pure Oil Research and Development Labs., Crystal Lake, Ill. *Box 126, Park Ave., Cary, Ill.*
WESSELY, H. W., B.S. (Illinois Tech.) Physicist, Gaertner Scientific Corp., Chicago, Ill. *4249 N. Meade Ave. (34)*
WESSON, J. R., M.A. (Vanderbilt) Asst. Prof., Birmingham Southern Coll., Birmingham, Ala.
WEST, C. S., JR., B.S. (Southern Methodist) Asst. to the Actuary, Guardian International Life Insurance Co., Dallas, Tex. *4016 McKinney Ave., Apt. 16 (4)*
WESTERN, D. W., Ph.D. (Brown) Asso. Prof., Franklin and Marshall Coll., Lancaster, Pa.
WESTHAFER, R. L., Ph.D. (Ohio State) Asso. Prof., New Mexico Coll. of A. and M. A., State College, N.M.
WESTON, K. W., Student, Univ. of Wisconsin, Madison, Wis. *R.R. 3, Box 138, Waukesha, Wis.*
WETROGAN, NATHAN, B.A. (Brooklyn) *1952 S. St., S.E., Washington, D.C.*
WETZIG, C. U., A.M. (Texas) Head of Dept., A. and M. Coll., Magnolia, Ark.
WEXLER, CHARLES, Ph.D. (Harvard) Prof., Arizona State Coll., Tempe, Ariz. *1215 Maple Ave.*
WEYL, F. J., Ph.D. (Princeton) Office of Naval Research, U.S. Navy
WHEELER, ANNA PELL (Mrs.), Ph.D. (Chicago) Emeritus Prof., Bryn Mawr Coll., Bryn Mawr, Pa. *819 Montgomery Ave.*
WHEELER, C. H. III, Ph.D. (Johns Hopkins) Prof., Univ. of Richmond, Richmond, Va.
WHEELER, H. E., Student, Northwestern Univ., Evanston, Ill. *4539 S. Greenwood Ave., Chicago 15, Ill.*
WHELAN, A. MARIE, Ph.D. (Johns Hopkins) Asst. Prof., Hunter Coll., New York, N.Y.
WHITE, A. E., M.S. (Purdue) Prof., Kansas State Coll., Manhattan, Kan.
WHITE, G. W., M.A. (California) Prof., College of the Pacific, Stockton, Calif.
WHITE, J. H., M.A. (Columbia) Asst. Prof., U.S. Naval Academy, Annapolis, Md.
WHITE, MARION B., Ph.D. (Chicago) *1949 Woodlyn Rd., Pasadena 7, Calif.*
WHITE, MARY, Student, Regis Coll., Weston 93, Mass. *15 Wolcott Ave., Andover, Mass.*
WHITE, M. E., M.A. (Columbia) Lt., U. S. Air Force
WHITE, P. A., Ph.D. (Virginia) Asso. Prof., Univ. of Southern California, Los Angeles, Calif. *3821 Degnan Blvd. (8)*
WHITE, R. L., M.A. (U.C.L.A.) Teacher, Woodrow Wilson High School, Los Angeles, Calif. *4223 Collis Ave. (32)*
WHITE, T. J., M.A. (Rice) Research Asst., Los Alamos Scientific Lab., Los Alamos, N.M.
WHITEMAN, A. L., Ph.D. (Pennsylvania) Asso. Prof., Univ. of Southern California, Los Angeles 7, Calif.
WHITFORD, A. E., A.M. (Wisconsin) Dean, Coll. of Liberal Arts, Alfred Univ., Alfred, N.Y.
WHITFORD, D. E., Ed.M. (Harvard) Asso. Prof., Poly. Inst. of Brooklyn, Brooklyn, N.Y. *34 Livingston St. (2)*
WHITING, MABEL G., A.M. (Oberlin) Retired Head of Dept., Junior Coll., Santa Ana, Calif. *506 E. Chesnut Ave.*
WHITMAN, E. A., A.M. (Pittsburgh) Asso. Prof., Carnegie Inst. of Tech., Pittsburgh, Pa. *521 Locust St. (18)*
WHITMAN, LOIS B. (Mrs.) B.A. (U.C.L.A.) Teacher, Fairfax High School, Los Angeles, Calif. *14242 Hart St., Van Nuys, Calif.*
WHITMAN, P. M., Ph.D. (Harvard) Mathematician, Applied Physics Lab., Johns Hopkins Univ., Silver Spring, Md.
WHITMORE, R. M., M.A. (Texas) Asso. Prof., Southwestern Univ., Georgetown, Tex.
WHITNEY, ANNE M., Ph.D. (Pennsylvania) Asst. Prof., Goucher Coll., Towson 4, Md. *106 Alleghany Ave. (4)*
WHITNEY, D. R., Ph.D. (Ohio State) Asst. Prof., Ohio State Univ., Columbus, Ohio. *39 Tibet Rd. (2)*
WHITTED, J. A., A.M. (Southwestern) Emeritus Prof., Ohio Northern Univ., Ada, Ohio. *75 Forest Ave., Delaware, Ohio*
WHYBURN, G. T., Ph.D. (Texas) Prof., Univ. of Virginia, Charlottesville, Va.
WHYBURN, W. M., Ph.D. (Texas) Prof., Univ. of North Carolina, Chapel Hill, N.C.
WICHT, M. C., M.A. (Vanderbilt) Prof., North Georgia Coll., Dahlonega, Ga.
WICKER, B. R., Ph.D. (Iowa) Head of Dept., Loyola Univ. of Los Angeles, Los Angeles, Calif.
WIDDER, D. V., Ph.D. (Harvard) Prof., Harvard Univ., Cambridge 38, Mass. *30 Gould Rd., Arlington, Mass.*
WIGGIN, EVELYN P., Ph.D. (Chicago) Prof., Randolph-Macon Woman's Coll., Lynchburg, Va.
WILANSKY, ALBERT, Ph.D. (Brown) Asst. Prof., Lehigh Univ., Bethlehem, Pa.
WILBANKS, T. F., M.A. (Louisiana) Prof., Southwestern Louisiana Inst., Lafayette, La. *130 Smith St.*

WILCOX, L. R., Ph.D. (Chicago) Asso. Prof., Illinois Inst. of Tech., Chicago 16, Ill.
WILCOX, MARY E., M.A. (Southwestern U.) Asst. Prof., Southwestern Univ., Georgetown, Tex. *1202 East 12th St.*
WILDER, R. L., Ph.D. (Texas) Research Prof., Univ. of Michigan, Ann Arbor, Mich. *1617 Cambridge Rd.*
WILDERMUTH, R. B., A.M. (Ohio State) Prof., Capital Univ., Columbus, Ohio. *2285 E. Mound St.*
WILEY, F. B., Ph.D. (Chicago) Prof., Ashland Coll., Ashland, Ohio. *Box 476, Granville, Ohio*
WILKINS, J. E., JR., Ph.D. (Chicago) Sr. Mathematician, Nuclear Development Assoc., New York 23, N.Y. *29 Wayne Ave., White Plains, N.Y.*
WILKINS, P. D., M.S. (Case) Prof., Bates Coll., Lewiston, Maine. *420 College St.*
WILKS, S. S., Ph.D. (Iowa) Prof., Princeton Univ., Princeton, N.J.
WILLERDING, MARGARET F., Ph.D. (St. Louis) Asst. Prof., Harris Teachers Coll., St. Louis, Mo. *5225 Walsh St. (9)*
WILLEY, MAUD, A.M. (Mills) *Route 10, Box 268-A, Fort Worth, Tex.*
WILLIAMS, ANNIE J., M.A. (North Carolina) Teacher, Julian S. Carr Junior High School, Durham, N.C. *2021 Sprunt St.*
WILLIAMS, A. R., Ph.D. (California) Emeritus Asst. Prof., Univ. of California, Berkeley, Calif.
WILLIAMS, B. B., B.A. (Howard Payne) Instr., Howard Payne Coll., Brownwood, Tex.
WILLIAMS, C. S., JR., M.A. (Oklahoma) Capt., U. S. Army
WILLIAMS, ERNEST, Ph.D. (Michigan) Prof., Alabama Poly. Inst., Auburn, Ala. *452½ Magnolia Ave.*
WILLIAMS, G. A., A.M. (California) Prof., Oregon State Coll., Corvallis, Ore. *306 N. 32nd St.*
WILLIAMS, G. T., *74 Sussex Rd., Elmont, L.I., N.Y.*
WILLIAMS, J. F., M.A. (Tennessee) Asst. Prof., Memphis State Coll., Memphis, Tenn. *3581 Rockwood Ave.*
WILLIAMS, K. P., Ph.D. (Princeton) Prof., Indiana Univ., Bloomington, Ind.
WILLIAMS, L. B., S.M. (Chicago) Asst. Prof., Reed Coll., Portland 2, Ore.
WILLIAMS, L. E., M.A. (Chicago) Head of Dept., Huntingdon Coll., Montgomery, Ala. *9 College Court, (6)*
WILLIAMS, MABEL, M.A. (Texas) Teacher, Tyler Jr. Coll., Tyler, Tex. *513 S. Chilton*
WILLIAMS, MARIE M., A.M. (Minnesota) Research Associate, Univ. of Minnesota, Minneapolis 14, Minn. *527 Fullerton Pkwy., Chicago 14, Ill.*
WILLIAMS, MARY E., A.M. (Kentucky) Asso. Prof., Skidmore Coll., Saratoga Springs, N.Y.
WILLIAMS, W. L., Ph.D. (Chicago) Prof., Univ. of South Carolina, Columbia, S.C.
WILLIAMS, W. L. G., Ph.D. (Chicago) Prof., McGill Univ., Montreal, Que., Can.
WILLIAMSON, C. O., Ph.D. (Chicago) Prof., College of Wooster, Wooster, Ohio. *1141 Beall Ave.*
WILSON, A. H., Ph.D. (Chicago) Emeritus Prof., Haverford Coll., Haverford, Pa.
WILSON, CHARLES CRAIG, M.A. (Loyola) Instr., Undergraduate Div., Univ. of Illinois, Chicago, Ill. *179 N. Leamington Ave. (44)*
WILSON, E. B., Ph.D. (Yale) Emeritus Prof., School of Public Health, Harvard Univ., Boston, Mass. *695 Huntington Ave. (15)*
WILSON, G. H., Ph.D. (Pennsylvania) Retired Asst. Prof., Physics, Drexel Inst., Philadelphia, Pa. *Buckingham Valley, Pa.*
WILSON, HAZEL S. (Mrs. L. T.), Ph.D. (Cornell) *421 N. Pelham Rd., Jacksonville, Ala.*
WILSON, J. A., M.S. (Case) Asst. Prof., Baldwin-Wallace Coll., Berea, Ohio. *1016 Quilliams Rd., Cleveland Heights 21, Ohio*
WILSON, J. C., B.S. (Oklahoma A. and M.) Trainee, Wolverine Insurance Co., Battle Creek, Mich.
WILSON, J. L., B.A. (Wabash) *610 North 28th St., Belleville, Ill.*
WILSON, R. H., JR., Ph.D. (Pennsylvania) Asst. Prof., Univ. of Louisville, Louisville 8, Ky.
WILSON, R. L., Ph.D. (Wisconsin) Asst. Prof., Univ. of Tennessee, Knoxville, Tenn.
WILSON, W. E., Ph.D. (Iowa) President, South Dakota School of Mines and Tech., Rapid City, S.D.
WILSON, W. H., Ph.D. (Illinois) Asso. Dean, Coll. of Arts and Sciences, Univ. of Florida, Gainesville, Fla.
WINEGEART, R. B., *Box 576, Jonesville, La.*
WINER, L. M., Student, Swarthmore Coll., Swarthmore, Pa.
WING, G. M., Ph.D. (Cornell) Staff Member, Los Alamos Scientific Lab., Los Alamos, N.M. *939 Iris St., Apt. 22*
WINGER, R. M., Ph.D. (Johns Hopkins) Prof., Univ. of Washington, Seattle, Wash.
WINN, W. F., B.S. in E.E. (Oklahoma) Military Instr., U. S. Army Signal Corps. *18 Peters Pl., Red Bank, N.J.*
WINNIS, W. H., Student, Iona Coll., New Rochelle, N.Y. *67 West 87th St., New York 24, N.Y.*
WINSTON, CLEMENT, Ph.D. (Pennsylvania) Principal Industrial Economist, War Production Board, Washington, D.C. *1420 Tuckerman St. (11)*

WIRSCHING, FLORENCE A., M.A. (Northwestern) *729 Third St., Secaucus, N.J.*
WIRSHUP, A. D., M.A. (Columbia) Teaching Fellow, Oregon State Coll., Corvallis, Ore.
WIRTH, H. P., Ph.D. (N.Y.U.) Asso. Prof., City Coll. of New York, New York, N.Y.
WITTY, W. H., B.A. (Mississippi U.) *Winona, Miss.*
WOHL, SONIA S., M.S. (Columbia) Asst. Librarian, Rockefeller Inst. for Medical Research, New York 21, N.Y. *1553 Minford Pl., Bronx 60, New York, N.Y.*
WOHLER, E. H., M.A. (Toledo) Asst. Prof., Bowling Green State Univ., Bowling Green, Ohio
WOHLFORD, E. G., B.S. (Kent State) Lt., U. S. Army
WOLF, HERBERT, M.A. (North Carolina) Grad. Student, Univ. of North Carolina, Chapel Hill, N.C.
WOLF, J. J., B.A. (George Washington) Grad. Student, George Washington Univ., Washington, D.C. *443 Decatur St. N.W. (11)*
WOLF, LOUISE A., Ph.D. (Wisconsin) Asso. Prof., Univ. of Wisconsin at Milwaukee, Milwaukee, Wis. *3700 S. 116th St. (14)*
WOLFE, ALBERTA, M.S. (Iowa State) Asst. Prof., Miami Univ., Oxford, Ohio
WOLFE, C. H., A.M. (Ohio Wesleyan) Supt., Danbury High School, Lakeside, Ohio
WOLFE, H. E., Ph.D. (Indiana) Prof., Indiana Univ., Bloomington, Ind. *812 S. Fess Ave.*
WOLFE, J. M., Ph.D. (N.Y.U.) Asst. Prof., Brooklyn Coll., Brooklyn 10, N.Y.
WOLFE, R. S., A.M. (Washington) Instr., Northwestern Univ., Evanston, Ill.
WOLINSKY, ALBERT, Ph.D. (Vienna) Instr., New York Univ., New York, N.Y. *56 Madison Rd., Scarsdale, N.Y.*
WOLLAN, G. N., M.S. (Iowa U.) Asst. Prof., Memphis State Coll., Memphis 11, Tenn.
WOLLAN, R. S., B.A. (Luther C.) Grad. Student, Univ. of Georgia, Athens, Ga. *1220 S. Milledge Ave.*
WONG, Y. K., Ph.D. (Chicago) Princeton Univ., Princeton, N.J.
WOOD, C. E., M.A. (Columbia) Teacher, North Syracuse High School, North Syracuse, N.Y. *362 10th Ave., Paterson 4, N.J.*
WOOD, FREDERICK, Ph.D. (Wisconsin) Dean, Coll. of Arts and Sciences, Univ. of Nevada, Reno, Nev.
WOOD, F. E., Ph.D. (Chicago) Asso. Prof., Univ. of Oregon, Eugene, Ore.
WOOD, H. A., Ph.D. (M.I.T.) Supervisor of Dynamic Analysis, Chance Vought Aircraft, Dallas, Tex. *6424-A Bandera Ave. (5)*
WOOD, P. G., B.Sc. (London) Mathematics Master, Eastleigh County High School, Hampshire, England. *2-D, Velmore Estate, Eastleigh, Hants, England*
WOOD, R. J., B.S. (St. Mary's) Instr., Edison High School, San Antonio, Tex. *501 W. Mulberry St. (1)*
WOOD, W. D., M.A. (Oberlin) Grad. Student, Purdue Univ., Lafayette, Ind.
WOODBRIDGE, MARGARET Y., Ph.D. (N.Y.U.) Instr., Brooklyn Coll., Brooklyn, N.Y. *169 Columbia Heights (2)*
WOODLE, R. G., JR., B.S. (Drury) Instr., Missouri School of Mines and Met., Rolla, Mo. *1103-A Oak St.*
WOODS, DALE, M.S. (Oklahoma A. and M.) Instr., Texas Western Coll., El Paso, Tex.
WOODS, ROSCOE, Ph.D. (Illinois) Asso. Prof., Univ. of Iowa, Iowa City, Iowa. *517 S. Lucas St.*
WOODSON, G. F., JR., A.M. (Ohio State) Prof., Central State Coll., Wilberforce, Ohio
WOODWARD, H. E., A.M. (Texas Tech.) Asst. Prof., Texas Tech. Coll., Lubbock, Tex.
WOOLARD, E. W., Ph.D. (George Washington) Asst. Dir. Nautical Almanac, U. S. Naval Observatory, Washington, D.C. *1232 30th St. N.W. (7)*
WORMSER, ARTHUR, Dr. of Engg. (Berlin) Product Engr., Infilco, Inc., Tucson, Ariz. *639 N. Columbus Blvd.*
WORTHAM, A. W., M.S. (Oklahoma A. and M.) Chance Vought Aircraft, Dallas, Tex. *3016 Dutton Dr. (8)*
WORTHINGTON, EUPHEMIA R., Ph.D. (Yale) Emeritus Asst. Prof., Univ. of California at Los Angeles, Los Angeles 24, Calif.
WORTHINGTON, L. G., M.A. (North Texas S.C.) Head of Dept., Tarleton State Coll., Stephenville, Tex. *1178 W. Jones*
WRAY, W. D., Ph.D. (Cornell) Mathematician, Armed Forces Security Agency, Washington 25, D.C. *R.D. 1, Box 127-B, McLean, Va.*
WREDE, R. C., JR., B.S. in E.E. (Miami U.) *325 West, University Apts., Bloomington, Ind.*
WREN, F. L., Ph.D. (Chicago) Prof., George Peabody Coll., Nashville 4, Tenn.
WRENCH, J. W., Ph.D. (Yale) Mathematician, David Taylor Model Basin, Navy Dept., Washington 7, D.C. *4711 Davenport St. N.W. (16)*
WRESTLER, FERNA E., A.M. (Kansas) Asst. Prof., Univ. of Wichita, Wichita, Kan.
WRIGHT, ALICE KELSEY (Mrs.), A.M. (Illinois) Asst. Prof., Southern Illinois Univ., Carbondale, Ill. *804 W. Main St.*
WRIGHT, C. B., Ph.D. (Pittsburgh) Prof., East Texas State Teachers Coll., Commerce, Tex. *1603 Bonham St.*

WRIGHT, E. W., Student, Univ. of Idaho, Moscow, Idaho. *Box 703, Burley, Idaho*
WRIGHT, FRANCES M., A.M. (Brown) Asst. Prof., Harpur Coll., Endicott, N.Y. *9 Minden Ave., Binghamton, N.Y.*
WRIGHT, H. A., Ph.D. (N.Y.U.) Prof., Transylvania Coll., Lexington, Ky.
WRIGHT, H. N., Ph.D. (California) President and Prof., City Coll. of New York, New York, N.Y. *3900 Greystone Ave. (63)*
WRIGHT, J. W., Student, Univ. of Maryland, College Park, Md. *3922 Gelston Dr., Baltimore 29, Md.*
WRIGHT, MARTIN, M.A. (Texas) Asso. Prof., Univ. of Houston, Houston, Tex. *1905 Idylwood Dr. (3)*
WRIGHT, T. M., M.A. (Michigan) Asso. Prof., Berea Coll., Berea, Ky.
WURSTER, MARIE A., Ph.D. (Chicago) Asst. Prof., Temple Univ., Philadelphia, Pa. *6635 McCallum St. (19)*
WYCKOFF, J. F., A.M. (Yale) Connecticut General Life Ins. Co., Hartford, Conn.
WYLIE, C. C., Ph.D. (Illinois) Prof., Astr., Univ. of Iowa, Iowa City, Iowa
WYLIE, C. R., JR., Ph.D. (Cornell) Prof., Univ. of Utah, Salt Lake City 1, Utah
WYMAN, MAX, Ph.D. (C.I.T.) Asst. Prof., Univ. of Alberta, Edmonton, Alberta, Can.
WYRE, JEAN M., M.A. (Oberlin) Grad. Student, Haverford Coll., Haverford, Pa. *582 E. Tuscarawas Ave., Barberton, Ohio*

YANARI, F. M., Student, George Pepperdine Coll., Los Angeles, Calif. *2707 S. Hobart Blvd. (18)*
YANNEY, B. F., Ph.D. (Chicago) Emeritus Prof., College of Wooster, Wooster, Ohio. *354 E Bowman St.*
YANOSIK, G. A., C.E. (N.Y.U.) Asso. Prof., New York Univ., New York, N.Y. *52 Greenvale Ave., Yonkers, N.Y.*
YARBROUGH, H. M., Ph.D. (Indiana) Head of Dept., Western Kentucky State Teachers Coll., Bowling Green, Ky. *Route No. 4*
YARNELLE, J. E., M.S. (Chicago) Prof., Hanover Coll., Hanover, Ind.
YATES, R. C., Ph.D. (Johns Hopkins) Lt. Colonel, U. S. Military Academy, West Point, N.Y.
YEAGER, E. N., M.S. (Notre Dame) Exec. Vice-Pres., Napoleon Products Co., Napoleon, Ohio. *323 W. Clinton St.*
YEATON, C. H., Ph.D. (Chicago) Prof., Oberlin Coll., Oberlin, Ohio. *189 Forest St.*
YEATON, MARIE M. (Mrs. C. H.), Ph.D. (Chicago) Retired Asso. Prof., Oberlin Coll., Oberlin, Ohio. *189 Forest St.*
YEDLIN, MELVIN, B.S. (Missouri) Automotive Design Engr., Detroit Arsenal, Centerline, Mich. *1405 Franklin, Berkley, Mich.*
YETT, F. R., B.S. (Texas) Teaching Fellow, Univ. of Texas, Austin, Tex. *2005 Indian Trail*
YIH, CHIA-SHUN, Ph.D. (Iowa) c/o Prof. Laurent Schwartz, Univ. of Nancy, Nancy, France
YOUDEN, W. J., Ph.D. (Columbia) Statistical Mathematician, National Bureau of Standards, Washington 25, D.C.
YOUDEN, W. W., Student, Brown Univ., Providence, R.I. *3810 47th St. N.W., Washington 16, D.C.*
YOUNG, D. M., JR., Ph.D. (Harvard) Mathematician, Aberdeen Proving Ground, Md. *Box 369, Aberdeen, Md.*
YOUNG, F. H., Ph.D. (Oregon) Mathematician, Bureau of Ordnance, U. S. Navy, Washington, D.C. *3339 17th St. N.W. (10)*
YOUNG, G. J., M.S. (Chicago) Technical Dir., Nuclear Development Associates, New York, N.Y. *80 Grand St., White Plains, N.Y.*
YOUNG, H. B., M.C.S. (Boston U.) Instr., Arkansas A. M. and N. Coll., Pine Bluff, Ark.
YOUNG, J. W., M.A. (Florida) Instr., Univ. of Florida, Gainesville, Fla.
YOUNG, MABEL M., Ph.D. (Johns Hopkins) Emeritus Prof., Wellesley Coll., Wellesley, Mass. *6 Norfolk Terrace*
YOUNG, P. M., Ph.D. (Ohio State) Prof., Kansas State Coll., Manhattan, Kan.
YOUNG, R. L., M.A. (Denver) Mathematician, Wright-Patterson A.F.B., Dayton, Ohio. *1611 E. 5th St. (3)*
YOUNG, R. W., M.A. (Lehigh) Prof., Henderson State Teachers Coll., Arkadelphia, Ark. *1031 Caddo St.*
YOUNGS, J. W. T., Ph.D. (Ohio State) *3397 51st Loop, Sandia Base, Albuquerque, N.M.*
YOUNKIN, W. G., Student, St. Ambrose Coll., Davenport, Iowa. *1214 E. Locust St.*
YOUSE, B. K., B.S. (Alabama Poly.) Grad. Asst., Univ. of Georgia, Athens, Ga. *Box 496, N. Webster, Ind.*
YOWELL, E. I., Ph.D. (Cincinnati) Emeritus Prof., Univ. of Cincinnati, Cincinnati, Ohio. *3127 Griest Ave. (8)*
YOZWIAK, B. J., M.S. (Pittsburgh) Asst. Prof., Youngstown Coll., Youngstown, Ohio. *505 Bryson St. (2)*

ZADER, G. C., M.A. (West Virginia) Asst. Prof., The Citadel, Charleston, S.C. *94 Ward Ave., Elkins, W. Va.*
ZAHLEN, JEAN-PIERRE, Director, Pythagore. *4 rue de L'acier, Differdange, Luxemburg*
ZANOLAR, A. J., M.S. (Catholic U.) President, St. Joseph's Coll., Collegeville, Ind.
ZANT, J. H., Ph.D. (Columbia) Prof., Oklahoma A. and M. Coll., Stillwater, Okla. *112 Admiral Rd.*
ZARING, W. M., A.B. (Kentucky Wesleyan) Grad. Student, Univ. of Kentucky, Lexington, Ky. *117 Woodland Ave.*
ZARISKI, OSCAR, Ph.D. (Rome) Prof., Harvard Univ., Cambridge 38, Mass.
ZAUSTINSKY, E. M., A.B. (U.C.L.A.) Grad. Student, Univ. of Southern California, Los Angeles, Calif. *930 East Del Mar St., Pasadena 5, Calif.*
ZEGERS, REV. T. A., M.S. (Fordham) Chm. of Dept., St. Peters Coll., Jersey City, N.J. *2652 Hudson Blvd.*
ZEIGLER, R. K., Ph.D. (Iowa) *108 West Newcomb Rd., Oak Ridge, Tenn.*
ZEITLIN, DAVID, B.Ch.E. (Minnesota) Instr., Univ. of Minnesota, Minneapolis, Minn.
ZELDIN, S. D., Ph.D. (Clark) Asso. Prof., Massachusetts Inst. of Tech., Cambridge, Mass.
ZEMLIN, R. A., B.A. (Coll. of St. Thomas) Asst., Ohio State Univ., Columbus, Ohio. *5207 Zenith S., Minneapolis 10, Minn.*
ZEMMER, J. L., JR., Ph.D. (Wisconsin) Asst. Prof., Univ. of Missouri, Columbia, Mo.
ZERBE, H. M., M.S. (Pennsylvania State) Asst. Prof., Pennsylvania State Coll., Hazleton, Pa. *R.D. 1, Box 110-G, Hazleton, Pa.*
ZIEBUR, A. D., Ph.D. (Wisconsin) Instr., Ohio State Univ., Columbus, Ohio
ZILBER, J. A., A.M. (Harvard) Instr., Johns Hopkins Univ., Baltimore 18, Md.
ZIMMERBERG, H. J., Ph.D. (Chicago) Asso. Prof., Rutgers Univ., New Brunswick, N.J.
ZIMMERMAN, B. C., A.M. (St. Louis) *Cayo, British Honduras*
ZIMMERMAN, L. J., M.Sc. (Purdue) Asst. Prof., Goshen Coll., Goshen, Ind.
ZIPPIN, LEO, Ph.D. (Pennsylvania) Asso. Prof., Queens Coll., Flushing, L.I., N.Y.
ZIRKEL, E. J., Student, St. John's Univ., Brooklyn, N.Y. *91-16 211th Pl., Queens Village, L.I., N.Y.*
ZORA, V. A., B.S. (Pittsburgh) Chem. Process Engr., Blaw-Knox Const. Co., Pittsburgh, Pa. *321 Penn Ave.*
ZORN, M. A., Dr. res.mat. (Hamburg) Prof., Indiana Univ., Bloomington, Ind.
ZUBAY, E. A., A.M. (Minnesota) Instr., Drake Univ., Des Moines, Iowa
ZUCKER, ALBERT, Student, City Coll. of New York, New York, N.Y. *1560 Amsterdam Ave. (31)*
ZUCKER, M. N., Student, New York Univ., New York, N.Y. *76-10 137th St., Flushing, Queens, N.Y.*
ZUCKERMAN, H. S., Ph.D. (California) Asst. Prof., Univ. of Washington, Seattle 5, Wash.
ZYGMUND, ANTONI, Ph.D. (Warsaw) Prof., Univ. of Chicago, Chicago 37, Ill.

Total membership, October 15, 1949 3,748
Total membership, October 15, 1951 4,564

GEOGRAPHICAL DISTRIBUTION OF MEMBERS

UNITED STATES

ALABAMA

AUBURN.
Alabama Poly. Inst. Burnette, Case, Cox, Daughtry, DeLoach, Dietrich, Doner, Eaves, Fitzpatrick, Ikenberry, Locker, Matthews, Miles, Mitchell, Moss, Norton, Parker, Robinson, Rutledge, Sentell, Sturm, Thompson, Williams.
BIRMINGHAM. Rayl.
Birmingham-Southern Coll. Locke, Moore, Wesson.
Howard Coll. Ballard, Hess, Peeples.
FLORENCE. Culmer, Pennington.
FT. MCCLELLAN. Esary.
HUNTSVILLE. Hart, Stark.
JACKSONVILLE. Wilson.
LIVINGSTON. Killebrew.
MARION. Thomas.
MOBILE. Larguier, Mulcrone.
MONTEVALLO. Braswell, Jackson.
MONTGOMERY. Williams.
ST. BERNARD. Capesius.
TALLADEGA. Brothers, Houston.
TUSCALOOSA. Ainsworth.
UNIVERSITY.
Univ. of Alabama. Cathey, Henriksen, Hummel, Jones, Lewis, Mancill, Palmer, Quirmbach, Seebeck, Smith, Stapp, Thurston, Ward.

ALASKA

COLLEGE. Cashen, Dolney.

ARIZONA

AJO. Starch.
FLAGSTAFF. Butchart, Lampland.
PHOENIS. Fraser, Leithold, Mitchell.
PRESCOTT. LaRoe.
TEMPE. Fouch, Wexler.
THATCHER. Olpin.
TUCSON. Wormser.
Univ. of Arizona. Cato, Chin, Denton, Duncan, Graesser, Leonard, Meyer, Purcell, Trifan, Webb.

ARKANSAS

ARKADELPHIA. Seward.
Henderson State Teachers Coll. Foster, Goldsmith, Young.
CONWAY. Lane.
FAYETTEVILLE.
Univ. of Arkansas. Adkisson, Gundlach, Richardson, Schwartz, Shniad.
LITTLE ROCK. Hodges.
MAGNOLIA. Wetzig.
MONTICELLO. Garrett.
PINE BLUFF.
A. M. & N. Coll. Clark, Kyle, Young.
PRAIRIE. Buchanan.
SEARCY.
Harding Coll. Bagnetto, Pitner, Pryor.
SILOAM SPRINGS. Porter, Macdonald.

CALIFORNIA

ALHAMBRA. Freeborn.
ANGWIN. Durham.
AUBURN. Kusick.
BAKERSFIELD. Nielsen.
BELL. Niersbach.
BERKELEY. Jones, Tabor.
Univ. of California. Bade, Bernstein, Cunningham, Evans, Fay, Flanders, Free, Hughes, Jeeves, Kuznets, Lees, McDonald, Morrey, Noble, Robbins, Robinson, Seidenberg, Sperry, Stoneham, Williams.
CAMPO. Lehman.
CARMEL. Corbin.
CHINA LAKE. Crow, Maxfield, Reid, Ward.
CLAREMONT.
Pomona Coll. Elliott, Hamilton, Jaeger, Mathews, Walton.
CULVER CITY.
Hughes Aircraft Co. Aroian, Butter, Coleman, Edwards.
DAVIS.
Univ. of California at Davis. Alder, Arnold, Baker, Burdette, Burton, Fulton, Hayes, Roessler.
DOWNEY. Fischer, Oberg.
EL CAMINO. Jones.
EL SEGUNDO. Bacon, Cressy.
FRESNO.
Fresno State Coll. Bell, Dubisch, Ewy, Morris.
FULLERTON. Reynolds.
GLENDALE. Barton, Lillie, Lindsay.
HAWTHORNE.
Northrop Aircraft. Anderson, Fairchild, Gourrich, Pierce.
HAYWARD. Gregory, Rahn.
HERMOSA BEACH. Campbell.
LA CANADA. Coble.
LAGUNA BEACH. Graham.
LA JOLLA. McEwen.
LA MESA. Smith.
LA VERNE. Herbst.
LINDEN. Pollack.
LONG BEACH. Black, Gamzon, McClellan, Randolph.
LOS ANGELES. Bankoff, Carnes, Coffin, Duncan, Ernsberger, Floris, Gass, Gibson, Goldwater, Isaacs, Johnson, Matlack, McConnell, McFarland, Perry, Porges, Savit, Sister Rose Gertrude, Smith, Solloway, Urner, Wadkins, Wall, White, Whitman, Wicker.
George Pepperdine Coll. Breglio, Campbell,

Frederick, Miller, Olson, Paris, Rex, Smith, Thomas, Torgerson, Yanari.
Immaculate Heart Coll. Beenken, Collier, Sister Mary Charles.
Inst. for Numerical Analysis. Arens, Blanch, Blumenthal, Forsythe, Huskey, Kruse, Lanczos, Lehmer, Schoenberg, Southard, Wasow.
Los Angeles City Coll. Elder, Gold, Herrera, Howell, Kaelin, Marer, Matison, Nelson, Skolnik, Thompson, Throckmorton, Trigg.
Univ. of California at Los Angeles. Beckenbach, Bell, Daus, Edmundson, Gatt, Gilbert, Glazier, Halberg, Hestenes, Hodge, Hoel, Hunt, Mason, Mittman, Paige, Peterson, Puckett, Redheffer, Seugling, Sherwood, Sokolnikoff, Sorgenfrey, Steinberg, Strutton, Swift, Taylor, Tunell, Valentine, Worthington.
Univ. of Southern California. Busemann, Hyers, Kanarik, Robb, Rudin, Smith, Snapper, Steed, White, Whiteman, Zaustinsky.
MADERA. Fuller.
MANHATTAN BEACH. Davidson.
MARYSVILLE. Miller.
MODESTO. Osner, Rolfe.
MOFFETT FIELD. Byrd, Heaslet.
MONTEREY. Mewborn.
OAKLAND. Eggett, Hullinghorst, Noble, Sister Madeleine Rose, Sumner, Sylwester.
PACOIMA. James.
PALO ALTO. Cochran.
PASADENA. Cairns, Glenn, Lay, Rock, Sydnor, Taylor, White.
California Inst. of Tech. Apostol, Bell, Birchby, Bohnenblust, Dickson, Dilworth, Erdelyi, Lagerstrom, Michal, Thomsen, VanBuskirk, Ward, Wear.
PT. MUGU. Hartranft.
REDLANDS. Bruce, Curtiss.
Univ. of Redlands. Albert, Bechtolsheim, Kimme.
REDWOOD CITY. Sturges.
RICHMOND. Fish, Good.
SACRAMENTO. Graue, Kirby, Merkel, Moredock.
ST. MARY'S COLLEGE. Dominic.
SAN BERNARDINO. Paxton.
SAN DIEGO. Goss, Klauber, Mador, Rhodes, Stovall, Uhl.
San Diego State Coll. Fong, Harris, Livingston.
SAN FRANCISCO. de Regt, Dernham, Frank, Ivanoff, McClelland, McCready, Morelli, Sussman, Taussig, Waider, Wohlford.
City Coll. of San Francisco. Bass, Eilertsen, Hanson, McKenzie.
SAN JOSE. Clarke, Falkenstern.
San Jose State Coll. Bird, Jamison, Myers, Olds.
SAN LUIS OBISPO. Cook, Pursel.
SAN MATEO. Francis, Hoffman, Walker.
SAN RAFAEL. Beckwith.
SANTA ANA. Clucas, Whiting.
SANTA BARBARA. Kelly, Rauch.
SANTA MONICA. Adams, Lehman.
Douglas Aircraft Co. Ahlin, Morison, Sedney, Struble.
Rand Corp. Beraru, Chiappinelli, Davies, Gilvarry, Hastings, Shephard, Specht, Thompson, Wendel.
SEAL BEACH. Bell.
STANFORD.
Stanford Univ. Bacon, Benson, Bradt, Herriot, Huggins, Loewner, McKinsey, Polya, Royden, Sunseri, Szego, Vitousek, Weinstock.
STOCKTON. White.
SUSANVILLE. Kerns.
WHITTIER. Pyle.

CANAL ZONE

BALBOA. McNair.

COLORADO

ALAMOSA. Jones.
BOULDER. Bartram.
Univ. of Colorado. Barrick, Briggs, Britton, Buschman, Campbell, Cheney, DeVol, Gunning, Hanna, Holubar, Hutchinson, Jones, Kempner, Kendall, Kros, Marsh, McCrossen, McKenzie, Nelson, Powell, Rutland, Snively, Sperline, Stahl, Stockman, Tovani, Varner, Wagner, Walters.
COLORADO SPRINGS. Leavens, Ross.
Colorado Coll. Hansman, Leech, Sisam.
DENVER. Bailey, Charlesworth, Doremus, Gorsline, Hoffman, Howerton, Howie, Hurry, McIntosh, Ries.
Univ. of Denver. Carmichael, Garland, Gorrell, Gysland, Lewis, Noble, Peterson, Recht.
FT. COLLINS.
Colorado A. & M. Coll. Clark, Guard, Hayward, Madison, Staley.
GOLDEN.
Colorado School of Mines. Carpenter, Cook, Everett, Gutzman, Hebel.
GREELEY. Fisch, Patterson.
GUNNISON. Celauro, Cook.
HESPERUS. Reid.
LORETTO. Cook.
PUEBLO. Given, Vanderburgh.

CONNECTICUT

BRIDGEPORT. Ledford.
CANAAN. Lambert.
FAIRFIELD. Murray.
GUILFORD. Brown.
HARTFORD. Bronstein, Elston, Friedman, Hyde, Keffer, Loring, Pease, Wyckoff.
Trinity Coll. Dadourian, Dorwart, Grace, Klimczak, Nilson, Stewart, Werdelin.
MIDDLETOWN.
Wesleyan Univ. Arnold, Camp, Day, Howland.
MILFORD. Rosenbaum.
NEW BRITAIN. Ferry.
Teachers Coll. of Connecticut. Fuller, Spooner, Weeber.
NEW HAVEN. Albert, Seely.

Yale Univ. Ballou, Begle, Bernard, Burrows, Dube, Dunford, Hedlund, Hille, Klopp, Kovarik, Longley, Miles, Mills, Ore, Tracey, Uhler.
NEW LONDON. Bower, Ferguson, Spong, von Schelling.
PUTNAM. Sister Irene.
ROWAYTON. Lanckton.
STORRS.
Univ. of Connecticut. Bourne, Cheney, Eisenman, Landau, Sedgewick.
WATERTOWN. Gillette.
WEST HAVEN. Anderton.

DELAWARE

KINGS COLLEGE. Ries.
NEWARK.
Univ. of Delaware. Jones, Kaskey, Kerr, Lewis, McDougle, Rees, Webber.
WILMINGTON. Dunleavy, Thigpen.

DISTRICT OF COLUMBIA

WASHINGTON. Ahrendt, Berry, Braun, Certaine, Clark, Claytor, Cromwell, A. F. Davis, Ruth M. Davis, Draim, Federico, Fisher, Friedman, Goldberg, Greenwood, Greville, Hammar, Harman, Hooke, Itken, Kaplan, Kennedy, Kupperman, Lennahan, D. B. Lloyd, Olwen Lloyd, Lukacs, McCamman, Mertie, Milam, Miller, Miser, Moulton, Nelson, Nemenyi, Petrie, Pond, Ponds, Rasor, Rupp, Schell, Schult, Sendrow, Shapiro, Shenton, Sister Gabrielle Marie, Theilheimer, Thornton, Tyler, Walton, Wetrogan, Winston, Young.
Armed Forces Security Agency. Blum, Dresser, Dribin, Gleason, Murray, Schlauch, Wray.
Army Map Service. Carville, Chovitz, Going, Klafter, Millman, Rouleau.
Bureau of Census. Daly, Lucas, Spencer, Weiss.
Catholic Univ. of America. Dwyer, Feeney, Finan, Landry, Manogue, Moller, Mortell, Nesbeda, Ramler, Rice.
Coast & Geodetic Survey. Darling, Duerksen, Orlin, Schmid, Sollins, Stone, Thomas.
Dept. of the Army. Blanche, Getchell, Nowlan.
Dept. of Defense. Carlan, Coffin, Goepper, Honig, Terry.
Georgetown Univ. Oliphant, Smith, Sohon.
George Washington Univ. Johnston, Kullback, Marlow, Marsh, McCall, Mears, Taylor, Weida, Wolf.
National Bureau of Standards. Brooks, Cameron, Curtiss, Durfee, Hoffman, Lieblein, Newman, Park, Todd, Youden.
Naval Observatory. Edelson, Hertz, Watts, Woolard.
Naval Research Lab. Anderson, Doyle, Ericksen, Grad, Montroll, Rees, Shepherd, Solomon.
Navy Dept. Boron, Burington, Campaigne, Lancaster, Penney, Smith, Wrench.
Trinity Coll. Burkart, Sister Catherine Marie, Varnhorn.

FLORIDA

BRADENTON. Kretschmar.
CORAL GABLES.
Univ. of Miami. Foulis, Logsdon, MacNeish, Meyer, Rosenbaum, Swingle.
DAYTONA BEACH. Perkins.
DE LAND. Ashcraft, Rumble.
DELRAY BEACH. Hoffman, LeStourgeon.
FT. LAUDERDALE. Gregg.
GAINESVILLE.
Univ. of Florida. Blake, Bridgland, Conkling, Cowan, Dostal, Ellis, Findley, Gager, Gormsen, Hadlock, Hutcherson, Kokomoor, Lang, Lewis, Mason, McCarty, McInnis, Meyer, Morales, Morelock, Morrow, Parker, Patterson, Phipps, Pirenian, Ricupito, Rohde, Simpson, Smith, Thoro, Wilson, Young.
JACKSONVILLE. Cobb, Horton, Wallace.
JACKSONVILLE BEACH. Armstrong.
LAKELAND. Reinsch, Terry.
MADISON. Thornton.
MARIANNA. Cowan.
MIAMI. Sister Marie Grace, Smulin.
ORLANDO. Feltges, Ramsdell.
PANAMA CITY. Coulter.
ST. PETERSBURG. Burt, Cantwell, Scott, Story.
TALLAHASSEE. Clarke, Eide, Tinner.
Florida State Univ. Barnett, Brown, Goodner, Heath, Jamison, Larson, Smith, Taylor, Wade.
TAMPA.
Univ. of Tampa. Bolser, Kasriel, Rhodes, Sheppard.
WINTER PARK. Sewell.
Rollins Coll. Jones, Saute, Smythe.

GEORGIA

ATHENS. Hadnot.
Univ. of Georgia. Barrow, Beckwith, Bercos, Butz, Cohen, Conwell, Fort, Hill, Hoke, Huff, Levit, Newton, Stanley, Stephens, Walker, Wall, Wollan, Youse.
ATLANTA. Brown, Gokhale, Hobbs, Howe, Pearson, Steinberg.
Georgia Inst. of Tech. Bailey, Christian, Currie, Field, Fulmer, Garrett, Holton, Hook, Martin, Neisius, Perlin, Smith, Starrett, Wahab.
AUGUSTA. Franz, Robertson, Taylor.
CAMP GORDON. Gould, Long.
COLLEGEBORO. Moye.
DAHLONEGA. Barnes, Rogers, Wicht.
DECATUR.
Agnes Scott Coll. Futral, Gaylord, Robinson.
EMORY UNIVERSITY.
Emory Univ. Briggs, Chasen, Clark, Latimer, Messick, Partington.
FORT VALLEY. Pitts.
GAINESVILLE. Edwards.
KENSINGTON. Carlock.
LA GRANGE. Bailey.
MACON. Carlton, Plymale.
MILLEDGEVILLE. Nelson.
ROME. Hightower.

SAVANNAH. Blakeley.
VALDOSTA. Babcock, Moore.
WALESKA. Vann.
WARNER ROBINS. Burrell, Knight.

IDAHO

BOISE. Buck.
CALDWELL. Rankin.
MOSCOW. Botsford, Wright.
NAMPA. Tillotson.
POCATELLO. Olive, Scobert.

ILLINOIS

AURORA. Miksa, Tapper.
BELLEVILLE. Wilson.
BERWYN. Cherry.
BLOOMINGTON. Hunt, Kinney.
BLUE ISLAND. Snider.
CARBONDALE.
Southern Illinois Univ. Berberian, Black, Hall, Mark, McDaniel, Rodabaugh, Wright.
CARLINVILLE. Hatfield.
CARTHAGE. Boatman.
CHARLESTON.
Eastern Illinois State Coll. Hendrix, Ringenberg, Taylor, Van Deventer.
CHICAGO. Anthony, Arenson, Bird, Blake, Boardman, Butler, Campbell, Christman, Dunning, Ettinger, Gerst, Greer, Herlihy, Hicks, Hilton, Liolios, London, Mansfield, Merrill, Nauman, Nicolet, Plait, Poppen, Sachs, Schweitzer, Sister Mary Charlotte, Sister Mary Esther, Sister Mary Ferrer, Sister Mary Sylvester, Smith, Stewart, Swanson, Velek, Werkman, Wessely.
DePaul Univ. Caton, D'Arco, DeCicco, Fischer, Kawata, Merkes, Newman, Pachucki, Saastad, Svoboda.
Illinois Inst. of Tech. Ballard, Berman, Bibb, Boblak, Bott, Brown, Comfort, DeLany, Frank, Friedlen, Gertz, Goodman, Heyman, Horwitz, Kennedy, Krathwohl, Levin, McDowell, Menger, Mihalek, Miller, Pall, Peterhans, Pollak, Rapp, Reingold, Rosenberg, Sadowsky, Soglin, Sutter, Wilcox.
Roosevelt Coll. Gore, Johnson, Silber, Street.
Univ. of Chicago. Albert, Allen, Barnard, Bartky, Chern, Dinkines, Everett, Gaffney, Gottlieb, Graves, Guy, Hartung, Herstein, Hill, Jones, Kaplansky, Kruskal, Lane, MacLane, Macy, Meyer, Northrop, R. H. Oehmke, Theresa C. Oehmke, Opatowski, Oppenheim, Putnam, Rapoport, Reisel, Schilling, Serbyn, Sklar, Stone, Strauss, Weiner, Zygmund.
Univ. of Illinois. Alberti, Bailey, Berglund, Corliss, Davis, Feinstein, Frank, Grundman, Hartley, Hornacek, Lariviere, Nolan, Nowlan, Olsen, Ondrak, Scholomiti, Schwartz, Sears, Stelling, Turner, Wilson.
Wilson Jr. Coll. Burack, James, Kinney, Lange, Manheim, Patin, Rasmusen, Siedband.
Wright Jr. Coll. Buelow, Eulenberg, Georges, Kurzin, Moran.
CRYSTAL LAKE. Weslowski.
DANVILLE. Reiner.
DECATUR.
James Millikin Univ. Brown, Falvey, Kiefer, Ploenges.
DEKALB.
Northern Illinois State Teachers Coll. Anderson, Hellmich, Pickard, Stelford.
ELGIN, Burton, Peters.
ELMHURST. Baumgart, Cody.
ELSAH. Hooper.
EVANSTON. Bradfield, Christian, Crum, Pancoe.
Northwestern Univ. Barrer, Boas, Buell, Freeland, Hildebrandt, Hsiung, Ishihara, Karns, Kingsley, Kliphardt, Littlejohn, Moulton, Reid, Scott, Simmons, Wescott, Wheeler, Wolfe.
FREEPORT. Baumgartner.
GALESBURG.
Knox Coll. Lindstrum, Smyth, Stephens.
GRIGGSVILLE. Carmichael.
JACKSONVILLE. Hallerberg, Miller, Schumaker.
JOLIET. Dickson, Sister M. Claudia.
KANKAKEE. Gardner, Rice.
LAKE FOREST. Curtis.
LA SALLE. Muller.
LEBANON.
McKendree Coll. Nellie P. Miser, W. L. Miser, Stowell.
LEMONT. Teed.
LIBERTYVILLE. Hart.
LINCOLN. Balof.
MACOMB. Ayre, Schreiber.
MANITO. Van Orstrand.
MARISSA. Mathews.
MAYWOOD. Hildebrandt.
MONMOUTH. Beveridge, Cramer.
NAPERVILLE. Seybold.
NORMAL.
Illinois State Normal Univ. Atkin, Bey, Brown, Flagg, Flynn, McCormick, Mills, Norskog, Rine, Ullsvik.
PEORIA. Boose, Martens.
Bradley Univ. Gault, Helms, McGaughey, Moore.
QUINCY. Dines, Velesz.
RANTOUL.
Chanute Air Force Base. Clark, Neilson, Schwartz.
RIVER FOREST. Dobbin.
ROCKFORD. Oldenburger, Presnell, Varnum.
ROCK ISLAND.
Augustana Coll. Cederberg, Engstrom, Jensen, Nelson, Olmsted.
SANDWICH. Rumney.
SCOTT AIR FORCE BASE. Brown.
SPRINGFIELD. Trennt.
ST. LOUIS. Felling.
URBANA.
Univ. of Illinois. Armstrong, Atchison, Barton, Bateman, Beberman, Blyth, Bostick, Brannon, Cairns, Chanler, Clutterham, Fort, Fox, Fry, Gregory, Hattan, Hoersch,

Hohn, Hundertmark, Ketchum, Koken, Landin, Levy, Marquardt, Meserve, Miles, Mitchell, Moore, Peters, Pingry, Priest, Reiner, Schweppe, Scott, Snader, Taub, Tross, Vaughan, Walkley.
WHEATON. Boyce, Brandt.
WINNETKA. Humphrey.

INDIANA

BLOOMINGTON. Wrede.
Indiana Univ. Gustin, Haslam, Merrilees, Nickel, Peak, Porter, Thomas, Truesdell, Williams, Wolfe, Zorn.
COLLEGEVILLE. Zanolar.
CRAWFORDSVILLE.
Wabash Coll. Carscallen, Hughes, Mielke, Polley.
DANVILLE. Shartle.
EARLHAM. Long.
EAST CHICAGO. Burns.
ELKHART. Nicholls.
EVANSVILLE. Kronsbein.
FAIRMOUNT. Berg.
FT. WAYNE. Carr, Olson, Virts.
GARY. Leskow, Oursler.
GOSHEN. Hartzler, Zimmerman.
GREENCASTLE.
De Pauw Univ. Arnold, Edington, Greenleaf, Talkington.
HANOVER. Heren, Yarnelle.
INDIANAPOLIS. Chambers, Fuller, Hadley, McColgin, Sidebottom, Sister Gertrude Marie, Suter, Welchons.
Butler Univ. Beal, Crull, Gambill, Speas, Uhrhan, Vaughn, Wayne.
Naval Ordnance Plant. Dowds, Ekstrom, Hafner, Heyda.
LAFAYETTE.
Purdue Univ. Ayres, Black, Bolks, Bossler, Burr, Cheatham, Crain, Cuthill, De Jonge, Fry, Glacken, Golomb, Graves, Hazard, Hughes, Hull, Keller, Klinger, McCune, Miller, Neff, Newburg, Overman, Oxley, Robbins, Rosenthal, Shanks, Smith, Stone, Walker, Webster, Wood.
LOWELL. Newson.
MICHIGAN CITY. Copp.
MUNCIE. Brumfiel, Edwards, Grekila, Shively.
NORTH MANCHESTER. Dotterer.
NOTRE DAME. Sister Mariam Patrick.
Univ. of Notre Dame. Bell, Caparo, De Baggis, Fan, Ginivan, Grainger, Griesmer, Lee, Lindemann, McCarthy, Montgomery, Nastucoff, Otter, Peach, Ross, Sullivan, Wagner, Wainwright.
PAOLI. Bentley.
REYNOLDS. Erwin.
ST. MEINRAD. Knaebel.
SOUTH BEND. Higgins.
TERRE HAUTE. Martin, Shriner, Strong.
Rose Poly. Inst. Palmer, Ross, Sousley.
UPLAND. Draper.
VEVAY. Parrish.
WEST BADEN SPRINGS. Hausmann.

IOWA

AMES. McKelvey.
Iowa State Coll. Anderson, Bancroft, Bicknell, Block, Brandner, Daniells, Davis, Dickey, Feyerherm, Gouwens, Herr, Hinrichsen, Holl, Kaldenberg, Kirkham, Kreider, Lambert, Lieberknecht, Lindahl, McKelvey, Reeves, Robertson, Robinson, Smith, Thielman, Vinograde.
CEDAR FALLS. Kearney.
Iowa State Teachers Coll. Brune, Lankton, Lott, Schurrer, Trimble, Van Engen.
CEDAR RAPIDS. Bowersox.
CHEROKEE. Laposky.
DAVENPORT.
St. Ambrose Coll. Hratz, Weeg, Younkin.
DECORAH. Jacobsen.
DES MOINES.
Drake Univ. Canfield, Gillam, Harper, Li, Neff, Zubay.
DUBUQUE. Ernsdorff, Rothlisberger, Sister Mary Michail.
EPWORTH. Earhart.
FARNHAMVILLE. Qualley.
FAYETTE. Deming.
GRINNELL. McClenon, Rusk.
IOWA CITY. Price.
State Univ. of Iowa. Blair, Chittenden, Conkwright, Cosby, Craig, Hogg, Jones, Knowler, Lindstrom, Muhly, Nemmers, Sewell, Smiley, Woods, Wylie.
LAMONI. Jacobson.
LEMARS. Dipert.
MT. PLEASANT. Stein.
MT. VERNON. Davis, Moots.
SIOUX CITY. Bushyager, Sister M. DePazzi.
STORM LAKE. Roorda.
WAUKON. Hancock.
WAVERLY. Hoffman, Van Horn.

KANSAS

ATCHISON. Pretz, Sister Jeanette, Sister M. Helen, Walsh.
ATWOOD. Nyhoff.
BALDWIN CITY.
Baker Univ. Foreman, Garrett, Hester.
EMPORIA.
Kansas State Teachers Coll. Albert, Laird, Peterson, Tucker.
HAYS. Grabbe, Stopher, Toalson.
HESSTON. Driver.
HUTCHINSON. Chatlain.
LAWRENCE. Lake.
Univ. of Kansas. Babcock, Bell, Black, Brown, Chowla, Diamond, Dodge, Dougherty, Forman, Hsu, Jordan, Kruse, Odland, Pedrick, Pihlblad, Price, Rasmussen, Riley, Schatten, Scott, Smith, Stouffer, Ulmer.
LINDSBORG. Marm.
MANHATTAN.
Kansas State Coll. Arnold, Babcock, Faulkner, Firl, Franck, Furman, Hyde, Janes, Larney, Lewis, Mossman, Parker, Sanger, Stratton, White, Young.

NORTH NEWTON. Richert, Wedel.
OTTAWA. Bemmels.
PITTSBURG.
Kansas State Teachers Coll. Curfman, German, Green, Keegan, Kriegsman, Pike, Smith.
SALINA. Sister M. Nicholas.
STERLING. Thompson.
TOPEKA. Messick.
Washburn Univ. Eberhart, Greene, Martinson, Rolfs.
WAKEENEY. Gibson.
WICHITA. Houser, Longenecker.
Friends Univ. Means, Reagan, Swanson.
Univ. of Wichita. Hoare, Morlan, Nibarger, Read, Reagan, Wedel, Wrestler.
WINFIELD. Kruger.
XAVIER. Sister Ann Elizabeth.

KENTUCKY

BARBOURVILLE. Peters.
BEREA.
Berea Coll. Pugsley, Roberts, Wright.
BOWLING GREEN. Yarbrough.
CAMPBELLSVILLE. Graham.
DANVILLE. Robinson.
FT. KNOX. Schocken.
FRANKFORT. Ross.
GEORGETOWN. Cook, Hatfield.
LEXINGTON. Wright.
Univ. of Kentucky. Beck, Boyd, M. C. Brown, R. C. Brown, Campbell, Cooper, Cowling, Downing, Faith, Fields, Foster, Goodman, Hagen, Hays, Leser, Pence, Ripy, Royster, South, Tapscott, Ward, Zaring.
LONDON. Pettus.
LOUISVILLE. Berry, Bullitt, Ford, Schaeffer.
Univ. of Louisville. Moore, Musch, Sauter, Simester, Stevenson, Wilson.
MURRAY. Carman, Holmes.
OWENSBORO. Howard, Sister M. Laurine.
RICHMOND. Sandlin.
Eastern Kentucky State Coll. Helton, McGlasson, Park.
WILLIAMSBURG.
Cumberland Coll. Boswell, Compton, Vallandingham.

LOUISIANA

BATON ROUGE.
Louisiana State Univ. Freas, Karnes, Nichols, O'Quinn, Rees, Rickey, Rutt, Sanders, Townsend, Trammell.
HAMMOND.
Southeastern Louisiana Coll. Davis, McClimans, Tucker.
LAFAYETTE.
Southwestern Louisiana Inst. Buchanan, Dossey, Hoag, Loflin, Ohmer, Wilbanks.
LAKE CHARLES. Ford, Franciol.
MONROE, Kennedy.
NATCHITOCHES.
Northwestern State Coll. Church, Corley, Killen, Maddox, Shelton, Winegeart.
NEW ORLEANS. Haller, Mooney, Preble, Sister Miriam Francis, Stevens, Tullier.
Newcomb Coll. Beard, Many, Spencer, Weiss.
Tulane Univ. Baus, Duren, Griffin, Hersh. Horne, Huff, Kelley, Koch, Morrison, Pettis, Riess, Thomson, Timon, Wallace, Ward.
PINEVILLE. Donohoe, Temple.
RUSTON.
Louisiana Poly. Inst. Garrison, Schroeder, Smith, Temple.
SHREVEPORT. Hardin, McKnight, Palmer.
THIBODAUX. Davis.

MAINE

BRUNSWICK.
Bowdoin Coll. Christie, Hammond, Holmes, Korgen.
LEWISTON. Ramsdell, Wilkins.
ORONO.
Univ. of Maine. Comegys, Daly, Kimball.
WATERVILLE. Ashcraft, Combellack.

MARYLAND

ABERDEEN PROVING GROUND. Cumming, Davis, Dederick, Golub, Hart, Leutert, Lotkin, Murphy, Reitwiesner, Robinson, Rosenfeld, Squires, Sterrett, Taylor, Treuenfels, Walters, Young.
ANNAPOLIS. Bingley.
U. S. Naval Acad. Abbott, Bailey, Ball, Benac, Buikstra, Chambers, Clements, Currier, Gorman, Gras, Hammond, Kells, Kinsolving, Lamb, Lyle, Mann, Mayer, McLaughlin, Milkman, Milos, Moore, Niles, Paydon, Pejsa, Popow, Rector, Robinson, Scarborough, Stilwell, Stotz, Swafford, Thomas, Tierney, Tyler, White.
U. S. Naval Postgrad. School. Bleick, Campbell, Church, Faulkner, Jennings, Lockhart, Pulliam, Rawlins, Root, Torrance.
BALTIMORE. Blakiston, Cook, Eagle, Edwards, Foley, From, Huck, Johnson, Marrian, Morrel, Roman, Schoderbek, Sister Constantia, Sister Mary Cordia, Snyder, Stephens.
Goucher Coll. Lewis, Torrey, Whitney.
Johns Hopkins Univ. Carstoiu, Haviland, Lewis, Light, Morrill, Reed, Silva, Smith, Templeton, Zilber.
BETHESDA. Cheston, Wagner.
BRENTWOOD. Meade.
CHEVY CHASE. Cramer, Maple, Norden.
CHEWSVILLE. Waltz.
COLLEGE PARK.
Univ. of Maryland. Good, Greenspan, Hall, Jackson, Martin, Melvin, Rogers, Wright.
EMMITSBURG. Burke.
FREDERICK. Brown, Maloney.
FROSTBURG. Rissler.
GLYNDON. Clifford.
HAVRE DE GRACE. Reklis.
MIDDLE RIVER. Kravitz.
ST. MARGARETS. Kinsman.
SALISBURY. Cauffman.
SILVER SPRING. Cohen.
Johns Hopkins Univ. Brigham, Mitchell, Neumann, Rand, Sigley, Whitman.
Naval Ordnance Lab. Christian, Layton, Schultz, Schweizer.

SUITLAND. Morris.
TAKOMA PARK. Anderson.
WESTMINSTER. Spicer.
WOODSTOCK. Hennessey.

MASSACHUSETTS

AMHERST. Boutelle, Breusch, Brown, Miller.
Univ. of Massachusetts. Guhse, Halpern, Kissin, Rose, Wagner.
ANDOVER. Cobb.
BELMONT. Walters.
BOSTON. Betts, Downey, Gillman, Gould, Hall, Hemenway, Hoskins, Hubbard, Hueston, Hyder, Kaufman, Miller, Sister Laurentine Marie, Weaver.
Boston Univ. Alman, Johanson, McLaughlin, Mode, Ross, Swaffield, Syer.
Northeastern Univ. Brown, Cook, Spear, Wallace.
BROOKLINE. McCarthy, Stempnitzky.
CAMBRIDGE. Bothwell, Carpenter, Kingsbury.
Harvard Univ. Ahlfors, Beatley, Birkhoff, Brown, Coolidge, Emmons, Hansen, Huntington, Kneale, Kravetz, Mosteller, Muller, Neustadter, Newman, Rivlin, Rulon, Walsh, Widder, Wilson, Zariski.
Massachusetts Inst. of Tech. Baily, Cioffi, Douglass, Franklin, Goldstein, Harvey, Kehl, Kimber, Moon, Morse, Reich, Reissner, Ross, Salem, Thomas, Zeldin.
CHESTNUT HILL.
Boston Coll. Eiardi, Marcou, Murphy, O'Donnell.
CHICOPEE. Sister Teresa Marie.
CONCORD. Parke.
FITCHBURG. Bissinger, Haskins.
GROTON. Nash.
LAWRENCE. McIntosh.
LYNN. Taylor.
MEDFORD.
Tufts Coll. Clarkson, Lotz, Mergendahl, Ransom.
MILFORD. Dennison.
NEW BEDFORD. Robinson.
NEWTON CENTER. Solomont, Walsh.
NEWTON HIGHLANDS. Vincent.
NORTHAMPTON.
Smith Coll. Johnson, McCoy, O'Neill, Rambo.
NORTON.
Wheaton Coll. Cohen, Garabedian, Watt.
PITTSFIELD. Washburne.
ROXBURY. Berkofsky.
SPRINGFIELD. Breynaert.
SOUTHBRIDGE. Boeder.
SOUTH HADLEY.
Mount Holyoke Coll. Bates, Kiokemeister, Litzinger.
TYNGSBORO. Richmond.
WALPOLE. Adams.
WATERTOWN. Leonard.
WELLESLEY.
Wellesley Coll. Ayer, Russell, Stark, Young.
WESTON. O'Shea.
Regis Coll. Finigan, Looney, Sister Leonarda, White.
WILLIAMSTOWN.
Williams Coll. Agard, Jordan, Klein, Wells.
WORCESTER.
Clark Univ. Bumer, Melville, Stubbe.
Coll. of the Holy Cross. Burns, McBrien, McGillicuddy.
Worcester Poly. Inst. Brown, Cobb, MacCullough, Morley, Rice.

MICHIGAN

ADRIAN. Alexander, Sister M. Irma.
ALBION.
Albion Coll. Cox, Gillespie, Ingalls, Larsen.
ANN ARBOR. Rothe.
Univ. of Michigan. Al-Ghita, Anning, Bartels, Bradshaw, G. U. Brauer, Richard Brauer, Brown, Churchill, Coburn, Coe, Coleman, Copeland, Core, Craig, Crisler, Crispin, Darling, Deal, Dwyer, Fischer, Goode, Harary, Harris, Hay, Heater, Hildebrandt, Hind, Hopkins, Jones, Kaplan, Karpinski, Lee, Leisenring, LeVeque, Loh, Lohwater, Losey, Love, McLaughlin, Myers, Nyswander, Piranian, Rainich, Rainville, Rauch, Reade, Riddle, Rippe, Rosenberg, Roth, Rothe, Rouse, Running, Samelson, Stengle, Thrall, Tornheim, Tysver, Ullman, Wilder.
BATTLE CREEK. Wilson.
BAY CITY. Ewing, Saile.
BERRIEN SPRINGS. Specht.
CENTERLINE. Fishman, Yedlin.
DEARBORN. Edmonson, Maguire, Spencer.
DETROIT. Bagby, Grossman, Johnson, Nace, Poston, Reiber, Sister M. Leona, Sister Mary Paula, Trotzuk.
Bendix Aviation Corp. Banhagel, Kral, Patterson.
Univ. of Detroit. Campbell, Duggar, Eckstein, Hannan, Holzhauer, Johnston, Lancaster, MacNeil, Magda, Mansour, Markle, McCarthy, Mehlenbacher, Sartor, Sherwood, Smith, Sowul, Steinbach, Teodoro, Weiss, Weitzenhoffer.
Wayne Univ. Allen, Baldwin, Borgman, Brown, Chu, Coral, Epstein, Folley, Harrison, Loweke, Mandelbaum, Minas, Morrow, Nelson, Pixley, Scibiorski.
EAST LANSING.
Michigan State Coll. Barbour, Baten, Bell, Carr, Coy, Frame, Grove, Herzog, Hill, Kelly, Lapidus, Lowell, Mesner, Nordhaus, Parker, Payne, Plant, Powell, Sander, Stelson, Stewart, Sudborough, Sweetland, Wells.
FLINT. Curtis.
General Motors Inst. DeMoss, Grotts, Raker, Schaefer.
GRAND RAPIDS. Bellardo, Vandort.
GROSSE POINTE PARK. Tegels.
HILLSDALE. Mattson.
HOLLAND. Lampen.
HOUGHTON.
Michigan Coll. of Mining & Tech. Boggs, Park, Stipe, Vichich.
IRONWOOD. Field.

KALAMAZOO. Walton.
Western Michigan Coll. Bartoo, Beeler, Blair, Butler, Everett, Ford, Hannon.
LANSING. Bartels.
MARQUETTE. Boynton.
MILFORD. McNeal.
MT. PLEASANT.
Central Michigan Coll. Bye, Foust, Piechan, Pratt, Richtmeyer.
PLYMOUTH. Sister M. Virgilia.
SAULT STE. MARIE. Otis.
TRENARY. McClintock.
WILLOW RUN VILLAGE. Deniston.
YPSILANTI.
Michigan State Normal Coll. Erikson, Goings, Lindquist, Pate.
Univ. of Michigan. Bartman, Brown, Graney, Ritter.

MINNESOTA

BEMIDJI. Colson.
COLLEGEVILLE. Danzl, Kalinowski.
DULUTH. Cothran, McEwen, Sister M. Mercedes, Sister M. Prudentia.
FARIBAULT. Bailey.
MANKATO. Fleming, Horeni.
MINNEAPOLIS. Bonnell, Tani.
Univ. of Minnesota. Bearman, Brink, Brooke, Bussey, Cameron, Carlson, Doeringsfeld, Eggers, Fischer, Fulks, Gibbens, Hafstrom, Hall, Hart, Hartig, Hartman, Hatfield, Ito, D. A. Johnson, G. P. Johnson, Johnston, Kalisch, Kenner, Kirchner, Kirmser, Koehler, Laws, Loud, McCutcheon, Munro, Ohnsorg, Olmsted, Priester, Quaid Rosenbloom, Shumway, Stoner, Thorp, Turrittin, Wang, Warschawski, Wells, Williams, Zeitlin.
MOORHEAD. Mundhjeld.
NORTHFIELD. Carlson, Stanaitis.
Carleton Coll. Francis, Johnson, May, Wegner.
ONAMIA. John.
ROCHESTER. Dubbert.
ST. CLOUD. Berg.
ST. JOSEPH. Sister M. Joanne.
ST. PAUL. Beck, Berger, Blakely, Bracewell, Hill, Morgan, Sister Mary Seraphim, Thornton.
Coll. of St. Thomas. Bush, Godderz, Montgomery, Norris, Reuber, Sheridan, Smith, Taylor, Terami.
Macalester Coll. Anderson, Camp, Jaeger, Koenen.
ST. PETER.
Gustavus Adolphus Coll. Anderson, Kaufmanis, Swanson.
VIRGINIA. Henning.
WINONA. De La Salle, Lokensgard, Schrader, Sister Mary Leontius.

MISSISSIPPI

BLUE MOUNTAIN. Gillis.
CLEVELAND. Walters.
COLUMBUS. Erickson.
GULFPORT. Brooks.
HATTIESBURG.
Mississippi Southern Coll. Felder, Foote, Jones, Van Hook.
JACKSON. Babbitt, Ethridge, Knox, Reynolds.
MERIDIAN. Cleveland.
POPLARVILLE. Gilmore.
STATE COLLEGE.
Mississippi State Coll. Goen, Grimes, Hopkins, Kanter, Murray, Ollivier, Temple.
UNIVERSITY.
Univ. of Mississippi. Bickerstaff, Miller, Mitchell, Samuels, Spragens.
WINONA. Witty.

MISSOURI

CANTON. Blue.
CAPE GIRARDEAU. Michel, Owens.
COLUMBIA. Cosby, Kitches, Litterick.
Univ. of Missouri. Burcham, Cummings, Gaddum, Golub, Haynes, Hoagland, Lloyd, Lowery, Sandy, Sawyer, Stamey, Taam, Utz, Zemmer.
FAYETTE.
Central Coll. Barrow, Blattner, Denny, Helton.
FLAT RIVER. Galloway.
FULTON. Lacy.
JEFFERSON CITY. Bridger.
KANSAS CITY. C. B. Baxter, Mary M. Baxter, Doyle, Hannan, Pierson, Rosen, Sister M. Pachomia.
KIRKSVILLE. Jamison.
LIBERTY. Jones.
MARYVILLE. Lafferty.
ROLLA.
Missouri School of Mines. Erkiletian, Evans, Goodhue, Johnson, Lee, Pagano, Rankin, Woodle.
ST. CHARLES. Beasley, Karr.
ST. LOUIS. Gove, Proctor, Sister Mary Teresine, Van Schaack.
Harris Teachers Coll. Marth, McLean, Moore, Willerding.
St. Louis Univ. Ammann, Andrews, Bold, Collins, Golomski, Lanzano, Regan.
Washington Univ. Beesack, Haimo, Hirschman, Leighton, Lowe, Middlemiss, Nehari, Thron.
SPRINGFIELD. Fronabarger, Graves, H'Doubler, Owchar.
UNIVERSITY CITY. Feldman, Stephens.
WARRENSBURG. Akers, Brown.

MONTANA

BOZEMAN.
Montana State Coll. Antosiewicz, Hurst, Livers, Lowney, McMurdo, Uhrich.
BUTTE. Smith.
GARRISON, Canning.
HELENA. Topel.
MISSOULA.
Montana State Univ. Chatland, Jameson, Merrill, Ostrom.
NEIHART. Coffey.

NEBRASKA

CHADRON. Berry.
CRETE. Johnson.
GOTHENBURG. Janssen.
HASTINGS. Lowry.
KEARNEY. Larsen, Nelson.
LINCOLN. Blank, Gaba, Gass, Lenser, Ogden, Perisho.
Univ. of Nebraska. Armstrong, Basoco, Brenke, Buck, Camp, Clarke, Cox, Halfar, Heath, Leavitt, Mendelson, Moomaw, Pool, Ribeiro, Runge.
OMAHA. Becker, Gayton.
Univ. of Omaha. Doss, Earl, Rice.
Creighton Univ. Bettinger, Clarkson, Dansky.
WAYNE. Boyce.
YORK. Feemster.

NEVADA

RENO.
Univ. of Nevada. Beesley, Little, Wood.

NEW HAMPSHIRE

DURHAM. Peterson.
EXETER. Pennell.
Phillips Exeter Acad. Adkins, Funkhouser, Lynch.
HANOVER. Morgan.
Dartmouth Coll. Brown, Forsyth, Fraser, Mathewson, Perkins, Robinson, Silverman.
KEENE. Goodrich, Peters.
MANCHESTER. Lankalis, O'Leary.
PLYMOUTH. Smith.

NEW JERSEY

BAYONNE. Koren, Quinn.
BELMAR. Borsuk, Kassler.
BLOOMFIELD. Neuwirth, Oergel.
BOUND BROOK. Creely.
CALDWELL. Sister M. Anita.
CAMDEN. Vlachos.
CARLSTADT. Norman.
CLIFTON. Struyk.
CONVENT STATION. Sister Esther Maria.
EAST ORANGE. LePori, Lintvedt, Nordgaard, Seebald.
FT. MONMOUTH. Blasch, Norwood, Orr, Spears. Winn.
HIGHSTOWN. Harrison.
HILLSIDE. Koellner, Mandelbaum.
HOBOKEN. Murray.
Stevens Inst. of Tech. Lucas, Morrison, Reeks, Rose.
JERSEY CITY. Ayres, Levitt, Marks, Reckzeh.
St. Peter's Coll. Kruse, Mortola. Zegers.
LAKEWOOD. Wallick.
LAWRENCEVILLE. Kiernan, Kimball.
MADISON. Battin.
MAPLEWOOD. Hazeltine.
MURRAY HILL.
Bell Telephone Labs. Gray, Hamming, Lewis, Raisbeck, Schelkunoff, Shewhart.
NEWARK. Carberry, Cherlin, Hellman, Mosesson, Strock.
Newark Coll. of Engg. Jaffe, Mainardi, Molina, Vedova, Wasson.
NEW BRUNSWICK.
Rutgers Univ. Barlaz, Bender, Biser, Brown, Bunyan, Campopiano, Clark, Cohn, Connors, Eisenberg, Feldman, Firestone, Gabriel, Galbraith, Garfunkel, Grant, Hazard, Klein, Makarov, Meder, Nelson, Ott, Payton, Phelps, Starke, Vilkelis, Walter, Zimmerberg.
ORANGE. Sayles.
PATERSON. Daugherty.
PRINCETON. Meier, Mettler, Nickerson, Osborn.
Inst. for Advanced Study. Alexander, Anderson, Cohen, Goldstine, Green, Grosswald, Hu, Kelly, Klee, Martin, Montgomery, Morse, Protter, Veblen, von Neumann.
Princeton Univ. Artin, Blackett, Feller, Gilbert, Howard, Karlin, Kruskal, Langenhop, Lefschetz, Lord, Lyndon, Mattuck, Semple, Tucker, Tukey, Wilks, Wong.
RAMSEY. Stuckey.
RED BANK. Hummel.
SECAUCUS. Wirsching.
SOUTH ORANGE. Davis, Dorsey, Stanwick.
TEANECK. Rayher.
TRENTON. Shuster.
UPPER MONTCLAIR. Campbell.
New Jersey State Teachers Coll. Clifford, Davis, Humphreys, Kays, Mallory, Nichols, Sensale.
WEST POINT PLEASANT. McMurtrie.
WEST ORANGE. Edison.
WOODSTOWN. Darling.

NEW MEXICO

ALBUQUERQUE. Bauer, Carey, Mathany, Sloatman.
Sandia Corp. Boldyreff, Brunk, Buell, Calvert, Clark, DiBella, Ewing, Flanagan, Hassell, Landes, Mahuron, Malley, Miller, Reed, Schutzberger, Silverman, Vick, Youngs.
Univ. of New Mexico. Beach, Gentry, Healy, Hendrickson, Hildner, Lane, La Paz, Rogers.
HOLLOMAN. Price.
LAS CRUCES. Culpepper, Graves.
LAS VEGAS.
New Mexico Highlands Univ. Roberts, Rodgers, Slechticky.
LOS ALAMOS. Gammel.
Los Alamos Scientific Lab. Benson, Cherry, Donaldson, Hammer, Ingersoll, Oeder, Parter, Rechard, Riebe, Schlesinger, Sobczyk, Stark, Wall, White, Wing.
ROSWELL. Harp.
SANTA FE. Luke.
SOCORRO. Reece, Sanchez-Diaz.
STATE COLLEGE.
New Mexico Coll. of A. & M. A. Branson, Crouch, Kramer, Walden, Westhafer.

NEW YORK

ALBANY. Dumont, Gibson, Gorsline, Keefer, Newsom, Sister Noel Marie.
New York State Coll. for Teachers. Beaver, Birchenough, Butler, Lester, Luippold, Turner.
ALFRED.
Alfred Univ. Freund, Miller, Nevins, Polan, Rhodes, Seidlin, Whitford.
AURORA. Hollcroft, Rusk.
BALDWIN. Bowden, Grove, Mitchell.
BAYSIDE. Jordan.
BINGHAMTON. Greene, Thompson.
BRONX. Lipsey, Sister St. Thomas of Canterbury, Tucker.
BROOKLYN. Appuhn, Braverman, Byrne, Cascio, Epsell, First, Gerst, Goldman, Lavoie, Leeds, LeLeiko, Levine, Lonner, McKenna, Milgram, Miller, Moskowitz, Odin, Ritterman, Rush, Salkind, Sandler, Shapiro, Waite, Wallach, Wayne.
Brooklyn Coll. Barotz, Borofsky, Boyer, Brown, Cocuzza, Fleisher, Forman, Goldman, Griffin, Johnson, Karrass, Kennison, Kieval, Landers, Levenson, Maria, Moore, Osterberg, Prenowitz, Richardson, Shapiro, Singer, Smith, Wolfe, Woodbridge.
Poly. Inst. of Brooklyn. Abramowitz, Bodner, Carson, Forray, Foster, Harris, Hutchinson, Kalish, Klamkin, Kovitz, Kramer, Kramer-Lassar, Laudante, Lee, Morduchow, Russell, Terzuoli, Ullman, Whitford.
Pratt Inst. Beckman, Cowles, Helme, Moore, Thompson.
St. John's Univ. Archibald, Germino, Olbrich, Phelan, Pomilla, Sarno, Tolle, Wampole, Zirkel.
BUFFALO. Batt, Browne, Buchman, Burrill, Emerson, Farber, Green, Hand, Kowalewski, Newell, Orloff, Podmele, Potts, Rowley, Scholl, Sharpe, Sister Mary Michael, Steudle, Tidd, Walker, Welmers.
Cornell Aeronautical Lab. Duke, Lawrence, Trabka.
Univ. of Buffalo. Baeumler, Behrns, Feidner, Gehman, Gough, Grinstein, Larson, McArtney, Montague, Montgomery, Moulton, Niedrauer, Noller, Pound, Schaer, Schillo, Schneckenburger, Seiden, Stevens, Strebe, Warner.
CANISTEO. Longley.
CANTON. Boak.
St. Lawrence Univ. Bates, Limpert, Parker, Peters, Smith.
CATTARAUGUS. Beals.
CAZENOVIA. Howe.
CLEVELAND. Morenus.
CLINTON.
Hamilton Coll. Clelland, Gere, van Alstyne.
EAST MEADOW. Goldman.
ELMHURST. Bremer.
ELMIRA. Frankel, Suffa.
ELMONT. Williams.
ENDICOTT. Andrus, O'Connor.
Harpur Coll. Cary, Kent, Shear, Wright.
FARMINGDALE. Schriro, Sealander, Stern.
FAR ROCKAWAY. Moore, Schor.
FLUSHING. Bakst, Mann, Strobel.
Queens Coll. Archibald, Brown, Cohen, Cope, Dean, Eaton, Feld, Raudenbush, Sard, Sullivan, Zippin.
FOREST HILLS. Frank, Hertzig.
GARDEN CITY. Stein.
GENEVA.
Hobart & William Smith Colls. Beinert, Bligh, Durfee, Hubbs, Mosey.
GREAT NECK.
Sperry Gyroscope Co. Hutchison, Maile, McCarthy, McMahon.
HAMILTON.
Colgate Univ. Aude, Downie, Munshower, Wardwell.
HAYT CORNERS. Ford.
HEMPSTEAD.
Hofstra Coll. Charlesworth, Draudt, Hawthorne, Hinrichs, Hove, Jaeger, Marshall, Mott, Ollmann, Stabler.
ITHACA.
Cornell Univ. Agnew, Carver, Fuchs, Gunder, Hurwitz, Kac, Murdock, Na Nagara, Pollard, Robison, Rosser, Walker.
JAMAICA. Sister Augustine Maria.
KINGS POINT.
U. S. Merchant Marine Acad. Keyes, Nickl, Oberist.
LOCKPORT. Fountain, Kallett.
LONG ISLAND CITY. Day.
LOUDONVILLE. Hanhauser, Kuhn.
MACEDON. Hood.
MAMARONECK. Annis.
MEXICO. Reddick.
MINEOLA. Basil.
MT. KISCO. Morgan.
NEW LEBANON. Pflaum.
NEW PALTZ. Brereton, Swain.
NEW ROCHELLE. Kiely, Winnis.
NEW YORK. Alfieri, Beerman, Berg, Berger, Bernard Alfred, Boehm, Bornmann, Brock, Burgess, Coleman, Cornwell, Crane, Croci, D'Atri, Deutsch, Dodes, Finkel, Freeman, Getman, Glusman, Gray, Greaney, Grossman, Hammer, Harris, Heath, Hlavaty, Hobbs, Hydeman, Ingram, Jablonower, Joffe, Jonas, Katz, Keeler, Kelley, Kinney, LaSalle, LeGrange, Lewis, Lukens, May, Mayerson, Mirick, Molloy, Morrison, Murray, Nehrbas, Nordstrom, Ogawa, Oxley, Peiser, Perlstein, Pryzie, Roll, Rosenfeld, Ruderman, Ryan, Salerno, Sammet, Schwartz, Silversten, Singer, Sister Maria Loyola, Skelding, Smith, Solomon, Spohn, Steinhaus, Tang, Tom, Vitale, Walker, Weaver, Weyl, Wilkins, Wohl, Young.
Bell Telephone Labs. Clos, Fry, Jones, MacColl, Mead, Riordan.
City Coll. of New York. Bergmann, Chernofsky, Fagerstrom, Gill, Hibbard, Hinman, Hubert, Hurwitz, Linehan, MacEwen, Nathan, Post, Rensin, Robinson, Schach, Schwartz, Wirth, Wright, Zucker.

Columbia Univ. Abramson, Al-Dhahir, Aurora, Berkowitz, Bernhard, Bolton, Campbell, Douglas, Eilenberg, Eisinger, Fehr, Fite, Gentzler, Kasner, Kaufman, Levine, Littauer, Lorch, McGaughy, Mullins, Murray, Orshansky, Plithides, Reeve, Scheffe, Siceloff, Upton.
Cooper Union. Berndt, Chessin, Eastham, Lehmann, Miller.
Fordham Univ. Curran, Kirby, Kubis.
Hunter Coll. Anderson, Bradley, J. H. Bushey, Jewell H. Bushey, Cooper, Darkow, Eisele, Hill, Kutman, Landers, Tuller, Weisner.
International Business Machines. Herrick, Hurd, Johnson, Powell, Seeber.
New York Univ. Adler, Arzt, Berkowitz, Bernardi, Carl, Ciolkowski, Cooley, Cory, Courant, Edison, Ficken, Graham, Hirsch, Isaacson, F. W. John, Fritz John, Kinsella, Kline, Kolodner, Kranzer, Kushner, McConnell, Mulligan, Palladino, Payne, Peters, Putnam, Rehberg, Ritger, Roth, Rubin, Schlauch, Solky, Spiegelthal, Wahlert, Wendroff, Wolinsky, Yanosik, Zucker.
Yeshiva Univ. Block, Frank, Ginsburg, Hellerstein, Solomon.
NIAGARA FALLS.
Bell Aircraft Corp. Bates, Carroll, Diesen, Dixon, Heckroth, Kroeger, Kurland, Meyer, Miller, Mullan, Rapp, Saltarelli, Smurthwaite, Welmers.
NIAGARA UNIVERSITY.
Niagara Univ. Banks, Egan, Newell.
ONEONTA. Callahan, Sanford.
PINE CAMP. Williams.
PLATTSBURGH.
Champlain Coll. Bush, Eves, Fleshler, Kotler, Lane, Skinner.
POTSDAM. Larsson, Merrill.
POUGHKEEPSIE. Phillips.
Vassar Coll. Asprey, Baker, Dolciani, McDonald, Newton, Petschek, Wells.
ROCHESTER. Harding, Sister M. Dorothea.
Eastman Kodak Co. Arnold, Chesna, Foard, Marchand.
Univ. of Rochester. Atkins, Bagemihl, Barton, Bernstein, Betz, Danese, Emich, Gale, Gunderson, Pitt, Randolph, Schaefer, Seidel, Small, Watkeys.
ROMULUS. Casey.
ST. ALBANS. Deutsch.
ST. BONAVENTURE. Scheier.
SARATOGA SPRINGS. Williams.
SCHENECTADY. Linney, Poritsky.
Union Coll. Burkett, Fox, Holt, Maddaus, Malé, Morse, Snyder.
SPRINGVILLE. Abbey, Harrington.
STATEN ISLAND. Stinetorf, Waldman.
STONY BROOK. Berry.
SYRACUSE. Wood.
Syracuse Univ. Baum, Carroll, Cole, Gelbart, Haber, Hetzelt, Johnson, Kibbey, Ogilvy, Prentice, Rosskopf, Stokes.
TROY.
Rensselaer Poly. Inst. Allen, Beck, Burger, Campbell, Clemens, Guilford, Hendler, Huston, Jones, Maly, McKeon, Nash, Nickol, Spiegel, Vrooman, Warnock.
UTICA. Greeley, Palmer, Richards.
WEST POINT.
U. S. Military Acad. Bessell, Farnell, Jones, Matheson, Nicholas, Yates.
WHITE PLAINS. Sister Mary Benedicta.
WYOMING. Hartnell.

NORTH CAROLINA

CHAPEL HILL. Heinzman, Stallard.
Univ. of North Carolina. Brauer, Browne, Cameron, Clatworthy, Garner, Henderson, Hickerson, Hill, Hoyle, Jones, Lasley, Lee, Mackie, Medlin, Minton, Payne, Pignani, Robbins, Sullivan, Walker, Whyburn, Wolf.
CHARLOTTE. Hoyle, Jenkins, Nicholson.
DAVIDSON.
Davidson Coll. Martin, McGavock, Mebane, Peyton.
DURHAM. Browne, Parker, Pegram, Williams.
Duke Univ. Carlitz, Dressel, Elliott, Gergen, Hickson, McPherson, F. R. Olson, Virginia L. Olson, Patterson, Rankin, Reid, Roberts, Thomas.
FT. BRAGG. Morrill.
GREENSBORO.
Woman's Coll. of the Univ. of North Carolina. Barton, Lewis, Strong.
GREENVILLE. Phillips, Reynolds, Scott.
HENDERSONVILLE. Brown.
HICKORY. Dodson.
HIGH POINT. Adams.
KINSTON. Huggins.
MARS HILL. Howell.
RALEIGH. Downing.
North Carolina State Coll. Baker, Bullock, Carroll, Cell, Clayton, Levine, Lewis, Nahikian, Nolstad, Speece, Strobel, Watson.
SALISBURY. Ackerson, Dearborn.
TRYON. Dimick.
WARRENTON. Graham.

NORTH DAKOTA

GRAND FORKS.
Univ. of North Dakota. Hankerson, McBride, Peterson, Rognlie, Simonson, Staley.
MADDOCK. McPherson.
MAYVILLE. Forseth.
MINOT. Beckstrom.
STATE COLLEGE.
North Dakota Agric. College. Arena, Grimes, Hill, Smith, Stennes, Thompson.

OHIO

ADA. Harp, Whitted.
AKRON. Fuller, Hitchcock.
Univ. of Akron. Davis, Mauch, Ross, Selby.
ALLIANCE. Clark, Freese.
ASHLAND. Wiley.
ATHENS.
Ohio Univ. Denbow, Fisch, Hokanson, Marquis, Reed, Riggs, Starcher, Sterrett.

BARBERTON. Anderson, Ross.
BEREA.
Baldwin-Wallace Coll. Annear, Robb, Wilson.
BOWLING GREEN.
Bowling Green State Univ. Atkins, Cornell, Krabill, Mathias, Overman, Tinnappel, Wohler.
CHILLICOTHE. Clinton.
CINCINNATI. Pinzka, Sister Mary Edmund, Sister Mary Henrietta, Stechschulte.
Univ. of Cincinnati. Barnett, Berman, Brand, Feige, Herwitz, Justice, Lipsich, Lubin, Merriman, Moore, E. S. Smith, T. F. Smith, Szasz, Yowell.
CLEVELAND. Burwell, Dustheimer, Johnson, Joliat, Sister Mary Cleophas, Voronovich.
Case Inst. of Tech. Brown, Crowder, Daraugh, Green, Guenther, Leone, McCuskey, Morris, Nassau, Rinehart, Saltzer, Thomas, Topp.
Fenn Coll. Haskins, Higgins, Kelly, Oravec, Van Voorhis.
National Advisory Committee for Aeronautics. Arnoff, Brown, Hansen.
Western Reserve Univ. Boblett, Musselman, Shively, Simon.
COLUMBUS. Glabe, Heinke, Hoyle, Sealander, Sister Charles Anne, Weiss, Wildermuth.
Ohio State Univ. Adney, Alden, Bareis, Beatty, Caris, Davis, Fadell, Hall, Hanzel, Helsel, Jones, Juelich, Kuhn, Lazar, Lowry, Mickle, Miller, Morris, Myers, Naiditch, Pepper, Rado, Reichelderfer, Rickard, Ryser, Toops, Whitney, Zemlin, Ziebur.
DAYTON. Rathbun.
Wright-Patterson Air Force Base. Ely, Fettis, Foote, Gephart, Kraft, Loch, Millsaps, Rider, Stone, Tikson, Toney, Wang, Young.
A.A.F. Inst. of Tech. Astrachan, Carson, Downing, Gatewood, Holtom, Tear.
Univ. of Dayton. Bellmer, Jehn, Peckham, Schraut.
DEFIANCE. Godfrey, MacCullough.
DELAWARE. Crane, Rowland.
GAMBIER.
Kenyon Coll. Finkbeiner, Nikodym, Transue.
GRANVILLE.
Denison Univ. Carpenter, Kato, Ladner.
HIRAM. Clarke, Olney.
IRONTON. Sister M. Thomas a Kempis.
KENT.
Kent State Univ. Brooks, Brumfield, Cox, Dressler, Durig, Evans, Harshbarger, Iwanchuk, Jenkins, Johnson, Kaiser, Lowenstein, Manchester, Olson, Portmann, Seibel, Stapleford, Warren.
KENTON. Abbeduto.
LAKESIDE. Wolfe.
LANCASTER. Smart.
LISBON. Martin.
MARIETTA. Bennett, Sandt.
MIAMISBURG. Orr.
MT. ST. JOSEPH. Sister Agnes Therese, Sister Maria Corona.
MT. VERNON. Adams.
NAPOLEON. Yeager.
NEW CONCORD. Knight.
NEW LEXINGTON. Hoops.
NEW WASHINGTON. Cummins.
NORTH BALTIMORE. Blackall.
OBERLIN. Yeaton.
Oberlin Coll. Graff, Sinclair, Tuckerman, Vance, Yeaton.
OSBORN. Wells.
OXFORD. Tappan.
Miami Univ. Anderson, Bloom, Epperson, Hill, Johnston, Pollard, Spenceley, Wolfe.
PERRYSBURG. Strow.
SPRINGFIELD. Krueger, Tripp.
STEUBENVILLE. Depkovich, Emmert.
SYLVANIA. Sister M. Agneta.
TIFFIN. Menke.
TOLEDO. Ginther, Sister M. Constantia, Sister M. Mercedes, Sister Vincent de Paul.
Univ. of Toledo. Amos, Brandeberry, Calhoon, Cutler, Dancer, Davis, Shoemaker.
WESTERVILLE. Bamforth, Glover.
WILBERFORCE. Curry, Woodson.
WILMINGTON. Spinks.
WOOSTER.
Coll. of Wooster. Fobes, McClung, Renzema, Smyth, Williamson, Yanney.
YELLOW SPRINGS. Hamilton, Myatt.
YOUNGSTOWN. Malak, Yozwiak.

OKLAHOMA

ADA. Heimann.
BARTLESVILLE. Rice.
BETHANY. Greer.
BOWRING. Chouteau.
DURANT. Krattiger.
ENID. Smith.
FT. SILL. Brown.
GOODWELL. Murphy.
NORMAN. Nemecek.
Univ. of Oklahoma. Andree, Bernhart, Brixey, Chen, Court, Deal, Dragoo, Dubois, Grau, Gregory, Hassler, Hoffman, Huff, Huneke, John, LaFon, Levy, Lewis, McFarland, McKnelly, Spears, Springer.
NOWATA. Spurgeon.
OKEMAH. Lawrence.
OKLAHOMA CITY. Duncan, Meador, Pirrong.
PONCA CITY. Ritchie.
STILLWATER. Hamilton.
Oklahoma A. & M. Coll. Allen, Barnett, Burns, Caskey, Flanders, Hamilton, Johnson, McDole, Morrison, Scholz, Smith, Zant.
TULSA. Eisen, Johnston, Shreve, Webster.
Univ. of Tulsa. Burkhart, Carter, Gwin, Roth, Veatch.
WARNER. Harrison.
WEATHERFORD. Linscheid.
WILBURTON. Holland.

OREGON

BROOKS. Steinkamp.

CORVALLIS.
Oregon State Coll. Beaty, Clark, Gregory, Hoggatt, Hostetter, Li, Lonseth, Milne, Pearson, Poole, Price, Saunders, Wenner, Williams, Wirshup.
CULVER. Bunch.
EUGENE.
Univ. of Oregon. Chrestenson, Civin, Clemans, Ghent, Livingston, Moursund, Niven, Peterson, Shepherd, Stalley, Wood.
FOREST GROVE. Allhands, Price.
FREEWATER. Hill.
KLAMATH FALLS. Burkhalter.
LA GRANDE. Oesterle.
MCMINNVILLE. Dolan, Ramsey.
PORTLAND. Biggerstaff, Buschman, Keeler, Rall, Richmond.
Reed Coll. Griffin, Louise J. Rosenbaum, R. A. Rosenbaum, Williams.
SALEM.
Willamette Univ. Christensen, Laidlaw, Luther.

PENNSYLVANIA

AKRON. Loewen.
ALLENTOWN. Kunkel.
Muhlenberg Coll. Deck, Holt, Koehler.
ALTOONA. Herpel.
ANNVILLE. Ablett, Erickson.
BEAVER FALLS. Cleland, Justis.
BETHLEHEM. Rader.
Lehigh Univ. Ashbaugh, Chellevold, Cutler, Hailperin, Kenny, Lane, Latshaw, Pitcher, Raynor, Reynolds, Samoloff, Schecter, Shook, Smail, Stoll, Van Arnam, Wilansky.
BRADFORD. Francis.
BRISTOL. Van Sant.
BRYN MAWR. Atkinson.
Bryn Mawr Coll. Burton, Lehr, Oxtoby, Wheeler.
BUCKINGHAM VALLEY. Wilson.
CARLISLE.
Dickinson Coll. Ayres, Kuebler, Nelson.
CHAMBERSBURG. Johnson.
CHESTER. Helms.
COLLEGEVILLE.
Ursinus Coll. Clawson, Dennis, Manning.
CORAOPOLIS. Zora.
DENVER. Marburger.
EASTON. Sandwick.
Lafayette Coll. Benner, Cawley, Hatch, Keck, Kohlmeyer, Smith, Tartler.
ELIZABETHTOWN. Heilman.
ERIE. Kraus, Myers, Russell.
GREENSBURG.
Seton Hill Coll. Drake, Kane, Sister Marie Gertrude, Sister Mary Deborah.
GROVE CITY. Carpenter.
HARRISBURG. Hartman.
HAVERFORD.
Haverford Coll. Oakley, Wilson, Wyre.
HAZLETON. Zerbe.
HERSHEY. Lanz.
HUNTINGDON. Stayer.
IMMACULATA. Sister Mary Raphael.
INDIANA. Mahachek, Stright.
KUTZTOWN. Knedler.
LANCASTER.
Franklin & Marshall Coll. Haag, Holzinger, Meier, Western.
LATROBE. Heid, Seubert.
LEWISBURG.
Bucknell Univ. Gold, Miller, Polak, Richardson.
LOCK HAVEN. Smith.
LORETTO. Mino.
MCDONALD. Fisher.
MCKEESPORT. Sister M. Tarcisius, Taylor.
MCKEES ROCKS. Arnold.
MEADVILLE.
Allegheny Coll. Steen, Sturley, Tanimoto.
MILLERSVILLE. Boyer.
MORTON. Eggert.
PHILADELPHIA. Applebaum, Connelly, Constable, Durand, Eisenhart, Gudge, Haseltine, Hearn, Hilferty, Houghton, Keralla, Koch, Latshaw, Levy, Maddox, McDonough, Miller, Moliver, Neale, O'Conor, Oneil, Ridley, Rutledge, Russ, Slepin.
Drexel Inst. Bickel, Davis, McNeary, Rosen.
Frankford Arsenal. Cavalli, Feder, Mall.
Temple Univ. Hostinsky, Scarf, Wurster.
Univ. of Pennsylvania. Aissen, Bredt, Caris, Evans, Fine, Garber, Goheen, Gottschalk, Gurk, Hunt, Kamel, Kline, Lehner, Loev, Patterson, Richman, Rosenblum, Schafer, Schub, Wear.
PITTSBURGH. Buker, Calkins, Crow, Frankel, Hallett, Harmon, Leifer, McDermot, Randich, Sister M. Michael, Taylor, Truan.
Carnegie Inst. of Tech. Charnes, DiPrima, Hoover, Johnson, Lemke, Moskovitz, Neelley, Nodvik, Olds, Rosenbach, Sacks, Saibel, Tyler, Warner, Whitman.
Duquesne Univ. DiDonato, Dunkelberger, Hardy, Hnath, O'Donnell, Ostrofsky, Schwartz, Smith.
Univ. of Pittsburgh. Blumberg, Brady, Bryson, Christiano, Edwards, Gettig, Hovey, Knipp, Laush, Michalik, Mount, Myers, Sebesta, Taylor, Teats, Wells.
PLEASANTVILLE. Kerr.
POTTSVILLE. Townsend.
READING. Speicher.
RIDGEWAY. Bauser.
SCRANTON. Sister Mary Cormac.
Univ. of Scranton. Bartley, Rounds, Savaro.
SHARON. Greene.
SPRING GROVE. Martin.
STATE COLLEGE. Grindall.
Pennsylvania State Coll. Cogan, Cohen, Curry, Dickinson, Dunlap, Erskine, Friedman, Frink, Gordon, Gravatt, Hagen, Harrington, Johnson, Kocher, Konhauser, Krall, Ormsby, F. W. Owens, Helen B. Owens, Schuster, Sheffer.
SWARTHMORE.
Swarthmore Coll. Brinkmann, Dresden, Marriott, Winer.
TURTLE CREEK. Derflinger.

WASHINGTON.
Washington & Jefferson Coll. Bert, Long, Patterson, Shaub, Thomas.
WAYNE. Snyder.
WAYNESBURG. Howcroft, Moston.
WILKES-BARRE. Richards.

PUERTO RICO

MAYAGUEZ. Garcia.
SAN JUAN. Piza.

RHODE ISLAND

KINGSTON.
Rhode Island State Coll. Bender, Haggerty, Stauffer.
NEWPORT. Chase, Mendenhall, Sister M. Rose Agnes, Spaulding.
PORTSMOUTH. Madden, Taliaferro.
PROVIDENCE. Eddy, MacNeille, McKenney, Wehausen.
Brown Univ. Adams, Albert, Archibald, Bennett, Carlen, Gilman, Glauz, Heins, Lee, Levy, Lewis, Manning, Schatz, Smiley, Smith, Stewart, Youden.

SOUTH CAROLINA

CHARLESTON. DeFrancesco, Jones.
The Citadel. Dye, Folsom, Hair, Hutchison, Reves, Sutton.
CLEMSON.
Clemson Agric. Coll. Brewster, Brown, Bryan, Harden, Kirkwood, Lagrone, Park, Sheldon, Stanley, Vause, Wade.
CLINTON. Coleman, Sloan.
COLUMBIA. Dicks.
Univ. of South Carolina. Croxton, Douglas, E. A. Hedberg, Marguerite Z. Hedberg, Jackson, Lytle, Martin, Novak, Perkins, Rasor, Shuler, Betty R. Weber, W. W. Weber, Williams.
GREENVILLE. Blackwell, Mays.
GREENWOOD. Cothran, McLeod.
HARTSVILLE. Reaves, Saunders.
NEWBERRY. Martin.
ROCK HILL. Hahn.
SPARTANBURG. Jordan, Patten.
SULLIVAN'S ISLAND. Cleveland.
SUMTER. Burgess.

SOUTH DAKOTA

ABERDEEN. Mewaldt.
BROOKINGS.
South Dakota State Coll. Engebretson, MacDougal, Scholten, Walder, Wente.
HURON. Gantvoort.
MITCHELL. Knox.
RAPID CITY.
South Dakota School of Mines & Tech. Harbison, Richardson, Swanson, Wilson.
SIOUX FALLS. Lyche.
VERMILLION.
Univ. of South Dakota. Alkire, Anderson, Ekman, Hoover, Miller.
YANKTON. Howell.

TENNESSEE

CHATTANOOGA. Massey.
CHURCH HILL. Gregg.
CLARKSVILLE. Bright.
COOKEVILLE.
Tennessee Poly. Inst. Bruce, Moorman, Qualls.
FOUNTAIN CITY. Keller.
GREENEVILLE. Eddington.
HARROGATE. Bowling, Callahan.
IRON CITY. Carter.
JEFFERSON CITY. Sloan.
JOHNSON CITY. Carson, Jasper.
KNOXVILLE. Sanders.
Univ. of Tennessee. Blackstock, Bradshaw, Brown, Cooley, Eaves, Givens, Griffith, Lee, Marr, Snyder, Wilson.
MARTIN. Taylor.
MARYVILLE. Sisk.
MEMPHIS. Coker, Thomas, Tiller.
Memphis State Coll. Anderson, Kaltenborn, McBride, Williams, Wollan.
NASHVILLE. Clement, Dark, Lorch.
George Peabody Coll. for Teachers. Banks, Rice, Wren.
Tennessee A. & I. State Coll. Boswell, Gasaway, Sasser.
Vanderbilt Univ. Badgley, Blair, Boyce, Bridgforth, Bryant, Clark, Depew, Graham, Hyden, Lawrence, Lundberg, Martin, Ratner, Shanks, Stiles, Thurman, Voorhees.
OAK RIDGE. McClure, Zeigler.
Carbide & Carbon Chemical Corp. Hudson, Norton, Rankin, Rowe.
Oak Ridge National Lab. Briant, Coveyou, Edmonson, Householder, Hull, Perry, Sangren.
SEWANEE. Shotwell.

TEXAS

ABILENE. Burnam, Hays, Mullings, Tate.
AMARILLO. Davis, McCuan.
ARLINGTON. Howard.
AUSTIN. Dickerson, Schmied.
Univ. of Texas. Batchelder, Bright, Craig, Decherd, Dobbins, Ettlinger, Guy, Hurt, Ling, Lubben, Osborn, Vandiver, Wall, Yett.
BEAUMONT. Baldwin.
BELLAIRE. Harrje.
BORGER. Miculka.
BROWNSVILLE. de la Garza, Stavinoha.
BROWNWOOD. Johnson, Williams.
CANYON. Murray.
COLLEGE STATION.
Texas A. & M. Coll. Basye, Beeman, Daum, Gandy, Klipple, Luther, McCulley, Moore, Tittle, Wapple.
COMMERCE.
East Texas State Teachers Coll. Baker, Coffman, Riggs, Taylor, Wright.
CORSICANA. Stokes.
DAINGERFIELD. Kennedy.
DALLAS. Aronofsky, Grogan, Harris, Lee,

Mason, McNabb, Sommers, Sorrells, Stulken, Thomas, West.
Chance Vought Aircraft. Ailara, Ingle, Wood, Wortham.
Southern Methodist Univ. Derr, Koger, Marks, Mouzon, Pipes, Seale, Starr.
DENISON. Standerfer.
DENTON. Ashburn, Miller.
North Texas State Coll. Bradford, Brown, Cooke, Ellis, Hanson, Parrish.
EL PASO. Iturralde, Woods.
FT. BLISS. Dillon, Geis, Greene, Stamper, Willey.
FT. WORTH.
Texas Christian Univ. Bramblett, Colquitt, Etter, Goldbeck, Henrick, Morgan, Neely, Ramsey, Sherer, Shore, Welch.
GEORGETOWN. Whitmore, Wilcox.
HOUSTON. *Blau, Dubay, Howe, Pennington,* Rainbow, Slotnick.
Univ. of Houston. Gibney, Gray, Grover, Kissel, Newhouse, Rader, Rogers, Wright.,
Rice Inst. Bray, Dean, Lovett, MacLane, Ulrich.
HUNTSVILLE.
Sam Houston State Teachers Coll. Lane, Querry, Vick, Wall, Wells.
KINGSVILLE. Dorroh.
LUBBOCK.
Texas Tech. Coll. Fuller, Hazlewood, Heineman, May, Michie, Parker, Roberts, Rowland, Sparks, Underwood, Woodward.
MILFORD. Diamond.
NACOGDOCHES. Layton, Pinson.
ODESSA. Thorp.
ORANGE. Schwaller.
PLAINVIEW. McCoy.
RANDOLPH AIR FORCE BASE. Arend.
SAN ANTONIO. Glass, Horton, Jenke, Krentel, Reilly, Sister Mary of Mercy, Sister Presentation, Wood.
St. Mary's Univ. Braune, Casseb, Cicchese, Finck, Galvan P., Morgan, Payne, Rice, Roland, Schnepp, Torian, Weissler.
Trinity Univ. Banes, Cullwell, Newton, Rees.
SAN MARCOS.
Southwest Texas State Teachers Coll. Bernhard, Cude, Porter, Tulloch.
SEQUIN. Ross.
SHERMAN. Dean.
STEPHENVILLE. Redden, Worthington.
TEAGUE. Notley.
TEXARKANA. Glass.
TYLER. Williams.
UVALDE. Hardy.
WACO. McLachlan, Moore.
WICHITA FALLS. Adams.

UTAH

LOGAN. Hunsaker, Nelson.
OGDEN. Sorenson.
SALT LAKE CITY.
Univ. of Utah. Biesele, Hayes, Henriques, Horsfall, Pehrson, Thorne, Wylie.

VERMONT

BURLINGTON.
Univ. of Vermont. Bullard, Butterfield, Fishback, Larrivee, Swift, Vollmer.
MIDDLEBURY. Bowker, Hazeltine.
NORTHFIELD. Dix.
ORWELL. Harwood.
WINOOSKI. Albiser, Alliot.

VIRGINIA

ALEXANDRIA. Burington.
ARLINGTON. Jasper, Johnson, Maltenfort, Mela, Quinn.
Engineering Research Associates. Engstrom, McLynn, Tompkins.
ASHLAND. Blincoe, Simpson.
BLACKSBURG.
Virginia Poly. Inst. Davis, Hatcher, Horne, McFadden, O'Shaughnessy, Spencer.
CHARLOTTESVILLE. Ford.
Univ. of Virginia. Aylor, Botts, Linfield, Lowdenslager, McShane, Oglesby, Whyburn.
COLONIAL BEACH. Eliezer.
DAHLGREN.
U. S. Naval Proving Ground. Barnes, Bramble, Gleissner.
EMORY. Biggers, Graybeal.
FARMVILLE. Sutherland, Taliaferro.
FT. MEYER. White.
FREDERICKSBURG. Frick.
HAMPTON. Fields, Gordon.
HARRISONBURG. Grabner, Ikenberry.
HOLLINS COLLEGE. Freitag.
LANGLEY FIELD.
National Advisory Committee for Aeronautics. Moore, Pinkerton, Rennemann.
LEXINGTON. Paxton, Smith.
Virginia Military Inst. Byrne, Knox, Purdie, Saunders, Steward.
LYNCHBURG.
Randolph-Macon Woman's Coll. Humphreys, Larew, Wiggin.
MIDDLEBURG. Keppler.
NEWPORT NEWS. Raine.
NORFOLK. Norris.
PETERSBURG. Hunter.
QUANTICO. Carlan.
RICHMOND. Drew.
Univ. of Richmond. Gaines, Grable, Harris, Sleight, Wheeler.
SALEM. Carpenter, Suter.
STAUNTON. Taylor.
SWEET BRIAR. Lee, McGar.
WILLIAMSBURG.
Coll. of William & Mary. Calkins, Phalen, Stetson.

WASHINGTON

BELLINGHAM. Gelder, Johnston.
CHENEY. Bell.
EVERETT. Van Arkel.
LONGVIEW. Parks.
MT. VERNON. Good, McKeehan.

OLYMPIA. Cebula.
PULLMAN.
State Coll. of Washington. Brenner, Butler, Clement, Cooke, Hacker, Irwin, Klotz, Knebelman, Reinhardt, Salisbury, Vatnsdal.
RICHLAND. Anselone, Mayer.
SEATTLE. Beegle, Bond, Innis, Trueblood, Vopni.
Boeing Airplane Co. Ablow, Colvin, Gaskell, Jones, Rowland, Stone.
Univ. of Washington. Allendoerfer, Ball, Ballantine, Beaumont, Birnbaum, Chapman, Cramlet, Dekker, Fleming, Haller, Hewitt, Jerbert, Kingston, McFarlan, Peterson, Street, Winger, Zuckerman.
SPOKANE. Barnard, Bergman, Carlson.
Gonzaga Univ. Kneer, Murray, O'Day, Ryan, Thomas.
TACOMA. Goman.
VANCOUVER. Stair.
WAPATO. Ivanoff.
YAKIMA. Seamons.

WEST VIRGINIA

BETHANY. Tye.
CLEAR CREEK. Thompson.
COLUMBUS. La Rue.
ELKINS. Zader.
HUNTINGTON. Barron, Goins.
INSTITUTE. Aheart.
KEYSER. Iverson.
MONTGOMERY. Renton.
MORGANTOWN. Blum.
West Virginia Univ. Bauserman, Cunningham, Davis, Peters, Reynolds, Stewart, Vehse, Vest.
MOUNDSVILLE. Turner.
PHILLIPPI. Sellers.
WEST LIBERTY. Plummer.

WISCONSIN

APPLETON.
Lawrence Coll. Berry, Palmbach, Stewart.
BELOIT.
Beloit Coll. Conwell, Finch, Huffer.
EAU CLAIRE. Otteson, Seber.
KENOSHA. Ward.
LACROSSE. Adkins, Sister M. Mechtildis.
MADISON. Erickson.
Univ. of Wisconsin. Allen, Arnold, Bartz, Batho, Bing, Braunschweiger, Bruck, Buck, Cowell, Eberlein, Eide, Evans, Germain, Higgins, Ingraham, Kleene, Langer, Lehner, Lercher, Lufburrow, MacDuffee, March, Mayor, Renno, Rose, Schaeffer, Sokolnikoff, Swokowski, Thomas, Trump, Wagner.
MENOMONIE. Heigl, Rich.
MILTON. Loomer.
MILWAUKEE. Bigelow, Clark, Jautz, Jones, Norris, Overn, Sister M. Mirabella, Sister Mary Felice, Sister Mary Olive, Sister Mary Petronia.
Marquette Univ. Harding, Moeller, Otto, Pettit, Talacko, Weideman.
Univ. of Wisconsin in Milwaukee. Bardell, Battig, Kenney, Marden, Meyer, Parkinson, Spitzbart, Thompson, Vass, Wolf.
PLATTEVILLE.
State Teachers Coll. Bristow, Bullis, Harrell.
PLYMOUTH. Rusch.
RACINE. May.
RIVER FALLS. McLaughlin.
SUPERIOR. Flogstad, Smith.
WAUKESHA. Hopkins, Weston.
Carroll Coll. Dancey, Glander, Meadows.
WEST DE PERE.
St. Norbert Coll. Bahng, Berner, DeCleene, Finkbeiner, Vande Castle, Watermolen.

WYOMING

LARAMIE. Bellamy.
Univ. of Wyoming. Barr, Neubauer, Rechard, Schwid, S. R. Smith, W. N. Smith, Steen, Varineau.

CANADA

ALBERTA

CALGARY. Jagoe.
EDMONTON.
Univ. of Alberta. Campbell, Moser, Wyman.

BRITISH COLUMBIA

COURTENAY. Douglas.
VANCOUVER.
Univ. of British Columbia. Derry, Gage, Hill, James, Jennings, Leimanis, Moyls, Murdoch, Neilson, Simons, Swanson.
VICTORIA. Wallace.

MANITOBA

BRANDON. Petersen.
WINNIPEG. McEwen, Mendelsohn.

NEW BRUNSWICK

SACKVILLE. Crawford.

NOVA SCOTIA

WOLFVILLE. Snow.

ONTARIO

CHALK RIVER. Sharp.
HAMILTON.
McMaster Univ. Attridge, Bankier, Findlay, McCallion, Terry.
KINGSTON. Jeffery, Miller.
LONDON.
Univ. of Western Ontario. Cole, Henderson, Kingston.
NORLAND. Gale.

OTTAWA. Dube, Eastcott, Keyfitz, Lane, MacPhail.
SHARBOT LAKE. Ritchie.
TORONTO. Dobson, Grant, Pounder.
Univ. of Toronto. Beatty, Burk, Coleman, Coxeter, Giordmaine, Pounder, Robinson.

PRINCE EDWARD ISLAND

CHARLOTTETOWN. Roche.

QUEBEC

MONTREAL. Ayoub, Cunha, Frechette, Gauthier, Gough, Lalonde, Lessard, Rosenthall, Williams.
QUEBEC. Lemay, Pouliot.

SASKATCHEWAN

SASKATOON. Ferns, Miller.

OTHER FOREIGN COUNTRIES

ARGENTINA

BUENOS AIRES. Baidaff, Barral-Souto.

AUSTRALIA

BRISBANE. Raybould.

BELGIUM

ANTWERPEN. Luykx.
BRUXELLES. Errera.
MONS. Deaux.
ST. ANDRE-LEZ-BRUGES. Goormaghtigh.
ST. NIKLAAS. Van Bergen.

BRAZIL

SAO PAULO. Murnaghan.

BRITISH HONDURAS

CAYO. Zimmerman.

CEYLON

VADDUKODDAI. Lockwood.

CHILE

SANTIAGO. Moreno.

CHINA

AMOY. Tan.

COLOMBIA

MEDELLIN. de Greiff B.

CUBA

HAVANA. González, Novoa, Piñeiro, Rodríguez.
SANTIAGO. Muguercia.

ENGLAND

BIRMINGHAM. Redmond.
CAMBRIDGE. Mordell, Sigler.
EASTLEIGH. Wood.
ENGLEFIELD GREEN. McCrea.
HOVE. Priçe.
LONDON. Dalal.
OXFORD. Cohen, McVoy.
SHEFFIELD. Mirsky.

FRANCE

BOURG-LA-REINE. Minois.
LAON. Droussent.
NANCY. Yih.
PARIS. Cohen.
TENNIE. Thébault.

FRENCH OCEANIA

TAHITI ISLAND. Simonel.

GUATEMALA

GUATEMALA. Engel.

INDIA

BANGALORE. Madhava Rao.
BELGAUM. Sharma.
BIHAR. Subramu.
CUTTACK. Misra.
PANJAB. Aggarwal.

IRAQ

BAGHDAD. Al-Salam, Baez.

IRELAND

DUBLIN. Sandham, Swords, Synge.

JAPAN

TOKYO. Barnhart.

LEBANON

BEIRUT. Kennedy.

LUXEMBURG

DIFFERDANGE. Zahlen.

MEXICO

MONTERREY. Lifshitz.

NETHERLANDS

AMSTERDAM. Macon.
DELFT. Hyman, Korevaar.

NEW ZEALAND

AUCKLAND. Kania.
DUNEDIN. Martyn.

NORWAY

LEVANGER. Dybvik.

PAKISTAN

LAHORE. Nasir.

PANAMA

PANAMA CITY. Linares, Thullen.

PERU

AREQUIPA. Melgar.
LIMA. de Losada y Puga, Secada.

PHILIPPINES

MANILA. Fitzgerald, Perez.

SALVADOR

SAN SALVADOR. Mejia.

SCOTLAND

ABERDEEN. Erdös.
GLASGOW. O'Beirne.

SOUTH AFRICA

BLOEMFONTEIN. Arndt.

SPAIN

MADRID. Bachiller, Diaz.

SWEDEN

STOCKHOLM. Blom.

SWITZERLAND

BASLE. Ostrowski.
FRIBOURG. Bays.
ZURICH. Burckhardt.

SYRIA

ALEPPO. Babikian.

URUGUAY

MONTEVIDEO. Calcagno, Halmos.

VENEZUELA

CARACAS. Colmenares-Carrillo, Duarte, Michalup.

YUGOSLAVIA

SKOPLJE. Milosevich.
ZAGREB. Markovic.

BY-LAWS OF THE MATHEMATICAL ASSOCIATION OF AMERICA (INC.)

(As amended to December 1, 1951)

ARTICLE I—NAME, PURPOSE AND CORPORATE SEAL

1. This organization shall be known as

THE MATHEMATICAL ASSOCIATION OF AMERICA (INCORPORATED)

2. Its object shall be to assist in promoting the interests of mathematics in America, especially in the collegiate field, by holding meetings in any part of the United States or Canada for the presentation and discussion of mathematical papers, by the publication of mathematical papers, journals, books, monographs and reports, by conducting investigations for the purpose of improving the teaching of mathematics, by accumulating a mathematical library and by coöperating with other organizations whenever this may be desirable for attaining these or similar objects.

3. The Corporate Seal of the Association shall have inscribed thereon the name of the Association and the words "Corporate Seal—Illinois."

ARTICLE II—MEMBERSHIP

1. Any person who is interested in the field of collegiate mathematics shall be eligible for election to membership in the Association.

2. Election to membership shall be by vote of the Board upon written application from the individual seeking admission, endorsed by two members of the Association.

3. Those who were admitted to membership in The Mathematical Association of America (unincorporated) prior to October 1, 1920, and were in good standing as such on that date, were thereby admitted to membership in this Association (Incorporated).

ARTICLE III—BOARD OF GOVERNORS AND OFFICERS

1. The Officers of the Association shall be a President, a First Vice-President, a Second Vice-President, an Editor-in-Chief of the Official Journal (hereinafter called the "Editor"), a Secretary-Treasurer, and an Associate Secretary.

2. There shall be a Board of Governors (hereinafter called the "Board"), to consist of the Officers, the Ex-Presidents for terms of six years after the expiration of their respective presidential terms, and of additional elected members (hereinafter called "Governors"). It shall be the function of the Board to supervise all scholarly and scientific activities of the Association, to administer and control these activities, and to authorize expenditures of funds of the Association, except that at the demand of ten or more members of the Board, or at the demand of forty or more members of the Association, any proposal to alter or initiate a matter of policy shall be referred to the general membership of the Association for its decision. All members of the Board shall hold over until their respective successors are selected or appointed and qualify.

3. There shall be an Executive Committee advisory to the Board, and consisting of the President, the two Vice-Presidents, the Editor and the Secretary-Treasurer. It shall be the function of this Committee to review continually the policies and activities of the Association, to plan and organize new activities, to formulate in broad outline the programs of meetings and of publications, and in general to consider all matters of importance or of interest to the Association. This Committee shall prepare the agenda for meetings of the Board, and shall analyze the implications and aspects of all matters which are to come before the Board for decision. It shall present to the Board the viewpoints suggested by such analyses, as well as all such facts as may seem pertinent, or as may in any way facilitate the Board's work.

4. A statement regarding any proposed action of the Board which makes or alters a question of policy shall be published in the official journal, or notice of such proposed action shall be mailed to each member, before final action has been taken, so that members of the Association may make known to the Board their individual views.

5. The Board shall have authority to fill vacancies *ad interim* in any office, including vacancies in the Board, and to make any other appointments necessary for the transaction of the business of the Association.

6. At all meetings of the Board of Governors a quorum shall consist of not less than five (5) members and no business may be validly transacted at a meeting at which less than a quorum is present; *provided* that any meeting of the Board, whether or not a quorum be present, may be adjourned to a specified time and place by a majority of the members present without notice to the members at large other than announcement at such meeting. Informal action based on a mail ballot by the members of the Board, if ratified at a properly convened meeting of the Board, shall be as valid and effective as if originally authorized at such meeting.

7. There shall be a Finance Committee responsible to the Board; at the direction of the Board it shall receive and administer the funds of the Association, control its properties and investments, make its contracts, and exercise such powers as may be delegated to it by the Board. This committee shall consist of three members, of whom the Secretary-Treasurer shall be one.

8. (a) The Officers and Governors of the Association shall be elected in part by the Board, in part by the general membership, and in part by the membership in the Sections of the Association or by the membership in constituencies authorized by the Board for territory where Sections do not exist.

(b) The membership at large shall elect in alternate years respectively a President and a First Vice-President, each for a term of two years, and shall elect each year two Governors, for terms of three years.

(c) The membership in each Section shall elect triennially a Governor for a term of three years. For these elections, at least two nominations shall be submitted to the members by a committee appointed for that purpose by the Chairman of the Section.

(d) The Board shall elect at appropriate times by ballot and for the terms stated: a Second Vice-President for two years; an Editor, a Secretary-Treasurer, and an Associate Secretary, each for five years; and members of the Finance Committee (other than the Secretary-Treasurer) for four years.

(e) The President shall be ineligible for rëelection. The Vice-Presidents, the Editor, and the Governors shall be eligible for reëlection only after an interim equal to their respective terms of office.

(f) Elections by the Board shall be made from nomination by the Executive Committee. At least two nominations shall be made for each office to be filled in the case of the Second Vice-President and the members of the Finance Committee, and the Board may in any case reject all nominations made and call for a new list.

(g) The names of members to be printed upon the ballots, together with blank spaces in the case of elections by the general membership, shall be determined by a Nominating Committee to be appointed annually for that purpose by the President with the approval of the Board. Approximately six months before the date of the annual meeting all members shall be given an opportunity to nominate by mail a candidate for each office to be filled by the members for the ensuing year. Approximately one month before the annual meeting the Nominating Committee shall select a nominee for President out of the three persons who received the most votes for this office in the nominations; the Nominating Committee shall furthermore select two candidates for each other office to be filled by the members, one being the person who received the highest vote in the nominations and the other being selected from among the several nominees next in order. The election shall be by mail or in person and shall close on the day of the annual meeting.

9. The President shall be the Executive Officer of the Association, shall preside at all meetings of the Board of Governors and at the annual meeting of the Association. He shall have the usual duties pertaining to his office and such other duties as may from time to time be assigned him by the Board of Governors.

10. The Vice-Presidents shall, in the absence of the President, have and exercise the powers of the President, their order being determined alphabetically. The Board of Governors may assign to the Vice-Presidents such duties as may from time to time be determined.

11. The Secretary-Treasurer shall have the usual duties pertaining to the office of Secretary and of Treasurer, including the custody of the records of the Association and of its Corporate Seal, the keeping of minutes of the meetings of the Board of Governors and of the annual meeting and special meetings of members, and giving of due notice of all regular and special meetings of the Association and of the Board of Governors, and the supervision and safekeeping of the funds of the Association. The Secretary-Treasurer shall also have the duty of seeing that whenever Governors are elected, including the election of Governors to fill vacancies, a Certificate, under the Seal of the Association, giving the names of those elected and the term of their office, shall be recorded in the Office of the Recorder of Deeds for Cook County, Illinois. Such Certificates shall be signed by the Secretary-Treasurer and verified by oath of the President.

Article IV—Meetings

1. A meeting of the Association shall be held annually, at such time and place as the Board may direct. Special meetings of the Association may be called from time to time by the Board, or while the Board is not in session by the President of the Association, to be held at such time and place as may appear from the call.

2. The Board shall hold a meeting each year immediately preceding the annual meeting of the Association. Further meetings of the Board may be held from time to time at the call of the President or of any three (3) members of the Board.

3. Notice of any meeting of members of the Association shall be given by the Secretary-Treasurer at least thirty (30) days prior to the date set for each meeting. Notice of all meetings of the Board other than the regular meetings provided in Section 2 shall be given to each member of the Board at least fifteen (15) days prior to the date set therefor.

4. Any member of the Association or of the Board may waive notice with the same effect as if due notice had been given him.

5. At all meetings of the Association a quorum shall consist of not less than twenty-five (25) members and no business may be validly transacted at a meeting at which less than a quorum

is present; *provided* that any meeting of the Association, whether or not a quorum be present, may be adjourned to a specified time and place by a majority of the members present without notice to the members at large other than the announcement at such meeting.

6. Members may take part and vote in person or by proxy at all meetings of the Association.

ARTICLE V—SECTIONS

1. Any group of not less than ten (10) members of this Association may petition the Board for authority to organize a Section of the Association for the purpose of holding local meetings. The Board shall have power to specify the conditions under which such authority shall be granted.

2. The Association shall not be obligated to pay from its treasury any of the expenses of such Sections except as the Board may provide.

ARTICLE VI—OFFICIAL PUBLICATIONS

1. The Association shall publish an official journal, which shall be sent free to all members of the Association in accordance with Article VII.

2. The Board shall have full control of the publication and sale of the official journal and of all other official publications.

3. There shall be appointed by the Board a body of Associate Editors who shall give assistance in connection with the official journal.

4. The Board shall from time to time, as the need arises, make special provision for the management of any other official publications.

5. The Board shall fix the price of the official journal and of any other official publications of the Association, but in no case shall the journal be sold to non-members for less than the annual dues of individual members.

ARTICLE VII—DUES

1. Members of the Association shall pay an initiation fee of Two Dollars ($2) at the time of election. The Board of Governors may authorize the admission to membership of individuals and classes of applicants without payment of the admission fee.

2. The annual dues of each member shall be Four Dollars ($4), including a subscription to the official journal.

3. All dues shall be payable on the first of January of each year. Should the annual dues of any member remain unpaid beyond a reasonable time, his name shall be dropped from the list after due notice.

4. New members entering the Association after April 1 of any year shall have their dues prorated for the balance of the year, except when they desire to receive the full current volume of the the official journal.

5. Any member who because of age is no longer in active service, who is in good standing at the time of his retirement and who has been a member of the Association for twenty years, may, upon notifying the Secretary of said retirement, be exempt from the payment of dues, with the privilege of obtaining the official journal at an annual cost of one dollar.

ARTICLE VIII—AMENDMENTS TO THE ARTICLES OF ASSOCIATION AND BY-LAWS

1. Changes in the Articles of Association or amendments to the By-Laws may be made at any annual meeting of the Association, or at any adjourned session, thereof, or at any special meeting of the Association called for such purpose, by a two-thirds ($\frac{2}{3}$) vote of those present and entitled to vote; *provided* that due notice concerning such amendment shall have been printed in the official journal, or mailed to each member, at least one (1) month before the date of such meeting.

2. No changes in the Articles of Association shall have legal effect until a certificate thereof, verified by oath of the President and under Seal of the Association, attested by the Secretary-Treasurer, shall be filed in the office of the Secretary of State of the State of Illinois and recorded in the office of the Recorder of Deeds for Cook County, Illinois.

GEORGE BANTA PUBLISHING COMPANY, MENASHA, WISCONSIN